Theory of Menu Auction and Applications

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Abstract

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My doctoral dissertation contains three essays on menu auction and its related applications. The first chapter is a theoretical generalization of classical menu auction model, and the second and the third chapters are applications to a resource allocation problem and an industrial organization problem.

Menu auction (Bernheim and Whinston, 1986) is a first-price package auction with complete information. They show that every Nash equilibrium under some refinements always leads to an efficient outcome. Therefore, this becomes a natural efficiency benchmark for package auction designs (e.g., Ausubel and Milgrom 2002). Menu auction can also be viewed as a model of economic influence where the auctioneer is going to choose an action which affects bidders payoff so that each bidder tries to influence the outcome by monetary transfer to the auctioneer. This framework is widely adopted in political lobbying models where the special interest groups lobbying the government over trade policies (e.g., Grossman and Helpman 1994). However, the applicability is

limited by quasi-linear preferences and the absence of budget constraints. In my first chapter, "Menu Auctions with Non-Transferable Utilities and Budget Constraints", I extends Bernheim and Whinston's (1986) menu auction model under transferable utilities to a framework with non-transferable utilities and budget constraints. Under appropriate definitions of equilibria consistent with subgame perfection, it is shown that every truthful Nash equilibrium (TNE) is a coalition-proof Nash equilibrium (CPNE) and that the set of TNE payoffs and the set of CPNE payoffs are equivalent, as in a transferable utility framework. The existence of a CPNE is assured in contrast with the possible non-existence of Nash equilibrium under the definition by Dixit, Grossman, and Helpman (1997). Moreover, the set of CPNE payoffs is equivalent to the bidder-optimal weak core.

The second chapter relates menu auction to a resource allocation problem. Kelso and Crawford (1982) propose a wage-adjustment mechanism resulting in a stable matching between heterogeneous firms and workers. Instead of a benevolent social planner, in "Profit-Maximizing Matchmaker" (w. Hideo Konishi), we consider a profit-maximizing auctioneer to solve this many-to-one assignment problem. If firms can only use individualized price, then the auctioneer can only earn zero profit in every Nash equilibrium and the sets of stable assignments and strong Nash equilibria are equivalent. Otherwise, the auctioneer might earn positive profit even in a coalition-proof Nash equilibrium. This reinforces Milgroms (2010) argument on the benefit of using simplified message spaces that it not only reduces information requirement but also improves resource allocation.

The third chapter applies menu auction in an industrial organization problem. In "Choosing a Licensee from Heterogeneous Rivals" (w. Hideo Konishi and Anthony Creane), we consider a firm licensing its production technology

to rivals when firms with heterogeneous in production costs competing in a Cournot market. While Katz and Shapiro (1986) show that a complete transfer in duopoly can be joint-profit reducing, we show that it is always joint-profit improving provided that at least three firms remain in the industry after transfer. While transfers between similarly efficient firms may reduce welfare, the social welfare must increase if the licensor is the most efficient in the industry, contrast with Katz and Shapiro (1985) in the duopoly environment. This has an important implication in competition regulation. Then we investigate relative efficiency of the licensee under different licensing auction mechanisms. With natural refinement of equilibria, we show that a menu auction licensee, a standard first-price auction licensee, and a joint-profit maximizing licensee are in (weakly) descending order of efficiency.

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Contents

Chapter 1

Menu Auctions with Non-Transferable Utilities and Budget Constraints

1.1 Introduction

The menu auction model with transferable utilities introduced by Bernheim and Whinston (1986) is a complete-information principal-agent problem with multiple principals (bidders) and one agent (auctioneer), in which the auctioneer's action affects her own and bidders' payoffs. A menu auction game has two stages: in the first stage, each bidder simultaneously submits a bidding menu that is a list of contingent payments for each action to the auctioneer; in the second stage, given the submitted bidding menus, the auctioneer selects an action. Due to coordination problems among bidders, there are usually numerous Nash equilibria, many of which are implausible. Bernheim and Whinston (1986) propose truthful Nash equilibrium (TNE) as a refinement and prove that there is always a TNE in every menu auction game.[1] They show that every TNE is a coalition-proof Nash equilibrium (CPNE),[2] and that the set of TNE payoffs is equivalent to the set of CPNE payoffs and the bidder-optimal strong core.[3]

Although the menu auction game has been widely applied to political-economy models of economic influence,[4] Dixit, Grossman, and Helpman (1997) argue that assumptions of quasi-linear preferences and the absence of budget constraints in Bernheim and Whinston (1986) limit its applications in practice. Under quasi-linearity, the auctioneer does not care about the distribution of payoffs among bidders, and marginal utility of payment is always a constant.[5] Without budget constraints on bidders, it is hard to apply the model to situations with certain institutional restrictions on payments.[6] For these reasons, Dixit, Grossman, and Helpman (1997) relax the above two assumptions. Defining truthful Nash equilibrium (TNE) for generalized menu auction games, they show that every TNE is strongly Pareto efficient for the auctioneer and all bidders. However, their definition does not guarantee the existence of a TNE. Indeed, Example 1 discussed below illustrates that even Nash equilibrium may fail to exist under their definition. This paper proposes an alternative defini-

[1] A TNE is a Nash equilibrium where each bidder submits a truthful bidding menu such that the bidder obtains the equilibrium payoff for every other action whenever possible.

