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Boston College

The Graduate School of Arts and Sciences

Department of Economics

ESSAYS IN NETWORK ECONOMICS AND GAME THEORY

a dissertation

by

HI-LIN TAN

submitted in partial fulfillment of the requirements

for the degree of

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ESSAYS IN NETWORK ECONOMICS AND GAME THEORY

– DISSERTATION ABSTRACT –

by

HI-LIN TAN

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RICHARD TRESCH

This dissertation comprises three papers that are concerned with the implications of strategic interactions between a finite set of agents in private goods economies. One form of strategic behavior I consider arises in a social network when the consumption decisions of agents are influenced by those around them. The other form of strategic behavior I consider arises when agents bargain with one another.

The first paper focuses on undirected networks in which consumers care about the average of their neighbors' consumption. The main contribution is to show how social networks affect equilibrium prices. I show that if every consumer has the same number of neighbors, then each consumer's influence on the market is independent of the number of neighbors. Due to the tradeoff between more neighbors responding and less sensitive responses, greater network intensity may not result in greater average influence of all consumers. In addition, I show that a consumer who is central in the network may not have the highest influence on the market because of the need to consider not only the number of neighbors that he has or his distances to other

consumers, but also the number of neighbors that his neighbors have.

The second paper examines strategic consumption in a directed network. The main contribution is to show how directed networks affect equilibrium outcomes. I show how the critical and promising links, and the key players in a social network can be identified. In doing so, I introduce the impact centrality and reaction centrality measures, and show how these measures are used to determine the effects on aggregate centrality of removing any agent from the network, and of removing or adding any directed link.

The third paper considers bargaining under two-sided incomplete information in a market with multiple buyers and sellers, each with either high or low independent private values. I show that there exists a mechanism that guarantees efficient trading outcomes even when gains from trade are uncertain. The main contribution of this paper to show that a large number of traders is not necessary to guarantee efficient trading if there are at least as many sellers as there are buyers, and there is at least one low valuation buyer.

To my parents

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The earliest version of my first paper was written as part of Richard Tresch's course in public economics. I would like to express my gratitude for his extensive comments and suggestions.

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Paper 1

Prices in Networks

Abstract

When there is strategic complementarity of consumption between neighbors in a social network, we find that certain consumers may have a bigger impact than other consumers on the market demand and therefore the equilibrium price. The influence that a particular consumer has on the market demand depends on the network structure and the consumer's location in the network. This analysis may, for example, shed light on the segment of consumers that should be the target of selective advertisements or promotions.

1.1 Introduction

The recent decades have been marked by an increasingly interconnected world due largely to advances in communication and transportation technology. The ubiquitous cellular phone, the television, and the Internet have become almost indispensable to present day living. They have, to an unprecedented extent, enabled interaction among people who are physically separated. Cars and air travel have become much more accessible forms of transportation, while mass transit systems have become common features of urban living throughout the world. Political and economic developments in the European Union, Eastern Europe, and China have resulted in an explosion of emigration and travel. As a result of all these technological and social developments, people are becoming increasingly interconnected both within and between countries.

The consumption of fashion products, ostentatious products, gifts, game tickets, guns and other forms of expenditures tend to be influenced by social or cultural norms in a way that displays a high degree of conformism. For example, our decisions to purchase a vacation package may be influence by whether our friends are also going on the same trip, perhaps because the trip would be more enjoyable with some friends. A household might keep up with the purchases of only those other households it comes into social contact with and not the purchases of the rest of the other households. However, because of the overlap of social circles, households that are distantly connected through the network of neighbors can have an indirect, though somewhat diminished, effect on one another's purchases. It is useful to think of this interaction among consumers occurring within a social network since consumers would typically respond to only those other consumers in their social circle rather than to respond to all other consumers. A network structure allows much richer social interactions, rather than distinctly local or global interactions.

This paper investigates how equilibrium prices are affected by consumers interacting strategically with their neighbors in a social network. In order to abstract from issues of market power and to focus on consumer behavior, we consider an exchange economy comprising many agents endowed with two goods. One of the two goods involves strategic complementarity in consumption between neighbors in the social network. However because no agent has market power over any good, the markets for both goods are competitive. We use the vocabulary of graph theory and network games to describe the strategic interactions occurring within the social network. The intention is neither to explain why strategic complementarity in consumption occurs nor to explain the existence of a social network. Instead, we would like to focus on the consequences of such phenomena, that is, to determine how the network structure affects the market demand and the equilibrium price.

To examine the impact of a particular agent's increased demand for a good, we analyze the response of price to a change in the endowment of the agent under various network structures. We find that the increase in supply of a good may in fact raise its relative price. This effect depends on the structure of the network and the source of the supply increase. This result suggests that goods that involve strategic complementarity in consumption between neighbors may become more valuable even as they become more abundant.

The idea that a consumer's demand depends on the demands of other consumers has been explored variedly in the literature. For example, a consumer's demand for a good may depend on the aggregate demand or network externalities, that is, the number of consumers consuming [Duesenberry (1949); Leibenstein (1950); Becker (1991); Karni and Levin (1994); Corneo and Jeanne (1997); Grilo, Shy, and Thisse (2001); Amaldoss and Jain (2005)]. Alternatively, a consumer's demand may be affected by the demands of other consumers because his utility depends on how his consumption of the good ranks against that of all other consumers [Frank (1985); Hopkins and Kornienko (2004); Hopkins and Kornienko (2006)]. There are also models that incorporate both local and global interactions but treat the effects distinctly [Glaeser and Scheinkman (2002); Horst and Scheinkman (2005)]. However, these forms of social interactions do not take into consideration how consumers not directly connected can be influenced indirectly and mutually by other consumers via a network of social relations. There is a growing literature on network formation and network games, which show how the structure of networks affects equilibrium outcomes [Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2006)]. Discrete choice interactions have been analyzed in a network structure but not with reference to the price mechanism [Ioannides (2006)].

The rest of this paper is organized as follows. Section 1.2 presents the general model, which includes a discussion of a few prominent network structures. Section 1.3 examines the equilibrium in the minimum consumption model under the various network structures. Section 1.4 concludes. An appendix contains the proofs.

1.2 The General Model

1.2.1 Definitions

Given a set of agents $N = \{1, \ldots, n\}$, an undirected *network* g is a set of pairs of agents linked to each other. For any pair of agents i and j, $ij \in g$ indicates that i and j are linked in the network g.

A pair of agents are *neighbors* in a network g if and only if they are linked in the network g. The set of agents with at least one neighbor in the network g is $N(g) = \{j \in N : \exists \quad ij \in g\}$. The set of neighbors of agent i in the network g is $N_i(g) = \{j \in N : ij \in g\}$. The *degree* of agent i is $n_i(g) = |N_i(g)|$, the number of neighbors that agent i has in network g. Assume that every agent has at least one neighbor in the network g so that $N(g) = N$ and $N_i(g) \neq \{\phi\}$.

A *path* in the network g connecting agents i and j is a sequence of distinct neighbors $i_1, \ldots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \ldots, K-1\}$ with $i_1 = i$ and $i_K = j$. The *length* of a path connecting agents i and j is the number of links connecting agents i and j on that path.

A network is *connected* if there exists a path connecting any agent to any other agent in the network. A network $g' \subset g$ is a *component* of network g if it is a maximal connected subnetwork of network g. That is,

(a) if $i \in N(g')$, $j \in N(g')$, and $j \neq i$, then there exists a path in g' connecting i and j, and

(b) if $i \in N(g')$, $j \in N(g)$, $j \neq i$, and $ij \in g$, then $ij \in g'$.

The set of components of network g is $C(g)$, so that $g = \bigcup_{g' \in C(g)} g'$. Since neighbors are in the same component of a network, the set of neighbors of agent i in the component g' of the network g is equivalent to the set of neighbors of agent i in the network g. That is, $N_i(g') = N_i(g)$.

The *distance* between any pair of agents i and j in the same component g' is $d_{ij}(g')$, the length of the shortest path between the pair of agents. For any integer $k \geq 1$, the set of all other agents that are connected to agent i by a distance of k is $N_i^k(g') = \{j \in N(g') : j \neq i, d_{ij}(g') = k\}$. Hence, $N_i^1(g') = N_i(g')$. The cardinality of $N_i^k(g)$ is $n_i^k(g) = |N_i^k(g)|$. The *eccentricity* of agent i is $\epsilon_i(g') = \max_{j \in N(g')} d_{ij}(g')$, the maximum distance between agent i and any other agent in the same component g'.

The *radius* of a component is $\underline{d}(g') = \min_{ij \in g'} d_{ij}(g')$, the minimum eccentricity of any agent in the component. The *diameter* of a component is $\bar{d}(g') = \max_{ij \in g'} d_{ij}(g')$, the maximum eccentricity of any agent in the component. The *closeness* of an agent i in the component is $c_i(g') = \frac{1}{\sum_{j \in N(g') \setminus \{i\}} d_{ij}(g')}$, the reciprocal of the sum of distances to all other agents in the component. An agent is *central* in a component if its eccentricity is equal to the radius of the component. The *center* of a component is the set of all central agents. An agent is *peripheral* in a component if its eccentricity is equal to the diameter of the component.

1.2.2 Influence

There are two goods - 1 and 2. x_ℓ^i denotes agent i's consumption of good ℓ. ω_ℓ^i denotes agent i's endowment of good ℓ. Both goods are traded throughout the economy so no agent has market power over any good. The price of good 1, which is the numeraire, is normalized to one and so the price of good 2, p, is also the price of good 2 relative to good 1. The wealth level of agent i, $m^i = \omega_1^i + p\omega_2^i$, is endogenously determined by the equilibrium price and the pattern of endowments.

Agents have identical, continuous, strictly convex, and strongly monotone preferences over goods 1 and 2. Hence, each agent i's preferences can be represented by a strictly quasiconcave and twice continuously differentiable utility function $u^i(x_1^i, x_2^i, x_2^{N_i(g)})$, where $x_2^{N_i(g)}$ is the vector of good 2 consumptions by each of the agents in the set of agents $N_i(g)$. There is no restriction on whether the good 2 consumption of each neighbor is a positive or negative externality, that is, there is no restriction on the sign of $u^i_{x_2^j}$ per se, where $j \in N_i(g)$. For any given level of good 2 consumption by each of its neighbors, each agent chooses its consumption of goods 1 and 2 to maximize its utility subject to its budget constraint. Formally,

$$\forall \quad i \in N : \quad \max_{\{x_1^i, x_2^i\}} u^i(x_1^i, x_2^i, x_2^{N_i(g)}) \quad \text{s.t.} \quad x_1^i + px_2^i = m^i = \omega_1^i + p\omega_2^i$$

There is strategic complementarity in the consumption of good 2 between each pair of neighbors in that an agent would increase its consumption of good 2 if its neighbor does so, holding all other factors, including the price, constant. Formally,

$$\forall \quad i \in N, \quad j \in N_i(g) : \quad (\frac{\partial x_2^i}{\partial x_2^j})_{x_2, p, m^i} > 0$$

Since we seek to determine the effect of a particular agent's increased demand from a change in the agent's endowment, we assume that both goods are normal. Formally,

$$\forall \quad i \in N: \quad (\frac{\partial x_\ell^i}{\partial m^i})_{x_2,p} > 0$$

The above two conditions depend on ordinal properties of the utility function because the best response function, which is implicitly determined by the first order conditions, is invariant to a monotonic transformation of the utility function. The best response correspondences are in fact best response functions because the utility functions are strictly quasiconcave.

Definition 1 (Nash-Walrasian Equilibrium) (x_1^N, x_2^N, p) *is a* Nash-Walrasian Equilibrium *if it satisfies every agent's best response function and budget constraint, and the market for good* 2 *clears. That is,*

$$\begin{cases} \forall \quad i \in N: \quad u^i_{x_2^i} = pu^i_{x_1^i} \quad \textit{which implicitly determines} \quad x_2^i = x_2^i(x_2^{N_i(g)}, p, m^i) \\ \\ \forall \quad i \in N: \quad x_1^i + px_2^i = m^i = \omega_1^i + p\omega_2^i \\ \\ \sum_{i\in N} x_2^i = \sum_{i \in N} \omega_2^i \end{cases}$$

Lemma 1 (Demand Correspondences) *Given every agent's best response function and budget constraint, each agent's demand for each good is a correspondence of the relative price of the two goods and the wealth levels of every agent in its component of the network. That is,*

$$\forall \quad g' \in C(g), \quad i \in N(g'): \quad x_1^i \in x_1^i(p, m^{N(g')}) \textit{ and } x_2^i \in x_2^i(p, m^{N(g')})$$

Proof.

$$\forall \quad g' \in C(g), \quad i \in N(g'): \quad N_i(g) = N_i(g') \quad \text{and} \quad x_2^i = x_2^i(x_2^{N_i(g)}, p, m^i)$$

$$\Rightarrow x_2^i = x_2^i(x_2^{N_i(g')}, p, m^i)$$

$$\Rightarrow x_2^i \in x_2^i(x_2^{N_i^2(g')}, p, m^{\{i\} \bigcup N_i(g')})$$

$$\Rightarrow x_2^i \in x_2^i(x_2^{N_i^3(g')}, p, m^{\{i\} \bigcup N_i(g') \bigcup N_i^2(g')})$$

$$\vdots$$

$$\Rightarrow x_2^i \in x_2^i(x_2^{N_i^{\epsilon_i}(g')}, p, m^{\{i\} \bigcup N_i(g') \bigcup N_i^2(g') \bigcup \cdots \bigcup N_i^{\epsilon_i - 1}(g')})$$

$$\Rightarrow x_2^i \in x_2^i(p, m^{\{i\} \bigcup N_i(g') \bigcup N_i^2(g') \bigcup \cdots \bigcup N_i^{\epsilon_i}(g')})$$

$$\Rightarrow x_2^i \in x_2^i(p, m^{\{i\} \bigcup (\bigcup_{k=1}^{\epsilon_i(g')} N_i^k(g'))})$$

$$\Rightarrow x_2^i \in x_2^i(p, m^{N(g')})$$

$$\Rightarrow x_1^i \in x_1^i(p, m^{N(g')}). \quad \blacksquare$$

As long as there is a path connecting a pair of agents, their demands and therefore their incomes would affect each other's demand. Whether the demand correspondences are in fact demand functions depends on the ordinal curvature properties of the best response functions and their upper and lower bounds [Randon (2004)].

Without loss of generality, we can examine the effect of an increase in the endowment of good 2 held by an agent by a perturbation of ω_2^1, the endowment of good 2 held by agent 1.

Proposition 1 (Equilibrium Price) *Given a downward-sloping aggregate demand for good 2, its price is increasing in the endowment of an agent's endowment of the good if and only if any resulting increase in aggregate demand exceeds the increase in endowment.*

Proof.

$$\begin{cases} \sum_{\ell=1}^{2} u^i_{x^i_2 x^i_\ell} dx^i_\ell + \sum_{j\in N_i(g)} u^i_{x^i_2 x^j_2} dx^j_2 = p(\sum_{\ell=1}^{2} u^i_{x^i_1 x^i_\ell} dx^i_\ell + \sum_{j\in N_i(g)} u^i_{x^i_1 x^j_2} dx^j_2) + u^i_{x^i_1} dp \\ \\ dx^1_1 + pdx^1_2 + x^1_2 dp = dm^1 = pd\omega^1_2 + \omega^1_2 dp \\ \\ \forall \quad i\backslash 1: \quad dx^i_1 + pdx^i_2 + x^i_2 dp = dm^i = \omega^i_2 dp \\ \\ \sum_{i\in N} dx^i_2 = d\omega^1_2 \end{cases}$$

$$\Rightarrow \begin{cases} \sum_{\ell=1}^{2} (u^i_{x^i_2 x^i_\ell} - pu^i_{x^i_1 x^i_\ell}) dx^i_\ell + \sum_{j\in N_i(g)} (u^i_{x^i_2 x^j_2} - pu^i_{x^i_1 x^j_2}) dx^j_2 - u^i_{x^i_1} dp = 0 \\ \\ dx^1_1 = pd\omega^1_2 + (\omega^1_2 - x^1_2)dp - pdx^1_2 \\ \\ \forall \quad i\backslash 1: \quad dx^i_1 = (\omega^i_2 - x^i_2)dp - pdx^i_2 \\ \\ \sum_{i\in N} dx^i_2 = d\omega^1_2 \end{cases}$$

$$\Rightarrow \begin{cases} [u^1_{x^1_2 x^1_2} - pu^1_{x^1_1 x^1_2} - p(u^1_{x^1_2 x^1_1} - pu^1_{x^1_1 x^1_1})]dx^1_2 + \sum_{j\in N_1(g)} (u^1_{x^1_2 x^j_2} - pu^1_{x^1_1 x^j_2}) dx^j_2 \\ +[(\omega^1_2 - x^1_2)(u^1_{x^1_2 x^1_1} - pu^1_{x^1_1 x^1_1}) - u^1_{x^1_1}]dp = -p(u^1_{x^1_2 x^1_1} - pu^1_{x^1_1 x^1_1})d\omega^1_2 \\ \\ \forall \quad i\backslash 1: \quad [u^i_{x^i_2 x^i_2} - pu^i_{x^i_1 x^i_2} - p(u^i_{x^1_2 x^i_1} - pu^i_{x^i_1 x^i_1})]dx^i_2 + \sum_{j\in N_i(g)} (u^i_{x^i_2 x^j_2} - pu^i_{x^i_1 x^j_2}) dx^j_2 \\ +[(\omega^i_2 - x^i_2)(u^i_{x^i_2 x^i_1} - pu^i_{x^i_1 x^i_1}) - u^i_{x^i_1}]dp = 0 \\ \\ \sum_{i\in N} dx^i_2 = d\omega^1_2 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{dx_2^1}{d\omega_2^1} + \sum_{j\in N_1(g)} \frac{u^1_{x_2^1x_2^j} - pu^1_{x_1^1x_2^j}}{u^1_{x_2^1x_2^1} - pu^1_{x_1^1x_2^1} - p(u^1_{x_2^1x_1^1} - pu^1_{x_1^1x_1^1})} \frac{dx_2^j}{d\omega_2^1} \\ + \frac{(\omega_2^1 - x_2^1)(u^1_{x_2^1x_1^1} - pu^1_{x_1^1x_1^1}) - u^1_{x_1^1}}{u^1_{x_2^1x_2^1} - pu^1_{x_1^1x_2^1} - p(u^1_{x_2^1x_1^1} - pu^1_{x_1^1x_1^1})} \frac{dp}{d\omega_2^1} = -p \frac{u^1_{x_2^1x_1^1} - pu^1_{x_1^1x_1^1}}{u^1_{x_2^1x_2^1} - pu^1_{x_1^1x_2^1} - p(u^1_{x_2^1x_1^1} - pu^1_{x_1^1x_1^1})} \\ \\ \forall \quad i\backslash 1: \quad \frac{dx_2^i}{d\omega_2^1} + \sum_{j\in N_i(g)} \frac{u^i_{x_2^ix_2^j} - pu^i_{x_1^ix_2^j}}{u^i_{x_2^ix_2^i} - pu^i_{x_1^ix_2^i} - p(u^i_{x_2^ix_1^i} - pu^i_{x_1^ix_1^i})} \frac{dx_2^j}{d\omega_2^1} \\ + \frac{(\omega_2^i - x_2^i)(u^i_{x_2^ix_1^i} - pu^i_{x_1^ix_1^i}) - u^i_{x_1^i}}{u^i_{x_2^ix_2^i} - pu^i_{x_1^ix_2^i} - p(u^i_{x_2^ix_1^i} - pu^i_{x_1^ix_1^i})} \frac{dp}{d\omega_2^1} = 0 \\ \\ \sum_{i\in N} \frac{dx_2^i}{d\omega_2^1} = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{dx_2^1}{d\omega_2^1} - \sum_{j\in N_1(g)} (\frac{\partial x_2^1}{\partial x_2^j})_{x_2,p,m^1} \frac{dx_2^j}{d\omega_2^1} - (\frac{\partial x_2^1}{\partial p})_{x_2} \frac{dp}{d\omega_2^1} = p(\frac{\partial x_2^1}{\partial m^1})_{x_2,p} \\ \\ \forall \quad i\backslash 1: \quad \frac{dx_2^i}{d\omega_2^1} - \sum_{j\in N_i(g)} (\frac{\partial x_2^i}{\partial x_2^j})_{x_2,p,m^i} \frac{dx_2^j}{d\omega_2^1} - (\frac{\partial x_2^i}{\partial p})_{x_2} \frac{dp}{d\omega_2^1} = 0 \\ \\ \sum_{i\in N} \frac{dx_2^i}{d\omega_2^1} = 1 \end{cases}$$

where $(\frac{\partial x_2^i}{\partial p})_{x_2} = (\frac{\partial x_2^i}{\partial p})_{x_2,u^i} + (\frac{\partial x_2^i}{\partial m^i})_{x_2,p}(\omega_2^i - x_2^i) = (\frac{\partial x_2^i}{\partial p})_{x_2,m^i} + (\frac{\partial x_2^i}{\partial m^i})_{x_2,p}\omega_2^i$

which decomposes the substitution and income effects.

$\Rightarrow Ax = b$

where

$$A = \begin{bmatrix} 1 & -(\frac{\partial x_2^1}{\partial x_2^2})_{x_2,p,m^1} & \dots & -(\frac{\partial x_2^1}{\partial x_2^n})_{x_2,p,m^1} & -(\frac{\partial x_2^1}{\partial p})_{x_2} \\ -(\frac{\partial x_2^2}{\partial x_2^1})_{x_2,p,m^2} & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & -(\frac{\partial x_2^{n-1}}{\partial x_2^n})_{x_2,p,m^{n-1}} & \vdots \\ -(\frac{\partial x_2^n}{\partial x_2^1})_{x_2,p,m^n} & \dots & -(\frac{\partial x_2^n}{\partial x_2^{n-1}})_{x_2,p,m^n} & 1 & -(\frac{\partial x_2^n}{\partial p})_{x_2} \\ 1 & \dots & \dots & 1 & 0 \end{bmatrix}$$

with $\forall \quad j \notin N_i(g): \quad (\frac{\partial x_2^i}{\partial x_2^j})_{x_2,p,m^i} = 0$

$$\text{and } x = \begin{bmatrix} \frac{dx_2^1}{d\omega_2^1} \\ \vdots \\ \frac{dx_2^n}{d\omega_2^1} \\ \frac{dp}{d\omega_2^1} \end{bmatrix}, \quad b = \begin{bmatrix} p(\frac{\partial x_2^1}{\partial m^1})_{x_2,p} \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow \frac{dp}{d\omega_2^1} = \frac{|A_{n+1}|}{|A|}$

where A_{n+1} is the matrix formed by replacing the $(n+1)^{th}$ column of matrix A with the column vector b.

$\Rightarrow \frac{dp}{d\omega_2^1} = \frac{\frac{|A_{n+1}|}{M_{n+1,n+1}}}{\frac{|A|}{M_{n+1,n+1}}}$

where the minor $M_{n+1,n+1}$ of matrix A is the determinant of the matrix formed by removing row $n+1$ and column $n+1$ of matrix A.

$\Rightarrow \frac{dp}{d\omega_2^1} = -\frac{\sum\limits_{i \in N} (\frac{\partial x_2^i}{\partial \omega_2^1})_p - 1}{\sum\limits_{i \in N} \frac{dx_2^i}{dp}}. \quad \blacksquare$

This proposition is a generalization of that which would emerge if preferences were independent. It holds for all network structures.

Definition 2 (Influence) *Agent i's* influence *on aggregate demand is equal to* $\frac{\partial p}{\partial \omega_2^i}$, *the marginal effect of an increase in agent i's endowment of good 2 on the price of good 2.*

Influence is a measure of the centrality of an agent in the network. We would like to determine which agents in the economy have a higher influence on aggregate demand and how the network structure affects their influence.

