Two Essays in Economics

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Two Essays in Economics

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The thesis includes two essays. The first essay, *Inequality Moments in Estimation of Discrete Games with Incomplete Information and Multiple Equilibria,* develops a method for estimation of static discrete games with incomplete information, which delivers consistent estimates of parameters even when games have multiple equilibria. Every Bayes-Nash equilibrium in a discrete game of incomplete information is associated with a set of choice probabilities. I use maximum and minimum equilibrium choice probabilities as upper and lower bounds on empirical choice probabilities to construct moment inequalities. In general, estimation with moment inequalities results in partial identification. I show that point identification is achievable if the payoffs are functions of a sufficient number of explanatory variables with a real line domain and outcome-specific coefficients associated with them.

The second essay, *Tenancy Rent Control and Credible Commitment in Maintenance*, co-authored with Richard Arnott, investigates the effect of tenancy rent control on maintenance and welfare. Under tenancy rent control, rents are regulated within a tenancy but not between tenancies. The essay analyzes the effects of tenancy rent control on housing quality, maintenance, and rehabilitation. Since the discounted revenue received over a fixed-duration tenancy depends only on the starting rent, intuitively the landlord has an incentive to spruce up the unit between tenancies in order to "show" it well, but little incentive to maintain the unit well during the tenancy. The essay formalizes this intuition, and presents numerical examples illustrating the efficiency loss from this effect.

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Part I

Inequality Moments in Estimation of Discrete Games with Incomplete Information and Multiple Equilibria

1 Introduction

Empirical industrial organization literature often employs discrete games as a convenient tool, which allows to model interaction between economic agents and has an advantage of clear interpretation of structural parameters. However, the use of games in empirical applications requires a careful choice of appropriate econometric techniques. In particular, it is often necessary to address the problem of multiplicity of equilibria, which is common in discrete games.

The sources of equilibrium multiplicity and the solutions to this problem differ depending on the type of the game. In this essay, I suggest an estimator for a particular type of discrete games, namely static discrete games with incomplete information and uncorrelated unobservables. The proposed estimator delivers consistent estimates of parameters even when games have multiple equilibria. Robustness to the presence of multiple equilibria is achieved by using moment inequalities, which hold in any equilibrium, forming upper and lower bounds on empirical choice probabilities. In general, estimation using moment inequalities results in the set identification of parameters. I show that point identification is achievable if i) payoffs are functions of a sufficient number of explanatory variables with a real line domain and ii) these expanatory variables are associated with outcome-specific coefficients.

A static discrete game with incomplete information is a simultaneous move game, in which players choose between a finite number of actions and each player has complete knowledge of their own payoffs but only incomplete knowledge of other players' payoffs.

Every Bayes-Nash equilibrium is associated with a set of equilibrium choice probabilities, i.e. probabilities that a particular action is taken by a particular player in an equilibrium. With multiple equilibria in the game, there are several sets of equilibrium choice probabilities. I call an equilibrium choice probability extremal if it is a maximum or a minimum one for a given player and action. Any equilibrium choice probability should lie between the extremal ones. Also, an empirical choice probability, which is an estimate of the probability that a particular action is taken by a particular player, should lie between the respective extremal equilibrium choice probabilities. Empirical choice probabilities are not necessarily consistent estimates of equilibrium choice probabilities. If the sample contains data on outcomes of games where different equilibria were played, empirical choice probabilities are not the consistent estimates of equilibrium choice probabilities. Rather, they are weighted averages of estimates of equilibrium choice probabilities associated with different equilibria. I suggest to estimate the payoffs of the game by using moment inequalities, which require empirical choice probabilities to be smaller than maximal and greater than minimal equilibrium choice probabilities. Thus, even in the absence of consistent estimates of equilibrium choice probabilities it becomes possible to estimate the parameters of the payoff functions consistently.

There is a growing industrial organization literature that models strategic interaction of economic agents as discrete games and then takes them to data. The examples include the studies of entry in small monopoly markets (Bresnahan and Reiss (1990)), entry and quality choice of motels (Mazzeo (2002)), location choice by video retailers (Seim (2002)), timing of radio commercials (Sweeting (2008)), welfare impact of environmental regulations of the cement industry (Ryan (2006)), airlines' entry decisions (Aguirregabiria and Ho (2008))[1]. In empirical applications it is assumed that games of similar structure (with the same number of players and actions, same timing, and same information structure) are played in many different markets/moments of time. The economist observes some characteristics of the players in each of these instances but does not fully know the payoffs. Usually it is assumed that payoffs have a specific parametric form that depends on observables. Then the problem

[1]See Berry and Reiss (2006) for the review of the empirical industrial organization literature employing discrete games.

is to estimate the parameters of the payoffs knowing the observable characteristics of the players and assuming the structure of the game. Essentially, the economist deals with a series of games that have the same structure but different payoffs.

The choice of information type of the game is made by the economist. In some cases, economic agents are likely to know so much about each other that the game of complete information should be employed. In other cases, it is plausible that agents possess important confidential information about their own payoffs, and incomplete information framework is more appealing. As pointed out by Bajari, Hong, Krainer, and Nekipelov (2007), a discrete game of incomplete information is a generalization of a single agent discrete choice model such as logit or probit, bringing in the strategic interaction between agents. Additionally, games of incomplete information may have some technical advantages, such as guaranteed existence of equilibrium in pure strategies (Seim (2006) discusses these issues in the context of static games and Doraszelski and Satterthwaite (2007) in the context of dynamic games).

Payoffs in discrete games are usually modelled as functions of some observable characteristics and of unobservable characteristics, which will be denoted ε. In games of complete information ε is assumed to be unobservable to the economist but known to all players of the game. In games of incomplete information unobservable characteristics ε_k of player k are assumed to be unknown not only to the economist, but also to other players.

Both games of complete and incomplete information may have multiple equilibria. However, the structure of the equilibria is different in two types of games, as well as the source of the multiplicity of equilibria. In games of complete information, a Nash equilibrium in pure strategies is characterized by a profile of actions. In contrast, in games of incomplete information strategies are mappings from unobservables ε to actions, and equilibrium strategies in a Bayes-Nash equilibrium are threshold-type rules. In games of complete information, several equilibria may exist for some realizations of unobservables ε, and there can be such realizations of ε that the equilibrium is unique. In games of incomplete information, the number of equilibria is exactly the same for all realizations of ε, but there can be a unique equilibrium for some realizations of observables and multiple equilibria for others.

Multiplicity of equilibria in games of incomplete information may lead to inconsistent estimates of equilibrium choice probabilities if the economist does not know from which equilibrium each observation comes. Often economists do not have this information.

There are several approaches to avoid the multiplicity problem in discrete games. First, it may be possible to concentrate on a particular equilibrium (e.g., Jia (2008) considers extremal equilibria, Mazzeo (2002) specifies the order of moves of the players, Berry (1992) chooses the equilibrium which maximizes joint profits of the firms, etc.) One more approach is to assume that all data come from the same equilibrium (e. g., Aguirregabiria and Mira (2007), Pesendorfer and Schmidt-Dengler (2008), Pakes, Berry, and Ostrovsky (2007)). In this case the economist may not know much about the equilibrium structure and relies on data to tell which equilibrium is played, in contrast to the first approach.

Yet another route is to model the selection mechanism as parameterized by some probability λ (which can be a function of some exogenous variables) and estimate this probability along with other parameters (Sweeting (2008)). A version of this approach is to find a variable that shifts markets from one equilibrium to another (Bajari, Hong, and Ryan (2008), and Bajari, Hong, Krainer, and Nekipelov (2007)). Else one may be able to find a feature of the game, which is invariant across all equilibria and build the estimation around this feature (Bresnahan and Reiss (1990), Tamer (2003), Aradillas-Lopez and Tamer (2008))[2].

This essay contributes to the latter line of research with respect to static games of incomplete information. My approach is to provide an estimation procedure which is robust to the presence of multiple equilibria and remains valid without knowledge of 'equilibrium shifters'. In general, the refusal to assume that data come from a single equilibrium leads to less efficient estimates than in case of other approaches discussed above but the benefit is no misspecification at the stage of modelling the selection mechanism or assumptions about the equilibrium selection. This is a particular case of the tradeoff between efficiency and robustness.

The essay has the following plan. Section 2 discusses the equilibrium structure in discrete games with incomplete information. Section 3 describes the related econometric problem of payoffs estimation in discrete games, proposes estimation strategy, and states identification conditions. Section 4 presents Monte Carlo analysis of the suggested estimator and Section 5 concludes.

[2]See Berry and Tamer (2007) for an overview of various strategies used for identification in entry games.

2 A discrete game with incomplete information

First, I describe a general case of a static discrete game with incomplete information and its equilibrium structure, and then I illustrate the nature of equilibrium multiplicity in games with incomplete information using an example of a binary game with two players.

2.1 General case

In empirical studies in industrial organization literature economists sometimes model interaction between economic agents as a game and then estimate the payoffs of this game. For this purpose one needs to observe similar situations of interaction repeatedly. The most studied example of such a case in the literature is firm entry in isolated geographic markets. Clearly, two such markets, as well as their actual and potential entrants, will most certainly differ in their attributes. Still, one may be willing to assume that the entry in both markets can be described by two games that have the same structure but differ in their payoff values. For brevity sake, further I refer to such related games as repetitions of a game.

Consider several repetitions of a game, indexed by $n = 1, 2, ..., N$ (in the entry example above each repetition corresponds to a single market). Alternatively, one may think that a series of N related games is observed. The relation between these games is described below.

In each repetition of the game there are K players, indexed by k. Each of the players simultaneously chooses one of the $M + 1$ actions, $a_k \in \{0, 1, ..., M\}$. An action profile of a game is described by vector $a = (a_1, a_2, ..., a_K)$, which belongs to the set of all possible action profiles $A = \{0, 1, ..., M\}^K$.

A payoff of player k in repetition n of the game is modelled as a sum of two components:

$$u_{k,n}(a) = v_{k,n}(a) + \varepsilon_{k,n}(a_k).$$

The first of them, $v_{k,n}(a)$, depends on the actions of all players of the game and is perfectly known to them. The second component, $\varepsilon_{k,n}(a_k)$ is assumed to come independently from distribution $F(\cdot)$. $\varepsilon_{k,n}(a_k)$ depends only on actions of player k and is known only to him before he makes a move in the game. One may interpret $\varepsilon_{k,n}(a_k)$ as a piece of private information about the payoffs associated with a particular action a_k that player k in game n receives before choosing an action. Other players know only the distribution of $\varepsilon_{k,n}$.

Abusing notation, I sometimes refer to $\varepsilon_{k,n}$ instead of $\varepsilon_{k,n}(a_k)$. Also, it is convenient to introduce $\varepsilon_n = (\varepsilon_{1,n}, \varepsilon_{2,n}, ..., \varepsilon_{K,n})$, which contains all private information for game n, and $\varepsilon_{-k,n}$, which contains private information for all players except player k. Similarly, I refer to $a_{-k} = (a_1, ..., a_{k-1}, a_{k+1}, ...a_k)$ and to $A_{-k} = \{0, 1, ..., M\}^{K-1}$, a set, to which a_{-k} belongs.

Without knowledge of $\varepsilon_{-k,n}$, player k cannot perfectly predict actions to be chosen by other players and has to resort to his expectation (belief) to observe particular actions of other players a_{-k}. I denote this belief by $\pi_{k,n}(a_{-k})$. The expected payoff of player k from choosing a particular action a_k is

$$U_{k,n}(a_k) = \sum_{a_{-k} \in A_{-k}} [\pi_{k,n}(a_{-k}) v_{k,n}(a_k, a_{-k})] + \varepsilon_{k,n}(a_k) \tag{1}$$

where $\pi_{k,n}(a_{-k})$ is the belief held by player k that the actions a_{-k} will be chosen by other players in game n. Each player chooses an action maximizing his expected payoff given his beliefs $\pi_{k,n}(a_{-k})$ about choice probabilities of other players in game n:

$$\delta_{k,n} = \arg \max_{a_k \in \{0,1,...,M\}} U_{k,n}(a_k). \tag{2}$$

Like a_k, $\delta_{k,n}$ takes values from the set $\{0, 1, ..., M\}$ but, unlike a_k, $\delta_{k,n}$ is a payoff-maximizing action in game n, not an arbitrary action. In any Bayes-Nash equilibrium, the beliefs $\pi_{k,n}(a_{-k})$ should be consistent with the (ex ante) probabilities to choose a particular action (therefore, all players should have the same beliefs, so the π's subscript k is dropped below):

$$\begin{aligned} \pi_n(a_k) &= \Pr[\delta_{k,n} = a_k], \\ \forall k &\in \{1, 2, ..., K\}, \quad \forall a_k \in \{0, 1, 2, ..., M\}. \end{aligned} \tag{3}$$

Notice that the right-hand side of (3) is a function of beliefs $\pi_n(a_{-k})$, as they affect the choice of the payoff-maximizing action $\delta_{k,n}$. Thus, one can treat (3) as a system of equations that should be satisfied in any Bayes-Nash equilibrium and can be solved for $\pi_n(a)$. Further,

	$a_2 = 0$		$a_2 = 1$	
$a_1 = 0$	$(0,$	$0)$	$(0,$	$v_{2,0} - \varepsilon_2)$
$a_1 = 1$	$(v_{1,0} - \varepsilon_1,$	$0)$	$(v_{1,1} - \varepsilon_1,$	$v_{2,1} - \varepsilon_2)$

Table 1: Payoffs in a binary game with two players

I use a bar to denote the equilibrium beliefs: $\bar{\pi}(a)$.

Depending on the form of $F(\cdot)$ as well as on the values of the payoffs, the system (3) may have a unique or multiple solutions. I assume that there is some unobservable to the economist variable(s) η, which does not affect payoffs but allows players to coordinate on a particular equilibrium in a given repetition of the game. Suppose there are E_n equilibria in repetition n of the game. Then for each $k = 1, 2, ..., K$ there exist E_n equilibrium beliefs $\{\bar{\pi}_{n,(1)}(a_k), \bar{\pi}_{n,(2)}(a_k), ..., \bar{\pi}_{n,(E_n)}(a_k)\}$, and among those one may choose the minimum and maximum equilibrium beliefs for each particular player-action combination:

$$\begin{aligned} \bar{\pi}_{n,\min}(a_k) &= \min_{q \in \{1,2,...,E_n\}} \bar{\pi}_{(q)}(a_k), \\ \bar{\pi}_{n,\max}(a_k) &= \max_{q \in \{1,2,...,E_n\}} \bar{\pi}_{(q)}(a_k). \end{aligned} \tag{4}$$

I use this fact later in the estimation procedure.

2.2 Example: 2×2 game with incomplete information

Using a simple game, I illustrate the issues with multiple equilibria in games of incomplete information. Consider a game similar to the entry games studied by Bresnahan and Reiss (1990, 1991), Tamer (2003), Sweeting (2008), and Aradillas-Lopez (2008). To simplify the exposition, I suppress the subscript n throughout this section and normalize the payoffs by setting one payoff of each player equal to zero (this leaves the action choices unaffected). Also, I slightly simplify the notation compared to the general case.

There are two players, each making a choice between two actions, $a_k \in \{0, 1\}$. The players' payoffs are shown in Table 1.

Variables ε_k, $k = 1, 2$, are observed only by player k and unobserved by the other player; ε_k comes independently from a distribution with the cdf $F(\cdot)$, which is known to both players. As ε's are private information in this setup, players cannot perfectly predict the actions of each other. They hold expectations about the behavior of the other based on their knowledge of the distribution of ε's. I denote the belief of player k that the other player chooses action 1 by π_{-k}. As this is a game with only two actions, player k's belief that the other player chooses action 0 is equal to $1 - \pi_{-k}$.

Taking into account the unobservables ε_k, the expected payoff of player k from choosing action 1 is $v_{k,1}\pi_{-k} + v_{k,0}(1 - \pi_{-k}) - \varepsilon_k$. If $a_k = 0$ the expected payoff of player k is 0. Therefore, the player k chooses action 1 if and only if

$$v_{k,1}\pi_{-k} + v_{k,0}(1 - \pi_{-k}) - \varepsilon_k > 0.$$

The requirement that in a Bayes-Nash equilibrium the belief π_k should be equal to the ex ante probability that player k chooses action 1 results in the following system of equations:

$$\pi_1 = F(v_{1,0} + (v_{1,1} - v_{1,0})\pi_2) \tag{5}$$

$$\pi_2 = F(v_{2,0} + (v_{2,1} + v_{2,0})\pi_1).$$

Thus, the decision of player k which action to choose depends on his belief about the behavior of the other player, and vice versa. The system (5) states that the beliefs of the players must be mutually consistent.

Even in this simple example, existence of multiple solutions to (5) is a general case. For example, Sweeting (2008) shows that for a logit distribution of unobservables there would be up to three symmetric equilibria in a version of this game. This fact highlights the importance of the assumption about the common knowledge of $F(\cdot)$. Nevertheless, if unobservables are distributed uniformly, there is only one equilibrium (Bajari et al., 2007).

Equation in the system (5) may be interpreted as two 'best-response' functions that show how the probability to choose action 1 by one player depends on the probability to choose action 1 by the other player, $\pi_1(\pi_2)$ and $\pi_2(\pi_1)$. In Figure 1 several possible equilibrium configurations are demonstrated in the space of beliefs (π_1, π_2).

Panels B and C show situations when the equilibrium is unique. Notice that in this example the best response are continuous monotone functions. As a result, when one of the

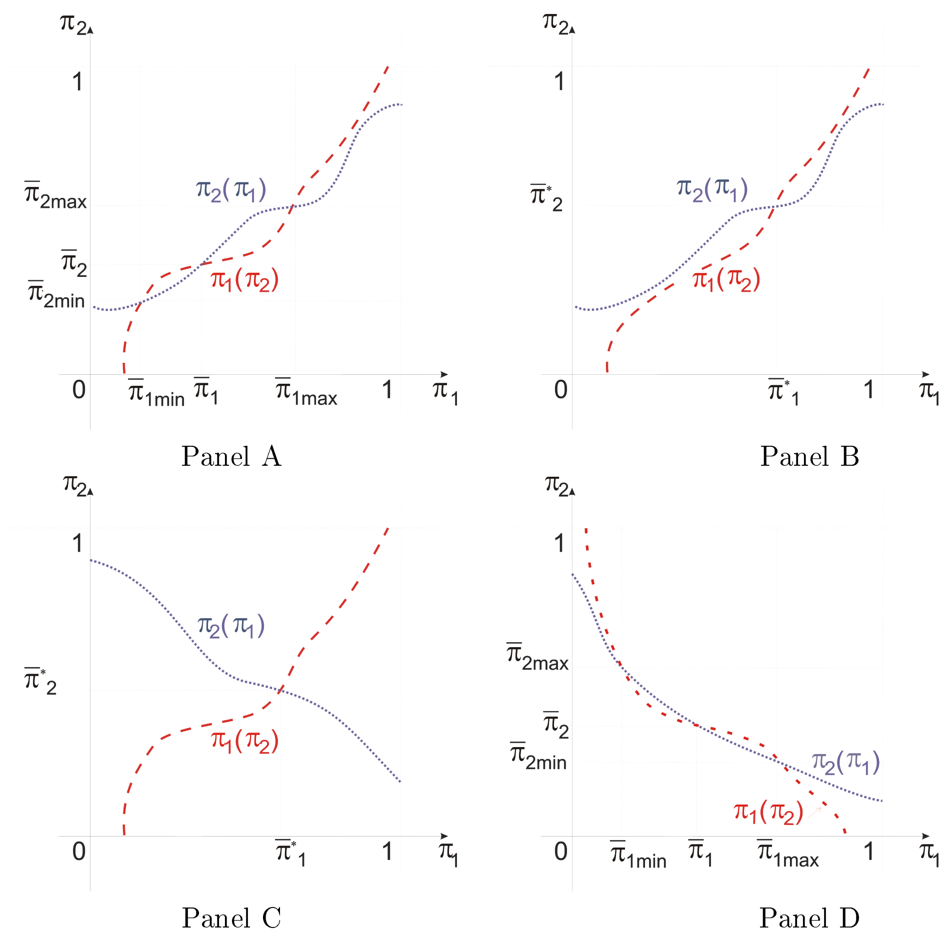

Figure 1: Different equilibrium configurations in games with two players and two actions

best responses is increasing and the other is decreasing, only one equilibrium is possible. When both best responses are either increasing, or decreasing (the former case is shown in Panels A and B, and the latter in Panel D), the number of equilibria depends on the distribution $F(\cdot)$ and payoffs. In panel A there are three equilibria, while in Panel B there is a unique equilibrium labelled by $\bar{\pi}^* = (\bar{\pi}_1^*, \bar{\pi}_2^*)$.