[2] A CPNE is a Nash equilibrium immune to every credible joint deviation by any subset of bidders, where credibility of a coalitional deviation is recursively defined.

[3] Bernheim and Whinston (1986) do not mention the term "core" directly. However, following the auction literature, a coalitional game among the auctioneer and bidders can be defined from a menu auction game. An allocation is in the weak core if there exists no other allocation that weakly improves all members in a coalition and strictly improves some members in the coalition. The strong core is defined similarly but requires strict improvements on all members in the coalition. The bidder-optimal strong core is a strong core allocation and there is no other strong core allocation that weakly improves all bidders and strictly improves some bidders.

[4] In particular, Grossman and Helpman (1994) popularize strategic lobbying models.

[5] In a public good provision problem, the government (auctioneer) may care about how much each one contributes to the project, and the income effect is usually not independent of the level of public good provided.

[6] For example, in United States, there are legal restrictions on political contributions.

tion that guarantees the existence of equilibrium and fully characterizes the sets of TNEs and CPNEs.

One of the key consequences of imposing budget constraints on Bernheim and Whinston's (1986) definition is that when budget constraints are binding, bidders cannot provide additional incentive to induce a favorable outcome among several actions to which the auctioneer is indifferent.[7] Dixit, Grossman, and Helpman (1997) overcome this problem by implicitly assuming that budget constraints are never binding when bidders consider possible deviations. However, when the budget constraint is binding, some sort of "optimism" by the bidder is required to justify a deviation when the auctioneer is indifferent. Unfortunately, this optimism is the very reason that Nash equilibrium fails to exist. Therefore, we need an alternative definition of equilibrium that implies risk aversion on the part of the budget-constrained bidders: they are not willing to deviate when the new outcome, depending on the particular action eventually chosen by the auctioneer after the deviation, could be worse than the existing outcome even though there are better outcomes that could be chosen by the auctioneer.

Unlike Bernheim and Whinston (1986), in our model, the strong core might be empty (Example 1), and even if it is non-empty, Example 3 shows a TNE under our definition may not be strongly Pareto efficient, in contrast with Dixit, Grossman, and Helpman (1997). As the difference is driven by binding budget constraints, it is natural to modify the strong core, which we call the Budget-Constraint core (BC-core), and the bidder-optimality by requiring a strict improvement from a budget-unconstrained bidder.[8] Theorem 1 shows the main

[7]This is different from standard assumptions in principal-agent models where bidders can always offer infinitesimally more to break ties.

[8]This is parallel to the alternative definition of Nash equilibrium: budget-constrained bidders by themselves are unable to induce favorable outcomes.

result of this paper that every TNE is a CPNE and the set of TNE payoffs, the set of CPNE payoffs, the bidder-optimal BC-core, and the bidder-optimal weak core are equivalent. With indispensability of private good (Mas-Collel 1977), the equivalence of the bidder-optimal weak core and the bidder-optimal strong core is reestablished, which coincides with Bernheim and Whinston (1986) (Corollary 2).

The extension to non-transferable utilities and budget constraints opens the door for new applications. For example, we can now deal with lobbying models without monetary transfers. Lobbies often reward politicians not by campaign contributions but by political support during elections. Since most elections are winner-take-all, marginal payoff of political support is non-linear, which is hard to capture through quasi-linearity. Moreover, the political support provided by any lobby is often limited, so budget constraints are needed to allow reasonable predictions.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 presents the results. Section 4 concludes. The proof of the main theorem (Theorem 1) is relegated to the appendix.