1.2.3 Network Structures

Network structures can be categorized into those which are regular and those which are irregular. Since there are very many possible regular and irregular networks, we focus our attention on a few prominent network structures. The aim is to show how aggregate demand and prices are affected by whether the network is regular or not, and by the particular network structure in question. Within the class of regular networks, we consider the complete network and the ring network. Within the class of irregular networks, we consider the star network and the line network.

Regular Networks

A network is *regular* if all agents have the same number of neighbors. That is,

$$\forall \quad i \in N : \quad n_i(g) = r \quad \text{where } r \text{ is a non-negative integer}$$

A network is *complete* if all agents are linked to one another. Hence, for any agent, every other agent is a neighbor of the agent. This network structure in effect describes the case of global interactions. Formally,

$$g = \{ij : i \in N, j \in N, j \neq i\}$$

$$\Rightarrow \begin{cases} \forall \quad i \in N : \quad \epsilon_i(g) = 1 \\ \forall \quad i \in N : \quad N_i(g) = \{j \in N : j \neq i\} \\ \forall \quad i \in N : \quad n_i(g) = n - 1 \end{cases}$$

A network is a *ring* if there is a single cycle through all agents. Hence every agent has a pair of neighbors. Without loss of generality, assume that agent 1 is linked to agent 2, which is, in turn, linked to agent 3, and so on until agent n. In addition, agent n is linked to agent 1, thereby completing the single cycle. Formally,

$$g = \{ij : i \in N, j \in N, j = i \pm 1\} \bigcup \{1n\}$$

$$\forall \quad i \in N : \quad \epsilon_i(g) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

$\forall \quad k \leq \frac{n-1}{2}$:

$$N_i^k(g) = \begin{cases} \{j \in N : j = i \pm k\} \bigcup \{i + (n-k)\} & \text{if} \quad i = 1, \ldots, k \\ \{j \in N : j = i \pm k\} & \text{if} \quad i = k+1, \ldots, n-k \\ \{j \in N : j = i \pm k\} \bigcup \{i - (n-k)\} & \text{if} \quad i = n-k+1, \ldots, n \end{cases}$$

$\forall \quad k = \frac{n}{2}$:

$$N_i^k(g) = \begin{cases} \{j \in N : j = i \pm k\} \bigcup \{i + (n-k)\} & \text{if} \quad i = 1, \ldots, k \\ \{j \in N : j = i \pm k\} \bigcup \{i - (n-k)\} & \text{if} \quad i = n-k+1, \ldots, n \end{cases}$$

$$\forall \quad i \in N : \quad n_i(g) = 2$$

Irregular Networks

Turning to the class of irregular networks, we first consider the star network, which is a maximally centralized network. A network is a *star* if it has one central agent and all other agents are linked only to the central agent. Without loss of generality, assume that the central agent is agent 1. Formally,

$$g = \{ij : i = 1, j \in N \backslash \{1\}\}$$

$$\forall \quad i \in N: \quad \epsilon_i(g) = \begin{cases} 1 & \text{if} \quad i = 1 \\ 2 & \text{if} \quad i \in N\backslash\{1\} \end{cases}$$

$$N_i(g) = \begin{cases} N\backslash\{1\} & \text{if} \quad i = 1 \\ \{1\} & \text{if} \quad i \in N\backslash\{1\} \end{cases}$$

$$N_i^2(g) = \begin{cases} \{\phi\} & \text{if} \quad i = 1 \\ N\backslash\{1, i\} & \text{if} \quad i \in N\backslash\{1\} \end{cases}$$

$$n_i(g) = \begin{cases} n-1 & \text{if} \quad i = 1 \\ 1 & \text{if} \quad i \in N\backslash\{1\} \end{cases}$$

A network is a *line* if all agents form a single acyclic path. Without loss of generality, assume that the path connects agent 1 to agent n through all other agents. It would identical to the ring but for the absence of a link between agent 1 and agent n. Formally,

$$g = \{ij : i \in N, j \in N, j = i \pm 1\}$$

$$\forall \quad i \in N: \quad \epsilon_i(g) = \max\{n - i, i - 1\}$$

$$\forall \quad i \in N, \quad \forall \quad k \geq 1: \quad N_i^k(g) = \{j \in N : j = i \pm k\}$$

$$n_i(g) = \begin{cases} 1 & \text{if} \quad i \in \{1, n\} \\ 2 & \text{if} \quad i \in N\backslash\{1, n\} \end{cases}$$

1.3 The Minimum Consumption Model

In order to impose more structure on each agent's best response function, we consider a specific model that introduces two sets of assumptions. First, assume that agents have identical Cobb-Douglas preferences symmetric in both goods. This assumption is consistent with the requirement that both goods are normal. Second, assume that every agent needs to consume an amount of good 2 that exceeds the fraction $\alpha \in [0,1)$ of the average good 2 consumption of its neighbors. This is similar to the Stone-Geary utility [Stone (1954); Geary (1950-1951)], where consumption needs to exceed a certain parameterized minimum level, except that the minimum level of consumption here is not parameterized but is endogenously determined. Formally,

$$\forall \quad i \in N: \quad \max_{\{x_1^i, x_2^i\}} x_1^i \Big(x_2^i - \alpha \frac{1}{n_i(g)} \sum_{j \in N_i(g)} x_2^j\Big) \quad \text{s.t.} \qquad x_1^i + p x_2^i = m^i = \omega_1^i + p\omega_2^i$$

As a result, the best response function of each agent is linear in the good 2 consumption of each of its neighbors:

$$\forall \quad i \in N: \quad x_2^i = \frac{1}{2p}\Big(m^i + \alpha p \frac{1}{n_i(g)} \sum_{j \in N_i(g)} x_2^j\Big)$$

$$\Rightarrow \forall \quad i \in N, \quad j \in N_i(g): \quad \Big(\frac{\partial x_2^i}{\partial m^i}\Big)_{x_2,p} = \frac{1}{2p}, \quad \Big(\frac{\partial x_2^i}{\partial x_2^j}\Big)_{x_2,p,m^i} = \frac{\alpha}{2n_i(g)}$$

This further implies that, holding the price and the demands of other agents constant, an agent would increase its consumption of good 2 by half the increase of its endowment of the good. Since the agent's increase in demand is independent of its own wealth and the consumption levels of the agent's neighbors, this ensures that, before taking into consideration the reaction of other agents, every agent responds symmetrically to an increase in its own endowment. Another implication is that, holding all other factors constant, the response of an agent to the good 2 consumption of one of its neighbors is decreasing in the number of neighbors that the agent has. This also simplifies the situation because the response is hence independent of the price, the agent's wealth, and the consumption levels of the agent's neighbors.

1.3.1 Regular Networks

Proposition 2 (Regular Networks) *In the minimum consumption model with a regular network, the price of good* 2 *is decreasing in the endowment of good* 2 *held by any agent and independent of the number of neighbors that every agent has.*

Proof.

$$\forall \quad i \in N: \quad x_2^i = \frac{1}{2p}(m^i + \alpha p \frac{1}{r} \sum_{j \in N_i(g)} x_2^j)$$

$$\sum_{i \in N} m^i + \alpha p \frac{1}{r} \sum_{i \in N} \sum_{j \in N_i(g)} x_2^j = 2p \sum_{i \in N} \omega_2^i$$

$$\sum_{i \in N} m^i + \alpha p \frac{r}{r} \sum_{i \in N} x_2^i = 2p \sum_{i \in N} \omega_2^i$$

$$\sum_{i \in N} \omega_1^i + p \sum_{i \in N} \omega_2^i + \alpha p \sum_{i \in N} \omega_2^i = 2p \sum_{i \in N} \omega_2^i$$

$$p = \frac{1}{1-\alpha} \frac{\sum_{i \in N} \omega_1^i}{\sum_{i \in N} \omega_2^i}$$

$$\forall \quad i \in N: \quad \frac{dp}{d\omega_2^i} < 0. \quad \blacksquare$$

In a regular network, since every agent has the same number of neighbors, an agent i's response to the an increased demand by any neighbor $j \in N_i(g)$, $(\frac{\partial x_2^i}{\partial x_2^j})_{x_2,p,m^i}$, is the same for every agent. Hence, the response of aggregate demand to an increase in the endowment of good 2 is independent of the number of agents in the economy because the multiplier $\frac{1}{1-\sum_{j \in N_i(g)} (\frac{\partial x_2^i}{\partial x_2^j})_{x_2,p,m^i}}$ is independent of the number of neighbors that every agent has. The greater the number of neighbors, the greater the number of neighbors responding to the increase in demand for good 2 by agent i. However, at the same time, every agent responds proportionately less to the increase in demand because every agent has more neighbors. The two opposing effects on the multiplier exactly counteract each other.

Corollary 1 (Regular Networks) *In the minimum consumption model with a regular network, every agent has the same level of influence on aggregate demand.*

This follows from Proposition 2.

Since no agent has a higher influence than any other agent in a regular network, this suggests that no segment of consumers would be favored over any other in being the target of advertisements or promotions.

Complete Networks

Since complete networks are regular networks, according to Proposition 2, the price of good 2 is decreasing in the endowment of good 2.

From the best response function of every agent, we have:

$$Ax = b$$

where $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ with $a_{ij} = \begin{cases} 2 & \text{if} \quad i = j \\ -\frac{\alpha}{n-1} & \text{if} \quad i \neq j \end{cases}$

$$x = \begin{bmatrix} x_2^1 \\ \vdots \\ x_2^n \end{bmatrix}, \quad b = \begin{bmatrix} \frac{m^1}{p} \\ \vdots \\ \frac{m^n}{p} \end{bmatrix}$$

Solving for x, we obtain the demand function for each agent:

$$\forall \quad i \in N: \quad x_2^i = \frac{1}{p}\left[\frac{2(n-1)-(n-2)\alpha}{4(n-1)-2(n-2)\alpha-\alpha^2} m^i + \frac{\alpha}{4(n-1)-2(n-2)\alpha-\alpha^2} \sum_{j \in N_i(g)} m^j\right]$$

In the absence of the strategic complementarity in consumption of good 2 between each pair of neighbors, that is, if $\alpha = 0$, then each agent's demand for each good depends only on its own income. However, with the strategic complementarity between neighbors, each agent's consumption of each good depends not just on its own income but also on the income of all other agents.

From the demand functions, we have:

$$\sum_{i \in N} \left(\frac{\partial x_2^i}{\partial \omega_2^1}\right)_p = \frac{1}{2-\alpha} < 1.$$

Hence, the increase in aggregate demand for good 2 is less than the increase in endowment of the good, and the extent of which is independent of the number of agents.

Ring Networks

Since ring networks are regular networks, according to Proposition 2, the price of good 2 is decreasing in the endowment of good 2.

From the best response function of every agent, we have:

$$Ax = b$$

where $A = \begin{bmatrix} 2 & -\frac{\alpha}{2} & 0 & \dots & 0 & -\frac{\alpha}{2} \\ -\frac{\alpha}{2} & 2 & -\frac{\alpha}{2} & 0 & \dots & 0 \\ 0 & -\frac{\alpha}{2} & 2 & -\frac{\alpha}{2} & \ddots & \vdots \\ \vdots & 0 & -\frac{\alpha}{2} & 2 & \ddots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & -\frac{\alpha}{2} \\ -\frac{\alpha}{2} & 0 & \dots & 0 & -\frac{\alpha}{2} & 2 \end{bmatrix} \in \mathbb{R}^{n\times n}$, a circulant matrix,

$$x = \begin{bmatrix} x_2^1 \\ \vdots \\ x_2^n \end{bmatrix}, \quad b = \begin{bmatrix} \frac{m^1}{p} \\ \vdots \\ \frac{m^n}{p} \end{bmatrix}$$

Solving for x, we obtain the demand function for each agent:

$$\forall \quad i \in N : \quad x_2^i = \frac{1}{p}[\frac{C_{1,1}}{|A|}m^i + \sum_{k=1}^{\epsilon_i(g)} \frac{C_{k+1,1}}{|A|} \sum_{j \in N_i^k(g)} m^j]$$

where the cofactor $C_{i,j}$ of matrix A is $(-1)^{i+j}$ times the determinant of the matrix formed by removing row i and column j of matrix A.

An agent's sensitivity of demand to another agent's income depends on the distance between the pair of agents. The greater the distance between the pair of agents, the less sensitive an agent's demand is to the other agent.

1.3.2 Irregular Networks

Star Networks

Turning to irregular networks, we begin with star networks. From the best response function of every agent, we have:

$$Ax = b$$

where $A = \begin{bmatrix} 2 & -\frac{\alpha}{n-1} & \dots & \dots & -\frac{\alpha}{n-1} \\ -\alpha & 2 & 0 & \dots & 0 \\ \vdots & 0 & 2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -\alpha & 0 & \dots & 0 & 2 \end{bmatrix} \in \mathbb{R}^{n \times n}$, $x = \begin{bmatrix} x_2^1 \\ \vdots \\ x_2^n \end{bmatrix}$, $b = \begin{bmatrix} \frac{m^1}{p} \\ \vdots \\ \frac{m^n}{p} \end{bmatrix}$.

Solving for x, we obtain the demand function for each agent:

$$x_2^i = \begin{cases} \frac{1}{p}\left(\frac{2}{4-\alpha^2}m^i + \frac{\alpha}{4-\alpha^2}\frac{1}{n-1}\sum_{j \in N \setminus \{1\}} m^j\right) & \text{if} \quad i = 1 \\ \frac{1}{p}\left(\frac{4(n-1)-(n-2)\alpha^2}{4-\alpha^2}\frac{1}{2(n-1)}m^i + \frac{\alpha}{4-\alpha^2}m^1 + \frac{\alpha^2}{4-\alpha^2}\frac{1}{2(n-1)}\sum_{k \in N_i^2(g)} m^k\right) & \text{if} \quad i \in N \setminus \{1\} \end{cases}$$

A peripheral agent's demand for good 2 is more sensitive to the central agent's income than the income of other peripheral agents because the central agent is a neighbor whereas the other peripheral agents are connected by a distance of two.

Proposition 3 (Star Network) *In the minimum consumption model with a star network, the price of good* 2 *is increasing in the endowment of good* 2 *held by the central agent if the fraction* α *is large enough.*

Proof. Combining every agent's demand function for good 2 and the market clearing condition for good 2, we can solve for the price of good 2:

$$p = \frac{(n-1)[2+(n-1)\alpha]\omega_1^1+[2(n-1)+\alpha]\sum_{j\in N_1(g)}\omega_1^j}{(n-1)[2-(n-1)\alpha-\alpha^2]\omega_2^1+[2(n-1)-\alpha-(n-1)\alpha^2]\sum_{j\in N_1(g)}\omega_2^j}$$

$$\frac{dp}{d\omega_2^1} > 0 \quad \text{if} \quad \alpha > \frac{-(n-1)+\sqrt{(n-1)^2+8}}{2}. \quad \blacksquare$$

The central agent has all other agents as neighbors and so has all other agents responding to its increase in demand for good 2. In addition, these other agents respond sensitively to the central agent's increased demand for good 2 because the central agent is their only neighbor. As the number of peripheral agents approaches infinity, the number of neighbors the central agent has approaches infinity and so the critical value of α decreases and approaches zero.

In contrast, the price of good 2 is decreasing in the endowment of good 2 held by a peripheral agent. This is because a peripheral agent has only one agent, the central agent, responding to its increase in demand. In addition, the central agent does not respond sensitively to the peripheral agent's demand increase because it has many other neighbors as well. As the number of peripheral agents increases, the number of neighbors for each peripheral agent remains constant but the number of agents connected to each peripheral agent by a distance of two increases.

Corollary 2 (Star Network) *In the minimum consumption model with a star network, the central agent has a higher influence than all other agents.*

This follows from Proposition 3.

Since the central agent also has the highest closeness, the analysis of the minimum consumption model with a star network seems to suggest that an agent's closeness is the key to its relative influence over market demand in the network. However, we shall see, in the analysis of line networks, that closeness does not in fact determine an agent's influence.

Line Networks

From the best response function of every agent, we have:

$$Ax = b$$

where $A = \begin{bmatrix} 2 & -\alpha & 0 & \dots & \dots & 0 \\ -\frac{\alpha}{2} & 2 & -\frac{\alpha}{2} & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -\frac{\alpha}{2} & 2 & -\frac{\alpha}{2} \\ 0 & \dots & \dots & 0 & -\alpha & 2 \end{bmatrix} \in \mathbb{R}^{n\times n}$, a tridiagonal matrix,

$$x = \begin{bmatrix} x_2^1 \\ \vdots \\ x_2^n \end{bmatrix}, \quad b = \begin{bmatrix} \frac{m^1}{p} \\ \vdots \\ \frac{m^n}{p} \end{bmatrix}.$$

Proposition 4 (Line Network) *In the minimum consumption model with a line network, the price of good* 2 *is increasing in the endowment of good* 2 *held by a non-peripheral agent if the fraction* α *is large enough.*

Proof. From the best response of every agent and the market clearing condition, the equilibrium price is implicitly determined by:

$$-\begin{vmatrix} A & b \\ \iota^T & 0 \end{vmatrix} = |A| \sum_{i\in N} \omega_2^i$$

where ι is an n-vector of ones.

Assume n=3

$$p = \frac{(4+\alpha)(\omega_1^1+\omega_1^3)+(4+4\alpha)\omega_1^2}{(4-\alpha-2\alpha^2)(\omega_2^1+\omega_2^3)+(4-4\alpha-2\alpha^2)\omega_2^2}$$

$$\text{For} \quad i=2: \quad \frac{\partial p}{\partial \omega_2^i} > 0 \quad \text{if} \quad 4-4\alpha-2\alpha^2 < 0 \iff \alpha > 0.732$$

Assume n=4

Agent i's demand function for good 2:

$$x_2^i = \begin{cases} \frac{1}{p}\left[\frac{(32-6\alpha^2)m^i+\alpha(16-2\alpha^2)m^{N_i(g)}+4\alpha^2 m^{N_i^2(g)}+\alpha^3 m^{N_i^3(g)}}{64-20\alpha^2+\alpha^4}\right] & \text{if} \quad i \in \{1,n\} \\ \frac{1}{p}\left[\frac{(32-4\alpha^2)m^i+\alpha(8-\alpha^2)m^{i-1}+8\alpha m^{i+1}+2\alpha^2 m^{i+2}}{64-20\alpha^2+\alpha^4}\right] & \text{if} \quad i=2 \\ \frac{1}{p}\left[\frac{(32-4\alpha^2)m^i+\alpha(8-\alpha^2)m^{i+1}+8\alpha m^{i-1}+2\alpha^2 m^{i-2}}{64-20\alpha^2+\alpha^4}\right] & \text{if} \quad i=3 \end{cases}$$

Although non-peripheral agents 2 and $(n-1)$, which are the agents next to the peripheral agents, each have a pair of neighbors, the income of their peripheral neighbor has a smaller marginal effect on their demand for good 2 than the income of their non-peripheral neighbor.

$$p = \frac{(8+2\alpha-\alpha^2)(\omega_1^1+\omega_1^4)+(8+6\alpha-\frac{1}{2}\alpha^3)(\omega_1^2+\omega_1^3)}{(8-2\alpha-4\alpha^2+\frac{1}{4}\alpha^4)(\omega_2^1+\omega_2^4)+(8-6\alpha-5\alpha^2+\frac{1}{2}\alpha^3+\frac{1}{4}\alpha^4)(\omega_2^2+\omega_2^3)}$$

$$\forall \quad i \in \{2,3\}: \quad \frac{\partial p}{\partial \omega_2^i} > 0 \quad \text{if} \quad 8-6\alpha-5\alpha^2+\frac{1}{2}\alpha^3+\frac{1}{4}\alpha^4 < 0 \iff \alpha > 0.828$$

Assume n=5

Agent i's demand function for good 2:

$$x_2^i = \begin{cases} \frac{1}{p}\left[\frac{(16-4\alpha^2+\frac{1}{8}\alpha^4)m^i+\alpha(8-\frac{3}{2}\alpha^2)m^{N_i(g)}+\alpha^2(2-\frac{1}{4}\alpha^2)m^{N_i^2(g)}+\frac{1}{2}\alpha^3 m^{N_i^3(g)}+\frac{1}{8}\alpha^4 m^{N_i^4(g)}}{32-18\alpha^2+\alpha^4}\right] \\ \frac{1}{p}\left[\frac{(16-3\alpha^2)m^i+\alpha(4-\frac{3}{4}\alpha^2)m^{i-1}+\alpha(4-\frac{1}{2}\alpha^2)m^{i+1}+\alpha^2 m^{i+2}+\frac{1}{4}\alpha^3 m^{i+3}}{32-12\alpha^2+\alpha^4}\right] \\ \frac{1}{p}\left[\frac{(16-3\alpha^2)m^i+\alpha(4-\frac{3}{4}\alpha^2)m^{i+1}+\alpha(4-\frac{1}{2}\alpha^2)m^{i-1}+\alpha^2 m^{i-2}+\frac{1}{4}\alpha^3 m^{i-3}}{32-12\alpha^2+\alpha^4}\right] \\ \frac{1}{p}\left[\frac{(4-\frac{1}{2}\alpha^2)^2 m^i+\alpha(4-\frac{1}{2}\alpha^2)\sum_{j\in N_i(g)} m^j+\alpha^2(1-\frac{1}{8}\alpha^2)\sum_{k\in N_i^2(g)} m^k}{32-12\alpha^2+\alpha^4}\right] \end{cases}$$

if $i \in \{1, n\}$, $i = 2$, $i = 4$, and $i = 3$ respectively.

$$p = \frac{(16+4\alpha-3\alpha^2-\frac{1}{2}\alpha^3+\frac{1}{8}\alpha^4)(\omega_1^1+\omega_1^5)+(16+12\alpha-2\alpha^2-\frac{3}{2}\alpha^3)(\omega_1^2+\omega_1^4)+(16+8\alpha-\alpha^3-\frac{1}{4}\alpha^4)\omega_1^3}{(16-4\alpha-9\alpha^2+\frac{1}{2}\alpha^3+\frac{7}{8}\alpha^4)(\omega_2^1+\omega_2^5)+(16-12\alpha-10\alpha^2+\frac{3}{2}\alpha^3+\alpha^4)(\omega_2^2+\omega_2^4)+(16-8\alpha-12\alpha^2+\alpha^3+\frac{5}{4}\alpha^4)\omega_2^3}$$

$\forall \quad i \in \{2, 4\}$:

$\frac{\partial p}{\partial \omega_2^i} > 0 \quad \text{if} \quad 16 - 12\alpha - 10\alpha^2 + \frac{3}{2}\alpha^3 + \alpha^4 < 0 \iff \alpha > 0.851.$

For $\quad i = 3$:

$\frac{\partial p}{\partial \omega_2^i} > 0 \quad \text{if} \quad 16 - 8\alpha - 12\alpha^2 + \alpha^3 + \frac{5}{4}\alpha^4 < 0 \iff \alpha > 0.927.$

Assume n=6

$\forall \quad i \in \{2, 5\}$:

$\frac{\partial p}{\partial \omega_2^i} > 0 \quad \text{if} \quad 32 - 24\alpha - 22\alpha^2 + 5\alpha^3 + 3\alpha^4 - \frac{1}{8}\alpha^5 - \frac{1}{16}\alpha^6 < 0 \iff \alpha > 0.856$

$\forall \quad i \in \{3, 4\}$:

$\frac{\partial p}{\partial \omega_2^i} > 0 \quad \text{if} \quad 32 - 16\alpha - 24\alpha^2 + \frac{5}{2}\alpha^3 + \frac{7}{2}\alpha^4 - \frac{1}{16}\alpha^6 < 0 \iff \alpha > 0.952$

Assume n=7

$\forall \quad i \in \{2, 6\} : \frac{\partial p}{\partial \omega_2^i} > 0 \quad \text{if} \quad \alpha > 0.857$

$\forall \quad i \in \{3, 5\} : \frac{\partial p}{\partial \omega_2^i} > 0 \quad \text{if} \quad \alpha > 0.959$

For $i = 4 : \frac{\partial p}{\partial \omega_2^i} > 0 \quad \text{if} \quad \alpha > 0.979$

Assume n=8

$\forall \quad i \in \{2, 7\} : \frac{\partial p}{\partial \omega_2^i} > 0 \quad \text{if} \quad \alpha > 0.857$

$\forall \quad i \in \{3, 6\} : \frac{\partial p}{\partial \omega_2^i} > 0 \quad \text{if} \quad \alpha > 0.961$

$\forall \quad i \in \{4, 5\} : \frac{\partial p}{\partial \omega_2^i} > 0 \quad \text{if} \quad \alpha > 0.986$

Assume n=9

$\forall \quad i \in \{2, 8\} : \frac{\partial p}{\partial \omega_2^i} > 0 \quad \text{if} \quad \alpha > 0.858$

$\forall \quad i \in \{3, 7\} : \frac{\partial p}{\partial \omega_2^i} > 0 \quad \text{if} \quad \alpha > 0.961$

$\forall \quad i \in \{4, 6\} : \frac{\partial p}{\partial \omega_2^i} > 0 \quad \text{if} \quad \alpha > 0.988$

For $i = 5 : \frac{\partial p}{\partial \omega_2^i} > 0 \quad \text{if} \quad \alpha > 0.994$, and so on. ∎

The critical value of α is increasing in the number of agents because every non-peripheral agent's influence is correspondingly reduced. This is because an additional agent linked to a peripheral agent decreases the hitherto peripheral agent's response to its hitherto only neighbor. This reduced response is translated across the line network to other agents.