3 Econometric problem

3.1 Observable variables

In cases when discrete games are employed in empirical work, a typical problem of the economist includes estimation of the deterministic components of the payoffs $v_{k,n}(a)$. Usually it is assumed that $v_{k,n}(a)$ is a function of some observable (state) variables $x_n = (x_{1,n}, x_{2,n}, ..., x_{K,n})$. Here $x_{k,n}$ is a row vector that contains observables affecting the payoffs of player k. These observable variables may include a constant, variables which differ across players (e.g., firm productivity), and variables common to all players in a given repetition of the game (e.g., market size). All observations of state variables available to the economist are denoted by x.

One approach to estimation of $v_{k,n}(a)$ is to treat it as a nonparametric function of observables x (see Bajari et al. (2007) for the discussion of this issue). However, even if the equilibrium is unique, nonparametric estimation of payoffs requires a formidable amount of data. Moreover, there is an identification problem as the number of payoffs $v_{k,n}(a)$ is larger than the number of available restrictions. As a result, additional assumptions about the structure of the problem are needed.

I follow an alternative approach to estimation of publicly known components of the payoffs $v_{k,n}(a)$, which is based on a parametric specification. I assume that a payoff $v_{k,n}(a)$ is a linear function of the observables:

$$v_{k,n}(a) \equiv v_{k,n}(a; x_{k,n}, \beta) = x_{k,n}\beta(a), \tag{6}$$

where $\beta(a) \in B$ is a vector of parameters, which depends on the profile of actions chosen by players, and B is a compact set. Thus, I assume that all coefficients on observable factors x may depend on action profile a chosen by players. Those elements of $\beta(a)$ that differ depending on a are called outcome-specific. For instance, in entry games and coordination games it is often assumed that actions of other players affect only a constant term of given player's payoff, which would make only one component of $\beta(a)$ outcome-specific. In my setup, I allow for multiple outcome-specific coefficients.

Besides x, the economist observes actions $\delta_{k,n}$, $k = 1, 2, ..., K$, and $n = 1, 2, ..., N$ of all players in all repetitions of the game. For brevity, all observations of actions stacked in a single vector will be denoted by δ.

3.2 Normalization of payoffs

For each agent, a payoff associated with action $a = 0$ is normalized to zero: $v_{k,n}(0, a_{-k}) = 0$ for $k = 1, 2, ..., K$. Indeed, adding the same constant to all payoffs of any agent does not affect the agent's choice of action. The action's choice depends only on the difference in expected payoffs from different actions, so the addition of a constant to all payoffs leaves the choice of action unaffected. Since only actions but not payoffs are observed, there is no way for the economist to pin down the level of payoffs and the normalization is necessary. The same logic applies to the variance of ε_n, which cannot be identified separately from the scale of $\beta(a)$. Consequently, it is necessary to normalize variance of ε_n unless it is already fixed (e.g., extreme value distribution).

3.3 Distributional assumptions

It is assumed that unobservables $\varepsilon_{k,n}$ are uncorrelated with the observables x_n and are independently and identically distributed with a cdf function $F(\cdot)$ known to the players and the economist. Aradillas-Lopez (2008) suggests an estimation procedure for the parameters of two-by-two games with incomplete information (under the assumption of unique equilibrium) with nonparametric distribution of the error term. Some exclusion restrictions and normalizations are used in this procedure. The potential presence of multiple equilibria complicates the situation significantly, and further I proceed with a parametric assumption about the distribution of the unobservables ε_n.

3.4 Estimation of the payoffs

The problem of the economist is to estimate β knowing x, δ, and the timing and structure of the game. Assuming that in all repetitions of the game players use equilibrium strategies implies the following set of equilibrium conditions:

$$\begin{aligned} \bar{\pi}_n(a_k) &= E[1\{\delta_{k,n} = a_k\}|x, \eta], \\ \forall k &\in \{1, 2, ..., K\}, \quad \forall a_k \in \{0, 1, 2, ..., M\} \end{aligned} \tag{7}$$

The economist observes actions $\delta_{k,n}$ of all players in all repetitions of the game. It would be a straightforward task to estimate β using the equilibrium conditions (7) if one observed the beliefs $\bar{\pi}_n$. Without them, one encounters the problem of potential multiplicity of solutions to (7). While one may estimate empirical choice probabilities, they will not necessarily be consistent estimates of equilibrium choice probabilities. In fact, if there are multiple equilibria in data, empirical choice probabilities are consistent estimates of some weighted average of equilibrium choice probabilities corresponding to different equilibria. Further I refer to this (population) weighted average as $E_\eta[1\{\delta_{k,n} = a_k\}|x]$ where the unobservable variable η essentially determines weights. As any weighted average of the equilibrium beliefs is, by definition, smaller than the largest equilibrium choice probability and larger than the smallest one, the following inequalities hold in any equilibrium:

$$\begin{gathered} \bar{\pi}_{\min}(a_k; x_n, \beta) - E_\eta[1\{\delta_k = a_k\}|x_n] \leq 0 \qquad , \\ E_\eta[1\{\delta_k = a_k\}|x] - \bar{\pi}_{\max}(a_k; x_n, \beta) \leq 0 \\ \forall k \in \{1, 2, ..., K\}, \quad \forall a_k \in \{0, 1, 2, ..., M \quad \} \end{gathered} \tag{8}$$

where $\bar{\pi}_{\min}(a_k; x, \beta)$ and $\bar{\pi}_{\max}(a_k; x, \beta)$ are defined in (4) and (6) is used as the definition of the publicly known component of the payoffs.

The inequalities (8) can be used to form moments

$$g(x, b) = \begin{pmatrix} g_{\min}(x, b) \\ g_{\max}(x, b) \end{pmatrix}, \tag{9}$$

where $g_{\min}(x,b)$ and $g_{\max}(x,b)$ are vectors of the size $(K \times M \times S_x)$ with S_x being the cardinality of set x (i.e., S_x is the number of repetitions of the game with different observables; if there are two markets with the same observables $x_n = x_r$, it means that there are two observations but only one unique game characterized by observables x_n, so in that case S_x is smaller than N). The elements of the vectors $g_{\min}(x,b)$ and $g_{\max}(x,b)$ have the following form:

$$\begin{aligned} g_{k,a_k,\min}(x_s,b) &= (\bar{\pi}_{\min}(a_k;x_s,b) - E_\eta[1\{\delta_k = a_k\}|x_s])_+ \quad (10) \\ g_{k,a_k,\max}(x_s,b) &= (E_\eta[1\{\delta_k = a_k\}|x_s] - \bar{\pi}_{\max}(a_k;x_s,b))_+ \\ &\text{for } k = 1,...,K, a_k = 1,...,M, s = 1,...,S_x \quad (11) \end{aligned}$$

where $(x)_+ = \max(0,x)$.

The solution to the following minimization problem contains all parameters b that satisfy (8):

$$\begin{aligned} B_0 &= \arg\min_{b \in B} Q(x,b) \quad (12) \\ \text{where } Q(x,b) &= g(x,b)'g(x,b). \end{aligned}$$

Here B_0 may be a set estimate that contains true parameter β along with other $b \in B$ satisfying (10).

Applying the analog principle to the optimization problem stated above, I suggest the following two-stage semiparametric procedure.

In the first stage, the empirical choice probabilities $\hat{\pi}$ are estimated non-parametrically. One can use various techniques for this purpose, including sieve method, kernel method, or local polynomial approximation method. It should be expected that parameter estimates will be sensitive to the quality of the first-stage estimates $\hat{\pi}$. Given the non-parametric nature of these estimates, the larger the number of state variables in the model, the harder it will be to obtain sufficiently precise first-stage estimates. This is an important practical restriction on the size of games that can be estimated by the proposed method.

At the second stage, the sample analogs of (10) are formed with $\hat{\pi}$'s standing for

$E_\eta[1\{\delta_k = a_k\}|x_s]$:

$$\begin{aligned} \hat{g}_n(x,b) &= \begin{pmatrix} \hat{g}_{\min}(x,b) \\ \hat{g}_{\max}(x,b) \end{pmatrix} \qquad (13) \\ \hat{g}_{\min}(x,b) &= (\bar{\pi}_{\min}(a_k;x,b) - \hat{\pi}(a_k;x))_+ \\ \hat{g}_{\max}(x,b) &= (\hat{\pi}(a_k;x) - \bar{\pi}_{\max}(a_k;x,b))_+ \end{aligned}$$

and the sample analog $\hat{Q}(x,b)$ of $Q(x,b)$ is minimized to obtain the (possibly) set estimates $\hat{B}_0$. The consistency of the proposed estimator will not be affected and its efficiency may be improved if moments are weighted. In particular, it seems reasonable to assign large weight to inequalities that are associated with small difference between maximal and minimal equilibrium beliefs and vice versa. Indeed, if $\bar{\pi}_{\min}(a_k;x_s,b) = 0$ and $\bar{\pi}_{\max}(a_k;x_s,b) = 1$ for some specific x_n and b, the inequalities employing these equilibrium beliefs do not provide any information at all, because whatever empirical choice probabilities $\hat{\pi}(a_k;x_s)$ are, the inequalities in question will not be violated. Thus, such moments can be safely discarded by assigning them zero weight. On the contrary, if $\bar{\pi}_{\min}(a_k;x_s,b) = \bar{\pi}_{\max}(a_k;x_s,b)$ the respective inequalities essentially amount to equality moments and should be given high weights in estimation. I suggest using the estimator, which is similar to described above but instead of (13) it employs weighted inequality moments:

$$\begin{aligned} \hat{g}_{w,\min}(x_s,b) &= (\bar{\pi}_{\min}(a_k;x_s,b) - \hat{\pi}(a_k;x_s))_+(1 - (\bar{\pi}_{\max}(a_k;x_s,b) - \bar{\pi}_{\min}(a_k;x_s,b)))\quad(14) \\ \hat{g}_{w,\max}(x_s,b) &= (\hat{\pi}(a_k;x_s) - \bar{\pi}_{\max}(a_k;x_s,b))_+(1 - (\bar{\pi}_{\max}(a_k;x_s,b) - \bar{\pi}_{\min}(a_k;x_s,b))), \\ k &= 1,...,K,\ \ s = 1,2,...,S_x,. \end{aligned}$$

In practical implementation of the estimators described in this section there is a necessity to calculate $\bar{\pi}_{\min}(a_k;x,b)$ and $\bar{\pi}_{\max}(a_k;x,b)$ for each value of b. There is no explicit formula for them, and, in general, finding all solutions to (7) is a difficult problem. While more detailed discussion of alternative methods to find $\bar{\pi}_{\min}(a_k;x,b)$ and $\bar{\pi}_{\max}(a_k;x,b)$ is beyond the scope of this essay, it should be pointed out the Bajari et al. (2007b) describe the procedure to find all equilibria for the case of the extreme value distribution of unobservables

ε, using the 'all-solutions' Homotopy continuation method.

3.5 Parameter identification

Despite the fact that, in general, B_0 is a set estimate and is obtained using the inequality moments, it can be a point estimate in some circumstances. Indeed, it is possible that under true parameters β for some values of state variables x_s there is a unique equilibrium. In this case the relevant inequalities in (10) 'tighten', effectively acting as equalities. Therefore, if the economist knew which x_s characterize games with unique equilibria, he could use these observations, essentially avoiding the problem of the potential multiplicity of equilibria. But without knowledge of true β it is impossible to select such x_s. Indeed, $\bar{\pi}_{\min}(a_k; x_s, b)$ and $\bar{\pi}_{\max}(a_k; x_s, b)$ depend on b, and in general it can be the case that $\bar{\pi}_{\min}(a_k; x_s, \beta) = \bar{\pi}_{\max}(a_k; x_s, \beta)$ but $\bar{\pi}_{\min}(a_k; x_s, b) < \bar{\pi}_{\max}(a_k; x_s, b)$. Still, if in the sample there are observations of games with unique equilibria, it might be possible to achieve point identification of β.

Definition. It is said that β is identified relative to $b \in B$ if there exists such $x' \in X$, $k \in \{1, ..., K\}$, and $a_k \in A_k$ that

$$\bar{\pi}_{\max}(a_k; x', b) < \bar{\pi}_{\min}(a_k; x', \beta)$$

or

$$\bar{\pi}_{\min}(a_k; x', b) > \bar{\pi}_{\max}(a_k; x', \beta).$$

Indeed, if this condition is satisfied, then it means that any weighted average of the extremal equilibrium beliefs under parameter β is either smaller than the minimum equilibrium belief or greater than the maximum equilibrium belief under b at least for one action of one player. Therefore, provided that a consistent estimate of $E_\eta[1\{\delta_k = a_k\}|x_s]$ is available, it will fall beyond the bounds suggested by using the candidate parameter b, which is inconsistent with the hypothesis that b is the true data generating parameter.

It is hard to find conditions that would guarantee identification of β using the above definition as there are no closed form expressions for the extremal equilibrium beliefs, but the task becomes easier if there are unique equilibria under both β and b. Then it would be sufficient to show that

$$\bar{\pi}_{\max}(a_k; x', b) = \bar{\pi}_{\min}(a_k; x', b) \neq \bar{\pi}_{\min}(a_k; x', \beta) = \bar{\pi}_{\max}(a_k; x', \beta). \tag{15}$$

Thus, if it is possible to find such x' that (15) is satisfied, then β is identified relative to b. It appears that if there is i) a sufficient number of state variables (which depends on the number of players and number of actions in the game) that have sufficiently large domain and support and ii) coefficients on these state variables are outcome-specific, it is possible to find such x'. The requirement of having sufficient number of outcome-specific coefficients is strong. Essentially it demands to make specific assumptions about the payoffs that go beyond linear structure. Nevertheless, such assumptions may be quite plausible in many cases. More importantly, the smaller the difference between these outcome-specific coefficients, the larger the domain of explanatory variables should be to identify them.

To understand how it is possible to choose such x' that there is a unique equilibrium in the relevant repetitions of the game, consider a choice probability for action a_k if unobservables have extreme value distribution (which allows to integrate out unobservables from the expression for an equilibrium choice probability):

$$\bar{\pi}(a_k; x_s, \beta) = \frac{\exp(\sum_{a_{-k} \subset A_{-k}} \bar{\pi}(a_{-k}; x_s, \beta) v_{k,s}(a_k, a_{-k}; x_{k,s}, \beta))}{1 + \sum_{a_j \in A} \exp(\sum_{a_{-k} \subset A_{-k}} \bar{\pi}(a_{-k}; x_s, \beta) v_{k,n}(a_j, a_{-k}; x_{k,s}, \beta))}, \tag{16}$$

$$\forall k \in \{1, 2, ..., K\}, \quad \forall a_k \in \{0, 1, 2, ..., M\}, \ s = 1, 2, ..., S_x,$$

and notice that if the righthand side of (16) does not depend on $\bar{\pi}(a_{-k})$ then only one value value of $\bar{\pi}(a_k)$ will satisfy (16). If one can find such x' that $\bar{\pi}(a_{-k})$ cancel out in sufficient number of equations in system (16), it will guarantee the uniqueness of equilibrium. Given my specification of the payoffs, the coefficients on $\bar{\pi}(a_{-k})$ in (16) are some linear combinations of the values of the state variables x and parameters b. Setting them equal to zero, one obtains a system of linear equations that can be solved for x. There is at least one solution if there is a sufficient number of variables (which translates into the exclusionary restrictions) and these variables have a sufficiently large domain for the system to have a solution. Using

	$a_2 = 0$		$a_2 = 1$	
$a_1 = 0$	0,	0	0,	$\nu_2\beta_\nu + \xi_2\beta_{\xi,0}$ $+\omega\beta_{\omega,0} - \varepsilon_2$
$a_1 = 1$	$\nu_1\beta_\nu + \xi_1\beta_{\xi,0}$ $+\omega\beta_{\omega,0} - \varepsilon_1$,	0	$\nu_1\beta_\nu + \xi_1\beta_{\xi,1}$ $+\omega\beta_{\omega,1} - \varepsilon_1$,	$\nu_2\beta_\nu + \xi_2\beta_{\xi,0}$ $+\omega\beta_{\omega,1} - \varepsilon_2$

Table 2: Payoffs in a binary game with two players

this system, one can determine precisely what the domain should be, given the compact space of parameters B. In practice, the economist would know the domain and support for the available observables and, using the same system, could determine which b's may be distinguished from each other.

For the purposes of my proof, I divide all observables into three groups: $x = [\nu, \xi, \omega]$ (parameters β are split into β_ν, β_ξ, and β_ω, respectively). Observables ν include those variables that do not have outcome-specific coefficients on them at all. For example, in an often employed specification for an entry game payoffs where only a constant term depends on the actions of the opponents, all other variables would be grouped in ν. Both ξ and ω include variables that have outcome-specific coefficients. The difference between them is that the second group, ξ, includes player- and game-specific variables, while the third group ω includes only game-specific variables (such as market size, for instance). Only ξ and ω are used in my proof. The reason is that if a variable has the same coefficient whatever the profile of actions is, it implies that this variable is not interacted with the beliefs about the actions of other players because whatever they do, the effect of this variable on the payoff does not change. As my proof relies on 'varying' some state variables to find a game with a unique equilibrium, variables from ν are of no use for this purpose. It is also important to note that having only an outcome-specific constant is not enough, because it does not vary.

Here I provide only a proposition regarding point identification in a binary game with two players and relegate a more general case for extreme value distribution of unobservables $\varepsilon_{n,k}(a_k)$, K players and $M + 1$ actions to Appendix A as the exposition becomes extremely cumbersome in the general case.

Continuing with the example discussed in Section 2.2 and the specification of the payoffs as described above, the payoff matrix now looks as specified in Table 2.

The system that determines equilibrium beliefs π_k, $k = 1, 2$ that player k chooses action 1 takes the following form:

$$\pi_1 = F(\nu_1\beta_\nu + \xi_1\beta_{\xi,0} + \omega\beta_{\omega,0} + (\xi_1(\beta_{\xi,1} - \beta_{\xi,0}) + \omega(\beta_{\omega,1} - \beta_{\omega,0}))\pi_2) \quad (17)$$

$$\pi_2 = F(\nu_2\beta_\nu + \xi_2\beta_{\xi,0} + \omega\beta_{\omega,0} + (\xi_2(\beta_{\xi,1} - \beta_{\xi,0}) + \omega(\beta_{\omega,1} - \beta_{\omega,0}))\pi_1).$$

This system has a unique solution if either of the following equations holds:

$$\begin{aligned} \xi_1(\beta_{\xi,1} - \beta_{\xi,0}) + \omega(\beta_{\omega,1} - \beta_{\omega,0}) &= 0, \\ &\text{or} \\ \xi_2(\beta_{\xi,1} - \beta_{\xi,0}) + \omega(\beta_{\omega,1} - \beta_{\omega,0})) &= 0. \end{aligned} \quad (18)$$

Indeed, if, for example, $\xi_1(\beta_{\xi,1} - \beta_{\xi,0}) + \omega(\beta_{\omega,1} - \beta_{\omega,0}) = 0$, then π_1 does not depend on π_2, resulting in a unique solution for π_2.

Proposition 1. Suppose that in a simultaneous binary game of incomplete information with two players indexed by $k = 1, 2$ and two actions characterized by a profile $a \in \{0, 1\} \times \{0, 1\}$, the payoffs of the players are as specified in Table 2 where ν_1, ν_2, ξ_1, ξ_2, and ω are row vectors of observables and the Bayes-Nash concept of equilibrium is applied. If vectors i) ξ_1, and ξ_2 have at least one continuously distributed element (with domain on R) associated with outcome-specific coefficients each, ii) ω has at least one element associated with outcome-specific coefficient, and iii) ν_1 contains at least one continuously distributed element (with domain on R) associated with a non-zero coefficient, then for any b that satisfies the outcome-specific requirement as stated above, $b \in B$, $b \neq \beta$, there exists a vector of observables $x \in X$ such that

$$\begin{aligned} \pi_{k,\max}(x, b) &< \pi_{k,\min}(x, \beta) \\ &\textit{or} \\ \pi_{k,\min}(x, b) &> \pi_{k,\max}(x, \beta), \\ k &= 1, 2 \end{aligned} \quad (19)$$

and, therefore, β is identified relative to $b \in B$.

While the proof of the proposition is given in the Appendix A, the reasoning behind this result goes as follows. Three explanatory variables required by the proposition are necessary i) to guarantee a unique equilibrium under the true parameter β, ii) to guarantee a unique equilibrium under any candidate parameter b and iii) to guarantee that these two equilibria do not coincide (the outcome-specific coefficients on these variables are necessary to insure that variation in them indeed affects the equilibrium beliefs).

If one is willing to assume some minimal difference in the outcome-specific coefficients on the explanatory variables required for the proposition, then one could limit the required domain of the explanatory variables. If an equilibrium is unique, then the upper and lower bounds on observed probabilities in (8) coincide, and unless they are the same under two different parameters, these parameters can be empirically distinguished from one another. With a unique equilibrium, a frequency of an action should be almost surely equal to the bounds under the true parameter (given unique equilibrium). If under every other candidate parameter b the bounds do not coincide with those under the true parameter β, the difference between the estimate of the expected frequency of actions and bounds will be minimized at true β.