1.2 The Model

The model follows Dixit, Grossman, and Helpman (1997). There are N bidders and an auctioneer (denoted by 0). The auctioneer chooses an action from a finite set A.[9] Bidder $i \in N$ submits a bidding menu $T_i : A \to \mathbb{R}_+$ to the auctioneer such that $0 \leq T_i(a) \leq \omega_i(a)$ for each $a \in A$, where $\omega_i(a)$ is the highest possible amount of contingent payment for action a. An important

[9] This assumption is made for ease of exposition only. All of our results hold when A is a compact set.

difference from Bernheim and Whinston (1986) is that bidder i faces budget constraint $\omega_i(a)$ when the auctioneer chooses action $a \in A$. Another departure is the relaxation of quasi-linear preferences: (1) the auctioneer's payoff function $U_0(a, (T_i(a))_{i\in N})$ is continuous and strictly increasing in $T_i(a)$ for all $a \in A$ and $i \in N$, and (2) bidder i's payoff function $U_i(a, T_i(a))$ is continuous and strictly decreasing in $T_i(a)$ for all $a \in A$. A **menu auction game** $\Gamma \equiv (N, (U_i, \omega_i)_{i\in N}, (U_0, A))$ is a two-stage complete information game such that all bidders submit bidding menus simultaneously in stage 1 and the auctioneer chooses an action in stage 2. Let $\mathcal{T}_i \equiv \{T_i : 0 \leq T_i(a) \leq \omega_i(a) \text{ for all } a \in A\}$, the collection of bidding menus of bidder i, and $\mathcal{T} \equiv (\mathcal{T}_i)_{i\in N}$, the collection of bidding menus of N bidders. An **outcome** of a menu auction game Γ is (a, T) where $a \in A$ and $T \equiv (T_i)_{i\in N} \in \mathcal{T}$. Define $M(T) \equiv \arg\max_{a\in A} U_0(a, T(a))$, the auctioneer's **best response set** given bidding menus T and $m(T) \equiv \max_{a\in A} U_0(a, T(a))$, the corresponding payoff.

Definition 1. An outcome (a^*, T^*) is a **Nash equilibrium** in Γ if and only if (i) $T^* \in \mathcal{T}$, (ii) $a^* \in M(T^*)$, (iii) for all $i \in N$ there exists no $\tilde{T}_i \in \mathcal{T}_i$ and $\tilde{a} \in M(\tilde{T}_i, T^*_{-i})$ such that (a) $U_i(\tilde{a}, \tilde{T}_i(\tilde{a})) > U_i(a^*, T^*_i(a^*))$ and (b) $\tilde{T}_i(\tilde{a}) < \omega_i(\tilde{a})$.

Condition (iii-b) $\tilde{T}_i(\tilde{a}) < \omega_i(\tilde{a})$ deserves further explanation as this is an important difference between this paper and Dixit, Grossman, and Helpman (1997).[10] Without condition (iii-b), bidders are assumed to be *optimistic* in the sense that bidder i would deviate to $\tilde{T}_i$ when it is *possible* to gain from deviation, without worrying about whether there *might* be another unfavorable action $\hat{a} \in M(\tilde{T}_i, T^*_{-i})$ with $U_i(a^*, T^*_i(a^*)) > U_i(\hat{a}, \tilde{T}_i(\hat{a}))$ to be chosen by

[10]Bernheim and Whinston (1986) adopt Definition 1 without condition (iii-b) since they do not have budget constraints.

the auctioneer. Without budget constraints, this optimism is not restrictive because any bidder could resolve the indifference of the auctioneer by paying infinitesimally more, as in standard principal-agent models.[11] However, a budget-constrained bidder cannot pay more to persuade the auctioneer, so there is no way to ensure that the auctioneer will choose the desirable action. Therefore, omitting condition (iii-b) implicitly assumes this kind of optimism, which restricts the set of Nash equilibria. Example 1 below shows that such an optimism might lead to the non-existence of Nash equilibrium.

Remark. It can be shown that condition (iii) requires that at a Nash equilibrium no bidder is able to *convincingly* persuade the auctioneer to choose another action. Hence, Definition 1 can be stated equivalently as follows: An outcome (a^*, T^*) is a **Nash equilibrium** in Γ if and only if (i) $T^* \in \mathcal{T}$, (ii) $a^* \in M(T^*)$, (iii) for all $i \in N$ there exists no $\tilde{T}_i \in \mathcal{T}_i$ and $\tilde{a} = M(\tilde{T}_i, T^*_{-i})$ such that $U_i(\tilde{a}, \tilde{T}_i(\tilde{a})) > U_i(a^*, T^*_i(a^*))$.

Example 1. Consider $N = \{1, 2\}$ and $A = \{a_1, a_2\}$. Assume quasi-linear preferences such that for all $a \in A$, $U_i(a, T(a)) = V_i(a) - T_i(a)$ with $\omega_i(a) = 2$ for all $i \in N$ and $U_0(a, T(a)) = V_0(a) + \sum_{i \in N} T_i(a)$ where

	a_1	a_2
$V_1(a)$	6	1
$V_2(a)$	1	6
$V_0(a)$	0	0

[11] When the auctioneer is indifferent between a and $\tilde{a}$, a bidder can pay $\varepsilon > 0$ more to induce one of outcomes.