Corollary 3 (Line Network) *In the minimum consumption model with a line network, the agents with the k^{th} highest influence are the non-peripheral agents connected to the nearest peripheral agent by a distance of k.*

The critical value of α is increasing in an agent's distance to the nearest peripheral agent but is asymptotic to a value less than one.

In the minimum consumption model with a line network, the influence of an agent on aggregate demand is increasing in the agent's degree but decreasing in the degree of his neighbor. An agent with a higher degree has more other agents responding to the agent's increase in demand. This response is greater if the agents responding have a lower degree. For example, in the six-agent line network, the central agents (3 and 4) have the same degree and in fact a higher measure of closeness than agents 2 and 5. However, agents 2 and 5 each have a neighbor that has only one link and so the neighbor provides a greater response to each of agents 2 and 5. This is because, in the minimum consumption example, an agent responds to the average of its neighbors. Hence, the importance of an agent on market demand does not depend on its centrality per se.

This analysis suggests that sellers of a product should want to focus their advertising and promotional efforts on those consumers who have a relatively high influence on the consumption of others, specifically those with many neighbors who in turn have few neighbors themselves. A highly centralized network, such as a star, offers sellers the opportunity to focus their promotional attempts at the central agent, which has a large impact on market demand.

1.4 Conclusion

The analysis in this paper shows that where consumers tend to conform with the average purchases of their neighbors in a social network, unless the network structure is fairly regular, certain consumers, by virtue of their location in the network, would tend to have a greater influence on aggregate demand than other consumers. Contrary to what one might expect, it is not an agent's closeness or degree per se that matters for its relative influence on market demand. Instead, the number of neighbors of an agent's neighbors also matters for the agent's influence. Since promotional efforts are costly, producers would be better off focusing their efforts on those consumers who have a relatively large influence on the purchase of others.

We have considered only connected networks in the analysis. Within the model in this paper, we can also analyze how prices are affected when networks are disconnected, comprising a number of components.

We have considered on a general equilibrium model to focus on the effect of consumer behavior on prices. We can also incorporate firm behavior by considering a partial equilibrium model. A variety of industry structures can be considered in this context, including strategic interaction among firms.

It may also be possible to consider directed networks. Directed networks may be especially relevant when considering the impact of celebrities and other prominent figures whose consumption patterns are observed by many in the general public, but who do not in turn observe the consumption patterns of those who observe their consumption patterns.

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Paper 2

Critical and Promising Links in a Social Network

Abstract

In the context of consumers interacting strategically in a directed social network, I measure the importance of each consumer, each directed link in the network, and each potential directed link that could be added to the network. A consumer's *impact centrality* measures the effect of a unit exogenous change in his demand on equilibrium aggregate demand. A consumer's *reaction centrality* measures the effect of a unit exogenous change in every consumer's demand on the consumer's equilibrium demand. I show how the effect on aggregate centrality of removing or adding a directed link from one consumer to another depends on the impact centrality of the latter, the reaction centrality of the former, and feedback effects from the latter to the former in the original network.

2.1 Introduction

Some of our consumption decisions may be influenced by some but not all of those around us. Whether due to cultural norms, social rivalry, or information sharing, a consumer may be inclined to increase or decrease his consumption when another consumer increases his.[1] Furthermore, the extent of this interdependence may be asymmetric and may vary across consumers.[2] It is useful to think of these strategic interactions as forming a directed social network among consumers. Some businesses have information about how their consumers are connected to one another. Wireless companies like Cingular and Verizon, Voice over Internet Protocol companies like Skype, webmail providers like Yahoo and MSN, social networking service providers like MySpace and Facebook, video sharing providers like YouTube, and photo sharing providers like Flickr have information about how their customers are linked to one another.[3] Businesses using multi-level marketing know how their independent contractors are connected to one another in the network. These contractors also consume the products that they sell to their social contacts.

Which consumer is the most influential in terms of being able to generate the greatest increase in aggregate demand? In light of the information available, businesses may find it more profitable to focus their marketing efforts on only certain consumers

[1] For the special cases of a consumer's demand for a good depending on either aggregate demand or network externalities, that is, the number of consumers consuming it, see for example Duesenberry (1949), Leibenstein (1950), Becker (1991), Karni and Levin (1994), Corneo and Jeanne (1997), Grilo, Shy, and Thisse (2001).

[2] For example, a fashion victim may be inclined to increase his consumption of a product in response to a rise in a trend setter's consumption of the product, but the trend setter may be inclined to decrease his consumption of the product in response to a rise in the fashion victim's consumption of the product.

[3] For example, wireless companies may be able to use phone records to estimate the extent to which a customer would reciprocate a phone call from a particular phone number. Webmail providers may have information about the length and frequency of emails exchanged between each pair of email addresses. Social networking service providers may have information about the propensity for a user to accept an invitation to download an application extended by the user's friend.

instead of all consumers, or allow other businesses the opportunity to target advertisements at specific customers.[4] Which consumer reacts the most to a change in aggregate demand? Policymakers may also find it better to focus on only certain consumers their campaign against a socially undesirable good.[5] Which link is most important to the average influence of consumers? Policymakers may seek to lower the aggregate consumption of a socially undesirable good by hampering only certain links if such efforts are costly. Which consumer is key to the average influence of consumers? Policymakers may similarly seek to lower the aggregate consumption of a socially undesirable good by targeting their efforts at certain consumers that would make the greatest overall difference to these consumption spillovers.

Consider the Nash Equilibrium of a network game involving a finite set of consumers with interdependent general utility functions and a budget constraint each. First, I show that the effect of a unit exogenous change in a consumer's demand on the equilibrium aggregate demand is equal to the consumer's *impact centrality*. Second, I show that the effect of a unit exogenous change in every consumer's demand on a consumer's equilibrium demand is the consumer's *reaction centrality*. Next, I consider changes to the structure of a network brought about by either the removal of a directed link or the addition of a directed link. I show that the reduction in aggregate centrality by removing a directed link is equal to the directed link's *arc importance*, and the increase in aggregate centrality by adding a directed link is equal to the directed link's *arc impact*. A directed link's arc importance or arc impact depends on the weight of the link, the reaction centrality of the consumer from whom the link begins, the impact centrality of the consumer to whom the link ends, and the feedback effects from the

[4]For example, an application provider in Facebook obtains access to information about a Facebook user when the user downloads the application.

[5]There is evidence to suggest that adolescent smoking may involve peer effects [Barber, Bolitho, and Bertrand (1999), Centers for Disease Control and Prevention (1994)].

the latter to the former in the original network. In particular, if there are no strategic substitutes, then removing a directed link results in a smaller reduction in aggregate centrality if this feedback effect is greater. In contrast, adding a directed link results in a greater increase in aggregate centrality if this feedback effect is greater. Finally, I show that the reduction in aggregate centrality by removing a consumer from the network is equal to the consumer's *intercentrality*. The consumer's intercentrality is the sum of the consumer's impact centrality and the reduction in other consumers' impact centrality from the removal of the directed links that begin or end with the consumer who is deleted from the network.

I shall now discuss the three papers in the literature that are the most closely related to this.[6] Ballester, Calvo-Armengol, and Zenou (2006) show that in an undirected network assuming linear-quadratic utility function, each agent's contribution to the aggregate equilibrium action is in proportion to his *Bonacich centrality*. They also seek to determine the *key player* in an undirected network assuming linear-quadratic utility function, there being no strategic substitutes, and all arcs having equal weight. They propose a measure of each agent's importance in a network based on the extent to which removal of the agent from the network changes the equilibrium aggregate action. The agent with the highest intercentrality is the agent whose removal causes the biggest fall in equilibrium aggregate action. In contrast, I assume general C^2 utility functions and do not restrict all arcs to having equal weight. In addition, my analysis is not restricted to undirected networks and there being no strategic substitutes. Further, I provide a proof and a characterization of the an agent's intercentrality that is not limited to the case of an undirected network. Finally, I consider the ef-

[6] For seminal references to centrality measures in social networks, see Katz (1953), Bonacich (1987). Other papers related to this include Bramoullé and Kranton (2007), Bloch and Quérou (2008), Galeotti (2008), Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2008).

fects of adding and removing directed links, which are not addressed in Ballester, Calvo-Armengol, and Zenou (2006).

Tan (2006) examines how equilibrium prices are affected by strategic complementarity of consumption in an undirected network. In a general equilibrium model with Cobb-Douglas utility function and agents responding to the average of their neighbors' consumption, he shows that an increase in the supply of a good involving strategic complementarity in consumption between neighbors cannot raise its relative price if all agents have the same number of neighbors. However, an increase in the supply of such a good through an increase in the endowment of the central agent may raise the relative price of this good in a star network if strategic complementarity is sufficiently strong. In contrast, I focus here on the identifying the critical and promising links, and the key player in arbitrary network structures and with general strategic interactions and utility functions.

In a general equilibrium model with Cobb-Douglas utility function, Ghiglino and Goyal (2008) illustrate with examples that a shift from social segregation to social integration reduce the welfare of the poor but increase the welfare of the rich. They also show that equilibrium prices and consumption allocations can be expressed as a function of network centrality. When agents respond to the aggregate of their neighbors' consumption, the addition of undirected links to the network increases the sum of centralities and thus equilibrium prices. When agents respond to the average of their neighbors' consumption, they show that equilibrium prices and consumption allocations do not depend on the network if endowments are identical. However, in contrast with this paper, Ghiglino and Goyal (2008) limit their attention to networks that are undirected, where there are no strategic substitutes, and agents have a specific form of utility function. Further, I consider the removal and addition of directed links, and the removal of agents from the network.

The rest of this paper is organized as follows: Section 2.2 presents the model and discusses impact centrality; Section 2.3 discusses reaction centrality; Section 2.4 discusses the addition and deletion of arcs; Section 2.5 discusses the deletion of agents; Section 2.6 concludes with some possible extensions.

2.2 Impact Centrality

Consider a non-empty finite set of agents $N = \{1, \ldots, n\}$ each with continuous, strictly convex, and strongly monotone preferences over two normal goods x and y. Good x is the good that induces strategic interactions and good y is the numeraire good. Each agent i's preferences can be represented by a strictly quasiconcave and twice continuously differentiable utility function $u_i(\mathbf{x}, y_i)$, where $\mathbf{x} = (x_1, \ldots, x_n) = (x_i, \mathbf{x}_{-i})$. Each agent i simultaneously chooses x_i and y_i to solve the consumer optimization problem

$$\max_{\{x_i, y_i\}} u_i(\mathbf{x}, y_i) \quad \text{subject to} \qquad px_i + y_i \leq w_i, \qquad x_i \geq 0, \qquad y_i \geq 0,$$

where p is the price of x and w_i is the income of agent i.

Definition 1 $(\mathbf{x}^*, \mathbf{y}^*)$ is an interior *Nash Equilibrium* if

$$(i) \qquad \frac{\partial u_i(\mathbf{x}^*, y_i^*)}{\partial x_i} = p\frac{\partial u_i(\mathbf{x}^*, y_i^*)}{\partial y_i}, \qquad \forall i \in N,$$

$$(ii) \qquad px_i^* + y_i^* = w_i, \qquad \forall i \in N.$$

The best response correspondences are in fact best response functions because the utility function is strictly quasiconcave. Each agent's best response function, which is implicitly determined by the agent's first order condition, is invariant to a monotonic transformation of the utility function.

For any $i \neq j$, let $g_{ij} = (\partial x_i / \partial x_j)_{\mathbf{x}_{-\{i,j\}}, \mathbf{y}_{-\{i,j\}}, p, \mathbf{w}}$ be the marginal effect of agent j's consumption of x on agent i's consumption of x ceteris paribus.[7] For any agent $i \neq j$, agent j's consumption of x is a strategic complement to agent i's consumption of x if $g_{ij} > 0$, and a strategic substitute to agent i's consumption of x if $g_{ij} < 0$. An *arc* from j to i with weight g_{ij} is an ordered pair of agents (j, i). There exists an arc (j, i) in the digraph if and only if $g_{ij} \neq 0$ for any $j \neq i$. For any $i = j$, let $g_{ij} = 0$.

The network of strategic interactions between agents can be described by a *weighted digraph*, which consists of the set of agents N, a finite set of arcs

$$A = \{(j, i) : \quad j \in N, \quad i \in N, \quad j \neq i, \quad \text{and} \quad g_{ij} \neq 0\}$$

in the digraph, and the set of weights for each arc. The weighted digraph can be represented by the *weighted adjacency matrix* $\mathbf{G} = [g_{ij}]$, which is a zero-diagonal square matrix. The *spectral radius* of $\mathbf{G}$ is $\rho(\mathbf{G}) = \max\{|\lambda_i(\mathbf{G})| : i \in N\}$, where λ_i is the ith eigenvalue of $\mathbf{G}$.

Definition 2 Consider any digraph $\mathbf{G}$ such that $\rho(\mathbf{G}) < 1$. The vector of *impact centralities* is

$$\mathbf{a}(\mathbf{G}) = \mathbf{1}^T \mathbf{M}(\mathbf{G})$$

[7] Note that g_{ij} holds constant the consumption bundles of all other agents other than i and j, not just prices and incomes. It depends on ordinal properties of the utility function.

where $\mathbf{M}(\mathbf{G}) = (\mathbf{I} - \mathbf{G})^{-1} = [m_{ij}(\mathbf{G})]$ is an n-square matrix, and $\mathbf{I}$ is the n-square identity matrix.

The vector of impact centralities $\mathbf{a} = (a_1, \ldots, a_n)$ is comprised of each agent's impact centrality. Let $\mathbf{e}_k$ be the standard basis vector with 1 for its kth component and 0 for every other component. Then agent k's impact centrality $a_k(\mathbf{G}) = \mathbf{1}^T\mathbf{M}(\mathbf{G})\mathbf{e}_k$ is the kth component of the vector of impact centralities. It is simply the sum of the kth column of $\mathbf{M}(\mathbf{G})$, that is $a_k(\mathbf{G}) = \sum_{i \in N} m_{ik}(\mathbf{G})$.

The Nash Equilibrium should be interpreted as the steady state result of an adjustment process in which agents simultaneously adjust their consumption choice each period by choosing a best response to their neighbor's consumption choice in the previous period. The condition $\rho(\mathbf{G}) < 1$ is required to ensure stability through the convergence of the power sequence of $\mathbf{G}$ since $\lim_{s \to \infty} \mathbf{G}^s = \mathbf{0}$ if and only if $\rho(\mathbf{G}) < 1$.[8] It does not restrict the value of any particular entry in $\mathbf{G}$. Instead, it requires that agents, as a whole, do not react too sensitively to a change in any agent's consumption of x. The matrix $\mathbf{G}$ depends on ordinal conditions and so it is potentially observable.

A seller may be interested in targeting advertisements or other promotion efforts at a select group of consumers that have a bigger impact on aggregate demand. Hence, I consider the effect of a unit exogenous change in agent k's demand on equilibrium aggregate demand, holding constant prices and the incomes of all other agents.[9] The exogenous change in agent k's demand is induced by a change in agent k's income in order to keep the utility function unchanged. Let $\sigma_k = (\partial x_k / \partial w_k)_{\mathbf{x}_{-k}, \mathbf{y}_{-k}, p, \mathbf{w}_{-k}}$ be agent k's marginal propensity to consume x ceteris paribus. Note that $\sigma_k > 0$ for

[8]If instead this convergence condition does not hold, then aggregate demand for x would not converge to an equilibrium in response to an exogenous change in some agent's demand for x.

[9]Note that the subscripts on partial derivatives indicate the variables that are being held constant.

all k because x is a normal good. Let $\mu_{ik} = (\partial x_i / \partial w_k)_{p,\mathbf{w}_{-k}}$ be agent k's marginal propensity to consume x, holding constant only prices and the incomes of all other agents.

Theorem 1 *Consider any digraph* $\mathbf{G}$ *such that* $\rho(\mathbf{G}) < 1$. *Then*

$$\sum_{i \in N} \frac{\mu_{ik}}{\sigma_k} = a_k(\mathbf{G})$$

for any k.

Proof. Linearize the system about the Nash Equilibrium $(\mathbf{x}^*, \mathbf{y}^*)$. Set $\mathrm{d}p = 0$ and substitute out $\mathrm{d}y_i$ to obtain for each i

$$\mathrm{d}x_i - \sum_{j \neq i} g_{ij} \mathrm{d}x_j = \sigma_i \mathrm{d}w_i \tag{2.1}$$

since

$$g_{ij} = \frac{-(\partial^2 u_i/\partial x_j \partial x_i - p\partial^2 u_i/\partial x_j \partial y_i)}{\partial^2 u_i/\partial x_i^2 - p\partial^2 u_i/\partial x_i \partial y_i - p(\partial^2 u_i/\partial y_i \partial x_i - p\partial^2 u_i/\partial y_i^2)}$$

and

$$\sigma_i = \frac{-(\partial^2 u_i/\partial y_i \partial x_i - p\partial^2 u_i/\partial y_i^2)}{\partial^2 u_i/\partial x_i^2 - p\partial^2 u_i/\partial x_i \partial y_i - p(\partial^2 u_i/\partial y_i \partial x_i - p\partial^2 u_i/\partial y_i^2)}.$$

Set $\mathrm{d}w_k = 1$ and $\mathrm{d}w_i = 0$ for all $i \neq k$. Apply the Implicit Function Theorem to obtain

$$(\mathbf{I} - \mathbf{G})\mu_k = \sigma_k \mathbf{e}_k \tag{2.2}$$

where $\mu_k^T = (\mu_{1k}, \ldots, \mu_{nk})$.

Next, I show that if $\rho(\mathbf{G}) < 1$, then $\det(\mathbf{I} - \mathbf{G}) > 0$ and so $\mathbf{I} - \mathbf{G}$ is nonsingular. Note that, for all i, $\mathbf{G}\mathbf{v} = \lambda_i(\mathbf{G})\mathbf{v}$ for some nonzero vector $\mathbf{v}$. This implies that $(\mathbf{I} - \mathbf{G})\mathbf{v} = (1 - \lambda_i(\mathbf{G}))\mathbf{v}$ and so $\lambda_i(\mathbf{I} - \mathbf{G}) = 1 - \lambda_i(\mathbf{G})$ for all i. Since $\rho(\mathbf{G}) < 1$, it must be that $-1 < \lambda_i(\mathbf{G}) < 1$ for all real eigenvalues of $\mathbf{G}$, and $0 < \lambda_i(\mathbf{I}-\mathbf{G}) < 2$ for all real eigenvalues of $\mathbf{I}-\mathbf{G}$. Note that $\det(\mathbf{I}-\mathbf{G}) = \prod_{i=1}^{n} \lambda_i(\mathbf{I}-\mathbf{G})$, and that complex eigenvalues of real matrices occur in complex conjugate pairs. Hence, $\det(\mathbf{I}-\mathbf{G}) > 0$.

Given that $\mathbf{I} - \mathbf{G}$ is nonsingular, the unique solution is $\mu_k = \sigma_k \mathbf{M}(\mathbf{G})\mathbf{e}_k$. The effect of a unit exogenous change in agent k's demand for x on each agent's equilibrium demand for x is

$$\sigma_k^{-1}\mu_k = \mathbf{M}(\mathbf{G})\mathbf{e}_k. \tag{2.3}$$

Sum over all agents to obtain $\sigma_k^{-1}\mathbf{1}^T\mu_k = \mathbf{1}^T\mathbf{M}(\mathbf{G})\mathbf{e}_k$.

Hence, $\sum_{i \in N} \mu_{ik}/\sigma_k = a_k(\mathbf{G})$. ■

That is, the effect of a unit exogenous change in agent k's demand for x on the equilibrium aggregate demand for x is equal to $a_k(\mathbf{G})$. The agent with the highest impact centrality has the highest impact on equilibrium aggregate demand. An agent's impact centrality is a multiplier that measures the number of times by which equilibrium aggregate demand for x would increase for a unit exogenous increase in the agent's demand for x. As the aggregate demand for x includes that of agent k, his exogenous increase in demand has an expansionary effect on equilibrium aggregate demand for x if $a_k(\mathbf{G}) > 1$, a contractionary effect if $a_k(\mathbf{G}) < 1$, and a neutral effect if $a_k(\mathbf{G}) = 1$.

Lemma 1 *Consider any digraph* $\mathbf{G}$ *such that* $\rho(\mathbf{G}) < 1$. *Then*

$$\frac{\mu_{ij}}{\sigma_j} = m_{ij}(\mathbf{G})$$

for all i *and* j.

Proof. According to (2.3), the effect of a unit exogenous change in agent j's demand for x on each agent's equilibrium demand for x is $\sigma_j^{-1}\mu_j = \mathbf{M}(\mathbf{G})\mathbf{e}_j$. Multiply both sides of the equation by the vector $\mathbf{e}_i^T$ to obtain $\sigma_j^{-1}\mathbf{e}_i^T\mu_j = \mathbf{e}_i^T\mathbf{M}(\mathbf{G})\mathbf{e}_j$. Hence, $\mu_{ij}/\sigma_j = m_{ij}(\mathbf{G})$. ■

That is, the effect of a unit exogenous change in agent j's demand for x on agent i's equilibrium demand for x is equal to $m_{ij}(\mathbf{G})$, which is the (i, j)th entry of $\mathbf{M}(\mathbf{G})$.

A *walk* of length s from agent j to agent i is an alternating sequence of agents and arcs of the form

$$i_0, (i_0, i_1), i_1, (i_1, i_2), i_2, \ldots, i_{s-1}, (i_{s-1}, i_s), i_s$$

where $j = i_0$ and $i = i_s$. Note that a walk may have repeated agents or arcs. Since g_{ij} is the marginal effect of agent j's demand on agent i's demand, $g_{ij}^{[s]}$, which is the (i, j)th entry of $\mathbf{G}^s$, is the marginal effect of agent j's demand on agent i's demand through all walks of length s from j to i. Since $\mathbf{M}(\mathbf{G}) = \sum_{s=0}^{\infty} \mathbf{G}^s$, $m_{ij}(\mathbf{G}) = \sum_{s=0}^{\infty} g_{ij}^{[s]}$ is the total of the marginal effects through all walks from j to i. Given that $\mathbf{a}(\mathbf{G}) = \mathbf{1}^T\mathbf{M}(\mathbf{G})$, $a_k(\mathbf{G})$ is the total of the marginal effects through all walks from k.