The idea of the proof is illustrated in Figure 2. Suppose that under some $x \in X$ there are three equilibria under both β and b. The panel A shows what happens when x' is considered instead of x. The probability of the first player to choose action 1 does not depend on π_2 and looks like a straight line on a graph.The Panel B of Figure 2 shows the what happens when one explanatory variable is chosen so that there is a unique equilibrium under b. The third regressor is used to guarantee that these two unique equilibria under different parameter values do not coincide.

Incidentally, if one is willing to assume some minimal difference between outcome-specific coefficients on the explanatory variable that are required for Proposition 2, one may limit the domain of the explanatory variables required in Proposition 1. Consider a further simplified example where ξ_1, ξ_2 and ω have one element each, and two parameters $\Delta_{\beta\xi}$ and $\Delta_{\beta\omega}$ that stand for the differences between outcome-specific coefficients (similar to $\beta_{\xi,1} - \beta_{\xi,0}$ and $\beta_{\omega,1} - \beta_{\omega,0}$). The conditions analogous to those used in the proof of Proposition 2 require

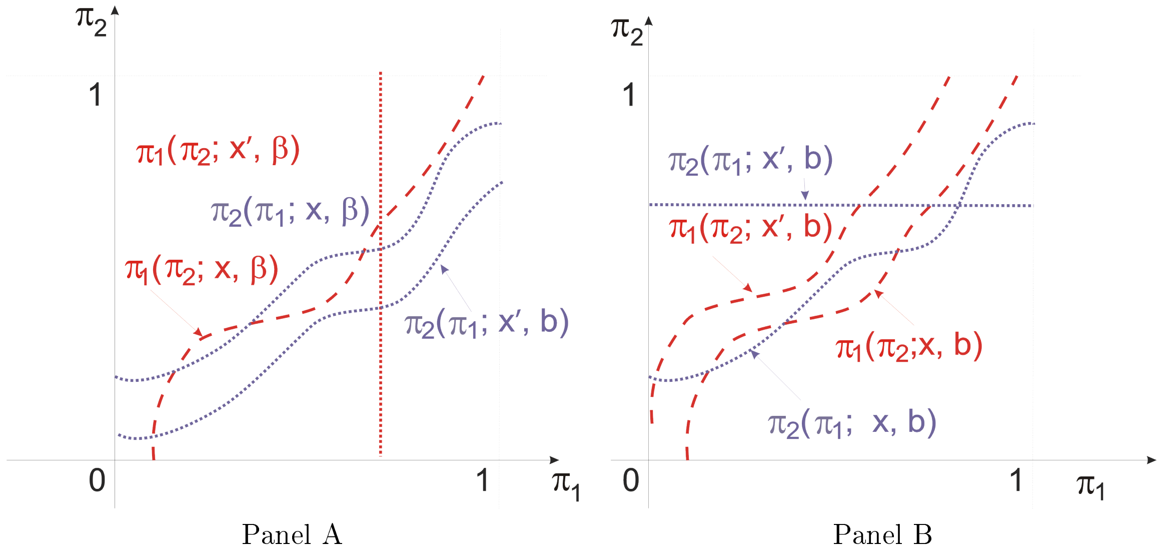

Figure 2

that

$$\xi_1 \Delta_{\beta\xi} + \omega \Delta_{\beta\omega} = 0$$
$$\xi_2 \Delta_{b\xi} + \omega \Delta_{b\omega} = 0.$$

Disregarding trivial (and useless for identification purposes) zero solution, it is clear that if domains of ξ_1, ξ_2 and ω are finite and $\Delta_{\beta\omega}$ is fixed, there is a limit on what absolute minimal $\Delta_{\beta\xi}$ can be.

When number of players and/or actions are greater than two, the nature of the requirements for a sufficient condition of point identification is similar but the number of necessary explanatory variables increases dramatically, as it is proportionate to both number of players and number of actions, and the same requirements about the outcome-specific coefficients apply.

3.6 Consistency

A technically inconvenient property of the suggested estimator is that its objective function is not continuous in the space of parameters B. Indeed, all moments include extremal beliefs $\bar{\pi}_{\min}(a_k; x, b)$ and $\bar{\pi}_{\max}(a_k; x, b)$ which are not continuous in B but may have jumps at the points where the number of equilibria changes. Together with the fact that, in general, the estimator provides set estimates, this makes the derivation of its asymptotic properties a very difficult problem. Here I only provide a proof of its consistency in case of point identification. Further I make use of a norm $\|\cdot\|$ on a finite-dimensional space.

Assumption 1. (Compactness.) The parameter space B is compact.

Assumption 2. (Identification.) β is point identified: there is a unique β such that

$$g(x, \beta) = 0$$

where $g(x, \beta)$ is defined in (9). Moreover, for any $\delta > 0$ and $B(\delta) = [b \in B : |b - \beta| \geq \delta]$

$$\inf_{b \in B(\delta)} ||g(x, b)|| > 0$$

Practically this assumption means that for any $b \neq \beta$, $b \in B$ at least one of the inequality conditions (10) is violated, i.e. at least one of the elements of the vector $g(x, \beta)$ is positive.

Assumption 3. (Random sampling.) The unobservables $\varepsilon_{k,n}(a_k)$, $n = 1, ..., \infty$, $k = 1, 2, ..., K$ are independent realizations from their distribution $F(\cdot)$.

Proposition 2. Assume that Assumptions 1-3 hold and the method of moments estimate $\hat{B}_0$ is non-empty for all N. Then

$$\lim_{N \to \infty} \sup_{b \in \hat{B}_0} ||b - \beta|| = 0, \text{ almost surely.}$$

The proof of Proposition 2 is given in Appendix B, and here I only discuss the importance of some of the assumptions and the main idea of the proof. The identification is the most crucial for the result. It says that for every parameter b different from the true parameter β there exists such x_n in the sample that either $\bar{\pi}_{\min}(a_k; x_n, b) - E_\eta[1\{\delta_{k,n} = a_k\}|x_n] > 0$ or $E_\eta[1\{\delta_{k,n} = a_k\}|x_n] - \bar{\pi}_{\max}(a_k; x_n, b) > 0$, which is a strong requirement. The compactness of the parameter space is a technical assumption. The random sampling assumption is much more restrictive but is maintained throughout this essay.

The logic of the proof is as follows. First, I prove a lemma that the uniform law of

	$a_2 = 0$		$a_2 = 1$	
$a_1 = 0$	$(0,$	$0)$	$(0,$	$\beta_{01}+\beta_{11}x_2+\beta_{21}z-\varepsilon_2)$
$a_1 = 1$	$(\beta_{01}+\beta_{11}x_1+\beta_{21}z-\varepsilon_1,$	$0)$	$(\beta_{02}+\beta_{12}x_1+\beta_{22}z-\varepsilon_1,$	$\beta_{02}+\beta_{12}x_2+\beta_{22}z-\varepsilon_2)$

Table 3: Matrix of payoffs in the game used for Monte Carlo experiments

large numbers holds in this problem, so that $\lim_{N\to\infty}\sup_{b\in B}||\hat{g}_n(x,b)-g(x,b)||=0$ almost surely. Then, I consider the compact sets $B(\delta)$ that expand as $\delta\to 0$. On each such set, by the identification assumption the infimum of $||g(x,b)||$ is greater than zero. Together with the ULLN it implies that infimum of $\hat{g}_n(x,b)$ is also greater than zero on $B(\delta)$ for sufficiently large N. It also follows that infimum of $\hat{Q}_n(x,b)=\hat{g}_n(x,b)'\hat{g}_n(x,b)$ is separated from zero for sufficiently large N. Therefore, $\hat{B}_0$ does not belong to $B(\delta)$ and should belong to $B-B(\delta)$, which contracts to β as $\delta\to 0$, which concludes the proof.

4 Monte Carlo evidence

In this section, I present the results of simulations illustrating the performance of the suggested estimator in a very simple setup. The model is a binary game with two players. The payoffs of the players are given in Table 3.

I consider three setups that differ in the number of unique x_n observations and in the version of estimator. In setup 1 1500 triples of (x_1,x_2,z) are randomly sampled and the empirical choice probabilities are estimated using the kernel method with normal density and bandwidths. In setups 2 and 3 a different approach to constructing a sample is taken. For each of the randomly chosen 150 x_n points I draw 10 ε_n per player so as to see how the first stage estimates affect the estimation. When there are several equilibria in the market, I randomize between the extremal ones. The domains of x_1 and x_2 are $[-80,80]$ and the domain of z is $[-5,5]$. In setup 2 I use an unweighted method of moments estimator, and in setup 3 I weight moments as described in (14). Table 4 shows the parameter values chosen for the simulation as well as the estimates.

Surprisingly, while the average absolute error in the first stage estimates thrice as high in kernel setup 1 as in frequency setups 2 and 3, it is not clear that the quality of estimates is significantly worse in the former case. On the contrary, it is comparable for most of the

	Setup 1	Setup 2	Setup 3	True
β_{01}	0.78	1.22	1.35	1
β_{11}	1.40	0.95	1.00	1.2
β_{22}	0.72	0.4	0.43	0.7
β_{02}	-1.81	-1.97	-1.99	-2
β_{12}	0.09	0.10	0.10	0.1
β_{22}	0.52	0.49	0.48	0.5
N	1500	1500	1500	
Percent of observations with multiple equilibria under true parameters	17%	18%	18%	
Average absolute error in empirical choice probabilities	0.042	0.014	0.014	

Table 4: True values of the parameters and the parameter estimates in 3 Monte Carlo setups

parameters, better for one of them, and worse for another one. Similarly, the performance of the estimator in setups 2 and 3 is similar with slight improvements for most of the parameters in setup 3 outweighed by its worse performance in case of β_{01}.

5 Conclusion

This essay suggests a method of estimation of a simultaneous discrete choice static game with incomplete information, which is robust to the presence of multiple equilibria in data. I suggest using conditions that hold in any equilibrium. Since these conditions have the form of inequalities, robustness usually comes at a price of point identification. Still, it appears that point identification may be achievable if there are outcome-specific coefficients on explanatory variables, whose number is proportional to the number of players and actions in the game. Further exploration of statistical properties of the proposed estimator is in order, as well as finding a way to incorporate heterogeneity and correlation between unobservables of the players.

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Appendix A

Proof of Proposition 1. The proof strategy is to show that for any parameter $b \in B$ it is possible to find such observables $\xi_1, \xi_2, \omega \in x, x \in X$ that three conditions hold simultaneously: the equilibrium under parameter β is unique; ii) the equilibrium under parameter b is unique; iii) the equilibria under parameters β and b are different:

$$\pi_{k\max}(x,\beta) = \pi_{k\min}(x,\beta) \neq \pi_{k\max}(x,b) = \pi_{k\min}(x,b),\ k \in \{1,2\}. \tag{20}$$

First the conditions are found that guarantee the uniqueness of the equilibrium under β and under b; then, it is checked that the unique equilibria under β and b are different.

Using (17), construct a system of equations that guarantees that equilibria under both β and b are unique:

$$\begin{aligned} \xi_1(\beta_{\xi,1} - \beta_{\xi,0}) + \omega(\beta_{\omega,1} - \beta_{\omega,0}) &= 0 \\ \xi_2(b_{\xi,1} - b_{\xi,0}) + \omega(b_{\omega,1} - b_{\omega,0}) &= 0 \end{aligned} \tag{21}$$

If ω has only one element and it is a constant, then (21) has at most one solution. If ω does not include a constant, the system (21) has (infinitely many) solutions if in the matrix

$$\begin{pmatrix} \beta_{\xi,1} - \beta_{\xi,0} & 0 & \beta_{\omega,1} - \beta_{\omega,0} \\ 0 & b_{\xi,1} - b_{\xi,0} & b_{\omega,1} - b_{\omega,0} \end{pmatrix}$$

the vectors $\beta_{\xi,1} - \beta_{\xi,0}$, $b_{\xi,1} - b_{\xi,0}$, $\beta_{\omega,1} - \beta_{\omega,0}$, and $b_{\omega,1} - b_{\omega,0}$ have at least one non-zero element each.

If it appears that given the chosen x

$$\pi_{k\max}(x,\beta) = \pi_{k\min}(x,\beta) = \pi_{k\max}(x,b) = \pi_{k\min}(x,b),\ k \in \{1,2\},$$

it is possible to change ν_1 so that it affects only the equilibrium probabilities of the first player.

Point identification in a game with K players and $M+1$ actions

Suppose that in a simultaneous game of incomplete information there are K players indexed by $k = 1, 2, ..., K$, taking one of $M+1$ actions characterized by a profile $a \in \{0, 1, ..., M\}^K$, and receiving private signals $\varepsilon_{k,n}$ distributed independently according to the extreme value distribution. The expected payoff of player k from taking action a_k is

$$\sum_{a_{-k} \subset A_{-k}} \bar{\pi}(a_{-k}; x_n, \beta) v_{k,n}(a_k, a_{-k}; x_{k,n}, \beta) + \varepsilon_{k,n}(a_k)$$

and the equilibrium choice probabilities satisfy the following conditions:

$$\begin{aligned} \bar{\pi}(a_k; x_n, \beta) &= \frac{\exp(\sum_{a_{-k} \subset A_{-k}} \bar{\pi}(a_{-k}; x_n, \beta) v_{k,n}(a_k, a_{-k}; x_{k,n}, \beta))}{1 + \sum_{\substack{a_j \in A \\ a_j \neq 0}} \exp(\sum_{a_{-k} \subset A_{-k}} \bar{\pi}(a_{-k}; x_n, \beta) v_{k,n}(a_j, a_{-k}; x_{k,n}, \beta))}, \qquad (22) \\ \forall k &\in \{1, 2, ..., K\}, \quad \forall a_k \in \{0, 1, 2, ..., M\}, \; n = 1, 2, ..., N. \end{aligned}$$

Let's arrange all observables into three groups: $x = [\nu, \xi, \omega]$ (parameters β are arranged, respectively, into β_ν, β_ξ, and β_ω). Observables ν include those variables that do not have outcome-specific coefficients at all. Both ξ and ω include variables that have outcome-specific coefficients, with ξ including player and game (market) specific variables and with ω including only game specific variables (such as market size). Then the expected payoff of player k from taking action a_k can be rewritten as follows:

$$\begin{aligned} & \sum_{a_{-k} \subset A_{-k}} \bar{\pi}(a_{-k}; x, \beta) v_{k,n}(a_k, a_{-k}; x_k, \beta) + \varepsilon_k(a_k) \qquad (23) \\ = \; & \nu_k \beta_\nu + \bar{\pi}(\tilde{a}_{-k}; x, \beta)(\xi_k \beta_\xi(\tilde{a}_{-k}) + \omega \beta_\omega(\tilde{a}_{-k}) + \varepsilon_k(a_k) \\ & + \sum_{\substack{a_{-k} \subset A_{-k} \\ a_{-k} \neq \tilde{a}_{-k}}} (\bar{\pi}(a_{-k}; x, \beta) - \bar{\pi}(\tilde{a}_{-k}; x, \beta)) \left[\xi_k(\beta_\xi(a_{-k}) - \beta_\xi(\tilde{a}_{-k})) + \omega(\beta_\omega(a_{-k}) - \beta_\omega(\tilde{a}_{-k})\right] \end{aligned}$$

where $\tilde{a}_{-k}$ is some action profile of all the players except for the kth. From the above formula it is clear that if

$$\begin{aligned} & \xi_k(\beta_\xi(a_{-k}) - \beta_\xi(\tilde{a}_{-k})) + \omega_n(\beta_\omega(a_{-k}) - \beta_\omega(\tilde{a}_{-k}) = 0 \\ & \text{for all } a_{-k} \subset A_{-k}, \; a_{-k} \neq \tilde{a}_{-k}, \end{aligned}$$

then the expected payoff of player k from choosing action a_k does not depend on the strategies of others. The form of (22) suggests that if there are (ξ_k, ω) that solve the system of L equations (L is the number of elements in A_{-k} minus one)

$$\begin{aligned} \xi_k(\beta_\xi(a_{-k}) - \beta_\xi(\tilde{a}_{-k})) + \omega(\beta_\omega(a_{-k}) - \beta_\omega(\tilde{a}_{-k})) &= 0, \\ a_{-k} &\subset A_{-k}, a_{-k} \neq \tilde{a}_{-k} \end{aligned} \tag{24}$$

then there is a unique vector of equilibrium beliefs $\bar{\pi}(a_k; x_n, \beta)$ for player k. Analogously, given parameters b, divided respectively into b_ν, b_ξ, and b_ω, there is a system of L equations similar to (24):

$$\xi_k(b_\xi(a_{-k}) - b_\xi(\tilde{a}_{-k})) + \omega(b_\omega(a_{-k}) - b_\omega(\tilde{a}_{-k})) = 0, \tag{25}$$

$$a_{-k} \subset A_{-k}, a_{-k} \neq \tilde{a}_{-k} \tag{26}$$

Suppose that i) vectors ξ_1 and ξ_2 have at least L (cardinality of A_{-k} minus one) and, if $K > 2$, ξ_k, $k = 3, ..., K$, each have at least $2L$ continuously distributed components (with domain on R) associated with outcome-specific coefficients, and denote vectors that contain only these components, respectively, $\tilde{\xi}_1$, $\tilde{\xi}_2$,..., $\tilde{\xi}_K$; ii) vector ω has at least one component associated with outcome-specific coefficient and denoted by $\tilde{\omega}$ and iii) vector ν_1 contains at least one continuously distributed element (with domain on R) element associated with non-zero coefficient and denoted by $\tilde{\nu}_1$. Denote the coefficients associated with $\tilde{\xi}_k$, $k = 1, 2, ..., K$, and $\tilde{\omega}$ in (24) by, respectively, $\tilde{\beta}_\xi(a_{-k}) - \tilde{\beta}_\xi(\tilde{a}_{-k})$ and $\tilde{\beta}_\omega(a_{-k}) - \tilde{\beta}_\omega(\tilde{a}_{-k})$ and in (25) by, respectively, $\tilde{b}_\xi(a_{-k}) - \tilde{b}_\xi(\tilde{a}_{-k})$ and $\tilde{b}_\omega(a_{-k}) - \tilde{b}_\omega(\tilde{a}_{-k})$. Let's rewrite (24) and (25) together in matrix form:

$$\begin{pmatrix} B_{\xi,1} & 0 & ... & 0 & 0 & B_{\omega,1} \\ 0 & B_{\xi,2} & ... & 0 & 0 & B_{\omega,2} \\ ... & & ... & 0 & ... & \\ 0 & 0 & ... & B_{\xi,K-1} & 0 & B_{\omega,K-1} \\ 0 & \tilde{B}_{\xi,2} & ... & 0 & 0 & \tilde{B}_{\omega,2} \\ & & ... & & & ... \\ 0 & 0 & ... & \tilde{B}_{\xi,K-1} & & ... \\ 0 & 0 & ... & 0 & \tilde{B}_{\xi,K} & \tilde{B}_{\omega,K} \end{pmatrix} \begin{pmatrix} \xi_{1,n} \\ \xi_{2,n} \\ ... \\ \xi_{K-1,n} \\ \xi_{K-,n} \\ \omega_n \end{pmatrix} = 0,$$

where matrix $B_{\xi,k}$ contains coefficients $\beta_\xi(a_{-k})-\beta_\xi(\tilde{a}_{-k})$, $\tilde{B}_{\xi,k}$ contains coefficients $\tilde{b}_\xi(a_{-k})-\tilde{b}_\xi(\tilde{a}_{-k})$, $B_{\omega,k}$ contains coefficients $\beta_\omega(a_{-k})-\beta_\omega(\tilde{a}_{-k})$, and $\tilde{B}_{\omega,k}$ contains coefficients $\tilde{b}_\omega(a_{-k})-\tilde{b}_\omega(\tilde{a}_{-k})$ with the rows corresponding to $a_{-k} \subset A_{-k}, a_{-k} \neq \tilde{a}_{-k}$. If $rank(B_{\xi,1}) = rank([B_{\xi,1}\ B_{\omega,1}])$, $rank(\tilde{B}_{\xi,K}) = rank([\tilde{B}_{\xi,K}\ \tilde{B}_{\omega,K}])$, $rank\left(\begin{bmatrix} B_{\xi,k} \\ \tilde{B}_{\xi,k} \end{bmatrix}\right) = rank\left(\begin{bmatrix} B_{\xi,1} & B_{\omega,1} \\ \tilde{B}_{\xi,k} & \tilde{B}_{\omega,1} \end{bmatrix}\right)$, then there is a unique equilibrium given parameters β and there is a unique equilibrium given parameters b. If they coincide for all actions and players, one can use ν_1 to affect equilibrium under β but not under b. This would guarantee that β can be distinguished from b.