No Nash equilibrium exists if condition (iii-b) is omitted, because no matter which action is chosen, one of two bidders will prefer another action.[12] However, there exists a Nash equilibrium with condition (iii-b). Consider T such that $T_1(a_1) = 2$, $T_1(a_2) = 0$, $T_2(a_1) = 0$ and $T_2(a_2) = 2$. Outcomes (a_1, T) and (a_2, T) are Nash equilibria.

Example 2 below shows that condition (iii-b) implies risk-averse behaviors of bidders.

Example 2. Consider $N = \{1, 2\}$ and $A = \{a_1, a_2, a_3\}$. Assume quasi-linear preferences such that for all $a \in A$, $U_i(a, T(a)) = V_i(a) - T_i(a)$ with $\omega_i(a) = 2$ for all $i \in N$ and $U_0(a, T(a)) = V_0(a) + \sum_{i \in N} T_i(a)$ where

	a_1	a_2	a_3
$V_1(a)$	6	1	4
$V_2(a)$	1	6	4
$V_0(a)$	0	0	-2

Consider T such that $T_1(a_1) = 2$, $T_1(a_2) = 0$, $T_1(a_3) = 2$, $T_2(a_1) = 0$, and $T_2(a_2) = T_2(a_3) = 2$. Outcomes (a_1, T), (a_2, T), and (a_3, T) are Nash equilibria. For outcome (a_3, T), suppose bidder 1 is considering whether to deviate from T_1 to $\tilde{T}_1(a_1) = 2$, $\tilde{T}_1(a_2) = \tilde{T}_1(a_3) = 0$. If bidder 1 is risk averse, the bidder would not deviate because although a_1 is more favorable than a_3, the auctioneer might choose a_2 and a_2 is less favorable than a_3. The argument is similar for bidder 2.

[12] Here we have $\tilde{T}_i = T_i$ for all $i \in N$. However, if one could slightly modify this example to include action a_3, which is never preferred by the auctioneer and any bidders, then one can have $\tilde{T}_i \neq T_i$ by altering $T_i(a_3)$.

Similar to Bernheim and Whinston (1986), there are usually a large number of Nash equilibria in a menu auction game due to coordination problems among bidders. They argue that not all of them are equally plausible and propose truthful Nash equilibrium (TNE) as a refinement. In this class of equilibrium, each bidder submits a bidding menu that mirrors the relative payoffs which the bidder attaches to various actions. However, in our model, bidders cannot pay more than budget constraints, so we have to accommodate those cases in our definition.

For bidder $i \in N$, a bidding menu T_i is a **truthful bidding menu relative to** payoff u_i if for all $a \in A$,

$$T_i(a) = \begin{cases} 0 & \text{if } U_i(a, 0) < u_i \\ \tau_i(a, u_i) & \text{if } U_i(a, \omega_i(a)) \leq u_i \leq U_i(a, 0) \\ \omega_i(a) & \text{if } u_i < U_i(a, \omega_i(a)) \end{cases}$$

where $\tau_i(a, u_i)$ is implicitly defined by $U_i(a, \tau_i(a, u_i)) = u_i$.[13] Denote $T_i^{u_i}$ to be the truthful bidding menu relative to payoff u_i and $T^u \equiv (T_i^{u_i})_{i \in N}$ to be the truthful bidding menus relative to payoffs $u = (u_i)_{i \in N}$. A TNE is a refinement on a Nash equilibrium such that all bidders choose truthful bidding menus relative to their equilibrium payoffs.

Definition 2. An outcome (a^*, T^*) is a **truthful Nash equilibrium (TNE)** in Γ if it is a Nash equilibrium and T^* are the truthful bidding menus relative to equilibrium payoffs $u^* = (U_i(a^*, T_i^*(a^*)))_{i \in N}$.

[13] Dixit, Grossman and Helpman (1997) define $T_i(a) = \min\{\omega_i(a), \max\{0, \tau_i(a, u_i)\}\}$ to be the truthful bidding menu relative to u_i. However, $\tau_i(a, u_i)$ may be undefined. For example, consider $A = \{0, 1\}$ and $U_i(a, T_i(a)) = a + (T_i(a) + 1)^{-1}$. It is clear that $\tau_i(1, 1)$ is unbounded.

Bernheim and Whinston (1986) argue that a TNE may be quite "focal" because truthful bidding menus are simple. A further support is that a bidder suffers no loss in using truthful bidding menus because there is always a truthful bidding menu in the set of best responses. Proposition 1 in the next section shows that this still holds. The strongest justification for TNE is the strong stable property: every TNE is a coalition-proof Nash equilibrium (CPNE) and the set of TNE payoffs is the same as the set of CPNE payoffs. A CPNE is a Nash equilibrium immune to any credible joint deviation by any subset of bidders, where credibility of a coalitional deviation is defined recursively. The main result of this paper (Theorem 1) shows that this important property is still true.