Agent k's *out-neighborhood* $N_k^+(\mathbf{G})$ is the set of all agents to each of whom there is an arc from k. The agents in k's out-neighborhood are his *out-neighbors*. Let $n_k^+(\mathbf{G})$ be the cardinality of k's out-neighborhood, that is the number of out-neighbors that k has. Agent k's *in-neighborhood* $N_k^-(\mathbf{G})$, *in-neighbors*, and the number of in-neighbors $n_k^-(\mathbf{G})$ can be defined similarly.

Theorem 2 *Consider any digraph* $\mathbf{G}$ *such that* $|g| < 1/r$, *where* r *is a nonnegative integer,* $n_k^+(\mathbf{G}) = r$ *for all* k, *and* $g_{ij} = g$ *for all* (j, i) *in* A. *Then* $a_k(\mathbf{G}) = 1/(1 - rg)$ *for all* k.

Proof. Use (2.2) and Lemma 1 to obtain

$$m_{kk}(\mathbf{G}) - \sum_{i \in N_k^-(\mathbf{G})} g_{ki} m_{ik}(\mathbf{G}) = 1$$

and

$$m_{jk}(\mathbf{G}) - \sum_{i \in N_j^-(\mathbf{G})} g_{ji} m_{ik}(\mathbf{G}) = 0$$

for all $j \neq k$.

Sum over the equations to obtain

$$\sum_{i \in N} [m_{ik}(\mathbf{G}) - \sum_{j \in N_i^+(\mathbf{G})} g_{ji} m_{ik}(\mathbf{G})] = 1.$$

Since all agents have an equal number of out-neighbors, $\sum_{j \in N_i^+(\mathbf{G})} g_{ji} = \sum_{j \in N_i^+(\mathbf{G})} g = n_i^+(\mathbf{G}) g = rg$. Hence, $\sum_{i \in N} (1 - rg) m_{ik}(\mathbf{G}) = 1$, and so $a_k(\mathbf{G}) = 1/(1 - rg)$. Note that the stability condition $\rho(\mathbf{G}) < 1$ requires that $|g| < 1/r$. ■

That is, in any digraph where all agents have the same number of out-neighbors and all arcs have equal weight, all agents have equal impact centrality, which is increasing in the number of out-neighbors and the arc weight but is independent of the number of agents. One might expect that every agent's impact centrality may be increasing in the number of agents since each agent may be able to affect more agents if there are more agents in the digraph. However, there are cycles, which give rise to walks of infinite length, in digraphs where all agents have the same positive number of out-neighbors. As a result, both the number of walks r^s of length $s \geq 0$ from each agent and the marginal effect g^s from a walk of length s are independent of n.

2.3 Reaction Centrality

Definition 3 Consider a weighted adjacency matrix $\mathbf{G}$ such that $\rho(\mathbf{G}) < 1$. The vector of *reaction centralities* is

$$\mathbf{b}(\mathbf{G}) = \mathbf{M}(\mathbf{G})\mathbf{1}.$$

The vector of reaction centralities $\mathbf{b}^T = (b_1, \ldots, b_n)$ is comprised of each agent's reaction centrality. Agent k's reaction centrality $b_k(\mathbf{G}) = \mathbf{e}_k^T\mathbf{M}(\mathbf{G})\mathbf{1}$ is the kth component of the vector of reaction centralities. It is simply the sum of the kth row of $\mathbf{M}(\mathbf{G})$, that is $b_k(\mathbf{G}) = \sum_{i \in N} m_{ki}(\mathbf{G})$, and measures the total of the marginal effects through all walks that end at k. Note that $\mathbf{b}(\mathbf{G})$ corresponds with the vector of Bonacich (1987) centralities defined in Ballester, Calvo-Armengol, and Zenou (2006) except that here $\mathbf{G}$ is not necessarily a matrix with every entry restricted to a value between 0 and 1.

Policymakers may be particularly concerned about those persons who react relatively more sensitively to higher consumption by other consumers. In this regard, I estimate the effect of a unit exogenous change in every agent's demand on an agent's equilibrium demand. This is a useful measure when the issue of interest is to determine which consumer responds most sensitively to a change in aggregate demand.

Theorem 3 *Consider any digraph* $\mathbf{G}$ *such that* $\rho(\mathbf{G}) < 1$*. Then*

$$\sum_{i \in N} \frac{\mu_{ki}}{\sigma_i} = b_k(\mathbf{G})$$

for any k*.*

Proof. From Lemma 1, $\mu_{ki}/\sigma_i = m_{ki}(\mathbf{G})$. Theorem 3 is obtained by summing over all agents i. ■

That is, the effect of a unit exogenous change in every agent's demand for x on agent k's equilibrium demand for x is equal to $b_k(\mathbf{G})$. The agent with the highest reaction centrality has the highest reaction to aggregate demand. An agent's reaction centrality is a multiplier that measures the number of times by which the agent's equilibrium demand for x would increase for a unit exogenous increase in every agent's demand for x. As every agent's demand for x includes that of agent k, the exogenous increase in demand of each agent has an expansionary effect on agent k's equilibrium demand for x if $b_k(\mathbf{G}) > 1$, a contractionary effect if $b_k(\mathbf{G}) < 1$, and a neutral effect if $b_k(\mathbf{G}) = 1$.

An *undirected graph* is a symmetric digraph, that is a digraph such that $g_{ij} = g_{ji}$ for all i and j.

Lemma 2 *Consider any undirected graph* $\mathbf{G}$ *such that* $\rho(\mathbf{G}) < 1$. *Then* $b_k(\mathbf{G}) = a_k(\mathbf{G})$ *for all* k.

Proof. If $g_{ij} = g_{ji}$ for all i and j, then $\mathbf{G}^T = \mathbf{G}$ and so $(\mathbf{I} - \mathbf{G})^T = \mathbf{I}^T - \mathbf{G}^T = \mathbf{I} - \mathbf{G}$. Since $\mathbf{M}(\mathbf{G}) = (\mathbf{I} - \mathbf{G})^{-1}$ and the inverse of a symmetric matrix is symmetric, $m_{ki}(\mathbf{G}) = m_{ik}(\mathbf{G})$ for all i and j. Hence, sum over all agents i to obtain $b_k(\mathbf{G}) = a_k(\mathbf{G})$. ■

That is, for any undirected graph, each agent's reaction centrality is equal to his impact centrality. This is because the total of the marginal effects through all walks that end at k is equal to the total of the marginal effects through all walks that begin at k. Hence, the reaction of an agent's equilibrium demand to the aggregate demand also reflects his impact on equilibrium aggregate demand. This implies that the agent who increases his equilibrium demand the most in reaction to a change in aggregate demand also causes the greatest increase in aggregate demand. Empirically, this result provides an additional restriction that may be useful for identification.

Lemma 3 *Consider any digraph* $\mathbf{G}$ *such that* $\rho(\mathbf{G}) < 1$ *and* $g_{ij} > 0$ *for all* (j, i) *in* A. *Then* $a_k(\mathbf{G}) \geq 1$ *and* $b_k(\mathbf{G}) \geq 1$ *for all* k.

Proof. If $g_{ij} > 0$ for all (j, i) in A, then $\mathbf{G} \geq \mathbf{0}$. Since the product of any nonnegative square matrix is also nonnegative, $\mathbf{G}^s \geq \mathbf{0}$ for any integer $s \geq 2$, and so $\mathbf{M}(\mathbf{G}) = \sum_{s=0}^{\infty} \mathbf{G}^s \geq \mathbf{I}$. Hence $a_k(\mathbf{G}) \geq 1$ and $b_k(\mathbf{G}) \geq 1$ for all k. ■

That is, if there are no strategic substitutes, then a unit exogenous increase in any agent's demand for x must result in at least a unit increase in the equilibrium aggregate demand for x. Similarly, a unit exogenous increase in every agent's demand for x must result in at least a unit increase in each agent's equilibrium demand for x. However, in general, an agent's impact centrality or reaction centrality may be less than one and even negative if there are strategic substitutes.

Theorem 4 *Consider any digraph* $\mathbf{G}$ *such that* $|g_{ij}g_{ji}| < 1$ *for any* $j \neq i$ *and* $n = 2$. *Then* $a_i(\mathbf{G}) > a_j(\mathbf{G})$ *if and only if* $b_i(\mathbf{G}) < b_j(\mathbf{G})$.

Proof. Since $m_{ii}(\mathbf{G}) = 1/(1 - g_{ij}g_{ji}) = m_{jj}(\mathbf{G})$, therefore $b_i(\mathbf{G}) - b_j(\mathbf{G}) = m_{ij}(\mathbf{G}) - m_{ji}(\mathbf{G}) = a_j(\mathbf{G}) - a_i(\mathbf{G})$. Hence $a_j(\mathbf{G}) > a_i(\mathbf{G})$ if and only if $b_i(\mathbf{G}) > b_j(\mathbf{G})$. ■

That is, if there are only two agents, then the agent who has the higher impact centrality must have the lower reaction centrality. Note that the stability condition $\rho(\mathbf{G}) < 1$ requires that $|g_{ij}g_{ji}| < 1$ for any $j \neq i$. However, an agent may have both a higher reaction centrality and a higher impact centrality than that of another agent if there are more than two agents.

Theorem 5 *Consider any digraph* $\mathbf{G}$ *such that* $|g| < 1/r$*, where* r *is a nonnegative integer,* $n_k^-(\mathbf{G}) = r$ *for all* k*, and* $g_{ij} = g$ *for all* (j,i) *in* A*. Then* $b_k(\mathbf{G}) = 1/(1-rg)$ *for all* k*.*

Proof. Consider a unit exogenous change in every agent's demand for x. The marginal effect through all walks of length 0 that end at k is equal to 1, the unit exogenous change in k's demand. Since all agents have an equal number of in-neighbors, the marginal effect through all walks of length 1 that end at k is equal to $\sum_{j \in N_k^-(\mathbf{G})} g_{kj} = \sum_{j \in N_k^-(\mathbf{G})} g = n_k^-(\mathbf{G})g = rg$, where r is a nonnegative integer. The marginal effect through all walks of length 2 that end at k is equal to $\sum_{j \in N_k^-(\mathbf{G})} g_{kj} rg = (rg)^2$. The marginal effect through all walks of length s that end at k is equal to $\sum_{j \in N_k^-(\mathbf{G})} g_{kj} (rg)^{s-1} = (rg)^s$. Hence, the total marginal effects through all walks that end at k is equal to $b_k(\mathbf{G}) = \sum_{s=0}^{\infty} (rg)^s = 1/(1-rg)$. Note that the stability condition $\rho(\mathbf{G}) < 1$ requires that $|g| < 1/r$. ■

That is, in any digraph where all agents have the same number of in-neighbors and all arcs have equal weight, all agents have equal reaction centrality, which is increasing in the number of in-neighbors and the arc weight but is independent of the number of agents. One might expect that every agent's reaction centrality may be increasing in the number of agents since each agent may be affected by more agents if there are more agents in the digraph. However, the explanation for this result is analogous to that for digraphs where all agents have the same positive number of out-neighbors and all arcs have equal weight. There are cycles, which give rise to walks of infinite length, in digraphs where all agents have the same positive number of in-neighbors. As a result, both the number of walks r^s of length $s \geq 0$ to each agent and the marginal effect g^s from a walk of length s are independent of n.

A digraph is *regular* if $n_k^+(\mathbf{G}) = n_k^-(\mathbf{G}) = r$ for all k, where r is a nonnegative integer.[10] Agent k's *neighborhood* is the union of his out-neighborhood and his in-neighborhood. The set of agents in k's neighborhood are his *neighbors*.

Corollary 1 *Consider any regular digraph* $\mathbf{G}$ *such that* $|g| < 1/r$ *and* $g_{ij} = g$ *for all* (j, i) *in* A. *Then* $a_k(\mathbf{G}) = b_k(\mathbf{G}) = 1/(1 - rg)$ *for all* k.

Proof. The proof follows from Theorems 2 and 5. ■

That is, in any regular digraph where all arcs have equal weight, all agents have equal impact centrality and reaction centrality, both of which are increasing in the number of neighbors and in the weight of each arc but is independent of the number of agents.

Example 6 Consider the regular digraph $\mathbf{G}$ in Figure 2.1 in which $r = 1$ and $g_{ij} = g$ for all (j, i) in A. From Corollary 1, every agent has reaction centrality and impact centrality that is equal to $1/(1 - g)$.

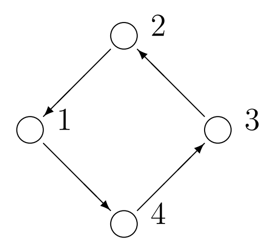

Figure 2.1: A regular digraph

If instead each agent responds to the average of his in-neighbors' demand for x such that $g_{ij} = g/r$ for all (j, i) in A. Then, $a_k(\mathbf{G}) = b_k(\mathbf{G}) = 1/(1-g)$ for all k. That is, all agents have equal impact centrality and reaction centrality, both of which are not only independent of the number of agents but also independent of the common number

[10]Note that a regular digraph is not necessarily an undirected graph.

of neighbors that each agent has. Consider a unit exogenous increase in any agent's demand for x. The marginal effect through all walks of length $s \geq 0$ is simply g^s. The greater the number of out-neighbors, the greater the number of out-neighbors responding to the increase in demand by an agent. However, at the same time, every agent responds proportionately less to the increase in demand because every agent also has more in-neighbors. The two opposing effects exactly offset each other. This suggests that, in contrast to Ballester, Calvo-Armengol, and Zenou (2006), an increase in network density due to an increase in every agent's number of neighbors may not increase the response of equilibrium aggregate demand to an increase in the demand of any one agent if the network remains regular and each agent is affected by the average consumption of his neighbors. With incomplete information of the network structure, Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2008) find that in every symmetric equilibrium, actions are increasing (decreasing) in the agent's number of neighbors if payoffs satisfy strategic complements (substitutes). However this result does not hold if there is complete information of the network structure or if agents respond to the average action of their neighbors rather than the aggregate action of their neighbors.

A *star graph* is an undirected graph with the only arcs being from one central agent to every other agent and from every other agent to the central agent, that is (j, i) in A if and only if either $j = k$ or $i = k$, where k is the central agent.

Theorem 7 *Consider any star graph* $\mathbf{G}$ *such that* $|g| < 1/\sqrt{n-1}$ *and* $g_{ij} = g$ *for all* (j, i) *in* A*. Then* $b_k(\mathbf{G}) = a_k(\mathbf{G}) = [1 + (n-1)g]/[1 - (n-1)g^2]$ *and* $b_j(\mathbf{G}) = a_j(\mathbf{G}) = (1 + g)/[1 - (n-1)g^2]$ *for any* $j \neq k$*.*

Proof. Note that $n_k^+(\mathbf{G}) = n_k^-(\mathbf{G}) = n-1$ and $n_j^+(\mathbf{G}) = n_j^-(\mathbf{G}) = 1$ for all $j \neq k$. First, consider a unit exogenous change in agent k's demand for x. The marginal effect through all walks of length 0 that begin at k is equal to 1, the unit exogenous change in agent k's demand for x. Since every other agent is an out-neighbor of k, the marginal effect through all walks of length 1 that begin at k is equal to $(n-1)g$. Since k is the only out-neighbor of the other agents, the marginal effect through all walks of length 2 that begin at k is equal to $(n-1)g^2$. The marginal effects through all walks of lengths 3 and 4 that begin at k are equal to $(n-1)^2 g^3$ and $(n-1)^2 g^4$ respectively. The marginal effect through all walks of length $s \geq 1$ that begin at k is equal to $(n-1)^{(s+1)/2} g^s$ if s is odd and $(n-1)^{s/2} g^s$ if s is even. Hence, the total marginal effects through all walks that begin at k is equal to

$$a_k(\mathbf{G}) = \sum_{s=0}^{\infty} [(n-1)g^2]^s + (n-1)g \sum_{s=0}^{\infty} [(n-1)g^2]^s = [1 + (n-1)g]/[1-(n-1)g^2].$$

Note that the stability condition $\rho(\mathbf{G}) < 1$ requires that $|g| < 1/\sqrt{n-1}$.

Next, consider a unit exogenous change in the demand for x of any agent $j \neq k$. The marginal effect through all walks of length 0 that begin at j is equal to 1, the unit exogenous change in agent j's demand for x. Since k is the only out-neighbor of j, the marginal effect through all walks of length 1 that begin at j is equal to g. Since every other agent is an out-neighbor of k, the marginal effect through all walks of length 2 that begin at j is equal to $(n-1)g^2$. The marginal effects through all walks of lengths 3 and 4 that begin at j are equal to $(n-1)g^3$ and $(n-1)^2 g^4$ respectively. The marginal effect through all walks of length $s \geq 1$ that begin at j is equal to $(n-1)^{(s-1)/2} g^s$ if s is odd and $(n-1)^{s/2} g^s$ if s is even. Hence, the total marginal effects through all walks that begin at any $j \neq k$ is equal to

$$a_j(\mathbf{G}) = \sum_{s=0}^{\infty}[(n-1)g^2]^s + g\sum_{s=0}^{\infty}[(n-1)g^2]^s = (1+g)/[1-(n-1)g^2].$$

Finally, since $\mathbf{G}$ is symmetric, from Lemma 2, $b_i(\mathbf{G}) = a_i(\mathbf{G})$ for all $i \in N$. ■

That is, for any star graph where all arcs have equal weight and $n \geq 3$, the central agent has the highest impact centrality and reaction centrality if $g > 0$ and the lowest impact centrality and reaction centrality if $g < 0$. Note that the convergence requirement $|g| < 1/\sqrt{n-1}$ implies that $1 + g > 0$ for any $n \geq 2$, and so $a_j(\mathbf{G})$ is increasing in n.

If instead each agent responds to the average of his neighbors' demand for x such that $g_{ij} = g/n_i^-(\mathbf{G})$ for all (j, i) in A. From a unit exogenous change in agent k's demand for x, the total marginal effect through all walks of length $s \geq 1$ that begin at k is equal to $(n-1)g^s$ if s is odd and g^s if s is even. Sum over all walks from k to obtain

$$a_k(\mathbf{G}) = \sum_{s=0}^{\infty} g^{2s} + (n-1)g\sum_{s=0}^{\infty} g^{2s} = [1+(n-1)g]/(1-g^2).$$

Note that the stability condition $\rho(\mathbf{G}) < 1$ requires that $|g| < 1$. From a unit exogenous change in agent j's demand for x, the total marginal effect through all walks of length s that begin at j is equal to $g^s/(n-1)$ if s is odd and g^s if s is even. Sum over all walks from j to obtain

$$a_j(\mathbf{G}) = \sum_{s=0}^{\infty} g^{2s} + [g/(n-1)]\sum_{s=0}^{\infty} g^{2s} = [1+g/(n-1)]/(1-g^2)$$

for all $j \neq k$. Hence, it is still the case that, for any $n \geq 3$, the central agent has the highest impact centrality and reaction centrality if $g > 0$ and the lowest impact centrality and reaction centrality if $g < 0$. However, now $a_j(\mathbf{G})$ is decreasing in n if $g > 0$ and increasing in n if $g < 0$.

All other agents are out-neighbors of the central agent. In addition, as the central agent is the only in-neighbor of the other agents, the other agents respond relatively sensitively to any increase in the central agent's demand for x. In contrast, the central agent is the only out-neighbor of each of the other agents. In addition, as all other agents are in-neighbors of the central agent, the central agent does not respond as sensitively to any increase in any of the other agents' demand for x.

Example 8 Consider the star graph $\mathbf{G}$ in Figure 2.2 in which $g_{ij} = g/n_i^-(\mathbf{G})$ for all (j, i) in A. The reaction centrality and impact centrality of agent 1, the central agent, is equal to $(1 + 3g)/(1 - g^2)$ compared to that of each of the other agents, which is equal to $(1 + g/3)/(1 - g^2)$.

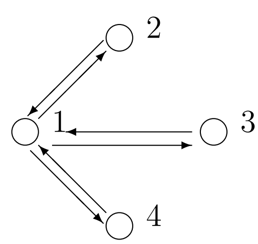

Figure 2.2: A star graph

A *acyclic digraph* is a digraph without a cycle so that every walk in the digraph has distinct agents. The agents in a acyclic digraph can be indexed in a *acyclic ordering* such that $i < j$ for each (j, i) in A. Hence, any acyclic digraph can be represented by a strictly upper triangular matrix $\mathbf{G}$. Since a strictly upper triangular matrix is nilpotent, all its eigenvalues are zero and so the convergence condition $\rho(\mathbf{G}) < 1$ is always satisfied for any acyclic digraph. Note that every acyclic digraph has at least one agent who has no out-neighbors and at least one agent who has no in-neighbors. A *tournament* is an asymmetric and complete digraph, that is, for every $j \neq i$, either (j, i) or (i, j) in A but not both. A *acyclic tournament* is a tournament without a cycle. A tournament is acyclic if and only if it is transitive. A digraph is transitive if, for every distinct i, j, and k in N, (k, j) and (j, i) in A implies that (k, i) in A. Since a acyclic tournament is a binary relation that is transitive, antisymmetric, and total, it represents, in the context of a social network, a linear order on N ranked according to the number of out-neighbors that each agent has.

Theorem 9 *Consider any acyclic tournament* $\mathbf{G}$ *such that* $g_{ij} = g$ *for all* (j, i) *in* A. *Then* $a_k(\mathbf{G}) = (1+g)^{n_k^+(\mathbf{G})}$ *and* $b_k(\mathbf{G}) = (1+g)^{n_k^-(\mathbf{G})}$ *for all* k.

Proof. In the acyclic ordering of agents, $\mathbf{G}$ is a strictly upper triangular matrix with $g_{ij} = g$ for all $i < j$ and $g_{ij} = 0$ for all $i \geq j$. First, consider any pair of agents j and $i \geq j$. Since there is no walk from any agent j to any agent $i > j$, $m_{ij}(\mathbf{G}) = 0$ for $i > j$. Since $m_{ij}(\mathbf{G}) = 1$ for $i = j$, sum over all agents $i \geq j$ to obtain $\sum_{i \geq j} m_{ij}(\mathbf{G}) = 1$, and sum over all agents $j \leq i$ to obtain $\sum_{j \leq i} m_{ij}(\mathbf{G}) = 1$. Next, consider any pair of agents j and $i < j$. The number of ways to choose, regardless of order, $s-1$ agents from among the $j-i-1$ agents that are indexed between i and j is $\binom{j-i-1}{s-1}$. Hence, the number of walks of length $s \geq 1$ from any agent j to any agent $i < j$ is equal to $\binom{j-i-1}{s-1}$ where $1 \leq s \leq j-i$. Note that there are no walks of length $s > j-1$ from any j to any $i < j$. Therefore, the total of the marginal effects through all walks that begin at j and end at $i < j$ is $\sum_{s=1}^{j-i} \binom{j-i-1}{s-1} g^s = g \sum_{s=0}^{j-i-1} \binom{j-i-1}{s} g^s$. Apply the binomial theorem to obtain $m_{ij}(\mathbf{G}) = g(1+g)^{j-i-1}$ for $i < j$. Sum over all agents $i < j$ to obtain

$$\sum_{i<j} m_{ij}(\mathbf{G}) = \sum_{i=1}^{j-1} g(1+g)^{j-i-1} = g\sum_{i=0}^{j-2}(1+g)^{i} = (1+g)^{j-1} - 1,$$

and sum over all agents $j > i$ to obtain

$$\sum_{j>i} m_{ij}(\mathbf{G}) = \sum_{j=i+1}^{n} g(1+g)^{j-i-1} = g\sum_{j=0}^{n-i-1}(1+g)^{j} = (1+g)^{n-i} - 1.$$

Hence,

$$a_k(\mathbf{G}) = \sum_{i\in N} m_{ik}(\mathbf{G}) = (1+g)^{k-1} = (1+g)^{n_k^{+}(\mathbf{G})}$$

and

$b_k(\mathbf{G}) = \sum_{i\in N} m_{ki}(\mathbf{G}) = (1+g)^{n-k} = (1+g)^{n_k^{-}(\mathbf{G})}$. ■

That is, in the acyclic ordering of agents for any acyclic tournament where all arc weights are equal, the impact centrality of agent k is $1+g$ times that of agent $k-1$ and the reaction centrality of agent $k-1$ is $1+g$ times that of agent k.