Appendix B

Lemma. Assumptions 1 and 3 imply the uniform law of large numbers (ULLN):

$$\lim_{N\to\infty} \sup_{b\in B} ||\hat{g}_n(x,b) - g(x,b)|| = 0, \quad \text{a.s.}$$

where $\hat{g}_n(x,b)$ is defined in (13) and $g(x,b)$ is defined in (9).

Proof. Consider $b \in B$ and a particular element of vector $\hat{g}_n(x,b) - g(x,b)$, for example,

$\hat{g}_{\max}(x_n, b) - g_{\max}(x_n, b)$:

$$\begin{aligned} & \hat{g}_{\max}(x_n, b) - g_{\max}(x_n, b) \\ = \; & \hat{\pi}(a_k; x_n) - \pi_{\max}(a_k; x_n, b) - E_\eta[1\{\delta_{k,n} = a_k\}|x_n] + \pi_{\max}(a_k; x_n, b) \\ = \; & \hat{\pi}(a_k; x_n) - E_\eta[1\{\delta_{k,n} = a_k\}|x_n]. \end{aligned} \tag{27}$$

The similar cancellations occur within any element of $\hat{g}_n(x, b) - g(x, b)$, so neither of them depend on b. Therefore, $||\hat{g}_n(x, b) - g(x, b)||$ does not depend on b and depends only on true parameter β. From assumption of Random Sampling and the strong law of large numbers it follows that for any $\eta > 0$ there almost surely exists a finite N_η such that for $N > N_\eta$

$$\max_{b \in B_\eta} ||\hat{g}_n(x, b) - g(x, b)|| < \eta. \tag{28}$$

Due to (27), (28) is true for any $b \in B$. As $\eta \to 0$, the lemma's claim follows.

Proof of Proposition 2. This proof closely follows the proof of Theorem 1 (Ch. 7) in Manski (1987).

Recall that $B(\delta) = [b \in B : ||b - \beta|| \geq \delta]$. Fix δ and denote $\varepsilon = \inf_{b \in B(\delta)} ||g(x, b)||$. By the identification assumption, $\varepsilon > 0$. Given that and the fact that, by condition of ULLN proved in Lemma above,

$$\lim_{N \to \infty} \sup_{b \in B} ||\hat{g}_n(x, b) - g(x, b)|| = 0, \quad \text{a.s.},$$

it follows that there almost surely exists a finite N_2 such that if $N > N_1$, then

$$\inf_{b \in B(\delta)} ||\hat{g}_n(x, b)|| > \varepsilon/2.$$

Also, let's introduce an origin-preserving transformation $r(T) = T'T$ and $\eta = \inf_{||T|| > \varepsilon/2} r(T)$. By construction of the transformation, $\eta > 0$. Therefore, if $N > N_1$

$$\inf_{b \in B(\delta)} r\left[\hat{g}_n(x, b)\right] > \eta > 0.$$

From the identification assumption, ULLN, and the properties of $r(\cdot)$ it follows that there almost surely exists a finite N_2 such that if $N > N_2$, then

$$r\left[\hat{g}_n(x, b)\right] < \eta.$$

Therefore, for $N > \max(N_1, N_2)$

$$\hat{B}_0 \subset B - B(\delta).$$

Thus, if $\delta \to 0$, $B - B(\delta)$ contracts to β, which completes the proof.

Part II

Tenancy Rent Control and Credible Commitment in Maintenance

(co-authored with Richard Arnott)

1 Introduction

Tenancy rent control is a form of rent control in which rents are regulated within a tenancy but may be raised without restriction between tenancies; more specifically, the starting rent for a tenancy is unregulated but the path of nominal rents within a tenancy, conditional on the starting rent, is regulated, typically causing rents to rise less rapidly over the tenancy than they would in the absence of controls[3]. Many, perhaps most, jurisdictions around the world that previously had traditional first- and second-generation rent control programs (Arnott (1995)) have moved in the direction of tenancy rent control as a method of partial decontrol[4].

In jurisdictions that have stricter forms of rent control, tenancy rent control may be an attractive method of partial decontrol. Because the starting rent adjusts to clear the market, tenancy rent control does not generate the excess demand phenomena (such as key money, waiting lists, and discrimination) of stricter rent control programs, and should have less adverse effects on tenant mobility and the matching of households to housing units[5]. Tenancy

[3]This defines the "ideal type", which is what will be modelled in this paper. Many jurisdictions have forms of rent control that are intermediate between tenancy rent control, according to the above definition, and more traditional forms of rent control. In some, rent increases are regulated both within and between tenancies, but less severely between tenancies than within tenancies. In others, rent increases are unregulated between tenancies but are subject to a variety of regulatory provisions within a tenancy, such as a guideline rent increase (which allows rents to rise by a certain percentage per year) with a cost-pass through provision (which allows the landlord to apply for a rent increase above the guideline rent increase if justified by cost increases).

[4]Basu and Emerson (2000, 2003) and Arnott (2003) list some of these jurisdictions. Borsch-Supan (1996) models the current German system and Iwata (2002) the current Japanese system, both of which are termed "tenant protection" systems.

[5]There is a large literature on the adverse effects of rent control. Three particularly good papers that avoid polemical rent-control bashing are Frankena (1975), Glaeser and Luttmer (2003), and Olsen (1988).

rent control continues to provide sitting tenants with improved security of tenure; for one thing, rent regulation within tenancies precludes economic eviction; for another, because tenancy rent control, like other forms of rent control, provides landlords with an incentive to evict tenants, it is invariably accompanied by conversion (rehabilitation, demolition and reconstruction, and conversion to condominium) restrictions[6]. As well, tenancy rent control may be a politically attractive method of partial decontrol since it continues to provide rent protection to sitting tenants, who are typically the strongest opponents of decontrol. These benefits must be weighed against the costs. The most obvious costs are the tenant lock-in created by tenancy rent control and the unfairness of the preferential treatment of sitting tenants. There are also less obvious costs. The workability of tenancy rent control makes it more difficult to move to complete decontrol, should this be deemed desirable. Also, because a rent control administration is kept in place, it is relatively easy to return to harder controls should the political winds change. Landlords, fearing this, may curtail investment[7].

This paper focuses on another less obvious cost of tenancy rent control – its adverse effect on maintenance, construction, demolition and reconstruction, and rehabilitation. Pollakowski (1999) provides an empirical analysis of the effects of New York City's rent control system on housing maintenance there. Arnott and Johnston (1981) provides an informal, diagrammatic discussion of the effects of several rent control programs (though not tenancy rent control) on housing quality and maintenance. This paper will adapt the model of Arnott, Davidson, and Pines (1983) to examine how the application of tenancy rent control to a single atomistic landlord-builder affects his profit-maximizing behavior[8].

Assume, as we will throughout the paper in order to abstract from the tenant lock-in effect, that tenancy duration is exogenous. There are two conflicting intuitions concerning the effects of tenancy rent control on the atomistic landlord's behavior. A lay person with good economic intuition would probably argue that tenancy rent control gives the landlord

[6]Miron and Cullingworth (1983) and Hubert (1991) examine the effects of rent control on security of tenure.

[7]These less obvious costs are evident in the Ontario experience with rent control (e.g., Smith, 2003).

[8]Since the analysis is "very" partial equilibrium, it will ignore the effects of tenancy rent control on the level of rents and on other markets such as the labor market.

While the paper focuses on tenancy rent control, the techniques employed can be applied to examine the effects of other forms of rent control on the landlord's optimal program (indeed, Arnott and Johnston (1981) does so, albeit informally).

an incentive to spruce up his units between tenancies so that they "show" well and hence can be let at a higher starting rent, but little incentive to maintain the units well during tenancies since, after the starting rent has been agreed upon, maintaining well has no effect on the rent stream during the tenancy. An economist might however reasonably object that, with tenancy duration exogenous, there is nothing to prevent the landlord from following the program that is profit maximizing in the absence of tenancy rent control – which we shall term the *efficient program*. If the landlord follows this program, the tenant should be willing to pay as much over her tenancy as she would have for an uncontrolled unit. This line of reasoning suggests that, were it not for the tenancy lock-in, the landlord's profit-maximizing program would be unaffected by the application of tenancy rent control.

The resolution of the two conflicting intuitions lies in the ability of the landlord to credibly commit to the efficient program. If he is able to credibly commit to a maintenance program, he will credibly commit to the efficient program and the tenant will agree to pay the same in rent in discounted terms over the duration of the tenancy as in the absence of rent control. The landlord will therefore be making the same revenue and incurring the same costs as in the absence of rent control, and can surely do not better than this. If, however, the landlord is unable to credibly commit to pursuing the efficient program, once the lease is signed he has an incentive to pursue a different maintenance program, which we term the *opportunistic program*. Since the signing of the lease fixes the discounted rent the landlord will receive over the current tenancy, the only incentive he has to maintain is to improve the quality of the unit at the end of the lease, as this will increase the discounted rent he receives on subsequent tenancies. Compared to the efficient program, the opportunistic program entails both a reduction in average maintenance and a postponement of maintenance within a tenancy. Before the lease is signed, a prospective tenant should in this situation realize that under tenancy rent control the landlord will pursue the opportunistic rather than the efficient maintenance program and hence not be willing to pay as high a starting rent as she would if he were to pursue the efficient program.

The crux of the matter is therefore the landlord's ability, under tenancy rent control, to commit to a particular maintenance program. Three commitment mechanisms might be partially effective. The first is contracting on maintenance. One problem with this commitment mechanism is that, since maintenance is such an amorphous concept, maintenance clauses in the lease would be highly incomplete; for example, if the contract were to require the landlord to replace appliances every ten years, he might replace with appliances that are used and reconditioned or of minimal quality. Another problem is that it would be costly for a tenant to document that her landlord had not met the maintenance terms of the contract. The second commitment mechanism, reputation, is likely to be ineffective since the typical prospective tenant knows little or nothing about different landlords' maintenance performance when she is searching for a unit. The third mechanism, maintenance regulation, suffers from problems similar to those for contracting on maintenance. In our judgment, such commitment devices are generally ineffective, and in our analysis we shall assume them to be completely ineffective. The efficiency costs that we identify are reduced to the extent that these commitment mechanisms are indeed effective.

Section 2 analyzes the landlord's profit-maximizing program in the absence of rent control. Section 3 examines how tenancy rent control in the absence of credible commitment in maintenance distorts the profit-maximizing program. Section 4 provides some calibrated examples focusing on the magnitude of the efficiency loss caused by tenancy rent control. Finally, section 5 concludes.

2 The Profit-Maximizing Program without Rent Control

A competitive landlord owns a vacant lot of fixed area on which only a single unit of housing can be constructed[9]. Housing is durable and its quality is endogenous. Four quality-changing technologies are available: construction, maintenance, rehabilitation, and demolition. The economic environment is stationary over time and described by the quality-changing technologies, the rent function relating market rent to quality, and the interest rate. The maintenance technology is autonomous – the unit's rate of quality change depends on its current quality and the current level of maintenance expenditure but not on the unit's age *per se*. The landlord chooses the profit-maximizing program. Under these assumptions, phase plane analysis may be employed.

[9]The analysis can be extended to endogenize structural density (Arnott, Davidson, and Pines (1986)).

A rather thorough analysis of this problem is presented in Arnott, Davidson, and Pines (1983). Here we focus on a special – but also probably the most realistic – case, in which, in the absence of controls, at the beginning of the program it is profit maximizing to construct and downgrade. Three qualitatively different active programs may be profit maximizing:

1. Initial construction, followed by downgrading to saddlepoint quality (program $\mathcal{S}$).

2. A construction-downgrading-demolition cycle (program $\mathcal{D}$).

3. Initial construction followed by a downgrading-rehabilitation cycle (program $\mathcal{R}$).

2.1 Program $\mathcal{S}$

Under program $\mathcal{S}$, at time 0 the landlord constructs a single housing unit of quality q_c on his lot and then downgrades the unit asymptotically to saddlepoint quality q^S. Where $q(t)$ is quality at time t, $P(q)$ the exogenous rent function, $m(t)$ maintenance expenditure at time t, r the interest rate, α construction cost per unit of quality, $g(q,m)$ the depreciation function, and T the terminal time, the profit-maximizing program is the solution to

$$\max_{q_c, m(t)} \int_0^\infty \left(P\left(q\left(t\right)\right) - m\left(t\right)\right) e^{-rt} dt - \alpha q_c$$

$$s.t. \quad \begin{array}{rl} i) & \dot{q} = g\left(q, m\right) \\ ii) & q_c \equiv q\left(0\right) \quad \text{free} \\ iii) & \lim_{T \uparrow \infty} q\left(T\right) \quad \text{free} \end{array} \tag{29}$$

Note that quality is measured as some fraction of construction costs, and that tenant maintenance is not considered. We impose non-negativity conditions on q and m. Where $'$s denote derivatives and subscripts partial derivatives, we also impose reasonable restrictions on the functions P and g: i) $P(0) = 0$, $P'(q) > 0$ and $P''(q) < 0$; and ii) $g_q < 0$, $g_m(q,0) = \infty$, $g(q,0) < 0$, $g_m(q,\infty) = 0$, $g_m > 0$, $g_{mm} < 0$. Thus, rent increases with quality but at a diminishing rate; there are positive but diminishing returns to maintenance; holding fixed the rate of quality deterioration, more has to be spent on maintenance as quality increases; and with zero maintenance, the unit deteriorates. In our numerical examples, the first-order conditions of the $\mathcal{S}$ program will define a unique interior maximum.

We solve the problem using optimal control theory (Kamien and Schwartz (1991)). The current-value Hamiltonian corresponding to (29) is

$$\mathcal{H}^{\circ} = P\left(q\left(t\right)\right) - m\left(t\right) + \phi\left(t\right) g\left(q\left(t\right), m\left(t\right)\right), \tag{30}$$

where $\phi(t)$ is current-value co-state variable on $\dot{q} = g\left(q, m\right)$. The first-order condition[10] for maintenance is

$$-1 + \phi\left(t\right) g_m\left(q\left(t\right), m\left(t\right)\right) = 0. \tag{31}$$

Since $\phi(t)$ is the marginal value of quality at time t, and $g_m(q(t), m(t))$ the amount by which quality is increased by an extra dollar's expenditure on maintenance, ϕg_m is the marginal benefit from maintenance. Thus, at each point in time, maintenance should be such that marginal benefit equals marginal cost. The conditions imposed on g_m guarantee that there is a unique, interior optimal level of maintenance expenditure for all non-negative values of q and ϕ; thus, we may write $m = m(q, \phi)$ with $m_\phi > 0$. Inserting this function into (30) yields the maximized current-value Hamiltonian:

$$\mathcal{H}\left(q, \phi\right) = P\left(q\right) - m\left(q, \phi\right) + \phi g\left(q, m\left(q, \phi\right)\right). \tag{32}$$

The equation of motion of the co-state variable is

$$\dot{\phi} = r\phi - \mathcal{H}_q = r\phi - P^{'} - \phi g_q. \tag{33}$$

The assumptions thus far have not ruled out the possibility that the optimal saddlepoint program entails upgrading to saddlepoint quality via maintenance alone. We assume that the maintenance and construction technologies are such that the optimal saddlepoint program entails construction at the start of the program. The transversality condition with respect to q_c is then

$$\phi\left(0\right) = \alpha; \tag{34}$$

[10]Throughout the analysis we shall omit second-order conditions as we compare the profit-maximizing programs with and without rent controls, for which the second-order conditions will hold. We shall also omit non-negativity conditions. In the numerical examples of section 4, we explicitly verify that non-negativity conditions hold.

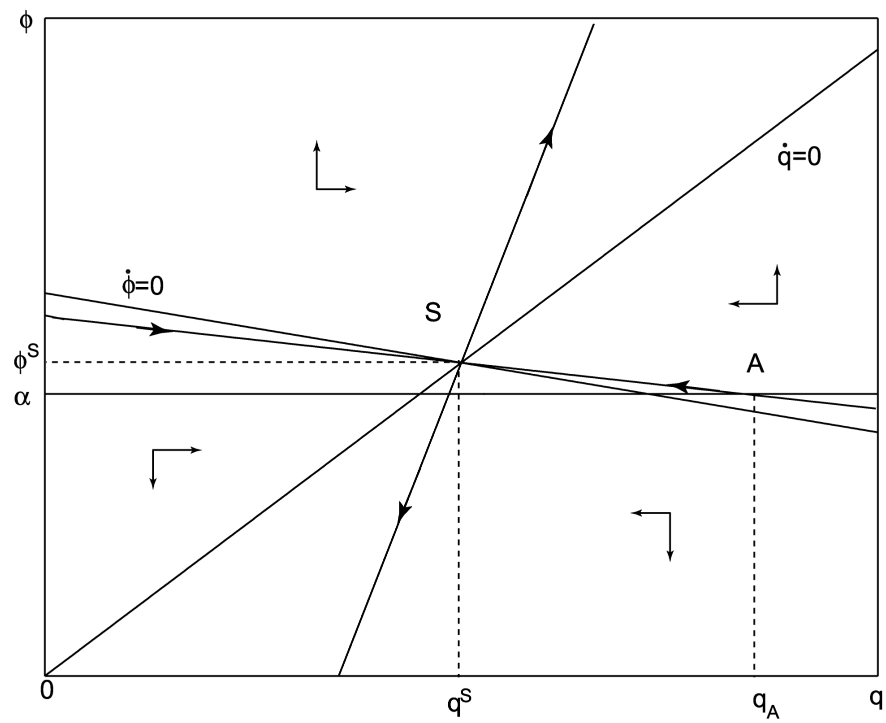


Figure 1: Phase plane for construction with downgrading to the steady state. Construction cost ($\alpha \cdot 10^5$) is \$60,000.

construction quality should be increased up to the point where the marginal value of quality equals its marginal cost.

We are now in a position to construct the phase plane corresponding to this program. We assume that: i) the $\dot{q} = 0$ locus is positively sloped; ii) the $\dot{\phi} = 0$ locus is negatively sloped; and iii) the $\dot{q} = 0$ locus and $\dot{\phi} = 0$ locus intersect in the positive orthant. Thus, there is a unique saddlepoint, $S = (q^S, \phi^S)$. We assume furthermore that $\phi^S > \alpha$, unless otherwise noted. Figure 1 displays a phase plane consistent with these assumptions. As is the case for all the figures, Figure 1 is drawn for the functional forms and parameters used in the series of numerical examples presented in Section 4.

We also have the infinite horizon transversality conditions associated with terminal quality and terminal time. Arnott, Davidson, and Pines (1983) proves that, under the assumptions made, these conditions imply that the optimal trajectory must terminate at the saddlepoint. Putting together the necessary conditions for optimality, we obtain that the $\mathcal{S}$ program entails construction at that quality at which the right stable arm intersects the ϕ

$= \alpha$ line, followed by downgrading along the stable arm to the saddlepoint.

For an autonomous optimal control problem with discounting, the value of the program at any time along an optimal trajectory equals the value of the Hamiltonian at that time divided by the interest rate:

$$V(t) = \frac{\mathcal{H}(q(t), \phi(t))}{r}.$$

The economic interpretation is that the value of the Hamiltonian gives the economic return per unit time from owning the program, which includes the net (of expenses and depreciation) earnings stream it generates plus capital gains, and competitive asset pricing requires that the net return per unit time from owning an asset equal the asset price times the discount rate.

With some abuse of notation, we denote the value of the maximized Hamiltonian at a point labeled X in the phase plane by $\mathcal{H}(X)$. The value of the program immediately after initial construction is then $\frac{\mathcal{H}(A)}{r}$, so that the value of the program immediately before initial construction, which is the value of the $\mathcal{S}$ program, is $V^{\mathcal{S}} = \frac{\mathcal{H}(A)}{r} - \alpha q_A$.