Formally, we define coalition-proof Nash equilibrium as follows. Given any non-empty subset of bidders $J \subseteq N$ and bidding menus $(T_i)_{i \in N \setminus J}$, a J-**component game** relative to $(T_i)_{i \in N \setminus J}$ is defined as $\Gamma \setminus (T_i)_{i \in N \setminus J} \equiv (J, (U_j, \omega_j)_{j \in J}, (\tilde{U}_0, A))$ where $\tilde{U}_0(a, (\tilde{T}_j(a))_{j \in J}) \equiv U_0(a, (\tilde{T}_j(a))_{j \in J}, (T_i(a))_{i \in N \setminus J})$.[14]

Definition 3. (i) An outcome (a^*, T_j^*) is a **coalition-proof Nash equilibrium (CPNE)** in $\Gamma \setminus T_{-j}$ if and only if it is a Nash equilibrium in $\Gamma \setminus T_{-j}$.

(ii-a) An outcome $(a^*, (T_j^*)_{j \in J})$ is **self-enforcing** in $\Gamma \setminus (T_i)_{i \in N \setminus J}$ if for all non-empty $S \subsetneq J$, $(a^*, (T_j^*)_{j \in S})$ is a CPNE in $\Gamma \setminus ((T_i)_{i \in N \setminus J}, (T_j^*)_{j \in J \setminus S})$.

(ii-b) An outcome $(a^*, (T_j^*)_{j \in J})$ is a **CPNE** in $\Gamma \setminus (T_i)_{i \in N \setminus J}$ if it is self-enforcing in $\Gamma \setminus (T_i)_{i \in N \setminus J}$, and there exists no other self-enforcing $(\tilde{a}, (\tilde{T}_j)_{j \in J})$ in $\Gamma \setminus (T_i)_{i \in N \setminus J}$ such that (α) $U_j(\tilde{a}, \tilde{T}_j(\tilde{a})) \geq U_j(a^*, T_j^*(a^*))$ for all $j \in J$, and (β) $U_{j'}(\tilde{a}, \tilde{T}_{j'}(\tilde{a})) > U_{j'}(a^*, T_{j'}^*(a^*))$ and $\tilde{T}_{j'}(\tilde{a}) < \omega_{j'}(\tilde{a})$ for some $j' \in J$.

[14]Therefore, an outcome $(a, (\tilde{T}_j)_{j \in J})$ in $\Gamma \setminus (T_i)_{i \in N \setminus J}$ gives the same payoffs to the auctioneer and all bidders in J as in an outcome $(a, (\tilde{T}_j)_{j \in J}, (T_i)_{i \in N \setminus J})$ in Γ.

Comparing the above definition with the one in Bernheim and Whinston (1986), the only difference is condition (ii-b-β): a strict improvement is needed from a *budget-unconstrained bidder.* It should not be too surprising because since a budget-constrained bidder cannot provide extra incentives to persuade the auctioneer to choose a favorable action, a group of budget-constrained bidders cannot provide extra incentives even if they act together.

1.3 Results

As suggested in the previous section, we will show that there is always a truthful bidding menu in the set of best responses.[15] Following our definition of Nash equilibrium, a bidding menu T_i is a bidder i's **best response** to other bidder's bidding menus T_{-i} if there exists $a \in M(T)$ such that there exists no $\tilde{T}_i \in \mathcal{T}_i$ such that $U_i(\tilde{a}, \tilde{T}_i(\tilde{a})) > U_i(a, T_i(a))$ with $\tilde{T}_i(\tilde{a}) < \omega_i(\tilde{a})$ and $\tilde{a} \in M(\tilde{T}_i, T_{-i})$.

Proposition 1. In every menu auction game Γ, for all $i \in N$, there exists a truthful bidding menu being a bidder i's best response.