2.4 Adding or Deleting Arcs

I consider in this section the addition of an arc to a digraph and the deletion of an arc from a digraph. I show that the effect of such changes to the network depend on the impact centrality and the reaction centrality of certain agents in the digraph.

Certain arcs may be considered more important than others in the sense that they play a key role in the transmission of consumption spill-overs through the network. A seller may wish to take care in preserving such arcs for a product she seeks to promote just as a policymaker may wish to hamper such arcs for a product that she seeks to discourage. Note that the effect of a unit exogenous change in every agent's demand for x on equilibrium aggregate demand for x is equal to the aggregate impact (or reaction) centrality $\sum_{i\in N} a_i(\mathbf{G}) = \sum_{i\in N} b_i(\mathbf{G})$. Let $\mathbf{G}_{-ij}$ represent the digraph obtained by deleting arc (j,i) from $\mathbf{G}$.

Lemma 4 *Consider any digraph* $\mathbf{G}$ *such that* $\rho(\mathbf{G}) < 1$. *Then*

$$b_j(\mathbf{G}) = b_j(\mathbf{G}_{-ij})[1 + g_{ij}m_{ji}(\mathbf{G})]$$

for all (j,i) *in* A.

Proof. From Lemma 1, the effect of a unit exogenous change in agent k's demand for x on agent j's equilibrium demand for x in $\mathbf{G}$ is equal to $m_{jk}(\mathbf{G})$. It is the marginal effect through all walks in $\mathbf{G}$ from k to j. Consider the removal from $\mathbf{G}$ of any (j,i) in A. The effect of a unit exogenous change in agent k's demand for x on agent j's equilibrium demand for x in $\mathbf{G}_{-ij}$ is equal to $m_{jk}(\mathbf{G}_{-ij})$. It is the marginal effect through all walks in $\mathbf{G}_{-ij}$ from k to j. The only difference between the walks from k to j in $\mathbf{G}$ and the walks from k to j in $\mathbf{G}_{-ij}$ is that the walks from k to j in $\mathbf{G}$ includes all walks from k to j that pass through (j,i). The difference is the marginal effect through all walks from k to j in $\mathbf{G}_{-ij}$, followed by the arc (j,i), and then all walks from i to j in $\mathbf{G}$. This is equal to $m_{ji}(\mathbf{G})g_{ij}m_{jk}(\mathbf{G}_{-ij})$. Hence $m_{jk}(\mathbf{G}) = m_{jk}(\mathbf{G}_{-ij})[1+g_{ij}m_{ji}(\mathbf{G})]$. Sum over all k in N to obtain $b_j(\mathbf{G}) = b_j(\mathbf{G}_{-ij})[1 + g_{ij}m_{ji}(\mathbf{G})]$. ■

Definition 4 Consider any digraph $\mathbf{G}$ such that $\rho(\mathbf{G}) < 1$. The *arc importance* of (j,i) in A is

$$c_{-ij}(\mathbf{G}) = \frac{g_{ij}b_j(\mathbf{G})a_i(\mathbf{G})}{1 + g_{ij}m_{ji}(\mathbf{G})}.$$

Theorem 10 *Consider any digraph* $\mathbf{G}$ *such that* $\rho(\mathbf{G}) < 1$. *Then*

$$\sum_{k \in N} [a_k(\mathbf{G}) - a_k(\mathbf{G}_{-ij})] = c_{-ij}(\mathbf{G})$$

for all (j, i) *in* A.

Proof. Deleting arc (j, i) from $\mathbf{G}$ affects the impact centrality of only those agents for whom there is a walk in $\mathbf{G}$ that begins at the agent and ends at j. The total marginal effect through all walks that end at j without passing through (j, i) in the process is equal to $b_j(\mathbf{G}_{-ij})$. The marginal effect through all walks that begin at i, including those that pass through (j, i), is equal to $a_i(\mathbf{G})$. Hence, the decrease in aggregate impact centrality from deleting arc (j, i) is equal to $\sum_{k \in N}[a_k(\mathbf{G}) - a_k(\mathbf{G}_{-ij})] = g_{ij} b_j(\mathbf{G}_{-ij}) a_i(\mathbf{G})$.

Now use Lemma 4 to obtain $\sum_{k \in N}[a_k(\mathbf{G}) - a_k(\mathbf{G}_{-ij})] = c_{-ij}(\mathbf{G})$. ■

That is, the decrease in aggregate impact centrality from deleting arc (j, i) is equal to $c_{-ij}(\mathbf{G})$. Hence, the change in aggregate impact centrality from deleting an arc is a function of the arc weight, the reaction centrality of the agent from whom the arc begins, the impact centrality of the agent to whom the arc ends, and the feedback effects through the arc. Note that each arc importance in any digraph $\mathbf{G}$ depends only on $\mathbf{M}(\mathbf{G})$ and thus can be determined without having to compute the aggregate impact centrality for each $\mathbf{G}_{-ij}$.

The *critical link* is the arc (j^*, i^*) the deletion of which results in the greatest decrease in aggregate impact centrality, that is $c_{-i^* j^*}(\mathbf{G}) \geq c_{-ij}(\mathbf{G})$ for all (j, i) in A.

Example 11 Consider the digraph $\mathbf{G}$ in Figure 2.3 in which $g_{ij} = 1/2$ for all (j, i) in A. Since every agent has one out-neighbor, from Theorem 2, all agents have impact centrality equal to 2. Removing $(4,2)$ would decrease aggregate impact centrality by as much as removing $(1,2)$ even though agent 4 has a higher reaction centrality than agent 1. This is because there is a walk from agent 2 to agent 4 but none from agent 2 to agent 1. However, neither arc is a critical link. Instead, the unique critical link in this example is $(2,3)$ because agent 2 has the highest reaction centrality.

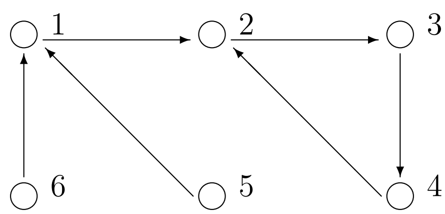

Figure 2.3: A digraph

Out of all the possible new arcs that can be added to a network, certain arcs may be considered more important than others in the sense that the addition of such new arcs could significant boost the transmission of consumption spill-overs through the network. A seller may wish to develop certain new links between her customers for a product she seeks to promote. Similarly, a policymaker may want to facilitate such links for a product that she seeks to discourage. Let $\mathbf{G}_{+ij}$ represent the digraph obtained by adding arc (j,i) with weight $g_{ij} = g \neq 0$ to $\mathbf{G}$.[11]

Lemma 5 *Consider any digraph* $\mathbf{G}$ *such that* $\rho(\mathbf{G}) < 1$*. Then*

$$a_i(\mathbf{G}) = a_i(\mathbf{G}_{+ij})[1 - g_{ij} m_{ji}(\mathbf{G})]$$

for all (j,i) *not in* A*.*

[11] Alternatively, if agents respond to the average consumption of their in-neighbors, it may be appropriate to set $g_{ij} = g/n_i^-(\mathbf{G})$ instead.

Proof. From Lemma 1, the effect of a unit exogenous change in agent i's demand for x on agent k's equilibrium demand for x in $\mathbf{G}$ is equal to $m_{ki}(\mathbf{G})$. It is the marginal effect through all walks in $\mathbf{G}$ from i to k. Consider the addition to $\mathbf{G}$ of any (j, i) not in A. The effect of a unit exogenous change in agent i's demand for x on agent k's equilibrium demand for x in $\mathbf{G}_{+ij}$ is equal to $m_{ki}(\mathbf{G}_{+ij})$. It is the marginal effect through all walks in $\mathbf{G}_{+ij}$ from i to k. The only difference between the walks from i to k in $\mathbf{G}$ and the walks from i to k in $\mathbf{G}_{+ij}$ is that the walks from i to k in $\mathbf{G}$ do not include all walks from i to k that pass through (j, i). This difference is the marginal effect through all walks from i to j in $\mathbf{G}$, followed by the arc from j to i, and then all walks from i to k in $\mathbf{G}_{+ij}$. This is equal to $m_{ki}(\mathbf{G}_{+ij})g_{ij}m_{ji}(\mathbf{G})$. Hence $m_{ki}(\mathbf{G}) = m_{ki}(\mathbf{G}_{+ij})[1 - g_{ij}m_{ji}(\mathbf{G})]$. Sum over all k in N to obtain $a_i(\mathbf{G}) = a_i(\mathbf{G}_{+ij})[1 - g_{ij}m_{ji}(\mathbf{G})]$. ■

Definition 5 Consider any digraph $\mathbf{G}$ such that $\rho(\mathbf{G}) < 1$. The *arc impact* of (j, i) not in A is

$$c_{+ij}(\mathbf{G}) = \frac{g_{ij}b_j(\mathbf{G})a_i(\mathbf{G})}{1 - g_{ij}m_{ji}(\mathbf{G})}$$

if $|g_{ij}m_{ji}(\mathbf{G})| < 1$.

Theorem 12 *Consider any digraph* $\mathbf{G}$ *such that* $\rho(\mathbf{G}) < 1$*. Then*

$$\sum_{k \in N} [a_k(\mathbf{G}_{+ij}) - a_k(\mathbf{G})] = c_{+ij}(\mathbf{G})$$

for all (j, i) *not in* A*.*

Proof. Adding arc (j, i) to $\mathbf{G}$ affects the impact centrality of only those agents for whom there is a walk in $\mathbf{G}$ that begins at the agent and ends at j. The total marginal effect through all walks that end at j without passing through (j,i) in the process is equal to $b_j(\mathbf{G})$. The marginal effect through all walks that begin at i, including those that pass through (j,i), is equal to $a_i(\mathbf{G}_{+ij})$. Hence, the increase in aggregate impact centrality from adding arc (j,i) is equal to $\sum_{k\in N}[a_k(\mathbf{G}_{+ij}) - a_k(\mathbf{G})] = g_{ij}b_j(\mathbf{G})a_i(\mathbf{G}_{+ij})$. Now use Lemma 5 to obtain $\sum_{k\in N}[a_k(\mathbf{G}_{+ij}) - a_k(\mathbf{G})] = c_{+ij}(\mathbf{G})$. ■

That is, the increase in aggregate impact centrality from adding arc (j,i) is equal to $c_{+ij}(\mathbf{G})$. Hence, the change in aggregate impact centrality from deleting an arc is a function of the arc weight, the reaction centrality of the agent from whom the arc begins, the impact centrality of the agent to whom the arc ends, and the feedback effects through the arc. Note that each arc impact in any digraph $\mathbf{G}$ depends only on $\mathbf{M}(\mathbf{G})$ and thus can be determined without having to compute the aggregate impact centrality for each $\mathbf{G}_{+ij}$.

The *promising link* is the arc (j^*,i^*) the addition of which, with weight $g_{ij} = g \neq 0$, results in the greatest increase in aggregate impact centrality, that is $c_{+i^*j^*}(\mathbf{G}) \geq c_{+ij}(\mathbf{G})$ for all (j,i) not in A.

Corollary 2 *Consider any undirected graph* $\mathbf{G}$ *such that* $\rho(\mathbf{G}) < 1$. *Then* $c_{-ij}(\mathbf{G}) = g_{ij}b_j(\mathbf{G})b_i(\mathbf{G})/[1 + g_{ij}m_{ij}(\mathbf{G})]$ *for all* (j,i) *in* A. *And if* $|g_{ij}m_{ji}(\mathbf{G})| < 1$, *then* $c_{+ij}(\mathbf{G}) = g_{ij}b_j(\mathbf{G})b_i(\mathbf{G})/[1 - g_{ij}m_{ij}(\mathbf{G})]$ *for all* (j,i) *not in* A.

Proof. The proof follows from Lemma 2. ■

Corollary 3 *Consider any digraph* $\mathbf{G}$ *such that* $\rho(\mathbf{G}) < 1$ *and* $g_{ij} > 0$ *for all* (j,i). *Then* $c_{-ij}(\mathbf{G}) \geq g_{ij} > 0$ *for all* (j,i) *in* A. *And if* $|g_{ij}m_{ji}(\mathbf{G})| < 1$, *then* $c_{+ij}(\mathbf{G}) \geq g_{ij} > 0$ *for all* (j,i) *not in* A.

Proof. If $g_{ij} > 0$ for all (j, i) in A, then $\mathbf{M}(\mathbf{G}) \geq \mathbf{I}$ and so $m_{ji}(\mathbf{G}) \geq 0$ for all (j, i) in A. The proof then follows from Lemma 3. ■

That is, if there are no strategic substitutes, then deleting any (j, i) in A results in a decrease in aggregate impact centrality that is greater or equal to the weight of (j, i) in A, and adding any (j, i) not in A results in an increase in aggregate impact centrality that is greater or equal to the weight of (j, i) not in A.

Note that $c_{-ij}(\mathbf{G})$ is decreasing in $m_{ji}(\mathbf{G})$ whereas $c_{+ij}(\mathbf{G})$ is increasing in $m_{ji}(\mathbf{G})$ if there are no strategic substitutes. That is, if there are no strategic substitutes, then removing an arc (j, i) results in a smaller reduction in aggregate impact centrality if the marginal effect through all walks from i to j is greater. This is because the greater the marginal effect through all walks from i to j, the smaller the marginal effect through all walks to j that do not pass through (j, i). In contrast, adding an arc (j, i) results in a greater increase in aggregate impact centrality if the marginal effect through all walks from i to j is greater. This is because the greater the marginal effect through all walks from i to j, the greater the marginal effect through all walks from i that pass through (j, i).

Corollary 4 *Consider any acyclic digraph* $\mathbf{G}$*. Then*

$$c_{-ij}(\mathbf{G}) = g_{ij} b_j(\mathbf{G}) a_i(\mathbf{G})$$

for all (j, i) *in* A*.*

Proof. Consider the removal of any (j, i) from $\mathbf{G}$. Since $\mathbf{G}$ has no cycles, there is no walk from i to j in $\mathbf{G}$. Hence, $m_{ji}(\mathbf{G}) = 0$ and so $c_{-ij}(\mathbf{G}) = g_{ij} b_j(\mathbf{G}) a_i(\mathbf{G})$ for all (j, i) in A. ■

Example 13 Consider the acyclic digraph $\mathbf{G}$ in Figure 2.4 in which $g_{ij} = 1/2$ for all (j, i) in A. Adding $(5, 3)$ with weight $g > 0$ would increase aggregate impact centrality more than by adding $(3, 5)$ with weight g even though agents 3 and 5 have equal impact centrality and reaction centrality in $\mathbf{G}$. This is because there is a walk from agent 3 to agent 5 but none from agent 5 to agent 3. However, $(5, 3)$ is not a promising link because of the relatively weak feedback effect from 3 to 5. Instead, the promising links in this example are $(3, 1)$, $(3, 2)$, $(4, 3)$, $(5, 4)$, $(6, 5)$, and $(7, 5)$, each with an arc impact equal to $4g/(1 - g/2)$. The critical links are $(3, 4)$ and $(4, 5)$ because agents 3, 4, and 5 have both high impact centrality and high reaction centrality in $\mathbf{G}$.

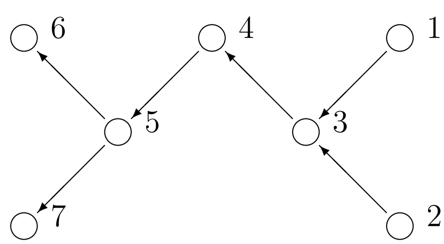

Figure 2.4: A acyclic digraph

Corollary 5 *Consider any acyclic tournament* $\mathbf{G}$ *such that* $g_{ij} = g$ *for all* (j, i) *in* A. *Then* $c_{-ij}(\mathbf{G}) = g(1 + g)^{n_j^-(\mathbf{G})+n_i^+(\mathbf{G})}$ *for all* (j, i) *in* A.

Proof. The proof follows from Theorem 9. ∎

Since $n_j^-(\mathbf{G}) + n_i^+(\mathbf{G}) = n - 1 + i - j$ in the acyclic ordering of agents, all the arcs represented on the superdiagonal of $\mathbf{G}$ are the critical links in $\mathbf{G}$ if $g > 0$. In contrast, arc $(n, 1)$ has the lowest arc importance because removing arc $(n, 1)$ affects only the impact centrality of n by the marginal effect of the walk $(n, 1)$.

Example 14 Consider the acyclic tournament $\mathbf{G}$ in Figure 2.5 in which $g_{ij} = g > 0$ for all (j, i) in A. From Corollary 5, the critical links in this example are $(4, 3)$, $(3, 2)$, and $(2, 1)$, each with an arc importance equal to $g(1 + g)^2$.

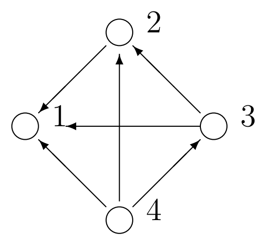

Figure 2.5: A acyclic tournament

2.5 Deleting Agents

I now consider the deletion of an agent from a digraph. Certain agents may be considered more important than others in the sense that they play a key role in the transmission of consumption spill-overs through the network. Removing an agent from a network may not only affect the aggregate impact centrality of the network directly by subtracting the agent's impact centrality but also affect the aggregate impact centrality of the network indirectly by affecting the impact centralities of other agents. Let $\mathbf{G}_{-i}$ represent the digraph obtained by deleting agent i from $\mathbf{G}$.

Lemma 6 *Consider any digraph* $\mathbf{G}$ *such that* $\rho(\mathbf{G}) < 1$*. Then*

$$b_i(\mathbf{G}) = m_{ii}(\mathbf{G})[1 + \sum_{j \in N_i^-(\mathbf{G})} g_{ij} b_j(\mathbf{G}_{-i})]$$

for all i*.*

Proof. From Lemma 1, the effect of a unit exogenous change in agent k's demand for x on agent i's equilibrium demand for x in $\mathbf{G}$ is equal to $m_{ik}(\mathbf{G})$. It is the marginal effect through all walks in $\mathbf{G}$ from k to i. Note that each of these walks in $\mathbf{G}$ from k to i must end with an arc from an in-neighbor of i to i himself. All the walks in $\mathbf{G}$ from k to i can thus be expressed as the total marginal effect of all walks in $\mathbf{G}_{-i}$ from k to each j in $N_i^-(\mathbf{G})$, followed by the walk (j, i), and then all the walks in $\mathbf{G}$ from i to himself. Hence $m_{ik}(\mathbf{G}) = m_{ii}(\mathbf{G}) \sum_{j \in N_i^-(\mathbf{G})} g_{ij} m_{jk}(\mathbf{G}_{-i})$. Sum over all $k \neq i$ to obtain $\sum_{k \neq i} m_{ik}(\mathbf{G}) = m_{ii}(\mathbf{G}) \sum_{j \in N_i^-(\mathbf{G})} g_{ij} b_j(\mathbf{G}_{-i})$. Add the term $m_{ii}(\mathbf{G})$ to both sides of the equation to obtain $b_i(\mathbf{G}) = m_{ii}(\mathbf{G})[1 + \sum_{j \in N_i^-(\mathbf{G})} g_{ij} b_j(\mathbf{G}_{-i})]$. ■

That is, the marginal effect through all walks that end at i is equal to the marginal effect through all walks that end at an in-neighbor of i, followed by the arc from the in-neighbor to i and followed by all walks from i to himself.

Definition 6 Consider any digraph $\mathbf{G}$ such that $\rho(\mathbf{G}) < 1$. Agent i's *intercentrality* is

$$c_i(\mathbf{G}) = \frac{b_i(\mathbf{G}) a_i(\mathbf{G})}{m_{ii}(\mathbf{G})}.$$

This definition of an agent's intercentrality corresponds with that of Remark 5 of Ballester, Calvo-Armengol, and Zenou (2006) for digraphs, except that here $\mathbf{G}$ is not necessarily a matrix with every entry restricted to a value between 0 and 1 and here arcs are not assumed to have all equal weight.

Theorem 15 *Consider any digraph* $\mathbf{G}$ *such that* $\rho(\mathbf{G}) < 1$*. Then*

$$\sum_{k \in N} [a_k(\mathbf{G}) - a_k(\mathbf{G}_{-i})] = c_i(\mathbf{G})$$

for all i*.*

Proof. The direct effect on the aggregate impact centrality of deleting agent i from $\mathbf{G}$ is the subtraction of i's impact centrality $a_i(\mathbf{G})$. In addition, there may be indirect effects on the aggregate impact centrality through the effects on the impact centralities of other agents in $\mathbf{G}$. These indirect effects are the marginal effects through all walks that pass through i from each $k \neq i$. Deleting agent i from $\mathbf{G}$ affects the impact centrality of only those agents for which there is a walk in $\mathbf{G}$ that begins at the agent and ends at an in-neighbor of i. Each such walk has to pass through an in-neighbor of i before it passes through i itself. For each j in $N_i^-(\mathbf{G})$, $b_j(\mathbf{G}_{-i})$ is equal to the marginal effect through all walks in $\mathbf{G}$ that end at j without passing through i in the process, and so the indirect effect on aggregate impact centrality from deleting i is equal to $\sum_{i \in N_i^-(\mathbf{G})} g_{ij} b_j(\mathbf{G}_{-i}) a_i(\mathbf{G})$. Since the reduction in aggregate impact centrality from deleting agent i from $\mathbf{G}$ is the sum of the direct and indirect effects,

$$\sum_{k \in N} [a_k(\mathbf{G}) - a_k(\mathbf{G}_{-i})] = a_i(\mathbf{G})[1 + \sum_{i \in N_i^-(\mathbf{G})} g_{ij} b_j(\mathbf{G}_{-i})].$$

Now use Lemma 5 to obtain $\sum_{k \in N} [a_k(\mathbf{G}) - a_k(\mathbf{G}_{-i})] = c_i(\mathbf{G})$. ■

That is, the change in aggregate impact centrality from deleting agent i is equal to $c_i(\mathbf{G})$. Hence, the change in aggregate impact centrality from deleting an agent is the sum of the agent's impact centrality and the reduction in other agents' impact centrality from the deletion of the arcs that begin or end with the agent who is deleted from the network. Note that each agent's intercentrality in any digraph $\mathbf{G}$ depends only on $\mathbf{M}(\mathbf{G})$ and thus can be determined without having to compute the aggregate impact centrality for each $\mathbf{G}_{-i}$.

The *key player* is the agent i^* the deletion of whom results in the greatest reduction to aggregate impact centrality, that is $c_{i^*}(\mathbf{G}) \geq c_i(\mathbf{G})$ for all i in N.

Corollary 6 *Consider any undirected graph* $\mathbf{G}$ *such that* $\rho(\mathbf{G}) < 1$. *Then* $c_k(\mathbf{G}) = [b_k(\mathbf{G})]^2/m_{kk}(\mathbf{G})$ *for all* k.

Proof. The proof follows from Lemma 2. ■

This result corresponds with that in Ballester, Calvo-Armengol, and Zenou (2006) for undirected graphs, except that here arc weights are not assumed to be all equal, and there can be both strategic complements and strategic substitutes.

Corollary 7 *Consider any digraph* $\mathbf{G}$ *such that* $\rho(\mathbf{G}) < 1$ *and* $g_{ij} > 0$ *for all* (j, i) *in* A. *Then* $c_k(\mathbf{G}) \geq 1$ *for all* k.

Proof. If $g_{ij} > 0$, then $\mathbf{M}(\mathbf{G}) \geq \mathbf{I}$ and so $m_{ii}(\mathbf{G}) \geq 1$. The proof then follows from Lemma 3. ■

That is, if there are no strategic substitutes, then removing any agent k from $\mathbf{G}$ would reduce the aggregate impact centrality by at least one.

Corollary 8 *Consider any acyclic digraph* $\mathbf{G}$. *Then* $c_i(\mathbf{G}) = b_i(\mathbf{G})a_i(\mathbf{G})$ *for all* i.