2.2 Program $\mathcal{D}$

Consider next program $\mathcal{D}$, which entails a construction-demolition cycle, where q_s is the starting quality for each cycle. The landlord's profit-maximizing program is the solution to[11]

$$\max_{q_s, q_T, T, m(t)} \frac{1}{1 - e^{-rT}} \left\{ \int_0^T \left(P\left(q\left(t\right)\right) - m\left(t\right) \right) e^{-rt} dt - \alpha q_s \right\}$$

$$\begin{aligned} & i) && \dot{q} = g(q, m) \\ s.t. \quad & ii) && q_s \equiv q(0) \text{ free} \\ & iii) && q_T \equiv q(T) \text{free} \\ & iv) && T \text{ free} \end{aligned} \tag{35}$$

Let $J(q_s, q_T, T)$ denote the maximized value of the expression in curly brackets, which is the present value of net revenue from a single cycle as a function of q_s, q_T, and T. Then (35)

[11]The analysis can be straightforwardly extended to treat demolition costs.

can be rewritten as

$$\max_{q_s, q_T, T} \quad \frac{1}{1-e^{-rT}} \quad J\left(q_s, q_T, T\right).$$

We assume that the $\mathcal{D}$ program entails construction at the beginning of each cycle. Eqs. (30) through (34) continue to apply. The transversality condition for q_T is

$$\phi(T)q(T) = 0, \tag{36}$$

which indicates that the building's quality should be run down until the optimal trajectory intersects one of the axes in the phase plane. If the optimal trajectory intersects the q-axis, as will be the case in all our numerical examples, the condition is that $\phi(T) = 0$; the building's quality should be run down until, at the end of the cycle, the marginal value of quality is zero. The transversality condition for T is

$$\mathcal{H}\left(q\left(T\right), \phi\left(T\right)\right) + r\alpha q_s = \mathcal{H}\left(q\left(0\right), \phi\left(0\right)\right); \tag{37}$$

the left-hand side is the marginal benefit from postponing demolition and reconstruction, the right-hand side the marginal cost. We can provide a useful geometric depiction of this transversality condition. Now,

$$\mathcal{H}\left(q\left(0\right), \phi\left(0\right)\right) - \mathcal{H}\left(q\left(T\right), \phi\left(T\right)\right) = \int_{q_T}^{q_s} \left(\mathcal{H}_q + \mathcal{H}_\phi \left(\frac{d\phi}{dq}\right)\right)^* dq,$$

where * indicates evaluation along a phase plane trajectory connecting the starting and end points. Since $(d\phi/dq)^* = \left(\frac{\dot{\phi}}{\dot{q}}\right)^*$ and $\mathcal{H}_\phi = \dot{q}$, using (33) the above expression reduces to

$$\mathcal{H}\left(q\left(0\right), \phi\left(0\right)\right) - \mathcal{H}\left(q\left(T\right), \phi\left(T\right)\right) = \int_{q_T}^{q_s} r\phi^*\left(q\right) dq. \tag{38}$$

Combining (37) and (38) gives

$$\alpha q_s = \int_{q_T}^{q_s} \phi^*\left(q\right) dq. \tag{39}$$

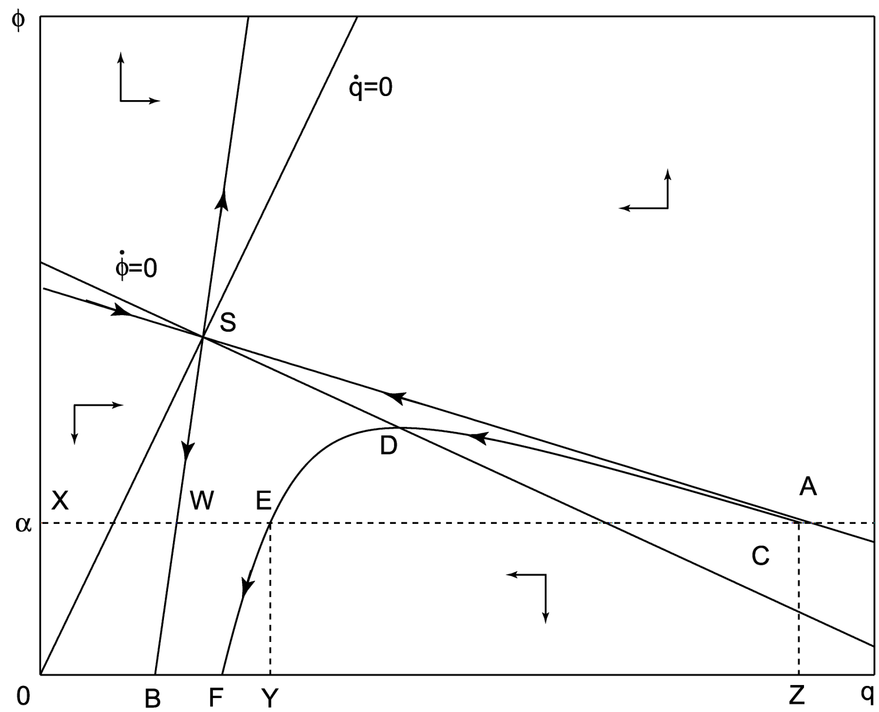


Figure 2: Phase plane for a construction-demolition cycle. Construction cost ($\alpha \cdot 10^5$) is \$30,000.

Figure 2 displays the phase plane for a $\mathcal{D}$ program. As drawn, the trajectory CDEF satisfies the three transversality conditions: it starts on $\phi = \alpha$, it terminates at $\phi = 0$, and it satisfies (37). Eq. (39) has the interpretation in the phase plane that the area under the optimal trajectory from the starting to the end point equals αq_s, that Area ZCDEF = Area OXCZ. Subtracting the common area ZCEF from both these areas gives the equivalent condition that Area CDE = Area OXEF. A necessary and sufficient condition for the existence of a trajectory that satisfies all three transversality conditions is that Area ASW > Area OXWB, where SWB is the unstable arm from the saddlepoint to its intersection with the q- or ϕ-axis, as the case may be. We refer to this as the $\mathcal{D}$-areas condition. If the $\mathcal{D}$-areas condition is satisfied, we say that a $\mathcal{D}$ program exists, and if it is not that a $\mathcal{D}$ program does not exist. Since increasing α decreases Area ASW and increases Area OXWB, there is a critical value of α, above which the $\mathcal{D}$-areas condition is not satisfied, and below which it is. Thus, a $\mathcal{D}$ program exists for construction costs below a critical level, but not otherwise.

If a $\mathcal{D}$ program exists, which is more profitable, the $\mathcal{D}$ program or the $\mathcal{S}$ program? We have already demonstrated that the value of the $\mathcal{S}$ program immediately prior to construction is $V^{\mathcal{S}} = \frac{\mathcal{H}(A)}{r} - \alpha q_A$. An analogous line of reasoning establishes that the value of the $\mathcal{D}$ program is $V^{\mathcal{D}} = \frac{H(C)}{r} - \alpha q_C$. Now, $\mathcal{H}_A - \mathcal{H}_C = \int_{q_C}^{q_A} \mathcal{H}_q(q,\alpha)\,dq = \int_{q_C}^{q_A}\left(r\alpha - \dot{\phi}\right)dq$ (from (33)). Thus, $\left(\frac{\mathcal{H}_A}{r} - \alpha q_A\right) - \left(\frac{\mathcal{H}_C}{r} - \alpha q_C\right) = \int_{q_C}^{q_A} -\dot{\phi}dq$ along $\phi = \alpha$, which can be seen to be negative. Thus, if a $\mathcal{D}$ program exists, it is more profitable than the $\mathcal{S}$ program. It can also be shown that if a $\mathcal{D}$ program does not exist, the optimal $\mathcal{S}$ program is more profitable than any construction-demolition cycle program. Thus, the construction-demolition cycle program is more profitable than the saddlepoint program when construction costs are below the critical value, and the saddlepoint program is more profitable than any construction-demolition cycle program when construction costs are above the critical value, which accords with intuition.

2.3 Program $\mathcal{R}$

The final option is a rehabilitation cycle, which entails constructing at quality q_c, downgrading to quality q_T, rehabbing up to quality q_s, and then repeating the downgrading-rehabilitation cycle. Discounted net rents

$$\max_{q_c,q_s,q_T,T_c,T,m(t)} \int_0^{T_c} \left(P\left(q\left(t\right)\right) - m\left(t\right)\right) e^{-rt}dt - \alpha q_c + \frac{e^{-rT_c}}{1-e^{-rT}}\left[\int_0^{T} \left(P\left(q\left(t\right)\right) - m\left(t\right)\right) e^{-rt}dt - R\left(q_s, q_T\right)\right],$$

are maximized with respect to q_c, q_s, q_T, T_c, T, and $m(t)$ where T_c is the length of time from construction to the first rehab, T the length of the rehabilitation cycle, and $R(q_s, q_T)$ the cost of rehabbing a unit of quality q_T to quality q_s. It is assumed that it remains profitable to construct initially, so that (31) through (34) continue to apply. The transversality conditions

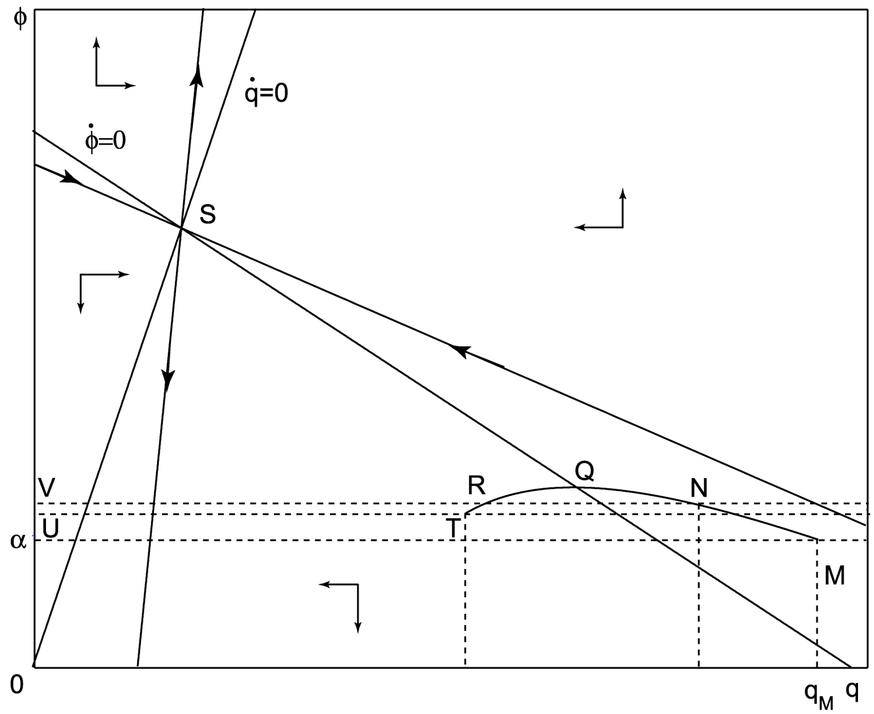


Figure 3: Phase plane for a rehabilitation cycle. Construction cost $(\alpha \cdot 10^5)$ is \$20,000.

are

$$
\begin{aligned}
q_c &: \phi_c = \alpha \\
T &: \mathcal{H}\left(q_T, \phi_T\right) + rR\left(q_s, q_T\right) = \mathcal{H}\left(q_s, \phi_s\right) \\
q_s &: \phi\left(0\right) = \phi_s = \frac{\partial R}{\partial q_s} \\
q_{T_c}, q_T &: \phi(T_c) = \phi(T) = \phi_T = \frac{\partial R}{\partial q_T} \\
T_c &: \mathcal{H}(q_{T_c}, \phi_{T_c}) = \mathcal{H}(q_T, \phi_T)
\end{aligned}
$$

In our numerical examples, we shall assume that the function $R(q_s, q_T)$ is strongly separable in q_s and q_T, i.e. $R(q_s, q_T) = R_1(q_s) - R_2(q_T)$. Figure 3 plots a configuration of the phase plane for which the rehabilitation cost function is linear in the two quality levels. Here too the timing transversality condition can be displayed as an equal areas condition, that Area NQR equals Area RTUV. Adapting the argument used in the previous two subsections, it can be shown that the value of the $\mathcal{R}$ program is $\mathcal{H}_M/r - \alpha q_M$.

Applying the same line of reasoning as in the previous subsection, it can be shown that if the $\mathcal{R}$ program exists, it is more profitable than the $\mathcal{S}$ program, and that if the $\mathcal{R}$ program does not exist, the $\mathcal{S}$ program is more profitable than any program entailing rehabilitation. It remains to compare the profitabilities of the $\mathcal{R}$ program and the $\mathcal{D}$ program, if both exist. Both start on the $\phi = \alpha$ line between where it intersects the right stable arm and the $\dot{\phi} = 0$ line. The argument employed in the previous subsection to prove that, if the $\mathcal{D}$ program exists, it is more profitable than the $\mathcal{S}$ program, can be adapted to prove that if both the $\mathcal{D}$ and the $\mathcal{R}$ program exist, the one which starts further to the left on the $\phi = \alpha$ line is the more profitable. An upward shift of the $R_1(q_s)$ function or a downward shift of the $R_2(q_T)$ function reduces the profitability of the $\mathcal{R}$ program relative to the $\mathcal{S}$ program and the $\mathcal{D}$ program.

In section 4 we shall present a series of related numerical examples, indicating different sets of parameter values for which each of programs $\mathcal{S}$, $\mathcal{D}$, and $\mathcal{R}$, are profit maximizing.

3 The Profit-maximizing Program with Tenancy Rent Control

We model tenancy rent control as a ceiling on the time path of rents over the duration of a tenancy, conditional on the starting rent[12]. Letting p_s denote the starting rent, u the length of time into the tenancy, and $F(p_s, u)$ (with $\partial F / \partial p_s > 0$) the rent control function – the maximum allowable rent u years into a tenancy, conditional on p_s – a tenancy rent control program imposes the constraint that $\widehat{P}(u) \leq F(p_s, u)$, where $\widehat{P}(u)$ is the rent charged by the landlord u years into the tenancy.

We shall examine the effects of tenancy rent control applied to a single housing unit when all other units are uncontrolled; the analysis is therefore partial equilibrium. We make a number of simplifying assumptions:

Assumption A.1. The length of a tenancy is exogenous at L.

This assumption is made for two reasons. First, we wish to abstract from the effect of tenancy rent control on tenancy duration, in order to focus on its effects on landlord

[12] There are tenancy rent control programs that restrict the percentage increase in rent from one year to the next. Under such a program, a landlord might find it profit maximizing to charge less than the maximum allowable rent increase for some time interval during a tenancy, in which case the ceiling on the time path of rents would thereafter be determined by the rent level at the time the percentage increase regulation again becomes binding. Thus, our modeling of tenancy rent control entails a simplification.

maintenance and conversion. Second, the assumption takes into account that tenancy rent control is invariably accompanied by restrictions on eviction[13]. Since tenancy rent control front-end loads rent over a tenancy, shorter tenancies are more profitable for landlords. In the absence of restrictions on eviction, tenancy rent control would therefore provide landlords with an incentive to evict tenants[14]. Under the assumption, the landlord is able to rehabilitate or to demolish-and-reconstruct only between tenancies.

Assumption A.2. The rent control function is such that the landlord finds it profit maximizing to charge the maximum controlled rent over the duration of a tenancy, i.e. $\widehat{P}(u) = F(p_s, u)$.

This assumption states that, under the opportunistic program, the time path of controlled rents over a tenancy are sufficiently "front-end loaded" relative to the time path of market rents that the tenancy rent control constraint binds strictly throughout the tenancy. While not primitive, this assumption greatly simplifies the analysis since otherwise the possibility would have to be considered that the rent control constraint binds over some quality intervals of a tenancy but not over others.

Assumption A.3. Tenants are identical.

Assumption A.4. Tenants face perfect capital markets and discount financial flows at the same rate as the landlord.

With identical tenants, the market rent as a function of quality adjusts so that a renter receives the same utility at all quality levels. Thus, under tenancy rent control, a tenant is indifferent between living in a controlled and uncontrolled unit if and only if the discounted value of controlled rents over the tenancy equals the discounted value of market rents for the same quality path, discounted at her discount rate. The assumption that the tenant's discount rate is the same as the landlord's is made to simplify the analysis.

Under the above assumptions, the opportunistic program is independent of the form of the rent control function. A proof runs as follows. Suppose that the profit-maximizing

[13] We use the term eviction to mean that the tenant is required to leave her unit even though she would prefer not to, rather than in the legal sense.

[14] Tenancy rent control rules out economic eviction (raising rents to force a tenant out) but at least in North America, where annual tenancies are the norm, a landlord can evict a tenant in some jurisdictions simply by choosing not to renew the lease, and in others by citing as just causes minor lease violations or his intention to lease the unit to a family member, convert it to owner occupancy, or rehabilitate it.

program with a particular rent control function has been solved for. Now modify the rent control function, holding constant the program but allowing the starting rents for each tenancy to adjust so that tenants remain indifferent between controlled and uncontrolled housing. The profitability of the program remains unchanged and the landlord cannot improve profitability by altering the program. Without ambiguity, we may then let $\hat{q}(u; q_s)$ denote the time path of quality over a tenancy under the opportunistic program, conditional on starting quality q_s. And the condition that, with the opportunistic program, over each tenancy the discounted value of controlled rents equals the discounted value of market rents may be written as

$$\int_0^L F(p_s, u)e^{-ru}du = \int_0^L P(\widehat{q}(u; q_s))e^{-ru}du.$$

Thus, under the above assumptions, it is the imposition of tenancy rent control rather than its severity[15] that matters since it is the imposition of tenancy rent control that undermines the credibility of the efficient program.

In the analysis of the previous section, without rent control, there were three qualitatively different optimal programs for the landlord, the $\mathcal{S}$ program, the $\mathcal{D}$ program, and the $\mathcal{R}$ program. The same three qualitatively different optimal programs are present under tenancy rent control.

3.1 Program $\widehat{\mathcal{S}}$

Program $\widehat{\mathcal{S}}$ under tenancy rent control is the analog of program $\mathcal{S}$ in the absence of rent control. Under our assumptions concerning the characteristics of the maintenance and construction technologies, program $\mathcal{S}$ entails construction followed by downgrading to steady-state quality. Program $\widehat{\mathcal{S}}$, too, entails construction followed by downgrading from one tenancy to the next, but maintenance follows a sawtooth pattern, increasing within each tenancy

[15] A tenancy rent control program is more severe than another if it permits a lower nominal percentage increase in rent every year during a tenancy.

Assumption A.2 is that the tenancy rent control program is sufficiently severe that the landlord finds it profit maximizing to charge the maximum controlled rent over the duration of the tenancy. If the tenancy rent control program is sufficiently "lax" that the landlord finds it profit maximizing to charge the maximum controlled rent over no portion of the tenancy, the program has no effect. Intermediate situations are analytically messy.

and then falling discontinuously from the end of one tenancy to the start of the next. The program converges to a steady-state tenancy maintenance cycle in which quality is highest at the beginning and end of each tenancy, rather than to a steady-state quality.

We decompose solution of the opportunistic program under tenancy rent control during a single tenancy into two stages. In the first stage, we solve the program taking as given not only the initial quality of the unit and the duration of the tenancy but also the terminal quality. In the second stage, we solve for the profit-maximizing terminal quality. The landlord decides on this program after the lease has been signed, and therefore after his discounted rent over the tenancy has been determined. The first-stage problem entails the minimization of discounted maintenance expenditures needed to achieve terminal quality, q_L, taking as given the starting quality, q_s, and the tenancy duration, L. This is an elementary optimal control program with a well-known solution. Define $J(q_s, q_L, L)$ to be the value of this program. We shall use three properties of the solution:

$$\partial J/\partial q_s = \phi(0) \qquad \partial J/\partial q_L = -\phi(L)e^{-rL} \qquad \dot{\phi} = r\phi - \phi g_q \tag{40}$$

where $\phi(t)$ is the current value of the co-state variable on $\dot{q} = g(q, m)$. The first solution property indicates that $\phi(0)$ is the marginal value of quality at the start of the tenancy, after the tenancy contract has been signed. The second indicates that $\phi(L)$ is the marginal value of terminal quality at terminal time, so that $\phi(L)e^{-rL}$ is the marginal value of terminal quality discounted to the beginning of the tenancy. Since the first stage of the problem entails deciding on the maintenance path over the tenancy, after the contract has been signed, we refer to ϕ as the marginal value of quality via maintenance or the *ex post* (viz., after the tenancy contract has been signed) marginal value of quality. The last solution property is that over a tenancy the marginal value of quality via maintenance grows[16] at the rate $r - g_q$ through the tenancy.