Proof. Consider T_i to be a best response to T_{-i} such that there exists $a \in M(T)$ such that there exists no $\tilde{T}_i \in \mathcal{T}_i$ such that $U_i(\tilde{a}, \tilde{T}_i(\tilde{a})) > U_i(a, T_i(a))$ with $\tilde{T}_i(\tilde{a}) < \omega_i(\tilde{a})$ and $\tilde{a} \in M(\tilde{T}_i, T_{-i})$. Consider a truthful bidding menu $T_i^{u_i}$ with $u_i = U_i(a, T_i(a))$. If $a = M(T_i^{u_i}, T_{-i})$,

[15]Strictly speaking, without knowing how the auctioneer chooses among payoff-equivalent actions, the set of best responses for a bidder is not well defined. Bernheim and Whinston (1986) argue (in their footnote 11) that such a problem disappears if payment has some smallest unit (however small). Milgrom (2005) argues that a bidding menu can be loosely defined as a best response of a bidder if for some $\varepsilon > 0$, the bidder will not choose another bidding menu assuming that the auctioneer considers the bidder is paying ε more on the bidders' favorable action when choosing an action, but the bidder's payoff is evaluated without paying ε more.

then $T_i^{u_i}$ is already a best response to T_{-i}. Hence, consider that there exists $\bar{a} \neq a$ with $\bar{a} \in M\left(T_i^{u_i}, T_{-i}\right)$. There are two cases: (Case 1) $T_i\left(\bar{a}\right) > T_i^{u_i}\left(\bar{a}\right)$: it implies $U_0\left(\bar{a}, T\left(\bar{a}\right)\right) > U_0\left(\bar{a}, T_i^{u_i}\left(\bar{a}\right), T_{-i}\left(\bar{a}\right)\right)$. As $\bar{a} \in M\left(T_i^{u_i}, T_{-i}\right)$, we have $U_0\left(\bar{a}, T_i^{u_i}\left(\bar{a}\right), T_{-i}\left(\bar{a}\right)\right) \geq U_0\left(a, T_i^{u_i}\left(a\right), T_{-i}\left(a\right)\right)$. Hence, $U_0\left(\bar{a}, T\left(\bar{a}\right)\right) > U_0\left(a, T_i^{u_i}\left(a\right), T_{-i}\left(a\right)\right) = U_0\left(a, T\left(a\right)\right)$, which contradicts $a \in M\left(T\right)$; (Case 2) $T_i\left(\bar{a}\right) \leq T_i^{u_i}\left(\bar{a}\right)$: it implies either $T_i^{u_i}\left(\bar{a}\right) = \tau_i\left(\bar{a}, u_i\right)$ or $T_i^{u_i}\left(\bar{a}\right) = \omega_i\left(\bar{a}\right)$ but both imply $U_i\left(\bar{a}, T_i^{u_i}\left(\bar{a}\right)\right) \geq u_i = U_i\left(a, T_i\left(a\right)\right)$ so that $T_i^{u_i}$ is at least as good as T_i. Therefore, $T_i^{u_i}$ is a best response to T_{-i}.$\square$

Following the auction literature, we can construct a coalitional game between the auctioneer and N bidders from a menu auction game Γ.[16] In Bernheim and Whinston (1986), every TNE is a CPNE and the set of TNE/CPNE payoffs is the bidder-optimal strong core. Theorem 1 will show this is still true with some modifications. As we have seen the presence of budget constraints requires alternative definitions of Nash equilibrium and CPNE, it is not surprising that we need alternative definitions of the core and bidder-optimality.[17]

In our model, a non-transferable utility coalitional game $\left(N \cup \{0\}, \left(\mathcal{U}_\Gamma\left(S\right)\right)_{S \subseteq N \cup \{0\}}\right)$ constructed from a menu auction game Γ is a coalitional game between the auctioneer and N bidders such that $\mathcal{U}_\Gamma\left(S\right)$ is the set of payoffs achievable by $S \subseteq N \cup \{0\}$ in Γ. Since bidders cannot generate meaningful payoffs without the auctioneer, define $\mathcal{U}_\Gamma\left(S\right) \equiv \{(u_i)_{i \in S} \in \mathbb{R}^S$: there exists $(a, T) \in A \times \mathcal{T}$ such that $u_0 = U_0(a, (T_i\left(a\right))_{i \in S})$ and $u_i = U_i\left(a, T_i\left(a\right)\right)$ for all $i \in S\}$ if

[16] This is different from the menu auction literature. Bernheim and Whinston (1986) do not mention "core". Laussel and Le Breton (2001) consider transferable utility coalitional games generated from menu auction games between bidders only.

[17] Day and Milgrom (2008) discuss the importance of the core and bidder-optimality in auction mechanisms (with transferable utilities). They argue that auctions selecting core allocations have the advantages that bidders have no incentive to merge bids, submit bids under other identities, or renege after the auction is conducted. Furthermore, if the selected allocation is in the bidder-optimal core, then bidders have minimal incentives to misreport among all core-selecting auctions and the auctioneer would not have incentive to disqualify bidders.