Proof. Consider the removal of any i from $\mathbf{G}$. Since $\mathbf{G}$ has no cycles, there is no walk of length $s \geq 1$ from i to itself. Hence, $m_{ii}(\mathbf{G}) = 1$ and so $c_i(\mathbf{G}) = b_i(\mathbf{G})a_i(\mathbf{G})$ for all i. ■

That is, for any acyclic digraph, every agent's intercentrality is simply the product his reaction centrality and impact centrality.

Corollary 9 *Consider any acyclic tournament* $\mathbf{G}$ *such that* $g_{ij} = g$ *for all* (j, i) *in* A. *Then* $c_i(\mathbf{G}) = (1 + g)^{n-1}$ *for all* i.

Proof. From Theorem 9, $c_i(\mathbf{G}) = (1+g)^{n_i^-(\mathbf{G})+n_i^+(\mathbf{G})}$ for all i. Since $n_i^-(\mathbf{G})+n_i^+(\mathbf{G}) = n-1$, hence $c_i(\mathbf{G}) = (1+g)^{n-1}$ for all i. ■

That is, for any acyclic tournament such where all arc weights are equal, all agents have equal intercentrality. Although the number of out-neighbors and the number of in-neighbors that each agent has varies across agents, all agents have an equal number of neighbors. In a acyclic tournament, an agent who contributes relatively more to aggregate impact centrality directly through a relatively higher impact centrality contributes relatively less to aggregate impact centrality indirectly through a relatively lower reaction centrality so that all agents make the same total direct and indirect contribution to aggregate impact centrality.

2.6 Extensions

The measures suggest that estimating $\mathbf{G}$ should be of considerable interest to both businesses and policy-makers. Where information of the weighted adjacency matrix is imperfect or unavailable but the adjacency matrix is known, interested parties may still find it useful to determine an estimate of the proposed measures by assuming appropriate functional forms for each g_{ij}. Future research may unravel important principles and relationships based on appropriate functional forms for g_{ij}, thus providing additional guidance without full knowledge of the weighted adjacency matrix.

Although this paper focuses on consumer networks and analyzes the effects of changes in demand, the arc importance, arc impact, and intercentrality measures employed here can be similarly applied to other social networks, which analyze action levels with linear-quadratic utility function.

Consider an undirected graph. An undirected link is a pair of arcs (j, i) and (i, j) for any $j \neq i$. Removing an undirected link would be equivalent to removing such a pair of arcs, and adding an undirected link would be equivalent to adding such a pairs of arcs. One can then determine the critical undirected links and the promising undirected links in an undirected graph.

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Paper 3

Bargaining under Two-Sided Incomplete Information: Efficient Outcomes with Multiple Traders

Abstract

Consider a market for an indivisible good comprising multiple buyers and at least as many sellers. Sellers and buyers each have independent private costs or valuations of two types. There is at least one low type buyer and this is common knowledge. I show that there exists a trading mechanism that fully implements in sequential equilibrium the social choice correspondence that satisfies ex post efficiency, ex post budget balance, and ex post individual rationality.

3.1 Introduction

I propose a solution to the problem of designing a trading mechanism to guarantee efficient outcomes, when there are multiple buyers and sellers with independent private values such that each trader either has a low or a high reservation price. I show that such a mechanism exists if there are at least as many sellers as there are buyers, and there is at least one buyer with a low reservation price. The implementation of efficient trades under these assumptions suggests that it is not necessary for there to be many buyers and many sellers in order for a market to guarantee efficient outcomes with two-sided incomplete information. The mechanism also provides a descriptive account of how efficient trades arise from a systematic and organized form of dynamic bargaining in which buyers outbid one another.

A key issue in the mechanism design literature concerns the difficulty of obtaining efficient outcomes when agents have private information and individual rationality constraints are binding [Fudenberg and Tirole (1991), page 245]. When an agent's value for a good is his private information, he may have an incentive to lie about his value for the good, or the behave as if his value for the good is different, in order to obtain more favorable terms of trade at the risk of not trading. However, as explained by Bolton and Dewatripont (2005) in the context of bilateral trading, the main difficulty is not to find incentive compatible prices per se, but to find incentive compatible prices that also satisfy individual rationality constraints. Budget balance is an important requirement as otherwise individual rationality can typically be satisfied by a sufficiently large transfer to ensure participation in the mechanism. Budget balance is also a natural requirement for bargaining problems where we expect transfers to be made between the agents who trade.

This paper relates to the literature on bargaining under two-sided incomplete information, which suggests that ex post efficient outcomes cannot generally be obtained for bilateral trading, and the multiplicity of equilibria is large. Fudenberg and Tirole (1983), Chatterjee and Samuelson (1983), Myerson and Satterthwaite (1983), Cramton (1984), Chatterjee and Samuelson (1987), Leininger, Linhart, and Radner (1989), Satterthwaite and Williams (1989), Cho (1990), Ausubel and Deneckere (1992), and Cramton (1992) considered the bargaining problem between a buyer and a seller for an indivisible good, when the buyer and the seller have independent private valuation or cost.

In the case of the buyer and the seller each having two potential types, Fudenberg and Tirole (1983) showed that there is a continua of pooling, separating, and hybrid equilibria with inefficient outcomes in a two-period bargaining game in which the seller makes two offers, or the traders alternate making offers. Chatterjee and Samuelson (1987) showed that there is a unique equilibrium with an inefficient outcome in an infinite-horizon bargaining game in which the traders alternate making offers. As in Fudenberg and Tirole (1983) and Chatterjee and Samuelson (1987), I consider the case of each trader having two potential types. However, I do not assume bargaining costs in the form of time discounting, and unlike Fudenberg and Tirole (1983), I do not assume that there is always potential gains from trade.

In the case of the buyer and the seller each having a continuum of types, Chatterjee and Samuelson (1983) showed that one cannot generally obtain ex post efficient outcomes through a double auction, in which the buyer and the seller simultaneously bid for the good and there is trade if the buyer's bid exceeds the seller's. More generally, Myerson and Satterthwaite (1983) showed that, if there is both a positive probability of gains from trade and a positive probability of no gains from trade, there exists no mechanism that can result in an ex post efficient allocation of the good while also satisfying interim individual rationality and ex post budget balance. Cramton (1984), Cho (1990), Ausubel and Deneckere (1992), and Cramton (1992) showed that there is multiple equilibria in an infinite-horizon bargaining game in which the seller makes all offers, or the traders alternate making offers. Satterthwaite and Williams (1989) showed that, if both traders can influence the price, no equilibrium of the double auction is ex ante efficient. In the case of uniform prior beliefs over types, Leininger, Linhart, and Radner (1989) showed that the set of pure strategy equilibria for a double auction is very large, and that the expected gains from trade may be as low as zero.

Although the incentive to misrepresent may diminish with a large number of traders, the literature suggests that ex post efficient outcomes cannot generally be obtained even when there is a large but finite number of traders. Wilson (1985), Gresik and Satterthwaite (1989), and Rustichini, Satterthwaite, and Williams (1994) considered the bargaining problem between a large number of buyers and a large number of sellers for an indivisible good, when the buyers and the sellers have independent private valuations or costs. Wilson (1985) showed that, if there are sufficiently many buyers and sellers, a double auction is interim efficient, in the sense that there is no other trading mechanism that is sure to increase every trader's expected gains from trade. His result assumed the existence of an equilibrium in symmetric strategies that are differentiable functions of reservation values and have uniformly bounded derivatives. Gresik and Satterthwaite (1989) showed that an ex ante efficient mechanism that satisfies interim individual rationality and ex post budget balance converges to ex post efficiency as the number of traders tends to infinity. Rustichini, Satterthwaite, and Williams (1994) showed that although a double auction results in multiple equilibria and inefficient outcomes, the extent of misrepresentation and thus the extent of inefficiency is small, when the number of traders is large.

In contrast with the conventional wisdom that approximate efficiency is possible only with a large number of traders, I show that appropriate restrictions on the preference domain ensure ex post efficiency even with a small number of traders, despite uncertainty over potential gains from trade.

The revelation principle provides a necessary condition for full implementation. It proposes that each equilibrium of any mechanism is equivalent to the truth-telling equilibrium of an incentive-compatible direct-revelation mechanism [Gibbard (1973); Green and Laffont (1977); Dasgupta, Hammond, and Maskin (1979); Myerson (1979)]. In the truth-telling equilibrium, no agent can do better by lying, if he anticipates that all other agents will be honest. The revelation principle implies that no mechanism can fully implement a social choice rule unless it is incentive compatible. However, the revelation principle does not address the problem of multiple equilibria arising from a direct-revelation mechanism. Incentive compatibility requires only that the truth-telling equilibrium be one equilibrium of the direct-revelation mechanism. But there may be other equilibria of the direct-revelation mechanism, which may not correspond to equilibria of the original mechanism. Hence, although the revelation principle is useful in determining if a social choice rule can be weakly implemented, it does not ensure that a social choice rule can be fully implemented. Although weak implementation requires only that every desired outcome be an equilibrium outcome, full implementation requires in addition that every equilibrium outcome be a desired outcome.

I consider a market for an indivisible good comprising multiple buyers and at least as many sellers. Sellers and buyers each have independent private costs or valuations of two types. There is at least one low type buyer and this is common knowledge. I show that there exists an extensive-form mechanism that fully implements in sequential equilibrium the social choice correspondence that satisfies ex post efficiency, ex post budget balance, and ex post individual rationality.

The implementing mechanism is a multi-stage game of observed actions and incomplete information. All buyers have an opportunity to indicate if they are wiling to buy at the low price. If no buyer indicates that he is willing to buy at the low price, then the game ends with no trade. If there are buyers who indicate that they are wiling to buy at the low price ("low price buyers"), then all sellers have an opportunity to indicate if they are willing to sell at the low price or higher ("low price sellers"). If there are sellers who indicate that they are willing to sell at the low price, then the low price buyers are randomly matched pairwise with the low price sellers. If there are at least as many low price sellers as there are low price buyers, then the game ends with the matched buyers are sellers trading at the low price. However, if there are more low price buyers than there are low price sellers, then all the unmatched low price buyers have an opportunity to indicate if they are willing to buy at the high price or lower. If there are unmatched low price buyers who indicate that they are willing to buy at the high price or lower ("high price buyers"), then all matched sellers are randomly matched pairwise with the high price buyers. But this results in low price buyers who were matched with these sellers to become unmatched. These low price buyers then have the opportunity to indicate if they are willing to buy at the high price or lower. The sellers, who had indicated that they were unwilling to sell at the low price, have an opportunity to indicate if they are willing to sell at the high price ("high price sellers") only if there are unmatched high price buyers in which case they would be randomly matched pairwise with the unmatched high price buyers.

The mechanism provides traders with the incentive to behave according to their preferences. It provides a low valuation buyer with an incentive to be a low price buyer because he would have a positive probability of being matched at the low price.

He has no incentive to be a high price buyer as he would obtain a negative payoff if he were to be matched at the high price. A high valuation buyer also has an incentive to be a low price buyer as he would otherwise not have an opportunity to trade. He has an incentive to be a high price buyer as he would have a positive probability of being matched at the high price. A low cost seller has an incentive to be a low price seller because the mechanism provides low price sellers with priority over the other sellers to be matched with high price buyers at the high price. The low cost seller also has to consider that her decision to be a low price seller would raise the probability of the game ending with her trading at the low price. However, she also takes into account the fact that the game may end before she has the opportunity to be a high price seller, and that she may face competition from other high price sellers if she does have the opportunity. A high cost seller in choosing to be a low price seller would increase her probability of being matched with a high price buyer at a high price. However, she would in doing so also increase the risk of the game ending with her trading at the low price. Furthermore, her potential gain from trading at the high price is almost nothing compared with her potential loss from trading at the low price. This prevents the high cost seller from mimicking the low cost seller.

The proof for the implementation of the desired social choice correspondence begins by showing that, if given the opportunity to do so in the game, each seller would indicate that she is willing to sell at the high price. Given this, I show that, if given the opportunity to do so in the game, each low valuation buyer would indicate that he is unwilling to buy at the high price, and each high valuation buyer would indicate that he is willing to buy at the high price. It appears that, if given the opportunity to do so in the game, a high cost seller may have an incentive to be a low price seller if she believes that this would give her a higher probability of being matched with a buyer at the high price. Furthermore, it appears that, if given the opportunity to do so in the game, a low cost seller may have an incentive to opt not to be a low price seller if she believes that this would increase the probability that there would be unmatched high price buyers. However, I show that a high cost seller does not have an incentive to be a low price seller, and a low cost seller does not have an incentive to opt not to be a low price seller. I then show that all high valuation buyers would choose to be low price buyers. Given this, I show that all low valuation buyers would also choose to be low price buyers.

If there are fewer sellers than there are buyers, and if sellers have prior beliefs that the probability of high valuation buyers is relatively high, then some low cost sellers may find it optimal not to be low price sellers, believing that it is likely that there will be relatively many high price buyers. Similarly, if it is not certain that there is at least one low valuation buyer, then some low cost sellers may find it optimal not to be low price sellers to lower the risk of causing the game to end with no high price buyers.

Since all traders have an incentive to behave according to their true preferences in the implementing mechanism, an alternative direct-revelation mechanism can be used to achieve the same equilibrium outcome. In this direct-revelation mechanism, all traders simultaneously announce their types to the principal who then makes the moves according to the equilibrium strategies in the implementing mechanism. This alternative mechanism avoids the risk of play off-the-equilibrium path resulting in renegotiation.

Section 3.2 describes the economic environment and the social choice rule that satisfies ex post efficiency, ex post budget balance, and ex post individual rationality. Section 3.3 describes the implementing mechanism and demonstrates that it implements the social choice rule in sequential equilibrium. Section 3.4 concludes with some possible extensions.

3.2 The Model

Consider a non-empty finite set of sellers $M = \{1, \ldots, m\}$ and a non-empty finite set of buyers $N = \{1, \ldots, n\}$. There are at least as many seller as there are buyers, that is $m \geq n$. Each seller has one unit of an indivisible homogeneous good for sale, and each buyer is looking to buy one unit of the good.

Outcomes are comprised of allocation of the goods available for sale and money transfers from buyers to sellers. For any seller j in M, let $t_j \in \mathbb{R}$ denote the transfer that seller j receives, and let $a_j \in \{0, 1\}$ denote the quantity of the good allocated away from seller j. The profile of transfers for sellers is $t = (t_1, \ldots, t_m)$ and the profile of allocation changes for sellers is $a = (a_1, \ldots, a_m)$. The set of transfer profiles for sellers is given by $\mathbb{R}^m$ and the set of allocation changes for sellers is given by $\{0, 1\}^m$. For any buyer i in N, let $\tilde{t}_i \in \mathbb{R}$ denote the transfer that buyer i pays, and let $\tilde{a}_i \in \{0, 1\}$ denote the quantity of the good allocated to buyer i. The profile of transfers for buyers is $\tilde{t} = (\tilde{t}_1, \ldots, \tilde{t}_n)$ and the profile of allocation changes for buyers is $\tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_n)$. The set of possible transfers for buyers is given by $\mathbb{R}^n$, and the set of possible allocation changes for buyers is given by $\{0, 1\}^n$.

Preferences are comprised of the valuations of buyers and the costs of seller. Sellers and buyers are risk neutral and have independent private costs or valuations. There are two types of sellers. For any seller j, let $c_i \in \{\underline{c}, \overline{c}\}$, where $\underline{c} < \overline{c}$, denote seller j's cost of the good. The prior probability of a low cost seller is equal to $q > 0$, and the prior probability of a high cost seller is equal to $1 - q > 0$. The profile of costs is $c = (c_1, \ldots, c_m)$, and the set of possible cost profiles is given by $\{\underline{c}, \overline{c}\}^m$. Let M_l denote the set of low cost sellers, and let M_h denote the set of high cost sellers. Let $\omega_j = t_j - c_j a_j$ denote seller j's quasilinear ex post payoff. There are two types of buyers but there is at least one low valuation buyer. Let buyer 1 be the low valuation buyer. Since valuations are private information, generally neither the other buyers nor the sellers can identify buyer 1.[1] For any other buyer i, let $v_i \in \{\underline{v}, \overline{v}\}$, where $\underline{v} < \overline{v}$, denote buyer i's valuation for the good. The prior probability of a low valuation buyer is equal to $p > 0$, and the prior probability of a high valuation buyer is equal to $1 - p > 0$. The profile of valuations is $v = (v_1, \ldots, v_n)$, and the set of possible valuation profiles is given by $\{\underline{v}, \overline{v}\}^n$. Let N_l denote the set of low valuation buyers, and let N_h denote the set of high valuation buyers. Let n_l denote the cardinality of N_l and let n_h denote the cardinality of N_h. Since there is at least one low valuation buyer, $n_l \geq 1$. Let $\tilde{\omega}_i = v_i \tilde{a}_i - \tilde{t}_i$ denote buyer i's quasiliner ex post payoff.

There are two reservation prices. For any low valuation buyer, the low reservation price $\underline{p}$ is the highest price for which he strictly prefers buying a unit of the good so that $\underline{v} - \underline{p} = \epsilon > 0$, where ϵ is infinitismally small. For any high valuation buyer, the high reservation price $\overline{p} > \underline{p}$ is the highest price for which he strictly prefers buying a unit of the good so that $\overline{v} - \overline{p} = \epsilon$. For any low cost seller, the low reservation price $\underline{p}$ is the lowest price for which she strictly prefers selling a unit of the good so that $\underline{p} - \underline{c} = \epsilon$. For any high cost seller, the high reservation price $\overline{p}$ is the lowest price for which she strictly prefers selling a unit of the good so that $\overline{p} - \overline{c} = \epsilon$. Note that $\overline{v} > \overline{c} > \underline{v} > \underline{c}$, which are the values for which establishing efficient trade is most difficult [Bolton and Dewatripont (2005)]. Since $\overline{v} > \underline{c}$ and $\overline{c} > \underline{v}$, there is both a

[1] The only exception is when there are only two buyers and buyer 2 is high type. In this case, although the sellers would still not be able to identify buyer 1, buyer 2 would be able to deduce that buyer 1 is low type.

positive probability of gains from trade and a positive probability of no gains from trade.

Definition 1 A *social choice correspondence* is a mapping $F : \{\underline{v}, \overline{v}\}^n \times \{\underline{c}, \overline{c}\}^m \to \{0,1\}^n \times \{0,1\}^m \times \mathbb{R}^n \times \mathbb{R}^m$.

That is, a social choice correspondence associates to each valuation profile and cost profile, a non-empty subset of feasible allocations and transfers.

Definition 2 A social choice correspondence is *ex post efficient* if every allocation $(\tilde{a}, a)$ of the social choice correspondence solves the social optimization problem

$$\max_{\tilde{a},a,\tilde{t},t} \sum_{i \in N} \tilde{\omega}_i(\tilde{a}_i, \tilde{t}_i) + \sum_{j \in M} \omega_j(a_j, t_j)$$

subject to

$$\sum_{i \in N} \tilde{a}_i = \sum_{j \in M} a_j$$

That is, a social choice correspondence is ex post efficient if every allocation of the social choice correspondence maximizes the total surplus of buyers and sellers subject to the allocation being feasible.

Definition 3 A social choice correspondence is *ex post budget balanced* if every transfer profile $(\tilde{t}, t)$ of the social choice correspondence satisfies

$$\sum_{i \in N} \tilde{t}_i = \sum_{j \in M} t_j.$$

Definition 4 A social choice correspondence is *ex post individually rational* if every allocation $(\tilde{a}, a, \tilde{t}, t)$ of the social choice correspondence satisfies

$$\tilde{\omega}_i(\tilde{a}_i, \tilde{t}_i) \geq 0$$

for all i in N, and

$$\omega_j(a_j, t_j) \geq 0$$

for all j in M.

Definition 5 The *Ideal Rule* is a social choice correspondence $W : V \times C \to \{0,1\}^n \times \{-1,0\}^m \times \mathbb{R}^n \times \mathbb{R}^m$ which satisfies ex post efficiency, ex post budget balance, and ex post individual rationality.

Definition 6 The Ideal Rule W is *implementable* in sequential equilibrium if there exists a mechanism, such that the set of sequential equilibrium outcomes equals the set of Ideal Rule outcomes in each possible state of the world.

The next section shows that there indeed exists a mechanim that implements the Ideal Rule in sequential equilibrium.

3.3 The Mechanism

The mechanism Γ^e is a multi-stage game of observed actions and incomplete information. At each stage of the game, buyers or sellers simultaneously choose from an action set of "yes" or "no" to trading at either $\underline{p}$ or $\overline{p}$. A buyer choosing "yes" to a price commits himself to buying a unit of the good at that price if he is matched with a seller at that price when the game ends. A buyer choosing "no" to a price commits himself to not buying a unit of the good at that price or higher, and is not required to take any further action. A seller choosing "yes" to a price commits himself to selling a unit of the good at that price or higher, and is not required to take any further action. A seller choosing "no" to a price commits herself to not selling a unit of the good at that price if she is matched to a buyer at that price when the game ends. Let $\underline{\gamma}$ and $\underline{\chi}$ denote the action of a buyer or seller choosing "yes" and "no" respectively to the price $\underline{p}$. Similarly, let $\overline{\gamma}$ and $\overline{\chi}$ denote the action of a buyer or seller choosing "yes" and "no" respectively to the price $\overline{p}$. All buyers and sellers observe the actions that have been chosen at the end of each stage, and there is perfect recall.

Stage 1: Each buyer simultaneously chooses from the action set $\{\underline{\gamma}, \underline{\chi}\}$. Let $\underline{N}$ denote the set of buyers who choose $\underline{\gamma}$ and let $\underline{n}$ denote the cardinality of $\underline{N}$.

1. If $\underline{n} = 0$, then the game ends with no trade.

2. If instead $\underline{n} > 0$, then the game proceeds to Stage 2.

Stage 2: Having observed the choices of all the buyers, each seller simultaneously chooses from the action set $\{\underline{\gamma}, \underline{\chi}\}$. Let $\underline{M}$ denote the set of sellers who choose y and let $\underline{m}$ denote the cardinality of $\underline{M}$.

1. If $\underline{m} = 0$, then all buyers in $\underline{N}$ are unmatched and the game proceeds to Stage 3.

2. If instead $\underline{m} > 0$, then all sellers in $\underline{M}$ are randomly matched pairwise with all buyers in $\underline{N}$.

   (a) If $\underline{n} \leq \underline{m}$, then the game ends with the matched buyers and sellers trading at the price of $\underline{p}$.

(b) If instead $\underline{n} > \underline{m}$, then there are unmatched buyers in $\underline{N}$ and the game proceeds to Stage 3.

Stage 3: Each unmatched buyer in $\underline{N}$ simultaneously chooses from the action set $\{\overline{\gamma}, \overline{\chi}\}$. Let $\overline{N}_1$ denote the set of buyers who choose $\overline{\gamma}$ at Stage 3 and let $\overline{n}_1$ denote the cardinality of $\overline{N}_1$.

1. If $\overline{n}_1 = 0$ and $\underline{m} = 0$, then the game ends with no trade.

2. If $\overline{n}_1 = 0$ and $\underline{m} > 0$, then the game ends with the matched buyers and sellers trading at the price of $\underline{p}$.

3. If $\overline{n}_1 > 0$ and $\underline{m} = 0$, then the game proceeds to Stage 5.

4. If $\overline{n}_1 > 0$ and $\underline{m} > 0$, then all buyers in $\overline{N}_1$ are randomly matched pairwise with sellers in $\underline{M}$.

   (a) If $\overline{n}_1 > \underline{m}$, then $\overline{n}_1 - \underline{m}$ buyers in $\overline{N}_1$ are unmatched, and $\underline{m}$ buyers who were matched with sellers at $\underline{p}$ become unmatched. The game proceeds to Stage 4.