The second stage of the solution of the opportunistic program entails the choice of q_L. To derive this, we work with a value function. Under tenancy rent control, the value of

[16]Suppose the landlord buys an extra unit of quality today at a price of ϕ. Instantaneously, he must make the competitive return on that unit, $r\phi$, and the return comprises two components, the capital gain, $\dot{\phi}$, minus the depreciation, $-\phi g_q$.

a housing unit is a function not only of quality but also of how much time remains in the current tenancy contract[17]. Let $\widehat{V}(q)$ denote the value of a housing unit of quality q between tenancies, and $Z(q_s)$ the revenue received over a tenancy contract, discounted to the beginning of the tenancy contract. The landlord decides on the maintenance program, and hence q_L, after signing the tenancy contract, and therefore after the revenue received over the tenancy has been determined. Then the value function for $\widehat{V}(q)$ may be written as

$$\widehat{V}(q_s) = Z(q_s) + max_{q_L}[J(q_s, q_L, L) + \widehat{V}(q_L)e^{-rL}]. \tag{41}$$

Terminal quality is chosen to maximize the expression in square brackets. The corresponding first-order condition is

$$\partial J/\partial q_L + \widehat{V}'(q_L)e^{-rL} = 0. \tag{42}$$

Comparing the second equation in (40) and (42) yields

$$\phi(L) = \widehat{V}'(q_L). \tag{43}$$

Differentiaing (41) with respect to q_s yields

$$\begin{aligned} \widehat{V}'(q_s) &= Z'(q_s) + \partial J/\partial q_s \qquad \text{(using the envelope theorem)} \\ &= Z'(q_s) + \phi(0). \end{aligned} \tag{44}$$

Eq. (44) requires some care in interpretation. $\widehat{V}'(q_s)$ is the *ex ante* (before the tenancy contract has been signed) marginal value of quality at the start of a tenancy, while $\phi(0)$ is the *ex post* (after the tenancy contract has been signed) marginal value of quality at the start of a tenancy. Eq. (44) indicates that, at starting quality, the *ex ante* marginal value of quality exceeds the *ex post* marginal value of quality by $Z'(q_s)$, marginal discounted revenue. Thus, there is a downward jump discontinuity in the marginal value of quality at the time

[17] Since the housing market remains competitive under rent control, it must still be the case that owning the program for an increment of time between u and $u+du$ within a tenancy provides income of $rV(q(u), u)$, where $V(q(u), u)$ is the market value of a controlled housing unit of quality q u units of time into a tenancy. From this relationship, the rent control function, and the boundary condition that $\widehat{V}(q_s) = V(q_s, 0)$, $V(q(u), u)$ may be calculated.

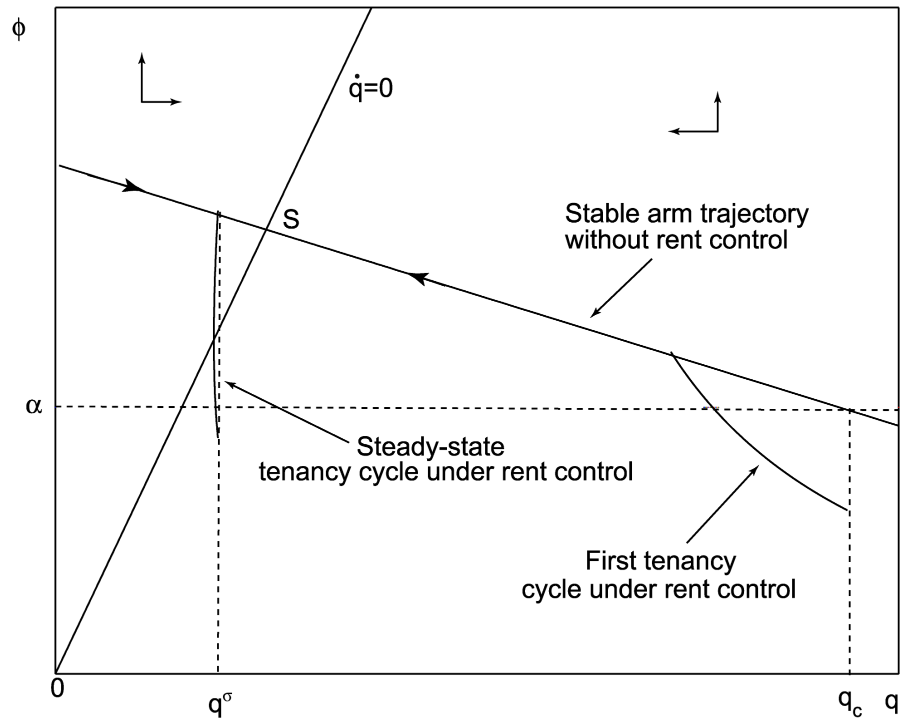


Figure 4: Phase plane for construction-downgrading to the steady-state cycle under rent control. Construction cost ($\alpha \cdot 10^5$) is \$40,000.

the lease is signed. Now return to (43). It states that, in contrast, the marginal value of quality immediately before the termination of the tenancy equals the marginal value of quality immediately afterwards, in both cases equaling the increase in the property price from a unit increase in terminal quality.

The value of the $\widehat{\mathcal{S}}$ program immediately prior to construction is

$$\widehat{V}^{\widehat{\mathcal{S}}} = \max_{q_c} \left[\widehat{V}(q_c) - \alpha q_c \right]. \tag{45}$$

Assuming an interior solution, the corresponding first-order condition for profit-maximizing construction quality is

$$\widehat{V}'(q_c) - \alpha = 0 \tag{46}$$

Comparing (44) and (46), for the first tenancy, since $q_c = q_s$,

$$\phi(0) = \alpha - Z'(q_c). \tag{47}$$

Construction occurs at that quality level, for which the *ex ante* marginal value of quality via construction equals the marginal cost, while the *ex post* marginal value of construction quality falls short of marginal construction cost by $Z'(q_c)$.

In the steady state, quality varies within a tenancy, but the starting and terminal qualities remain constant from one tenancy to the next. Let q_σ denote the optimal starting and terminal quality of a steady state cycle. Since in a steady-state tenancy $q_s = q_L = q_\sigma$,

$$\widehat{V}(q_\sigma) = \frac{1}{1 - e^{-rL}} \{Z(q_\sigma) + J(q_\sigma, q_\sigma, L)\}.$$

Figure 4 displays the phase diagram of the $\widehat{\mathcal{S}}$ program for the numerical example, and plots the optimal trajectory for two tenancies, the first tenancy that occurs immediately after construction and the steady-state tenancy. For comparison it also plots the optimal (stable arm) trajectory without rent control. With the depreciation function we employ, maintenance expenditures are positively related to ϕ and independent of q. The diminished incentive to maintain under tenancy rent control is reflected in the lower position, on average, of the optimal trajectory under tenancy rent control. The incentive under tenancy rent control to postpone maintenance expenditures towards the end of the tenancy is also evident.

3.2 Program $\widehat{\mathcal{D}}$

Program $\widehat{\mathcal{D}}$ under tenancy rent control is the analog of program $\mathcal{D}$ in the absence of rent control. Recall that, under our assumptions concerning the construction and maintenance technologies, program $\mathcal{D}$ entails constructing at a quality above saddlepoint quality, downgrading smoothly to demolition, and then repeating the cycle, which has an endogenous length of T. Recall, too, that if an optimal demolition program exists, it is more profitable than the optimal saddlepoint program. The program $\widehat{\mathcal{D}}$ differs from program $\mathcal{D}$ in two important respects. First, because of the assumed fixed duration of a tenancy under tenancy rent control, demolition can occur only between tenancies, so that the length of the demolition cycle must be some integer multiple of L. Thus, there are two types of cycles, the construction-demolition cycle and the maintenance cycle within each tenancy. Since terminal time is not, therefore, a continuous variable, there will not be a timing transversality condition. Instead, optimal cycle length can be computed by comparing the profit obtained when demolition occurs after every tenancy, after every second tenancy, and so on. Second, under tenancy rent control the commitment problem arises.

Our solution of the $\widehat{\mathcal{D}}$ program proceeds in two stages[18]. In the first stage, the profit-maximizing program is calculated conditional on the number of tenancies in a construction-demolition cycle. Let $\widehat{V}^n(q_c)$ denote the value of a housing unit that has just been constructed at quality q_c, conditional on n tenancies in the cycle, and $\mathcal{V}^n$ the value of the optimal program conditional on n tenancies within a demolition cycle. In the second stage, the corresponding profit levels are compared for different numbers of tenancies within the cycle. In this subsection, we ignore the complications that would arise if the non-negativity constraint on q would bind.

We start by solving for the optimal program, conditional on the unit being demolished after each tenancy. Once the tenancy contract has been signed, the landlord has no incentive to maintain. Spending on maintenance does not increase the revenue received over the tenancy and the value of the structure is zero at the end of the cycle since it is about to be demolished. The value of the program is

$$\mathcal{V}^1 = \max_{q_c} \left\{ \widehat{V}^1(q_c) - \alpha q_c \right\} = \max_{q_c} \frac{1}{1 - e^{-rL}} \{ Z(q_c) - \alpha q_c \},$$

from which the first-order condition for profit-maximizing construction quality is straightforward to obtain.

We now solve for the optimal program, conditional on the structure being demolished after two tenancies. It is profit maximizing for the landlord to spend nothing on maintenance during the second tenancy. Let superscript i on q denote the order of tenancy within a demolition cycle, so that q_L^1 is terminal quality for the first tenancy, for example. Then

$$\widehat{V}^2(q_c) = Z(q_c) + \max_{q_L^1}[J(q_c, q_L^1, L) + Z(q_L^1)e^{-rL}] + (\widehat{V}^2(q_c) - \alpha q_c)e^{-2rL}.$$

[18] Eqs. (41) - (47) apply to the demolition case as well. We proceed as we do in order to provide more insight into the economics, and to motivate the numerical solution algorithm we employ.

Thus,

$$\widehat{V}^2(q_c) = \frac{1}{1-e^{-2rL}}\{Z(q_c) + \max_{q_L^1}[J(q_c, q_L^1, L) + Z(q_L^1)e^{-rL}] - \alpha q_c e^{-2rL}\}.$$

Calculate first $Z(q_L^1)$. Then solve the maximization problem in square brackets, which yields q_L^1 as a function of q_c, from which an expression for $\widehat{V}^2(q_c)$ is obtained. Since the value of the program prior to construction, conditional on construction at quality q_c, is $\widehat{V}^2(q_c) - \alpha q_c$, the final step is to choose q_c to maximize $\widehat{V}^2(q_c) - \alpha q_c$.

This line of reasoning suggests an algorithm for solving for the profit-maximizing program with n tenancies during a construction-demolition cycle. Let $v^i(q_s^i, n)$ be the value of revenue net of maintenance expenditures received from the beginning of tenancy i until the structure is demolished, discounted to the beginning of tenancy i, conditional on q_s^i and the number of tenancies within a demolition cycle. Proceed by backward recursion[19]. First, calculate $v^n(q_s^n, n)$ ($= Z(q_s^n)$). Second, solve

$$\max_{q_s^n} J(q_s^{n-1}, q_s^n, L) + v^n(q_s^n, n)e^{-rL}.$$

Denote by $q_s^n(q_s^{n-1})$ the value of q_s^n that solves this maximization problem, as a function of q_s^{n-1}. Then

$$v^{n-1}(q_s^{n-1}, n) = Z(q_s^{n-1}) + J(q_s^{n-1}, q_s^n(q_s^{n-1}), L) + v^n(q_s^n(q_s^{n-1}), n)e^{-rL}.$$

Return to step 2, but replacing n by $n-1$, and $n-1$ by $n-2$. Proceed recursively backwards until $v^1(q_s^1, n)$ – the value discounted to construction time of the net revenue received over the life of the building as a function of $q_s^1 = q_c$, conditional on n tenancies – is obtained. Then[20]

$$\mathcal{V}^n = \max_{q_c} \frac{1}{1-e^{-rnL}}\{v^1(q_c, n) - \alpha q_c\}.$$

If the optimal number of tenancies is finite, then $n^* = \arg\max_n\{\mathcal{V}^n\}$, and $\mathcal{V}^{n^*}$ is the value

[19] This algorithm is inapplicable to the optimal saddlepoint program, since the optimal saddlepoint program contains an infinite number of tenancies.

[20] Alternatively, we may write $\widehat{V}^n(q_c) = v^1(q_c, n) + (-\alpha q_c + \widehat{V}^n(q_c))e^{-rnL}$, and obtain $\mathcal{V}^n$ as the value of $\widehat{V}^n(q_c) - \alpha q_c$ maximized with respect to q_c.

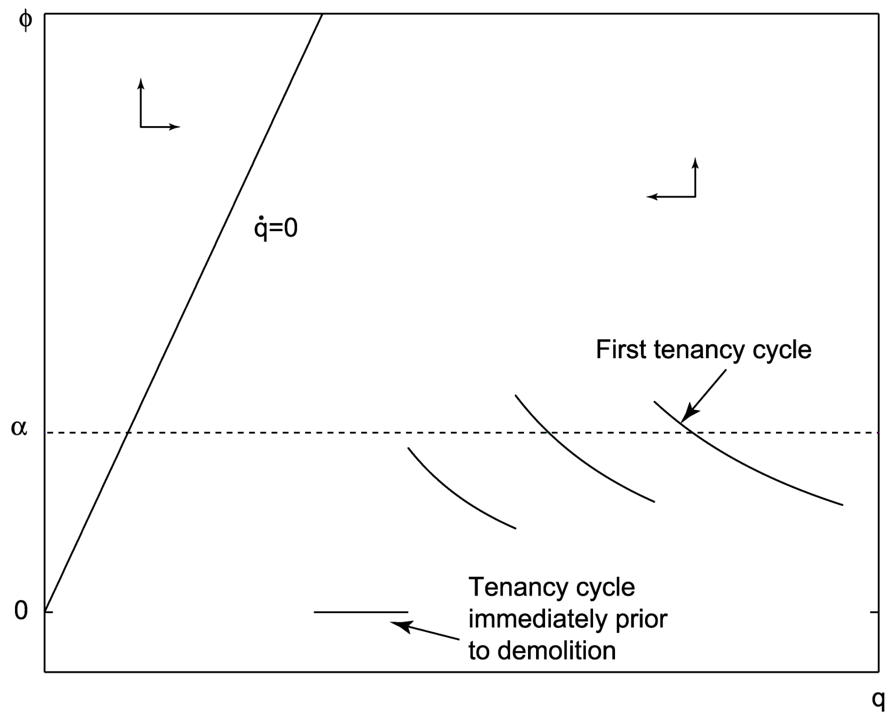


Figure 5: Phase plane for a demolition cycle under tenancy rent control. Construction cost $(\alpha \cdot 10^5)$ is \$30,000.

of the $\widehat{\mathcal{D}}$ program. If the optimal number of tenancies is infinite, we say that an optimal demolition program does not exist.

Figure 5 plots one cycle of the $\widehat{\mathcal{D}}$ program for the numerical example for which the profit-maximizing number of tenancies within a demolition cycle is four. Note that $\phi = 0$ throughout the last tenancy.

3.3 Program $\widehat{\mathcal{R}}$

Program $\widehat{\mathcal{R}}$ under rent control is the analog of program $\mathcal{R}$ in the absence of rent control. Recall that, under our assumptions concerning the construction and maintenance technologies, program $\mathcal{R}$ entails constructing at quality q_c above saddlepoint quality, downgrading the unit to quality q_T, upgrading it via rehabilitation to quality q_s, downgrading it along the original trajectory from q_s to q_T, and then repeating the rehabilitation cycle *ad infinitum*. We also showed that if program $\mathcal{R}$ exists, it is more profitable than program $\mathcal{S}$, and that, if both program $\mathcal{R}$ and program $\mathcal{D}$ exist, the one with the lower construction quality is the more profitable. Program $\widehat{\mathcal{R}}$ differs from program $\mathcal{R}$ in two respects. First, because under tenancy rent control rehabilitation is permitted only between tenancies and because tenancy duration is L, the period from initial construction to the first rehabilitation must be some integer multiple of L, as must the period between subsequent rehabilitations. Because of this, the starting and terminal quality of a rehabilitation cycle will in general vary from one rehabilitation to the next. Second, as with the other two rent control programs, downgrading does not occur smoothly because of the commitment problem.

In the optimal demolition program with rent control, all the cycles are the same. This is not in general true of the optimal rehabilitation program; the number of tenancies may be different for different rehabilitation cycles. In our numerical examples, however, since we assume that the marginal benefit of increasing quality via rehabilitation is independent of the quality level from which rehabilitation is undertaken, the first rehabilitation is followed by the stationary rehabilitation cycle. In this case, the construction of a solution algorithm is relatively straightforward. First, one solve for the opportunistic stationary rehabilitation cycle, conditional on one, two, etc. tenancies between rehabilitations, and then for the unconditional opportunistic stationary cycle. And second, solve for the optimal program up to the first rehabilitation, conditional on one, two, etc. tenancies to that point, and then for the unconditional optimal program.

Among the $\widehat{\mathcal{S}}$, $\widehat{\mathcal{D}}$, and $\widehat{\mathcal{R}}$ programs, the overall optimal program is the one with the highest value. The deadweight loss due to rent control is simply the difference between the value of the optimal program without rent control minus the value of the optimal program with rent control.

4 Numerical Examples

This section presents a series of related numerical examples with the aim of quantifying the effects of tenancy rent control. The efficiency cost caused by the commitment problem is of special interest.

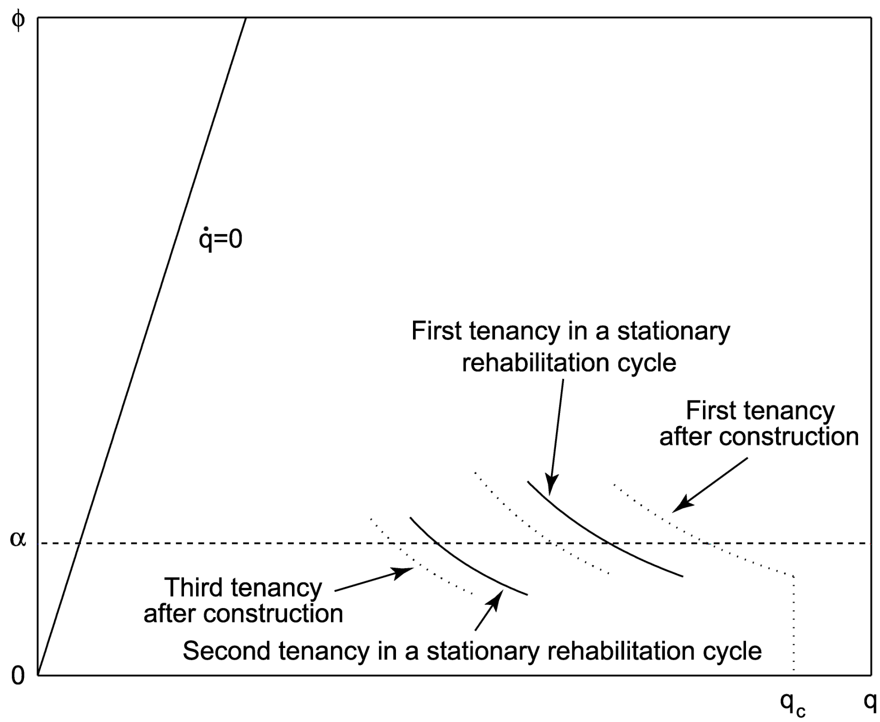


Figure 6: Phase plane for a rehabilitation cycle under tenancy rent control. Construction cost ($\alpha \cdot 10^5$) is \$20,000.

4.1 Choice of functional forms and parameters

We had hoped to draw on the empirical literature in our choice of functional forms and parameters. Unfortunately, there seem to be no empirical studies that have employed the Arnott, Davidson, and Pines (1983) conceptual framework as the basis for empirical analysis. As a result, we adopt the more modest goal of developing numerical examples whose parameters and functional forms are "reasonable". We choose the functional forms so as to obtain equations of motion that are the solutions to linear differential equations, as well as (for the case of rent control) closed-form value functions. And we choose the parameters to generate plausible results for the steady-state, demolition, and rehabilitation programs.

As in the theoretical analysis, we measure quality as proportional to construction costs. We assume the following functional forms for the rent function, the construction cost function, and the maintenance/depreciation function:

$$P(q) = eq - \frac{fq^2}{2} \qquad C(q) = \alpha \qquad \dot{q} = -\delta q + 2am^{1/2}$$

The rent equation generates a linear, downward-sloping marginal-willingness-to-pay-for-quality function. The maintenance/depreciation function is about the simplest possible. In the absence of maintenance, quality depreciates exponentially at the rate δ. A given level of maintenance expenditure slows down the rate of quality depreciation by an amount that is independent of quality, and there are diminishing returns to maintenance. The optimal expenditure on maintenance is given by $a^2\phi^2$; maintenance expenditure is therefore increasing in ϕ and independent of q. Substituting the expression for optimal maintenance into the depreciation function gives the maximized depreciation function,

$$\dot{q} = -\delta q + 2a^2\phi. \tag{48}$$

In the absence of rent control, these equations imply a co-state equation of the form

$$\dot{\phi} = (r+\delta)\phi - e - fq, \tag{49}$$

and with tenancy rent control[21],

$$\dot{\phi} = (r+\delta)\phi. \tag{50}$$

In the absence of rent control, these equations of motion correspond to a phase plane with a linear, upward-sloping $\dot{q} = 0$ line and a linear, downward-sloping $\dot{\phi} = 0$ line, whose intersection point, the saddlepoint is at

$$q^S = \frac{2a^2e}{\delta(r+\delta)+2a^2f} \qquad \phi = \frac{e\delta}{\delta(r+\delta)+2a^2f} \ .$$

With rent control, the $\dot{\phi} = 0$ line coincides with the q-axis, so that the $\dot{q} = 0$ and $\dot{\phi} = 0$ lines do not intersect in the interior of the phase plane.