$\{0\} \in S$, and $\mathcal{U}_\Gamma(S) = \{(u_i)_{i\in S} \in \mathbb{R}^S : u_i = \inf_{a\in A} U_i(a,0) \text{ for all } i \in S\}$ if $\{0\} \notin S$. To save notation, let $S_0 \equiv S \cup \{0\}$, a set comprising the auctioneer and all bidders in $S \subseteq N$, and $u_{S_0} \equiv (u_0, (u_i)_{i\in S})$, a list of their payoffs. A list of payoffs u_{N_0} is an **allocation** if $u_{N_0} \in \mathcal{U}_\Gamma(N_0)$.[18] An allocation u is **supported** by an outcome (a,T) if $u_0 = U_0(a, T(a))$ and $u_i = U_i(a, T_i(a))$ for all $i \in N$.

Definition 4. An allocation u is **weakly blocked** by S if there exists $\tilde{u}_S \in \mathcal{U}_\Gamma(S)$ such that (i) $\tilde{u}_i \geq u_i$ for all $i \in S$ and (ii) $\tilde{u}_i > u_i$ for some $i \in S$. An allocation u is in the **strong core** ($Score_\Gamma$) if it is not weakly blocked by any $S \subseteq N \cup \{0\}$.

In Bernheim and Whinston (1986), the strong core is non-empty and includes the set of TNE payoffs. However, this does not extend to our model. Example 1 shows that the strong core can be empty,[19] though it will be shown that there is always a TNE. Moreover, Example 3 below shows that even when the strong core is non-empty, there is an allocation supported by a TNE but not in the strong core.

Example 3. Consider $N = \{1, 2\}$ and $A = \{a_1, a_2, a_3\}$. Assume quasi-linear preferences such that for all $a \in A$, $U_i(a, T(a)) = V_i(a) - T_i(a)$ with $\omega_i(a) = 2$ for all $i \in N$ and $U_0(a, T(a)) = V_0(a) + \sum_{i\in N} T_i(a)$ where

	a_1	a_2
$V_1(a)$	6	6
$V_2(a)$	6	1
$V_0(a)$	0	2

[18] Without confusion, we drop the subscript N_0 when a list of payoffs is an allocation.

[19] In Example 1, every allocation, except those weakly blocked by some bidders only, is weakly blocked by allocations $(2, 4, 1)$ or $(2, 1, 4)$.

Consider T such that $T_1(a_1) = T_1(a_2) = 0$, $T_2(a_1) = 2$ and $T_2(a_2) = 0$. Outcomes (a_1, T) and (a_2, T) are TNEs. However, the allocation supported by (a_2, T) is not in the strong core.

As hinted above, it seems natural to modify the definition of weak blocking by taking budget constraints into account.

Definition 4. An allocation u is **BC-blocked** (Budget-Constraint blocked) by S if there exists $\tilde{u}_S \in \mathcal{U}_\Gamma(S)$ supported by an outcome $(\tilde{a}, \tilde{T})$ such that (i) $\tilde{u}_i \geq u_i$ for all $i \in S$, and (ii) either $\tilde{u}_0 > u_0$, or $\tilde{u}_i > u_i$ and $\tilde{T}_i(\tilde{a}) < \omega_i(\tilde{a})$ for some $i \in S \backslash \{0\}$. An allocation u is in the **BC-core** (Budget-Constraint core, $BCcore_\Gamma$) if it is not BC-blocked by any $S \subseteq N \cup \{0\}$.

Since the strong core and the weak core are equivalent in Bernheim and Whinston (1986), it is interesting to see how the BC-core is related to the weak core in our model.

Definition 5. An allocation u is **strongly blocked** by S if there exists $\tilde{u}_S \in \mathcal{U}_\Gamma(S)$ such that $\tilde{u}_i > u_i$ for all $i \in S$. An allocation u is in the **weak core** ($Wcore_\Gamma$) if it is not strongly blocked by any $S \subseteq N \cup \{0\}$.

At the first glance, one may conjecture that BC-blocking is more effective than strong blocking, but Proposition 2 shows that they are equivalent.