   (b) If $\overline{n}_1 \leq \underline{m}$, then all $\overline{n}_1$ buyers in $\overline{N}_1$ are matched, and $\overline{n}_1$ buyers who were matched with sellers at $\underline{p}$ become unmatched. The game proceeds to Stage 4.

Stage 4: Each buyer who was matched at $\underline{p}$ but became unmatched at Stage 3 simultaneously chooses from the action set $\{\overline{\gamma}, \overline{\chi}\}$. Let $\overline{N}_2$ denote the set of buyers who choose $\overline{\gamma}$ at Stage 4 and let $\overline{n}_2$ denote the cardinality of $\overline{N}_2$.

1. If $\overline{n}_2 = 0$ and there is at least one buyer who has chosen $\overline{\gamma}$ but is unmatched, then the game proceeds to Stage 5.

2. If $\overline{n}_2 = 0$ and all buyers who have chosen $\overline{\gamma}$ are matched, then the game ends with all matched buyers and sellers trading at the prices that they are matched.

3. If $\overline{n}_2 > 0$, then all buyers in $\overline{N}_2$ are randomly matched pairwise with any sellers in $\underline{M}$ who are matched with a buyer at $\underline{p}$.

    (a) If there are no sellers in $\underline{M}$ who are matched with a buyer at $\underline{p}$, then the game proceeds to Stage 5.

    (b) If there is at least one seller in $\underline{M}$ who is matched with a buyer at $\underline{p}$, then all buyers in $\overline{N}_2$ are randomly matched pairwise with those sellers in $\underline{M}$ who are matched with a buyer at $\underline{p}$, and Stage 4 iterates until either the game ends at Stage 4 or proceeds to Stage 5.

Stage 5: Each seller who is not in $\underline{M}$ simultaneously chooses from the action set $\{\overline{\gamma}, \overline{\chi}\}$. Let $\overline{M}$ denote the set of sellers who choose $\overline{\gamma}$ and let $\overline{m}$ denote the cardinality of $\overline{M}$.

1. If $\overline{m} = 0$, then the game ends with the matched buyers and sellers trading at the price at which they are matched, i.e. $\overline{p}$.

2. If $\overline{m} > 0$, then the sellers in $\overline{M}$ are randomly matched pairwise with those buyers who chose $\overline{\gamma}$ but are unmatched. The game then ends with the matched buyers and sellers trading at the price at which they are matched, i.e. $\overline{p}$.

Let $\underline{\tau}$ denote any terminal node in which seller j trades at $\underline{p}$ and let $\overline{\tau}$ denote any terminal node in which seller j trades at $\overline{p}$. Hence, $\omega_j(\underline{\tau}) = \epsilon$ and $\omega_j(\overline{\tau}) = 1+\epsilon$ for all j in M_l and $\omega_j(\underline{\tau}) = -1+\epsilon$ and $\omega_j(\overline{\tau}) = \epsilon$ for all j in M_h. Note that $\omega_j(\overline{\tau}) = 1+\omega_j(\underline{\tau})$ for any j in M.

Let S_i denote the set of all possible information states for i in Γ^e. Assume that information states are labelled such that S_i and S_j are disjoint sets whenever $i \neq j$, and let $S^* = \bigcup_{i \in N+M} S_i$ denote the union of all these sets. For any i and any information state s in S_i, the set of actions or moves D_s available to i at information state s is either $\{\underline{\gamma}, \underline{\chi}\}$ or $\{\overline{\gamma}, \overline{\chi}\}$. Let $\Delta(D_s)$ denote the set of all probability distributions over the set D_s.

Definition 7 A *behavioral-strategy profile* or *scenario*

$$\sigma = (\sigma_i)_{i \in N+M} = (\sigma_{i.s})_{i \in N+M, s \in S_i} \in \times_{s \in S^*} \Delta(D_s) = \times_{i \in N+M} \times_{s \in S_i} \Delta(D_s)$$

specifies a probability distribution over the set of possible moves for each possible information state of each buyer or seller.

A behavioral strategy $\sigma_i = (\sigma_{i.s})_{s \in S_i} \in \times_{s \in S_i} \Delta(D_s)$ for i specifies a probability for each possible move at every information state s of i. If s is an information state in which i chooses from the set of moves $\{\underline{\gamma}, \underline{\chi}\}$, then $\sigma_{i.s} = (\sigma_{i.s}(\underline{\gamma}), \sigma_{i.s}(\underline{\chi})) \in \Delta(D_s)$; if s is an information state in which i chooses from the set of moves $\{\overline{\gamma}, \overline{\chi}\}$, then $\sigma_{i.s} = (\sigma_{i.s}(\overline{\gamma}), \sigma_{i.s}(\overline{\chi})) \in \Delta(D_s)$. The move probability $\sigma_{i.s}(\underline{\gamma})$ denotes the conditional probability under scenario σ that i would choose move $\underline{\gamma}$ if the path of play reached a node that is controlled by i with information state s.

Let Y_s denote the set of nodes belonging to i with information state s in S_i. For any information state s of any i, the belief-probability distribution $\pi_{i.s} \in \Delta(Y_s)$ for i at information state s is a probability distribution over Y_s, the set of nodes labelled "$i.s$". For each node x in Y_s, $\pi_{i.s}(x)$ is the conditional probability that i would assign to the event that he was making a move at node x, when he knew that he was making a move at some node in Y_s.

Definition 8 A *beliefs vector*

$$\pi = (\pi_{i.s})_{i \in N+M, s \in S_i} \in \times_{s \in S^*} \Delta(Y_s) = \times_{i \in N+M} \times_{s \in S_i} \Delta(Y_s)$$

specifies a belief-probability distribution for each information state of each buyer or seller.

Consider any scenario σ and any two nodes x and y. If y follows x, then let $\bar{P}(y|\sigma, x)$ denote the conditional probability that the path of play would go through y after x if all buyers and sellers chose their moves according to scenario σ and if the play of the game started at x. It is the multiplicative product of all the chance probabilities and move probabilities specified by σ for the branches on the path from x to y. If y does not follow x, then let $\bar{P}(y|\sigma, x) = 0$.

Let Ω denote the set of all terminal nodes in Γ^e and let $\omega_i(y)$ denote the payoff to i at terminal node y. Then $U_i(\sigma|x) = \sum_{y \in \Omega} \bar{P}(y|\sigma, x)\omega_i(y)$ is the expected payoff to i if the play of the game began in node x and all buyers and sellers thereafter choose their moves as specified by σ.

Definition 9 Given a beliefs vector π, a scenario σ is *sequentially rational* for i at s with beliefs π iff

$$\sigma_{i.s} \in \text{argmax}_{\rho_s \in \Delta(D_s)} \sum_{x \in Y_s} \pi_{i.s}(x) U_i(\sigma_{-i.s}, \rho_s | x).$$

That is, σ is sequentially rational for i at information state s if and only if $\sigma_{i.s}$ would maximize i's expected payoff when a node in Y_s occurred in the path of play, given the belief probabilities that i would assign to the various nodes in Y_s when he learned that one of them had occurred, and assuming that all moves after this node would be determined according to σ.

For any node y and any scenario σ, the prior probability $\bar{P}(y|\sigma)$ of node y under the scenario σ is the probability that the path of play, starting at the root, will reach node y when all players choose their moves according to the scenario σ. Since $\bar{P}(y|\sigma) = \bar{P}(y|\sigma, x^0)$, where x^0 denotes the root or initial node of the tree, it is the multiplicative product of all the chance probabilities and move probabilities specified by σ for the branches on the path from the root to y.

Let $\times_{s \in S^*} \Delta^0(D_s)$ denote the set of all scenarios in which all move probabilities are positive. So if σ in $\times_{s \in S^*} \Delta^0(D_s)$, then $\bar{P}(y|\sigma) > 0$ for every node y, and there is a unique beliefs vector π that satisfies Bayes's formula

$$\pi_{i.s}(x) = \frac{\bar{P}(x|\sigma)}{\sum_{y \in Y_s} \bar{P}(y|\sigma)}, \qquad \forall x \in Y_s.$$

Definition 10 A beliefs vector π is *fully consistent* with a scenario σ if and only if there exists some sequence $(\hat{\sigma}^k)_{k=1}^{\infty}$ such that

$$\hat{\sigma}^k \in \times_{s \in S^*} \Delta^0(D_s), \qquad \forall k \in \{1, 2, 3, \dots\},$$

$$\sigma_{i.s}(d_s) = \lim_{k \to \infty} \hat{\sigma}^k_{i.s}(d_s), \qquad \forall i \in N, \qquad \forall s \in S_i, \qquad \forall d_s \in D_s,$$

$$\pi_s(x) = \lim_{k \to \infty} \frac{\bar{P}(x|\hat{\sigma}^k)}{\sum_{y \in Y_s} \bar{P}(y|\hat{\sigma}^k)}, \qquad \forall s \in S^*, \qquad \forall x \in Y_s.$$

That is, π is fully consistent with a scenario σ iff there exists scenarios that are arbitrarily close to σ and that assign strictly positive probability to every move, such that the beliefs vectors that satisfy Bayes's formula for these strictly positive scenarios are arbitrarily close to π [Kreps and Wilson (1982); Myerson (1991)].

Definition 11 A *sequential equilibrium* of Γ^e is any (σ, π) in $(\times_{s \in S^*} \Delta(D_s), \times_{s \in S^*} \Delta(Y_s))$ such that the beliefs vector π is fully consistent with σ and, with beliefs π, the scenario σ is sequentially rational for every buyer or seller at every information state [Kreps and Wilson (1982); Myerson (1991)].

A sequential equilibrium of this extensive-form game exists because it is finite [Kreps and Wilson (1982)]. In particular, the sets of traders, actions, and types are finite.

The rest of this section demonstrates that the mechanism implements the Ideal Rule in sequential equilibrium. Let $S_{j.5}$ denote the set of all possible information states for seller j in which she makes a choice in Stage 5.

Lemma 1 *Sequential rationality requires $\sigma_{j.s}(\overline{\gamma}) = 1$ for all j in M and for all s in $S_{j.5}$.*

Proof. If any seller chooses $\overline{\chi}$, then she does not trade and her payoff is 0. Since a seller who is not in $\underline{M}$ has the opportunity to choose $\overline{\gamma}$ only if there is a positive number of unmatched buyers who have chosen $\overline{\gamma}$ and all sellers in $\underline{M}$ are matched with buyers who have chosen $\overline{\gamma}$, a seller has a positive probability of being matched when she chooses $\overline{\gamma}$. Since all sellers have a positive payoff from trading at $\overline{p}$, every seller has a positive expected payoff from choosing $\overline{\gamma}$. Hence, sequential rationality requires $\sigma_{j.s}(\overline{\gamma}) = 1$ for all j in M and for all s in $S_{j.5}$. ■

Let $S_{i.34}$ denote the set of all possible information states for buyer i in which he makes a choice in Stage 3 or Stage 4.

Lemma 2 *Sequential rationality requires $\sigma_{i.s}(\overline{\chi}) = 1$ for all i in N_l, for all s in $S_{i.34}$, and $\sigma_{i.s}(\overline{\gamma}) = 1$ for all i in N_h, for all s in $S_{i.34}$.*

Proof. If a buyer chooses $\overline{\chi}$, then he does not trade and his payoff is 0. If a buyer chooses $\overline{\gamma}$ when there are sellers matched with buyers at $\underline{p}$, these sellers are randomly matched pairwise with the buyer and any other buyer who chooses $\overline{\gamma}$ at the same time. If a buyer chooses $\overline{\gamma}$ when there are no sellers matched with buyers at $\underline{p}$, since $n \leq m$, there has to be a positive number of sellers making a choice at Stage 5. From Lemma 1, all these sellers making a choice at Stage 5 will choose $\overline{\gamma}$ and be randomly matched pairwise with all these buyers choosing $\overline{\gamma}$ when there are no sellers matched with buyers at $\underline{p}$. Hence, regardless of whether or not there are sellers matched with buyers at $\underline{p}$, there is a positive probability that a buyer will be matched with a seller at $\overline{p}$ if he chooses $\overline{\gamma}$. Since a low valuation buyer has a negative payoff when he buys at $\overline{p}$ and a high valuation buyer has a positive payoff when he buys at $\overline{p}$, every low valuation buyer would choose $\overline{\chi}$ and every high valuation buyer would choose $\overline{\gamma}$. ■

Lemma 3 *Suppose that $\underline{n} = n > n_h > \underline{m}$. Then the game enters Stage 5 with $n_h - \underline{m}$ unmatched buyers and all sellers in $\underline{M}$ trade at $\overline{p}$.*

Proof. Since $\underline{n} > \underline{m}$, there is a positive number of unmatched buyers and $\underline{m}$ pairs of matched buyers and sellers. Since $n_h > \underline{m}$, there is a positive number of high valuation buyers who are unmatched. From Lemma 2, every high valuation buyer who is unmatched would choose $\overline{\gamma}$ and every low valuation buyer who is unmatched would choose $\overline{\chi}$. Hence, $\overline{n}_1 > 0$. If $\underline{m} \leq \overline{n}_1$, then all sellers in $\underline{M}$ are matched at $\overline{p}$ and so the expected payoff of any seller j is equal to $\omega_j(\overline{\tau})$. If instead $\underline{m} > \overline{n}_1$, then since there are $\overline{n}_1$ sellers who are matched at $\overline{p}$, this causes $\overline{n}_1$ buyers who were matched at $\underline{p}$ to become unmatched while $\underline{m} - \overline{n}_1$ buyers remain matched at $\underline{p}$. But

$n_h - \overline{n}_1$ high valuation buyers were matched at Stage 2. Given that $n_h > \underline{m}$, there were more high valuation buyers who were matched at Stage 2 than there are buyers who remain matched at $\underline{p}$. Hence, there is a positive number of high valuation buyers among the $\overline{n}_1$ buyers who have become unmatched and so $\overline{n}_2 > 0$. If $\underline{m} - \overline{n}_1 \leq \overline{n}_2$, then all the remaining sellers in $\underline{M}$ are matched at $\overline{p}$ and so the expected payoff of seller j is equal to $\omega_j(\overline{\tau})$. If instead $\underline{m} - \overline{n}_1 > \overline{n}_2$, then since $\overline{n}_2$ additional sellers are matched at $\overline{p}$, this causes $\overline{n}_2$ additional buyers who were matched at $\underline{p}$ to become unmatched while $\underline{m} - \overline{n}_1 - \overline{n}_2$ buyers remain matched at $\underline{p}$. But $n_h - \overline{n}_1 - \overline{n}_2$ high valuation buyers were matched at Stage 3. Given that $n_h > \underline{m}$, hence $\overline{n}_3 > 0$ and so on until all sellers in $\underline{M}$ are matched at $\overline{p}$. Since all sellers in $\underline{M}$ are matched at $\overline{p}$ and $n_h > \underline{m}$, there are $n_h - \underline{m}$ unmatched buyers when the game enters Stage 5. ■

Let $S_{j.1}$ denote the information state for seller j in Stage 2 after she observes $\underline{n} = n$ in Stage 1.

Lemma 4 *Suppose that $n > n_h > \underline{m}$ and seller j is in $S_{j.1}$. Then seller j's expected payoff from choosing $\underline{\gamma}$ is equal to $\omega_j(\overline{\tau})$ and her expected payoff from choosing $\underline{n}$ is equal to $(n_h - \underline{m} + 1)\omega_j(\overline{\tau})/(m - \underline{m} + 1)$.*

Proof. Consider first the expected payoff of any seller j when she chooses $\underline{\gamma}$. From Lemma 3, seller j would trade at $\overline{p}$ and so have an expected payoff equal to $\omega_j(\overline{\tau})$.

Consider now the expected payoff of seller j when she chooses $\underline{\chi}$. Let $\underline{M} - j$ denote the set of sellers who choose $\underline{\gamma}$ when seller j chooses $\underline{\chi}$. Note that the cardinality of $\underline{M} - j$ is equal to $\underline{m} - 1 \geq 0$. Given that $n_h > \underline{m} - 1$, from Lemma 3, the game enters Stage 5 with $n_h - \underline{m} + 1$ high valuation buyers who are unmatched. From Lemma 1, every seller not in $\underline{M} - j$ chooses $\overline{\gamma}$ at Stage 5. Since there are $m - \underline{m} + 1$ sellers not in $\underline{M} - j$, seller j's probability of trading at $\overline{p}$ is equal to $(n_h - \underline{m} + 1)/(m - \underline{m} + 1)$.

Hence, by choosing $\underline{n}$, seller j has an expected payoff of $(n_h-\underline{m}+1)\omega_j(\overline{\tau})/(m-\underline{m}+1)$.

■

Lemma 5 *Suppose that $n > \underline{m} \geq n_h$ and seller j is in $S_{j.1}$. If $n_h > 0$, then seller j's expected payoff from choosing $\underline{\gamma}$ is equal to $(n-\underline{m})/(n-n_h+1)\times n_h/\underline{m}+\omega_j(\underline{\tau})$ and her expected payoff from choosing $\underline{\chi}$ is equal to 0 if $\underline{m} > n_h$ and equal to $\omega_j(\overline{\tau})/(m-n_h+1)$ if $\underline{m} = n_h$. If $n_h = 0$, then seller j's expected payoff from choosing $\underline{\gamma}$ is equal to $\omega_j(\underline{\tau})$ and her expected payoff from choosing $\underline{\chi}$ is equal to 0.*

Proof. Suppose that $n_h > 0$. Consider the expected payoff of seller j when she chooses $\underline{\gamma}$. Since $\underline{n} > \underline{m}$, there is a positive number of unmatched buyers and $\underline{m}$ pairs of matched buyers and sellers. From Lemma 2, every high valuation buyer who is unmatched would choose $\overline{\gamma}$ and every low valuation buyer who is unmatched would choose $\overline{\chi}$. Since $\underline{m} \geq n_h$, there may or may not be any high valuation buyers who are unmatched. Conditional on $\underline{n} = n > \underline{m} \geq n_h > 0$, the probability that there are $\overline{n}_1$ high valuation buyers who are unmatched at $\underline{p}$ is equal to $\binom{n-\underline{m}}{\overline{n}_1}\binom{\underline{m}}{n_h-\overline{n}_1}/\binom{n}{n_h}$, which is the probability mass function of a hypergeometric distribution. Note that $0 \leq \overline{n}_1 \leq \min\{n-\underline{m}, n_h\}$. Conditional on $\overline{n}_1$ high valuation buyers being unmatched, the probability that a seller is matched with one of the $\overline{n}_1$ high valuation buyers at $\overline{p}$ is equal to $\overline{n}_1/\underline{m}$ and with probability $(\underline{m}-\overline{n}_1)/\underline{m}$ the seller is not matched with any of the $\overline{n}_1$ high valuation buyers at $\overline{p}$. Note that if $\overline{n}_1 = n_h$, then all high valuation buyers are matched at $\overline{p}$ and so any seller in $\underline{M}$ not matched at $\overline{p}$ would trade at $\underline{p}$. If $\overline{n}_1 < n_h$, then there may or may not be any high valuation buyers who are unmatched. The conditional probability that there are $\overline{n}_2$ high valuation buyers who are unmatched at $\underline{p}$ is equal to $\binom{\overline{n}_1}{\overline{n}_2}\binom{\underline{m}-\overline{n}_1}{n_h-\overline{n}_1-\overline{n}_2}/\binom{\underline{m}}{n_h-\overline{n}_1}$. Note that $0 \leq \overline{n}_2 \leq \min\{\overline{n}_1, n_h-\overline{n}_1\}$. The conditional probability that a seller is matched with one of the $\overline{n}_2$ high valuation buyers at $\overline{p}$ is equal to $\overline{n}_2/(\underline{m}-\overline{n}_1)$ and with probability $(\underline{m}-\overline{n}_1-\overline{n}_2)/(\underline{m}-\overline{n}_1)$ the seller is not matched with any of the $\overline{n}_2$ high valuation buyers at $\overline{p}$. If $\overline{n}_1+\overline{n}_2=n_h$, then all high valuation buyers are matched at $\overline{p}$ and the game ends. If $\overline{n}_1+\overline{n}_2<n_h$, then the game continues as there may or may not be any high valuation buyers who are unmatched. Note that if $\overline{n}_i=0$ for any i, then the game ends with buyers and sellers trading at the prices at which they are matched.

Hence, the expected payoff for any seller j in M_l when she chooses $\underline{\gamma}$ is equal to

$$\sum_{\overline{n}_1=0}^{\min\{n-\underline{m},n_h\}} \frac{\binom{n-\underline{m}}{\overline{n}_1}\binom{\underline{m}}{n_h-\overline{n}_1}}{\binom{n}{n_h}}[\frac{\overline{n}_1}{\underline{m}}\omega_j(\overline{\tau})+\frac{\underline{m}-\overline{n}_1}{\underline{m}}$$

$$\times[\sum_{\overline{n}_2=0}^{\min\{\overline{n}_1,n_h-\overline{n}_1\}} \frac{\binom{\overline{n}_1}{\overline{n}_2}\binom{\underline{m}-\overline{n}_1}{n_h-\overline{n}_1-\overline{n}_2}}{\binom{\underline{m}}{n_h-\overline{n}_1}}[\frac{\overline{n}_2}{\underline{m}-\overline{n}_1}\omega_j(\overline{\tau})+\frac{\underline{m}-\overline{n}_1-\overline{n}_2}{\underline{m}-\overline{n}_1}$$

$$\times[\sum_{\overline{n}_3=0}^{\min\{\overline{n}_2,n_h-\overline{n}_1-\overline{n}_2\}} \frac{\binom{\overline{n}_2}{\overline{n}_3}\binom{\underline{m}-\overline{n}_1-\overline{n}_2}{n_h-\overline{n}_1-\overline{n}_2-\overline{n}_3}}{\binom{\underline{m}-\overline{n}_1}{n_h-\overline{n}_1-\overline{n}_2}}[\frac{\overline{n}_3}{\underline{m}-\overline{n}_1-\overline{n}_2}\omega_j(\overline{\tau})+\frac{\underline{m}-\overline{n}_1-\overline{n}_2-\overline{n}_3}{\underline{m}-\overline{n}_1-\overline{n}_2}$$

$$\cdots\times[\sum_{\overline{n}_{z-1}=0}^{\min\{\overline{n}_{z-2},n_h-\sum_{i=1}^{z-2}\overline{n}_i\}} \frac{\binom{\overline{n}_{z-2}}{\overline{n}_{z-1}}\binom{\underline{m}-\sum_{i=1}^{z-2}\overline{n}_i}{n_h-\sum_{i=1}^{z-1}\overline{n}_i}}{\binom{\underline{m}-\sum_{i=1}^{z-3}\overline{n}_i}{n_h-\sum_{i=1}^{z-2}\overline{n}_i}}[\frac{\overline{n}_{z-1}}{\underline{m}-\sum_{i=1}^{z-2}\overline{n}_i}\omega_j(\overline{\tau})+\frac{\underline{m}-\sum_{i=1}^{z-1}\overline{n}_i}{\underline{m}-\sum_{i=1}^{z-2}\overline{n}_i}$$

$$\times[\frac{\binom{\overline{n}_{z-1}}{0}\binom{\underline{m}-\sum_{i=1}^{z-1}\overline{n}_i}{1}}{\binom{\underline{m}-\sum_{i=1}^{z-2}\overline{n}_i}{n_h-\sum_{i=1}^{z-1}\overline{n}_i}}\omega_j(\underline{\tau})+\frac{\binom{\overline{n}_{z-1}}{1}\binom{\underline{m}-\sum_{i=1}^{z-1}\overline{n}_i}{0}}{\binom{\underline{m}-\sum_{i=1}^{z-2}\overline{n}_i}{n_h-\sum_{i=1}^{z-1}\overline{n}_i}}[\frac{1}{\underline{m}-\sum_{i=1}^{z-1}\overline{n}_i}\omega_j(\overline{\tau})$$

$$+\frac{\underline{m}-\sum_{i=1}^{z-1}\overline{n}_i-1}{\underline{m}-\sum_{i=1}^{z-1}\overline{n}_i}\omega_j(\underline{\tau})]\dots]$$

$$=\frac{1}{\underline{m}\binom{n}{n_h}}\sum_{\overline{n}_1=1}^{\min\{n-\underline{m},n_h\}}\binom{n-\underline{m}}{\overline{n}_1}[\overline{n}_1\binom{\underline{m}}{n_h-\overline{n}_1}+\sum_{\overline{n}_2=1}^{\min\{\overline{n}_1,n_h-\overline{n}_1\}}\binom{\overline{n}_1}{\overline{n}_2}$$

$$\times[\overline{n}_2\binom{\underline{m}-\overline{n}_1}{n_h-\overline{n}_1-\overline{n}_2}+\sum_{\overline{n}_3=1}^{\min\{\overline{n}_2,n_h-\overline{n}_1-\overline{n}_2\}}\binom{\overline{n}_2}{\overline{n}_3}$$

$$\times[\overline{n}_3\binom{\underline{m}-\overline{n}_1-\overline{n}_2}{n_h-\overline{n}_1-\overline{n}_2-\overline{n}_3}+\sum_{\overline{n}_4=1}^{\min\{\overline{n}_3,n_h-\overline{n}_1-\overline{n}_2-\overline{n}_3\}}\binom{\overline{n}_3}{\overline{n}_4}$$

$$\cdots\times[\overline{n}_{z-2}\binom{\underline{m}-\sum_{i=1}^{z-3}\overline{n}_i}{n_h-\sum_{i=1}^{z-2}\overline{n}_i}+\sum_{\overline{n}_{z-1}=1}^{\min\{\overline{n}_{z-2},n_h-\sum_{i=1}^{z-2}\overline{n}_i\}}\binom{\overline{n}_{z-2}}{\overline{n}_{z-1}}$$

$$\times[\overline{n}_{z-1}\binom{\underline{m}-\sum_{i=1}^{z-2}\overline{n}_i}{n_h-\sum_{i=1}^{z-1}\overline{n}_i}+\binom{\overline{n}_{z-1}}{1}\binom{\underline{m}-\sum_{i=1}^{z-1}\overline{n}_i}{0}]\ldots]+\omega_j(\underline{\tau})$$

$$=\frac{n-\underline{m}}{n-n_h+1}\times\frac{n_h}{\underline{m}}+\omega_j(\underline{\tau}).$$

Next, consider the expected payoff of seller j from choosing $\underline{\chi}$ instead. If $\underline{m}>n_h$, then $\underline{m}-1\geq n_h$. From Lemma 2, every low valuation buyer would choose $\overline{\chi}$ and every high valuation buyer would choose $\overline{\gamma}$. Hence, there would be at most n_h buyers choosing $\overline{\gamma}$. Since each of these buyers choosing $\overline{\gamma}$ would be randomly matched pairwise with the $\underline{m}-1\geq n_h$ sellers who chose $\underline{\gamma}$, the game will not reach Stage 5 and so seller j would not trade and would have a payoff of 0. If instead $\underline{m}=n_h$, then $\underline{m}-1=n_h-1<n_h$. Since there are more high valuation buyers than there are sellers in $\underline{M}-j$, by Lemma 3, the game enters Stage 5 with $n_h-\underline{m}+1=1$ unmatched buyer. Since, from Lemma 1, every seller not in $\underline{M}-j$ chooses $\overline{\gamma}$ at Stage 5 and so seller j's expected payoff from choosing $\underline{n}_j$ is equal to $\omega_j(\overline{\tau})/(m-n_h+1)$. In contrast, seller j's expected payoff from choosing $\underline{\gamma}$ is equal to $(n-n_h)/(n-n_h+1)+\omega_j(\underline{\tau})$.