We take as our units of measurement years and hundreds of thousands of dollars. We

[21] Thus, both with and without rent control, the state and co-state equations are together a pair of linear first-order differential equations in q and ϕ. In the absence of rent control, substituting one into the other generates linear, second-order differential equations for q alone and ϕ alone. And with rent control, (50) is a linear, first-order differential equation in ϕ alone, and substituting the solution to (50) into (48) results in a linear, first-order differential equation in q alone.

start by setting the following parameters:

$$\delta = 0.03,\ r = 0.0375,\ a = 0.2121,\ e = 0.055,\ f = 0.005,\ \text{and}\ L = 10.$$

These parameters imply a saddlepoint quality of 2.0, saddlepoint maintenance of 0.02 ($2000 per year), saddlepoint rent of 0.10 ($10000 per year), and a value of the co-state variable (the marginal value of quality) at the saddlepoint of 0.667. α is varied across examples.

Our rehabilitation function has a very simple form: $R(q_s, q_T) = \beta_1 q_s - \beta_2 q_T = \beta_2(q_s - q_T) - (\beta_1 - \beta_2)q_s$, where $\beta_1 = 0.25$, $\beta_2 = 0.24$. Thus, besides a linear cost of quality upgrade, the landlord has to pay a fee proportional to the 'target' quality q_s.

4.2 Numerical solution procedures

The details of the numerical solution procedures employed are presented in the Appendix. Here we just describe in broad terms the general approaches. In the absence of rent control, the solution procedure centers on solving for the solution parameters of the second-order linear differential equation for ϕ, since everything else may be solved for once these parameters are obtained. One parameter is obtained from the initial condition that $\phi(0) = \alpha$. How the other parameter is determined depends on the type of program. In the case of the saddlepoint program, the second parameter is obtained from the ϕ-coordinate of the saddlepoint; in the case of the demolition program, the second parameter and the period of the demolition cycle are solved simultaneously from $\phi(T) = 0$ and the terminal time (or equal-areas) transversality condition; in the case of the rehabilitation program, the second parameter, as well as ϕ_s and ϕ_T, are solved simultaneously from the transversality conditions for ϕ_s, ϕ_T, and the terminal time transversality condition.

The approaches taken to solve the optimal programs with tenancy rent control are more complex. It is convenient to express the unknown parameters in the functions $\phi(t)$ and $q(t)$ in terms of $q(0)$ and $q(L)$. This allows us to obtain the discounted revenue received over a tenancy, $Z(q_0)$, and the net value of a tenancy cycle, $J(q_0, q_L, L)$. For program $\widehat{\mathcal{S}}$, we make a conjecture about the form of $\widehat{V}(q)$. Then, using (42) to find $q_L(q_s)$ and plugging it into (41), we apply the method of undetermined coefficients to solve for $\widehat{V}(q)$.

The final step is to find the construction quality q_c using (46). The solution algorithm for the demolition program with rent control was described in Section 3.2 and that for the rehabilitation program sketched in Section 3.3.

4.3 Examples without rehabilitation

In this subsection, we assume that rehabilitation is unprofitable and that α is not so high as to make initial construction unprofitable. In the absence of rent control, the optimal program is therefore either the optimal saddlepoint program or the optimal demolition program, with the saddlepoint program being optimal for α above 0.4166 and the demolition program for α below that level. With rent control, the optimal program entails either convergence to a steady-state cycle or a demolition program, with the former occurring when construction costs are high relative to maintenance. We proceed by lowering α from one example to the next.

- $\alpha = 0.695$

The fourth panel of Figure 7 displays the phase diagram for this example, both with and without rent control. The $\dot{\phi} = 0$ locus in the absence of rent control is shown as the dotted line; with rent control, it coincides with the q-axis. Recall that the level of maintenance is proportional to ϕ. In the absence of rent control, the optimal program entails construction at $q = 1.416$, followed by upgrading to steady-state quality, $q^S = 2.0$. Construction occurs at that quality at which the marginal value of quality, α, equals the marginal cost of construction. The value of the program is 0.751. With rent control, the optimal program entails a steady-state tenancy cycle, with construction at $q = 1.542$. As explained earlier, ϕ jumps downwards discontinuously immediately after a tenancy contract is signed, reflecting the commitment problem, and then rises continuously within the tenancy. With increasing maintenance over the tenancy, quality initially falls and then rises until it reaches construction quality by the end of the tenancy. The value of the program is 0.694. Thus, the efficiency loss due to tenancy rent control is 7.6% of the value of the uncontrolled program. Observe that the average quality of housing is lower under rent control, consistent with intuition.

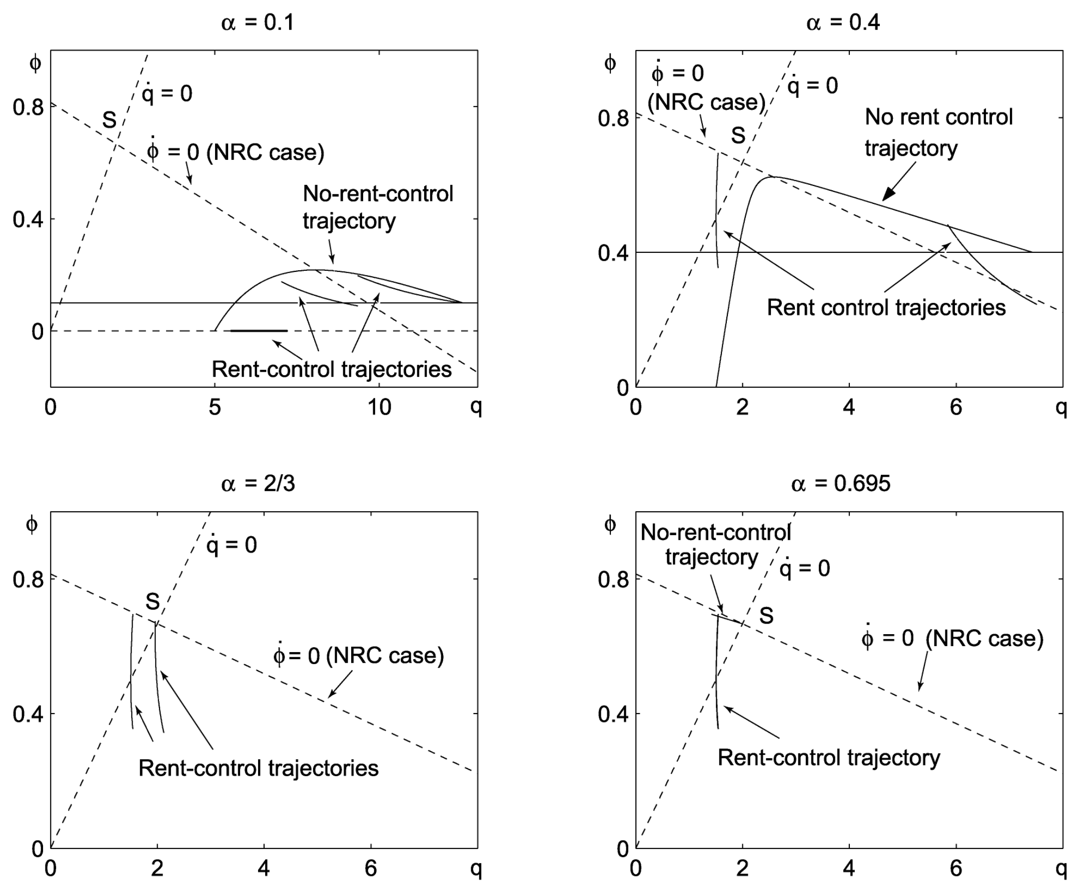


Figure 7: Phase planes with and without rent control. No rehabilitation.

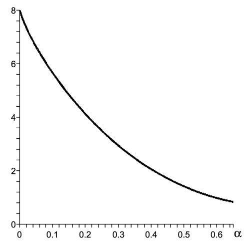
- $\alpha = 0.667$

The third panel of Figure 7 shows the optimal trajectories for this example without and with rent control. The optimal program in the absence of rent control entails constructing at saddlepoint quality and holding quality constant at that level. The value of the Hamiltonian at the saddlepoint is 0.08 (rent of 0.10 minus maintenance costs of 0.02 and of course no depreciation). Housing value is 2.133 and construction costs are 1.333, so that the value of the program prior to construction is 0.800 and the land to housing value ratio 0.375. Are these numbers reasonable? The "cap rate" (the percentage of net rent to value) is low, under the model's assumptions simply equaling the interest rate; if uncertainty and property taxes were considered, the cap rate would be reasonable. Maintenance expenditures are 0.94% of housing value, which accords broadly with the 1-percent rule that maintenance expenditures are typically about 1% of property value. The Figure shows two rent-control trajectories. The path on the right is for the first tenancy, that on the left for the steady-state tenancy. Construction occurs just above saddlepoint quality. Maintenance increases within each tenancy, but starting quality falls from one tenancy to the next, converging to steady-state starting quality below saddlepoint quality. The value of the program is 0.746, implying a deadweight loss due to tenancy rent control of 6.8% of value.

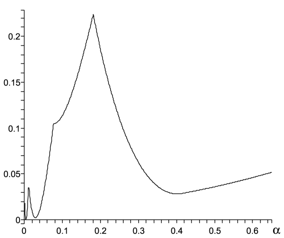
- $\alpha = 0.4$

It was noted earlier that, with the assumed functional forms and parameter values, in the absence of rent control the optimal demolition program is more profitable than the optimal saddlepoint program when α is below 0.4166. Thus, in this example, displayed in the second panel of Figure 7, the optimal program without rent control is a demolition cycle. Construction occurs at a quality considerably above saddlepoint quality. This is followed by downgrading to demolition quality, at which point the structure is demolished and the cycle exactly repeated. The value of the program is 2.058. In contrast, with tenancy rent control, convergence to a steady-state cycle remains optimal[22]. Construction occurs at high initial

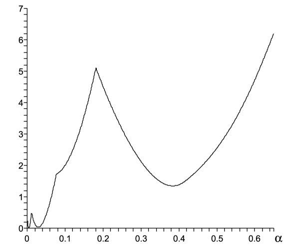
[22] The critical construction cost level below which the optimal program entails demolition is therefore lower with rent control than without. With $\alpha = 0.4$, the deadweight loss due to the commitment problem is therefore higher with the optimal demolition program than with the optimal steady-state program.



Panel A: Value of the optimal programs without rent control as a function of construction cost, $\$10^5$



Panel B: Absolute deadweight loss due to rent control, $\$10^5$



Panel C: Relative deadweight loss due to rent control, %

Figure 8: Values of optimal programs without rent control,
and deadweight loss due to rent control - Case without rehabilitation

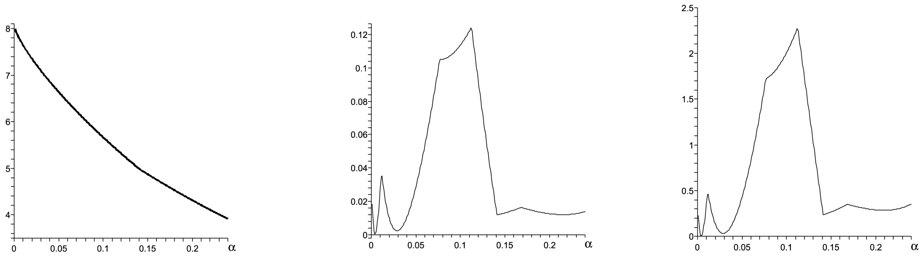
quality, followed by downgrading from one tenancy to the next (but with rising maintenance within each tenancy) converging to a steady-state cycle. The value of the program is 2.030, so that in this case the deadweight loss due to rent control is only 1.4% of the uncontrolled program value.

- $\alpha = 0.1$

In this example, shown in the first panel of Figure 7, construction is sufficiently cheap relative to maintenance that a demolition cycle is profit maximizing both with and without rent control. The range of qualities over a demolition cycle is similar for the two programs. The level of maintenance is lower under rent control at every quality level; as a result, depreciation is more rapid and the demolition cycle shorter. The values of the program without and with rent control are 5.663 and 5.549, respectively, implying a deadweight loss due to rent control of 2.0% of the value of the uncontrolled program.

Figure 8 focuses on the deadweight loss resulting from the application of rent control. Panel A shows the value of the optimal program without rent control as a function of α. There is a slope discontinuity in the value of this optimal program at $\alpha = 0.4166$, where the switch occurs between the range of qualities where the saddlepoint program is optimal and where the demolition cycle is optimal. There are several slope discontinuities in the value of the optimal program with rent control.[23] The one corresponding to the highest value of α corresponds to the switch point between the range of qualities for which the steady-state

[23]This function is not drawn since to the naked eye, it is hard to distinguish for that drawn in Panel A.



Panel A: Values of the optimal programs without rent control as a function of construction cost, $\$10^5$

Panel B: Absolute deadweight loss due to rent control, $\$10^5$

Panel C: Relative deadweight loss due to rent control, %

Figure 9: Values of optimal programs without rent control,
and deadweight loss due to rent control - Case with rehabilitation

cycle is optimal and for which the demolition cycle is optimal. The ones at lower values of α correspond to switch points for which different numbers of tenancies within a demolition cycle are optimal. Panel B shows the absolute loss in program value from the application of tenancy rent control, and Panel C the corresponding proportional loss.

4.4 Examples with rehabilitation

In examples with rehabilitation, we consider a limited range of α $(0 < \alpha < 0.24)$. Due to our choice of the functional form, for higher values of α construction becomes unreasonably expensive compared to rehabilitation. Figure 9 presents the value of the optimal program without rent control as a function of α, and the absolute and relative deadweight loss due to rent control.

For $\alpha \leq 0.112$ the optimal program with or without rent control is demolition (programs $\mathcal{D}$ and $\widehat{\mathcal{D}}$). Under rent control: for $\alpha \leq 0.011$, the $\widehat{\mathcal{D}}$ program has only one tenancy cycle between demolitions; for $0.011 < \alpha \leq 0.077$, two tenancy cycles; and for $0.077 < \alpha \leq 0.112$, three tenancy cycles. This explains the non-smoothness of deadweight loss when demolition is optimal. For $0.112 < \alpha \leq 0.141$, $\mathcal{D}$ is still the optimal program without rent control but under rent control rehabilitation is more profitable. For $0.141 < \alpha \leq 0.24$, the optimal program is rehabilitation with or without rent control. For $0.141 < \alpha \leq 0.168$, the $\widehat{\mathcal{R}}$ program entails three tenancies before the first rehab, while for $0.168 < \alpha \leq 0.24$ only two tenancies precede the first rehab. As a result, there is a 'kink' in panels B and C at

$\alpha = 0.168$. At these relatively low values of α steady-state programs are never optimal. The relative loss is limited and does not exceed 2.5% of the value of an optimal program. Absolute loss may reach $12,000 per unit per year.

5 Conclusion

In recent years an increasing number of jurisdictions around the world have adopted what has come to be known as tenancy rent control, typically as a method of partial decontrol of a previously stricter form of rent control. Under tenancy rent control, rents are controlled within a tenancy but are free to vary between tenancies. Tenancy rent control appears attractive, as a way of providing security of tenure to sitting tenants without the excess demand distortions created by stricter control programs. How attractive tenancy rent control in fact is depends on the magnitude of the distortions it creates. Since tenancy rent control typically results in the contract rent exceeding the market rent in the early years of the tenancy and falling short of it in later years, it provides an incentive for tenants to stay in their apartments longer than they otherwise would. In this paper we examined the effects of tenancy rent control on a landlord's choice of the quality path of his housing units, which includes his decisions on construction quality, maintenance, rehabilitation, and demolition and reconstruction, under the assumptions that tenancy duration is exogenous and that the controls are applied to only a single housing unit. We showed that the application of tenancy rent control gives rise to a potential commitment (or time inconsistency) failure. We contrasted two programs, the efficient program and the opportunistic program. The efficient program is the profit-maximizing program in the absence of rent control. The opportunistic program is the profit-maximizing program over a tenancy *once the tenancy contract has been signed*. The signing of the contract results in the present value of revenue from the tenancy being independent of the landlord's maintenance expenditure, and hence reduces his incentives to maintain. Before the tenancy contract is signed, the landlord would like to commit to following the efficient program, but none of the commitment mechanisms available – contract, reputation, and regulation – is likely to be very effective. In our analysis, we assumed that these mechanisms are completely ineffective, so that the landlord follows the opportunistic program. Building on the Arnott-Davidson-Pines model of housing quality and maintenance, we compared the properties of the efficient and opportunistic programs. Section 4 presented a series of related numerical examples, with the aim of quantifying the deadweight loss due to the commitment failure. For reasonable parameter values, we found that the deadweight loss is modest but not insignificant, ranging from zero to eight percent of the pre-control value of the program.

There are several open questions left for future research.

1. The paper considered the application of tenancy rent control to a single housing unit when the rest of the market is uncontrolled. How do the results change when the entire market is controlled?

2. The paper built on the Arnott-Davidson-Pines filtering model. Since there is no empirical work based on this model, the numerical examples used simple functional forms and "reasonable" parameter values. How would the results change if estimated functional forms were used instead?

3. The paper assumed, under tenancy rent control, that tenancy duration is exogenous. But, by front-end loading rents, tenancy rent control should increase tenancy duration. How important is this distortion compared to the commitment-in-maintenance distortion considered here, and how do the two distortions interact?

4. The paper noted that tenancy rent control improves security of tenure for tenants. What is the social value of doing so?

5. The paper compared the unrestricted market equilibrium to the market equilibrium under tenancy rent control. But since tenancy rent control has typically been employed as a method of partial decontrol, it is perhaps more relevant to ask: What is the magnitude of the efficiency gain when a stricter form of rent control is replaced by tenancy rent control?

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Appendix

In this appendix we find optimal programs without and with rent control assuming the functional forms used in our numerical examples. As was stated in Section 4, in the numerical examples we employ the following rent function, construction cost function, and maintenance/depreciation function:

$$\begin{aligned} &P(q) = eq - \frac{fq^2}{2} = 0.055q - \frac{0.005q^2}{2} \\ &C(q) = \alpha q \\ &\dot{q} = -\delta q + 2am^{1/2} = -0.03q + 2(0.045\phi^2)^{1/2}. \end{aligned}$$

No-rent-control programs

We start by solving the system of differential equations (20) and (21):

$$\dot{q} = -\delta q + 2a^2\phi,$$

$$\dot{\phi} = (r+\delta)\phi - e + fq.$$

This system can be reformulated as follows:

$$\ddot{\phi} - r\dot{\phi} - (2a^2 f + \delta(r+\delta))\phi + \delta e = 0,$$

$$q = \frac{\dot{\phi} + e - \phi(r+\delta)}{f}. \tag{51}$$

The solution to the second order differential equation for ϕ has the following form:

$$\phi(t) = C_1 e^{\gamma_1 t} + C_2 e^{\gamma_2 t} + B \tag{52}$$

where

$$B = \frac{\delta e}{2a^2 f + \delta(r+\delta)} = \phi^S,$$

$$\gamma_1 = \frac{r + \sqrt{r^2 + 4(2a^2 f + \delta(r+\delta))}}{2} \;,$$
$$\gamma_2 = \frac{r - \sqrt{r^2 + 4(2a^2 f + \delta(r+\delta))}}{2}.$$

With $\phi(t)$, we can find $q(t)$ using (51):

$$q(t) = \frac{1}{f}\left[C_1 e^{\gamma_1 t}(\gamma_1 - r - \delta) + C_2 e^{\gamma_2 t}(\gamma_2 - r - \delta) + e - B(r+\delta)\right].$$

Recalling that

$$q^S = \frac{2a^2 e}{\delta(r+\delta) + 2a^2 f}$$

and rearranging $\frac{e - B(r+\delta)}{f}$, we obtain that

$$q(t) = \frac{1}{f}[C_1 e^{\gamma_1 t}(\gamma_1 - r - \delta) + C_2 e^{\gamma_2 t}(\gamma_2 - r - \delta)] + q^S. \tag{53}$$

Whether the $\mathcal{S}$ or $\mathcal{D}$ program is optimal, the transversality condition (6) holds:

$$\phi(0) = \alpha.$$

Using this condition, we solve for C_2:

$$C_1 + C_2 + \phi^S = \alpha,$$
$$C_2 = \alpha - C_1 - \phi^S.$$

The other transversality condition that allows us to solve for C_1 is different for the $\mathcal{S}$ and $\mathcal{D}$ programs, which we consider in turn.

Program $\mathcal{S}$

The steady-state program implies that

$$\lim_{t\to\infty} q(t) = q^S \tag{54}$$
$$\lim_{t\to\infty} \phi(t) = \phi^S.$$

Notice that $\gamma_1 > 0$ while $\gamma_2 < 0$. Therefore, (54) can hold only if $C_1 = 0$. This condition completely defines $q(t)$ and $\phi(t)$:

$$\phi(t) = C_2 e^{\gamma_2 t} + \phi^S,$$

$$q(t) = \frac{1}{f} C_2 e^{\gamma_2 t}(\gamma_2 - r - \delta) + q^S,$$

$$C_2 = \alpha - \phi^S.$$

Program $\mathcal{D}$

To find C_1, we use the transversality condition $\phi(T) = 0$:

$$C_1 e^{\gamma_1 T} + C_2 e^{\gamma_2 T} + B = 0,$$

$$C_1 = -\frac{B + C_2 e^{\gamma_2 T}}{e^{\gamma_1 T}}.$$

The last unknown is T. It is determined by the equal-areas condition:

$$\mathcal{H}(T) = \mathcal{H}(0) - r\alpha q(0) \tag{55}$$

where

$$\mathcal{H}(t) = eq(t) - \frac{fq(t)^2}{2} + a^2\phi(t)^2 - \delta q(t)\phi(t). \tag{56}$$

Equation (55) involves sums of exponents of T, so it cannot be solved analytically. We find its solution numerically for a given value of α. It appears that this equation has two solutions in the region where T is positive. We choose the one that results in the higher value of the program.

Program $\mathcal{R}$

Our rehabilitation technology is $R(q_s, q_T) = 0.25q_s - 0.24q_T$. In this problem, there are two different pairs of laws of motion for q and ϕ, $\{q_c(t), \phi_c(t)\}$ for a tenancy immediately after construction, which we call a construction cycle, and the other, $\{q(t), \phi(t)\}$, for all subsequent tenancies, which we call rehabilitation cycles. Both pairs are described by (53) and

(52), respectively, but with different unknown constants, which we will denote as $\{C_{c1}, C_{c2}\}$ for a construction cycle and $\{C_1, C_2\}$ for rehabilitation cycles. We start by finding the laws of motion for rehabilitation cycles. The transversality conditions

$$\phi(0) = \frac{\partial R(q_s, q_T)}{\partial q_s} = 0.25,$$

$$\phi(T) = -\frac{\partial R(q_s, q_T)}{\partial q_T} = 0.24$$

allow us to solve for the unknown constants on which $q(t)$ and $\phi(t)$ depend. Then we find the optimal duration of the rehabilitation cycle T, using the equal-area condition:

$$(\mathcal{H}(0) - \mathcal{H}(T))/r = R(q(0), q(T)). \tag{57}$$

Here $\mathcal{H}(\cdot)$ is defined in (56) and depends on the laws of motion for the rehabilitation cycle. We solve this equation numerically using Maple 9.5 and obtain that the optimal duration of the rehabilitation cycle is (approx.) 16.61 years. We verify that there are no other solutions for positive T by examining behavior of the left-hand side and the right-hand side of (57). Notice that T does not depend on the cost of construction. Then we find the laws of motion $q_c(T)$ and $\phi_c(T)$ for the construction cycle using the following transversality conditions:

$$\phi_c(0) = \alpha,$$

$$\phi_c(T_c) = \phi(T).$$

Finally, we numerically solve for the length of the construction cycle T_c for each specific α using the following equation:

$$q_c(T_c) = q(T).$$

Programs with rent control

Under programs with rent control, the differential equation for ϕ is different from that without rent control. Solving the system (20) and (22)

$$\dot{q} = -\delta q + 2a^2\phi,$$

$$\dot{\phi} = (r+\delta)\phi,$$

we obtain the following solutions:

$$\phi(t) = c_1 e^{(r+\delta)t},$$

$$q(t) = \frac{2a^2 c_1}{r+2\delta} e^{(r+\delta)t} + c_2 e^{-\delta t}.$$

We solve for c_1 and c_2 in terms of initial and terminal quality of a tenancy cycle, q_s and q_L:

$$q(0) = \frac{2a^2 c_1}{r+2\delta} + c_2 = q_s,$$
$$c_2 = q_0 - \frac{2a^2 c_1}{r+2\delta} \quad .$$

Since the analytical solutions to programs with rent control contain quite messy expressions, we give only solutions for the values of parameters used in our numerical examples and round all values to the third digit.

$$\begin{aligned} q(t) &= \frac{q_L - q_s e^{-\delta L}}{e^{(r+\delta)L} - e^{-\delta L}} \left(e^{(r+\delta)t} - e^{-\delta t} \right) + q_0 e^{-\delta t} \\ &= 0.817(e^{0.0675t} - e^{0.03t})q_L + (-0.606 e^{0.0675t} + 1.606 e^{0.03t})q_s \quad , \\ \phi(t) &= c_1 e^{(r+\delta)t} = \frac{\left(q_L - q_s e^{-\delta L}\right)(r+2\delta)}{\left(e^{(r+\delta)L} - e^{-\delta L}\right) 2a^2} e^{(r+\delta)t} \\ &= (-0.656 q_s + 0.886 q_L) e^{0.0675t}. \end{aligned}$$

Recall that optimal maintenance is $m(t) = a^2\phi^2$. Thus, the value of a tenancy cycle is

$$J(q_s, q_L, L) = -\int_0^L m(t)e^{-rt}dt = -\int_0^L a^2\phi(t)^2 e^{-rt}dt.$$

It is straightforward to calculate the integral from the righthand side but the expression is cumbersome; for exposition purposes, we just say that $J(q_s, q_L, L)$ can be presented in the following form:

$$J(q_s, q_L, L) = G_1 q_s^2 + G_2 q_L^2 + G_3 q_s q_L.$$

where G_1, G_2, and G_3 are some known functions of parameters.

Using the definition of the rent function $P(\cdot)$, we also calculate the discounted present value of rent received over a tenancy:

$$Z(q_s, L) = \int_0^L [e\hat{q}(t; q_s) - \frac{f}{2}\widehat{q}(t; q_s)^2]e^{-rt}dt,$$

with

$$\hat{q}(t; q_s) = \frac{q_L(q_s) - q_s e^{-\delta L}}{e^{(r+\delta)L} - e^{-\delta L}} \left(e^{(r+\delta)t} - e^{-\delta t}\right) + q_s e^{-\delta t}$$

where a final quality of a cycle, $q_L(q_s)$, is optimally chosen and is a function of an initial quality of a cycle, q_s. The functional form of $q(t)$ implies that

$$Z(q_s, L) = B_1 q_s^2 + B_2 q_L(q_s)^2 + B_3 q_s q_L(q_s) + B_4 q_s + B_5 q_L(q_s).$$

Again, $\{B_i\}_{i=1}^5$ are some known functions of the parameters.

Program $\widehat{\mathcal{S}}$

In case of the $\widehat{\mathcal{S}}$ program, the problem of the landlord boils down to an infinite horizon dynamic programming problem, in which the state variable is the initial quality while the control variable is the terminal quality of a unit. Thus, we have the following Bellman equation:

$$\widehat{V}(q_s^i) = Z(q_s^i) + \max_{q_L^i}[J(q_s^i, q_L^i, L) + \widehat{V}(q_s^{i+1})e^{-rL}] \tag{58}$$

$$s.t. \quad q_s^{i+1} = q_L^i, \quad i = 1, 2, ... \text{ is the number of the tenancy cycle.}$$

We apply the 'guess-and-verify' method. Notice that J is quadratic in q_L^i. If q_L^i is a linear function of q_s^i, then Z is also quadratic in q_L^i. Notice also that q_L^i is a linear function of q_s^i if $\widehat{V}$ is quadratic. Thus, we make a guess that $\widehat{V}$ is quadratic:

$$\widehat{V}(q) = A_0 + A_1 q + A_2 q^2. \tag{59}$$

We find A_0, A_1 and A_2 by the method of undetermined coefficients. First we need to find q_L^i as a function of q_s^i. Assuming that $J(q_s^i, q_L^i, L) + \widehat{V}(q_s^{i+1})e^{-rL}$ is concave, we use the first order condition:

$$\frac{\partial}{\partial q_L^i}\left[J(q_s^i, q_L^i, L) + \widehat{V}(q_s^{i+1})e^{-rL}\right] = 2G_2 q_L^i + G_3 q_s^i + e^{-rL}A_1 + 2e^{-rL}A_2 q_L^i = 0,$$

$$q_L^i = -\frac{G_3 q_s^i + e^{-rL}A_1}{2(G_2 + e^{-rL}A_2)} \equiv K_1 q_s^i + K_2. \tag{60}$$

Substituting (59) and (60) into the Bellman equation (58) and suppressing the index for the cycle i, we obtain

$$\begin{aligned} A_0 + A_1 q_s + A_2 (q_s)^2 = {} & B_1 q_s^2 + B_2 (K_1 q_s + K_2)^2 + B_3 q_s (K_1 q_s + K_2) \\ & + B_4 q_s + B_5 (K_1 q_s + K_2) \\ & + G_1 q_s^2 + G_2 (K_1 q_s + K_2)^2 + G_3 q_s (K_1 q_s + K_2) \\ & + e^{-rL}[A_0 + A_1 (K_1 q_s + K_2) + A_2 (K_1 q_s + K_2)^2]. \end{aligned} \tag{61}$$

One can see that (61) is quadratic in q_s. We find the unknown constants A_0, A_1 and A_2 by rewriting (61) in the form

$$W_0 + W_1 q_s + W_2 q_s^2 = 0$$

and solving the system

$$
\begin{aligned}
W_0 &= 0 \\
W_1 &= 0 \\
W_2 &= 0
\end{aligned} \tag{62}
$$

for A_0, A_1 and A_2.

After some simplification and rounding, the system (62) can be rewritten as

$$A_2 + 0.336 + \frac{0.779}{1.375A_2 - 1.195} + \frac{0.473 - 0.539A_2}{(1.375A_2 - 1.195)^2} = 0$$

$$A_1 - 0.254 + \frac{0.175 + 1.213A_1}{1.375A_2 - 1.195} + \frac{0.734A_1 - 0.837A_1A_2}{(1.375A_2 - 1.195)^2} = 0$$

$$0.313A_0 + \frac{0.136A_1 + 0.472A_1^2}{1.375A_2 - 1.195} + \frac{0.285A_1^2 + 0.325A_2A_1^2}{(1.375A_2 - 1.195)^2} = 0.$$

This system of (cubic) equations has three solutions:

$$A_0 = 3.012,\ A_1 = -1.634,\ A_2 = 0.567,$$

$$A_0 = 34.012, A_1 = -12.235, A_2 = 0.861,$$

$$A_0 = 0.635,\ A_1 = 0.772,\ A_2 = -0.025.$$

Only the third solution results in a concave value function while other solutions have $A_2 > 0$. Indeed, one can check that the first and second solutions are spurious, since they result in convex $J(q_s^i, q_L^i, L) + \widehat{V}(q_s^{i+1})e^{-rL}$. We proceed further with the third solution

$$\widehat{V}(q) = 0.635 + 0.772q - 0.025q^2.$$

To complete the solution of the problem, we use the the first-order condition for the maximization of the value of the program:

$$\frac{d}{dq_0}\left(-\alpha q_0 + \widehat{V}(q_0)\right) = -\alpha + A_1 - 2A_2 q_0 = 0.$$

Therefore,

$$q_0 = \frac{A_1 - \alpha}{2A_2}.$$

Program $\widehat{\mathcal{D}}$

Recall that program $\widehat{\mathcal{D}}$ entails an infinite number of repetitions of a contruction-demolition cycle, each of which comprises of n tenancy cycles. Conditional on pursuing program $\widehat{\mathcal{D}}$, the problem of the landlord is not only to choose an optimal trajectory for each tenancy cycle but also to choose optimal n. Given n, the problem of the landlord is to find the optimal maintenance path and optimal construction quality. To find optimal maintenance, the landlord solves a finite-horizon dynamic programming problem similar to (58):

$$v(q_s^i) = Z(q_s^i) + \max_{q_L^i}[J(q_s^i, q_L^i, L) + v(q_s^{i+1})e^{-rL}], \tag{63}$$

$$\text{s.t.} \quad q_s^{i+1} = q_L^i,\ i = 1, 2, ..n \text{ is the number of the tenancy cycle,}$$

$$v(q_s^{n+1}) = 0.$$

Given our particular functional form, we show the solution for $n = 1$. $v(q_s^2) = 0$, so q_L^1 is a solution to the first-order condition:

$$\frac{d}{dq_L^1}\left[-0.328(q_0^1)^2 + 0.886 q_L^1 q_0^1 - 0.598(q_L^1)^2\right] = 0.$$

Thus,

$$q_L^1 = 0.741 q_0^1.$$

Given q_L^1,

$$v(q_s^1) = 0.4 q_s^1 - 0.016(q_s^1)^2.$$

Knowing $v(q_s^1)$, the landlord optimizes with respect to q_s^1 :

$$\max_{q_s^1}(-\alpha q_s^1 + v(q_s^1)), \tag{64}$$

which gives

$$q_s^1 = -31.31\alpha + 12.522. \tag{65}$$

The value of program $\widehat{\mathcal{D}}$ for $n = 1$ is

$$\begin{aligned}\widehat{V}^1(q_s^1) &= \frac{1}{1-e^{-rL}}(v^1(q_s^1, 1) - \alpha q_s^1) \\ &= -40.044\alpha + 16.016 - (0.051 + 3.198\alpha)(-31.31\alpha + 12.522).\end{aligned}$$

Following the same strategy we solve for $\widehat{V}^n(q_s^1)$ for n from 1 to 20.

One more complication we encounter is that for sufficiently high values of α (for $\alpha > 0.55$) the non-negativity condition $q \geq 0$ binds for an optimal $\widehat{\mathcal{D}}$ program. We say that the demolition program under rent control does not exist for $\alpha > 0.55$ given our choice of functional forms and parameters.

Program $\widehat{\mathcal{R}}$

The profit-maximizing rehabilitation program under tenancy rent control requires finding the sequence of initial and terminal qualities in each tenancy cycle that maximizes the landlord's net income stream and solves:

$$\begin{aligned}&\max_{\{q_s^i, q_L^i\}_{i=1}^{\infty}} [-\alpha q_s^1 + Z(q_s^1) + J(q_s^1, q_L^1, L) \\ &+ \sum_{i=2}^{\infty} e^{-(i-1)rL}(-(R(q_s^i, q_L^i))^+ + Z(q_s^i) + J\left(q_s^i, q_L^i, L\right))]\end{aligned}$$

where $(x)^+ = x$ if $x > 0$ and $(x)^+ = 0$ if $x \leq 0$. The superscripts on q stand for the number of the tenancy cycle. In this case the main problem is to guess the solution. We make two conjectures. First, consider the following value function:

$$W(q_s^1) = \max_{q_L^1, \{q_s^i, q_L^i\}_{i=2}^{\infty}} [-\alpha q_s^1 + Z(q_s^1) + J(q_s^1, q_L^1, L)$$
$$+ \sum_{i=2}^{\infty} e^{-(i-1)rL}(-(R(q_s^i, q_L^i))^+ + Z(q_s^i) + J\left(q_s^i, q_L^i, L\right))].$$

Our first conjecture is that $W(\cdot)$ is quadratic. This is suggested by the functional form of $Z(\cdot)$ and $J(\cdot, \cdot)$ which are quadratic. But even knowing that $W(\cdot)$ is quadratic is not sufficient to get the complete solution as there is another issue: when does the landlord rehabilitate and when not? We look for the program that has the following form:

$$-\alpha q_s^1 + \sum_{i=1}^{M} e^{-(i-1)rL} \left[Z(q_s^i) + J(q_s^i, q_L^i, L)\right] + e^{-MrL} \widehat{V}^K(q_L^M)$$

where

$$\widehat{V}^K(q_L^M) = \sum_{j=1}^{\infty} e^{-(j-1)rKL} Y(q_L^M; K)$$

and

$$Y(q_L^M; K) = -R(q_s^{M+1}, q_L^M) + \sum_{i=1}^{K} e^{-(i-1)rL} \left[Z(q_s^{M+i}) + J(q_s^{M+i}, q_L^{M+i})\right].$$

Thus we are looking for programs that have two parts, a 'non-stationary' and a 'stationary' one. A stationary part $\widehat{V}(q_L^M; K)$ consists of infinite repetition of the same cycle $Y(q_L^M; K)$, which starts with rehabilitation followed by K tenancy cycles without rehabilitation. The non-stationary part of the program is the initial part, which comprises M tenancy cycles without rehabilitation.

To find $V^K(q_L^M)$, we consider the following system:

$$V_1(q_L^0; K) = \max_{q_s^1}[-R(q_s^1, q_L^0) + V_2(q_s^1; K)], \tag{66}$$

$$V_2(q_s^1; K) = Z(q_s^1) + \max_{q_L^1}[J(q_s^1, q_L^1) + e^{-rL} V_3(q_L^1; K)], \tag{67}$$

...

$$V_{K+1}(q_s^K; K) = Z(q_s^K) + \max_{q_{LK}}[J(q_s^K, q_L^K) + e^{-rL} V_1(q_L^K; K)], \tag{68}$$

$$q_s^i = q_L^{i-1}, \quad i = 2, ..K.$$

Assuming that V_i, $i = 1, .., K+1$ is quadratic ($V_i(x; K) = A_i + B_i x + C_i x^2$), one can notice that we have two types of equations. Let us examine the optimal choices for each equation type. First, we consider maximization in equation (66).

$$\max_{q_{s1}} [-\beta_1 q_s^1 + \beta_2 q_L^0 + A_2 + B_2 q_s^1 + C_2 (q_s^1)^2].$$

Provided that $C_2 < 0$,

$$q_s^1 = \frac{\beta_1 - B_2}{2C_2},$$

i.e. q_s^1 is just a constant. Note that q_s^1 would be a constant when the rehabilitation function is additively separable in its two arguments. Additive separability of the rehabilitation function implies that as soon as the landlord finds it profitable to rehabilitate for the first time, the system loses memory about its history. This fact suggests that the solution indeed should contain a stationary cycle of the kind described above. Also, the loss of memory after rehabilitation implies that the non-stationary part of the solution may not contain rehabilitation and, therefore, necessarily consists of a sequence of tenancy cycles without rehabilitation.

Given that $V_i(x) = A_i + B_i x + C_i x^2$, it is straightforward to obtain the solutions to (67)-(68). We do not present the explicit solutions as they involve quite cumbersome expressions. Having obtained the solutions for optimal choices of the q's, we substitute them into the system(66)-(68) and construct a new system that has $3(K+1)$ equations in the coefficients on $V_i(\cdot)$, $i = 1, ..., K+1$. The properties of the system that we obtain are described in Table 1.

Fortunately, this system boils down to linear equations and has a solution. First, we solve for C_1, then for $C_{K+1}, C_K, ..., C_2$. Then we are able to solve for B_1 and combining this solution with solutions for C_1 we solve for $B_{K+1}, B_K, ..., B_2$. Substituting all these solutions into the the rest of equations involving A_i's, we obtain a system of linear equations that has exactly one solution. Having the solution for the stationary part of the problem, it is easy to solve the problem completely by working backwards starting from the stationary part.

	Equation for the coefficient on					
	q^0		q		q^2	
	Variables that enter the equation					
	linearly	nonlinearly	linearly	nonlinearly	linearly	nonlinearly
Eq. 1	A_1, A_{K+1}	B_2, C_2	B_1	–	C_1	–
Eq. 2	A_2, A_3	B_3, C_3	B_2	B_3, C_3	C_2	C_3
...	...	...	...	...	...	...
Eq. $K+1$	A_{K+1}, A_1	B_{K+1}, C_{K+1}	B_{K+1}	B_1, C_1	C_{K+1}	C_1

Table 1: Properties of the system of equations for the coefficients of value functions

To find the optimal program, the programs with $K, M = 1, 2, ..., 20$ were considered. It appears that, under the chosen values of parameters, the stationary part of the program has two tenancies in one rehabilitation cycle. The optimal number of tenancies in the non-stationary part depends on α and can be 2 or 3.

Depending on α, the value function for this program has the following form:

$$\widehat{V} = \begin{cases} 6.887 - 15.103\alpha + 10.554\alpha^2, \\ \text{if } 0.003 < \alpha \leq 0.168, \text{ (3 tenancies in a non-stationary cycle)} \\ 6.851 - 15.056\alpha + 11.527\alpha^2, \\ \text{if } 0.168 < \alpha \leq 0.25 \text{ (2 tenancies in a non-stationary cycle)} \end{cases}$$

(we do not consider $\alpha > 0.24$). It is clear why we have more non-stationary cycles for lower $\alpha:$ the lower the cost of construction, the higher the initial quality the landlord chooses and the longer it takes to downgrade to the quality where it is profitable to rehabilitate. Figure 6 shows the optimal trajectories for $\alpha = 0.2$.