Finally, suppose that $n_h=0$. Since $\underline{n}=n_l$, by Lemma 2, each unmatched buyer would choose $\overline{\chi}$. If seller j chooses $\underline{\gamma}$, she would trade at $\underline{p}$ and have an expected payoff equal to $\omega_j(\underline{\tau})>0$ if j in M_l and $\omega_j(\underline{\tau})<0$ if j in M_h. If seller j chooses $\underline{\chi}$ instead, she would not trade and her expected payoff would be equal to 0. ■

Lemma 6 *Suppose that $\underline{m}\geq n$ and seller j is in $S_{j.1}$. Then seller j's expected payoff from choosing $\underline{\gamma}$ is equal to $n\omega_j(\underline{\tau})/\underline{m}$ and her expected payoff from choosing $\underline{\chi}$ is equal* *to* 0.

Proof. If $\underline{n} = n \leq \underline{m}$, then the game ends and seller j's expected payoff from choosing $\underline{\gamma}$ is equal to $n\omega_j(\underline{\tau})/\underline{m} > 0$ if j in M_l and $n\omega_j(\underline{\tau})/\underline{m} < 0$ if j in M_h.

Consider now seller j's expected payoff from choosing $\underline{\chi}$ instead. If $n < \underline{m}$, then $n \leq \underline{m} - 1$ and the game ends with seller j not trading and getting a payoff of 0. If $n = \underline{m}$, then $\underline{m} - 1 = n - 1 < n$ and so there is one unmatched buyer. Since $n_l > 0$, $n_h \leq n - 1$. By Lemma 2, every low valuation buyer would choose $\overline{\chi}$ and every high valuation buyer would choose $\overline{\gamma}$. Hence, there would be no greater than $n - 1$ buyers choosing $\overline{\gamma}$. Since each of these buyers choosing $\overline{\gamma}$ would be randomly matched pairwise with the sellers who chose $\underline{\gamma}$, the game will not reach Stage 5 and so seller j would not trade and would have a payoff of 0. ■

Lemma 7 *Sequential rationality requires that* $\sigma_{j.s}(\underline{\gamma}) = 1$ *for all* j *in* M_l *and* s *in* $S_{j.1}$*, and* $\sigma_{j.s}(\underline{\chi}) = 1$ *for all* j *in* M_h *and* s *in* $S_{j.1}$*.*

Proof. By Lemmas 4, 5, and 6,

$$U_j(\sigma^*, [\underline{\gamma}]|x) = \begin{cases} \omega_j(\overline{\tau}) & \text{if} \quad n > n_h > \underline{m} \\ \frac{n-n_h}{n-n_h+1} + \omega_j(\underline{\tau}) & \text{if} \quad n > \underline{m} = n_h > 0 \\ \frac{n-\underline{m}}{n-n_h+1} \times \frac{n_h}{\underline{m}} + \omega_j(\underline{\tau}) & \text{if} \quad n > \underline{m} > n_h \\ \frac{n}{\underline{m}} \times \omega_j(\underline{\tau}) & \text{if} \quad \underline{m} \geq n \end{cases}$$

where σ^* is any behavioral-strategy profile in which each buyer chooses $\underline{\gamma}$, each unmatched buyer in N_h chooses $\overline{\gamma}$, each unmatched buyer in N_l chooses $\overline{\chi}$, and each seller chooses $\overline{\gamma}$; x is any of seller j's decision node at Stage 2 after observing $\underline{n} = n$.

$$U_j(\sigma^*, [\underline{\chi}]|x) = \begin{cases} \frac{n_h - \underline{m} + 1}{m - \underline{m} + 1} \times \omega_j(\overline{\tau}) & \text{if} \quad n > n_h > \underline{m} \\ \frac{1}{m - n_h + 1} \times \omega_j(\overline{\tau}) & \text{if} \quad n > \underline{m} = n_h > 0 \\ 0 & \text{if} \quad n > \underline{m} > n_h \\ 0 & \text{if} \quad \underline{m} \geq n \end{cases}$$

Consider the case $m \geq n > n_h > \underline{m}$. Since $\omega_j(\overline{\tau}) > 0$ for all j in M,

$$\frac{n_h - \underline{m} + 1}{m - \underline{m} + 1} \times \omega_j(\overline{\tau}) < \omega_j(\overline{\tau})$$

for all j in M.

Consider the case $n > \underline{m} = n_h > 0$. Since $n - n_h \geq 1$ and $n \leq m$,

$$\frac{1}{m - n_h + 1} \leq \frac{n - n_h}{n - n_h + 1} < 1.$$

Given that $\omega_j(\underline{\tau}) = \epsilon$, $\omega_j(\overline{\tau}) = 1 + \epsilon$ for all j in M_l, and $\omega_j(\underline{\tau}) = -1 + \epsilon$, $\omega_j(\overline{\tau}) = \epsilon$ for all j in M_h, hence

$$0 < \frac{1}{m - n_h + 1} \times \omega_j(\overline{\tau}) < \frac{n - n_h}{n - n_h + 1} + \omega_j(\underline{\tau})$$

if j in M_l and

$$\frac{1}{m - n_h + 1} \times \omega_j(\overline{\tau}) > 0 > \frac{n - n_h}{n - n_h + 1} + \omega_j(\underline{\tau})$$

if j in M_h.

Consider the case $n > \underline{m} > n_h$. Since $0 < (n-\underline{m})/(n-n_h+1) < 1$ and $0 \leq n_h/\underline{m} < 1$,

$$0 \leq \frac{n-\underline{m}}{n-n_h+1} \times \frac{n_h}{\underline{m}} < 1.$$

Given that $\omega_j(\underline{\tau}) = \epsilon$ for all j in M_l and $\omega_j(\underline{\tau}) = -1+\epsilon$ for all j in M_h, hence

$$\frac{n-\underline{m}}{n-n_h+1} \times \frac{n_h}{\underline{m}} + \omega_j(\underline{\tau}) > 0$$

for any j in M_l, and

$$\frac{n-\underline{m}}{n-n_h+1} \times \frac{n_h}{\underline{m}} + \omega_j(\underline{\tau}) < 0$$

for any j in M_h.

Consider the case $n \leq \underline{m}$.

$$\frac{n}{\underline{m}} \times \omega_j(\underline{\tau}) > 0$$

for any j in M_l and

$$\frac{n}{\underline{m}} \times \omega_j(\underline{\tau}) < 0$$

for any j in M_h.

Since $U_j(\sigma^*, [\underline{\gamma}]|x) > U_j(\sigma^*, [\underline{\chi}]|x)$ for any j in M_l, sequential rationality requires that $\sigma_{j.s}(\underline{\gamma}) = 1$ for all j in M_l and s in $S_{j.1}$.

The probability that there are i high valuation buyers among $n-1$ buyers is equal to $B(i;(n-1),(1-p)) = \binom{n-1}{i}p^{n-1-i}(1-p)^i$, which is the probability mass function of a binomial distribution. Similarly, the probability that there are j high cost sellers among $m-1$ sellers is equal to $B(j;(m-1),(1-q))$. Since sequential rationality requires that all low cost sellers choose $\underline{\gamma}$ if all buyers choose $\underline{\gamma}$, the remaining uncertainty, as perceived by a high cost seller, rests in the choices made by each of the other high cost sellers, if any. In this regard, the probability that k among j high cost sellers choose $\underline{n}$ is equal to $B(k; j, (1-\beta))$, where $0 \leq \beta \leq 1$.

Noting that $\underline{m} = (m-1-j) + (j-k) + 1 = m-k$, high cost seller g's expected payoff from choosing $\underline{\gamma}$ is equal to

$$\sum_{i=0}^{n-1} B(i; (n-1), (1-p))[\sum_{j=0}^{m-1} B(j; (m-1), (1-q))[\sum_{k=0}^{j} B(k; j, (1-\beta)) U_g(\sigma^*, [\underline{\gamma}]|x)]]$$

$$= \sum_{i=0}^{n-1} B(i; (n-1), (1-p))[\sum_{j=0}^{m-1} B(j; (m-1), (1-q))[\sum_{k=0}^{\min\{m-n,j\}} B(k; j, (1-\beta))$$

$$\times \frac{n}{m-k}(-1+\epsilon)]]$$

$$+ \sum_{i=0}^{n-2} B(i; (n-1), (1-p))[\sum_{j=m-n+1}^{m-1} B(j; (m-1), (1-q))[\sum_{k=m-n+1}^{\min\{m-1-i,j\}} B(k; j, (1-\beta))$$

$$\times(-1 + \frac{n-m+k}{n-i+1} \times \frac{i}{m-k} + \epsilon)]]$$

$$+ \sum_{i=1}^{n-1} B(i; (n-1), (1-p))[\sum_{j=m-i}^{m-1} B(j; (m-1), (1-q))[B((m-i); j, (1-\beta))$$

$$\times(-1 + \frac{n-i}{n-i+1} + \epsilon)]]$$

$$+ \sum_{i=2}^{n-1} B(i; (n-1), (1-p))[\sum_{j=m+1-i}^{m-1} B(j; (m-1), (1-q))[\sum_{k=m+1-i}^{j} B(k; j, (1-\beta))\epsilon]]$$

and her expected payoff from choosing $\underline{n}$ is equal to

$$\sum_{i=0}^{n-1} B(i; (n-1), (1-p))[\sum_{j=0}^{m-1} B(j; (m-1), (1-q))[\sum_{k=0}^{j} B(k; j, (1-\beta)) U_g(\sigma^*, [\underline{\chi}]|x)]]$$

$$= \sum_{i=1}^{n-1} B(i; (n-1), (1-p))[\sum_{j=m-i}^{m-1} B(j; (m-1), (1-q))[B((m-i); j, (1-\beta))$$

$$\times \frac{1}{m-i+1}\epsilon]]$$

$$+\sum_{i=2}^{n-1} B(i;(n-1),(1-p))[\sum_{j=m+1-i}^{m-1} B(j;(m-1),(1-q))[\sum_{k=m+1-i}^{j} B(k;j,(1-\beta))$$

$$\times \frac{i-m+k+1}{k+1}\epsilon]].$$

Since ϵ, but not $p > 0$ and $q > 0$, is infinitismally small, each high cost seller's expected payoff from choosing $\underline{\gamma}$ is strictly less than the expected payoff from choosing $\underline{\chi}$, that is

$$\sum_{i=2}^{n-1} B(i;(n-1),(1-p))[\sum_{j=m+1-i}^{m-1} B(j;(m-1),(1-q))[\sum_{k=m+1-i}^{j} B(k;j,(1-\beta))$$

$$\times \frac{m-i}{k+1}\epsilon]]$$

$$< \sum_{i=0}^{n-1} B(i;(n-1),(1-p))[\sum_{j=0}^{m-1} B(j;(m-1),(1-q))[\sum_{k=0}^{\min\{m-n,j\}} B(k;j,(1-\beta))$$

$$\times \frac{n}{m-k}(1-\epsilon)]]$$

$$+\sum_{i=0}^{n-2} B(i;(n-1),(1-p))[\sum_{j=m-n+1}^{m-1} B(j;(m-1),(1-q))[\sum_{k=m-n+1}^{\min\{m-1-i,j\}} B(k;j,(1-\beta))$$

$$\times(1-\frac{n-m+k}{n-i+1}\times\frac{i}{m-k}-\epsilon)]]$$

$$+\sum_{i=1}^{n-1} B(i;(n-1),(1-p))[\sum_{j=m-i}^{m-1} B(j;(m-1),(1-q))[B((m-i);j,(1-\beta))$$

$$\times(1-\frac{n-i}{n-i+1}-\epsilon+\frac{1}{m-i+1}\epsilon)]].$$

Hence, sequential rationality requires that $\sigma_{j.s}(\underline{\chi}) = 1$ for all j in M_h and s in $S_{j.1}$. ■

Let $S_{i.1}$ denote the information state in which buyer i chooses at Stage 1.

Lemma 8 *Sequential rationality requires $\sigma_{i.s}(\underline{\gamma}) = 1$ for all i in N_h and s in $S_{i.1}$.*

Proof. The payoff from choosing $\underline{\chi}$ is equal to 0 for any buyer. Consider the payoff from choosing $\underline{\gamma}$ instead for any high valuation buyer i. Note that from Lemma 1, the common belief is that any seller not in $\underline{M}$ would choose $\overline{\gamma}$ if the game proceeds to Stage 5, and given that $n \leq m$, any buyer choosing $\overline{\gamma}$ would be expected to end the game matched at $\overline{p}$. If $\underline{n} \leq \underline{m}$, then the game ends with buyer i trading at $\underline{p}$ and obtaining a payoff equal to $1 + \epsilon$. If $\underline{n} > \underline{m}$, then buyer i is either unmatched or matched at Stage 2. If he is unmatched at Stage 2, he will choose $\overline{\gamma}$ to trade at $\overline{p}$ and obtain a payoff equal to ϵ. If he is matched at Stage 2, he will either remain matched at $\underline{p}$, resulting in a payoff equal to $1 + \epsilon$, or subsequently unmatched and be allowed to choose $\overline{\gamma}$ and obtain a payoff equal to ϵ. Hence, sequential rationality requires that $\sigma_{i.s}(\underline{\gamma}) = 1$ for all i in N_h and s in $S_{i.1}$. ■

Lemma 9 *Sequential rationality requires $\sigma_{1.s}(\underline{\gamma}) = 1$ for all s in $S_{1.1}$.*

Proof. Consistency requires that $\pi_{1.s}(x) = (1-p)^{n-1}q^m$ for s in $S_{1.1}$, where x is the chance node associated with all buyers except buyer 1 being high type and all sellers being low type. Buyer 1's payoff from choosing $\underline{\chi}$ is equal to 0. From Lemma 8, sequential rationality requires that all high type buyers choose $\underline{\gamma}$. From Lemma 7, sequential rationality requires that all low type sellers choose $\underline{\gamma}$ if all buyers choose $\underline{\gamma}$. Since $n \leq m$, buyer 1's expected payoff from choosing $\underline{\gamma}$ is greater or equal to $(1-p)^{n-1}q^m\epsilon > 0$. Hence, sequential rationality requires that $\sigma_{1.s}(\underline{\gamma}) = 1$ for all s in $S_{1.1}$. ■

Lemma 10 *Sequential rationality requires $\sigma_{i.s}(\underline{\gamma}) = 1$ for all i in $N_l - \{1\}$ and s in $S_{i.1}$.*

Proof. Consistency requires that $\pi_{i.s}(x) = (1-p)^{n-2}q^m$ for s in $S_{i.1}$, where x is the chance node associated with all buyers except buyer i and buyer 1 being high type and all sellers being low type. Buyer i's payoff from choosing $\underline{\chi}$ is equal to 0. From Lemma 9, sequential rationality requires that buyer 1 chooses $\underline{\gamma}$. From Lemma 8, sequential rationality requires that all high type buyers choose $\underline{\gamma}$. From Lemma 7, sequential rationality requires that all low type sellers choose $\underline{\gamma}$ if all buyers choose $\underline{\gamma}$. Since $n \leq m$, buyer i's expected payoff from choosing $\underline{\gamma}$ is greater or equal to $(1-p)^{n-2}q^m\epsilon > 0$. Hence, sequential rationality requires $\sigma_{i.s}(\underline{\gamma}) = 1$ for all i in $N_l - \{1\}$ and s in $S_{i.1}$. ■

Lemma 11 *Sequential rationality requires $\sigma_{i.s}(\underline{\gamma}) = 1$ for all i in N and s in $S_{i.1}$.*

Proof. The proof follows from Lemmas 8, 9, and 10. ■

Theorem 16 *The Ideal Rule W is implementable in sequential equilibrium.*

Proof. From Lemma 11, all buyers would choose $\underline{\gamma}$ in Stage 1. Hence, the game would not end at Stage 1.

From Lemma 7, all low cost sellers would choose $\underline{\gamma}$ and all high cost sellers would choose $\underline{\chi}$ in Stage 2 having observed all buyers choosing $\underline{\gamma}$ in Stage 1. If $\underline{n} \leq \underline{m}$, then the game ends at Stage 2 with all buyers randomly matched with only low cost sellers. Hence, the set of sequential equilibrium outcomes equals the set of Ideal Rule outcomes in the state of the world in which $m_l \geq n$, i.e. there are at least as many low cost sellers as there are buyers.

From Lemma 2, all low valuation buyers would choose $\overline{\chi}$ and all high valuation buyers would choose $\overline{\gamma}$ if given the opportunity to choose at Stage 3 or Stage 4. If the game ends at Stage 3 with no trade because $\overline{n}_1 = 0$ and $\underline{n} > \underline{m} = 0$, then all buyers have low valuations and all sellers have high costs. Hence, the no trade sequential equilibrium outcome equals the Ideal Rule outcome in the state of the world in which $n = n_l$ and $m = m_h$, i.e. all buyers have low valuations and all sellers have high costs.

If the game ends at Stage 3 with the matched buyers and sellers trading at $\underline{p}$ because $\overline{n}_1 = 0$ and $\underline{n} > \underline{m} > 0$, then

1. all unmatched buyers have low valuations;
2. all unmatched sellers have high costs;
3. any high valuation buyer is matched; and
4. all matched sellers have low costs.

If the game ends at Stage 4 with matched buyers and sellers trading at the prices that they are matched because $\overline{n}_t = 0$ for any $t \in \{2, \ldots, n\}$ and all buyers who have chosen $\overline{\gamma}$ are matched, then

1. all unmatched buyers have low valuations;
2. all unmatched sellers have high costs;
3. all high valuation buyers are matched;
4. all matched sellers have low costs; and
5. all buyers matched at $\overline{p}$ have high valuations.

Given the above, if any game ends at Stage 3 or Stage 4 because $\overline{n}_t = 0$ for any $t \in \{1, \ldots, n\}$, $\underline{n} > \underline{m} > 0$, and any buyer who has chosen $\overline{\gamma}$ is matched, then there are more buyers than there are low cost sellers and there are at least as many low cost sellers as there are high valuation buyers. Hence, the set of sequential equilibrium outcomes equals the set of Ideal Rule outcomes in the state of the world in which $n > m_l$ and $m_l \geq n_h$, i.e. there are more buyers than there are low cost sellers and there are at least as many low cost sellers as there are high valuation buyers.

The game proceeds to Stage 5 only if

1. At Stage 3, $\overline{n}_1 > 0$ and $\underline{m} = 0$;
2. At Stage 4, $\overline{n}_t = 0$ for any $t \in \{2, \ldots, n\}$ and there is at least one buyer who has chosen $\overline{\gamma}$ but is unmatched; or
3. At Stage 4, $\overline{n}_t > 0$ for any $t \in \{2, \ldots, n\}$ and there are no sellers in $\underline{M}$ who are matched with a buyer at $\underline{p}$.

Hence, the game proceeds to Stage 5 only if $n_h > m_l$. From Lemma 1, all sellers who have chosen $\underline{\chi}$ would choose $\overline{\gamma}$ at Stage 5. Since $n \leq m$, there are high cost sellers who would choose $\overline{\gamma}$ at Stage 5, and so $\overline{m} > 0$. If the game ends at Stage 5 with matched buyers and sellers trading at $\overline{p}$, then

1. all unmatched buyers have low valuations;
2. all unmatched sellers have high costs;
3. all high valuation buyers are matched;
4. all matched buyers have high valuations;
5. all low cost sellers are matched; and
6. all low valuation buyers are unmatched.

Hence, the set of sequential equilibrium outcomes equals the Ideal Rule outcomes in the state of the world in which $n > m_l$ and $n_h > m_l$, i.e. there are more buyers than there are low cost sellers and there are more high valuation buyers than there are low cost sellers.

Since there exists a mechanism such that the set of sequential equilibrium outcomes equals the set of Ideal Rule outcomes in each possible state of the world, the Ideal Rule is implementable in sequential equilibrium. ■

3.4 Extensions

If $n \geq m$, then the Ideal Rule is implementable in similar fashion as long as there is at least one high cost seller, and this is common knowledge. The implementing mechanism would begin by requiring all sellers to choose from the set $\{\overline{\gamma}, \overline{\chi}\}$ instead of requiring all buyers to choose from the set $\{\underline{\gamma}, \underline{\chi}\}$.

An obvious extension is to find a mechanism that implements the Ideal Rule with more than two reservation prices. With more than two reservation prices, it becomes more difficult to ensure that buyers and sellers would indirectly reveal their true preferences even if it were common knowledge that there is at least one buyer at each reservation price.

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