Proposition 2. In every menu auction game Γ, we have

$$Wcore_\Gamma = BCcore_\Gamma.$$

Proof. By definition, if there is a strong blocking deviation for $u \in \mathcal{U}_\Gamma(N_0)$, then it is also a BC-blocking deviation for u because there is a strict improvement for the auctioneer.[20] Therefore, we have $Wcore_\Gamma \supseteq BCcore_\Gamma$. Now suppose $u \in Wcore_\Gamma$ but u is BC-blocked by S_0.[21] There exists $\tilde{u}_{S_0} \in \mathcal{U}_\Gamma(S_0)$ supported by an outcome $(\tilde{a}, \tilde{T}) \in A \times \mathcal{T}$ such that $\tilde{u}_i \geq u_i$ for all $i \in S_0$, and $\tilde{u}_j > u_j$ and $\tilde{T}_j(\tilde{a}) < \omega_j(\tilde{a})$ for some $j \in S$. Let $\tilde{S} \equiv S \backslash \{i \in S : \tilde{T}_i(\tilde{a}) = 0\}$ and $K \equiv \{j \in \tilde{S} : \tilde{u}_j > u_j$ and $\tilde{T}_j(\tilde{a}) < \omega_j(\tilde{a})\}$. There exists $\varepsilon_i > 0$ for all $i \in \tilde{S}$ such that $\bar{u}_{\tilde{S}_0} \in \mathcal{U}_\Gamma(\tilde{S}_0)$ supported by $(\tilde{a}, (\tilde{T}_{i'})_{i' \in N \backslash \tilde{S}}, (\bar{T}_i)_{i \in \tilde{S}}) \in A \times \mathcal{T}$ strongly blocks u by $\tilde{S}_0$ where for all $j \in K$, $\bar{T}_j(\tilde{a}) = \tilde{T}_j(\tilde{a}) + \varepsilon_j < \omega_j(\tilde{a})$ and $\bar{T}_j(a) = 0$ for all $a \in A \backslash \{\tilde{a}\}$, and for all $i \in \tilde{S} \backslash K$, $\bar{T}_i(\tilde{a}) = \tilde{T}_i(\tilde{a}) - \varepsilon_i > 0$ and $\bar{T}_i(a) = 0$ for all $a \in A \backslash \{\tilde{a}\}$.[22] Thus, $Wcore_\Gamma \subseteq BCcore_\Gamma$.$\square$

Dixit, Grossman, and Helpman (1997) prove that every TNE under their definition is strongly Pareto efficient for the auctioneer and all bidders.[23] However, Example 3 shows that (a_1, T) weakly Pareto dominates (a_2, T), both of which are TNEs. This arises because budget-constrained bidders are unable to provide incentives to the auctioneer to induce a Pareto improvement. Therefore, we have to incorporate the implication of budget constraints into the bidder-optimality.[24]

[20] If a strong deviation comes from some bidders only, then it is easy to construct a BC-blocking deviation by those bidders.

[21] If the BC-blocking deviation comes from some bidders only, then it is also a strong blocking deviation by some of those bidders. Similarly, if the BC-blocking deviation comes from the auctioneer only, it is also a strong blocking deviation by the auctioneer. Both cases contradict $u \in Wcore_\Gamma$.

[22] If $\tilde{S}$ is empty, then $\tilde{u}$ BC-blocks u by the auctioneer only so that $\tilde{u}$ also strongly blocks u.

[23] An allocation u is strongly Pareto efficient for the auctioneer and all bidders if there exists no $\tilde{u} \in \mathcal{U}_\Gamma(N_0)$ such that $\tilde{u}_i \geq u_i$ for all $i \in N_0$, and $\tilde{u}_i > u_i$ for some $i \in N_0$.

[24] The standard definition of the bidder-optimality is strongly Pareto efficiency for all bidders without taking budget constraints into account.

Definition 6. An allocation u is in the **bidder-optimal BC-core** ($\overline{BCcore_\Gamma}$) if there exists no $\tilde{u} \in BCcore_\Gamma$ supported by an outcome $(\tilde{a}, \tilde{T})$ such that $\tilde{u}_i \geq u_i$ for all $i \in N$, and $\tilde{u}_i > u_i$ and $\tilde{T}_i(\tilde{a}) < \omega_i(\tilde{a})$ for some $i \in N$. The **bidder-optimal strong core** ($\overline{Score_\Gamma}$) and the **bidder-optimal weak core** ($\overline{Wcore_\Gamma}$) are defined similarly.

While $\overline{Score_\Gamma}$ may be empty, Proposition 3 below shows that $\overline{BCcore_\Gamma}$ and $\overline{Wcore_\Gamma}$ are always non-empty. As Theorem 1 shows that $\overline{BCcore_\Gamma}$ is equivalent to the set of TNE/CPNE payoffs, the existence of a TNE/CPNE is assured.

Proposition 3. In every menu auction game Γ, the bidder-optimal BC-core is non-empty.

Proof. Scarf (1967) proves that in a coalitional game $(N \cup \{0\}, (\mathcal{U}_\Gamma(S))_{S \subseteq N \cup \{0\}})$ if for all $S \subseteq N \cup \{0\}$, $\mathcal{U}_\Gamma(S)$ is comprehensive and closed, and satisfies balancedness, and $\{u_S \in \mathcal{U}_\Gamma(S) : u_i \geq \sup \mathcal{U}_\Gamma(\{i\}) \text{ for all } i \in S\}$ is non-empty and bounded, then $Wcore_\Gamma \neq \emptyset$. It is easy to check that all conditions are satisfied. As $Wcore_\Gamma$ is compact and dominance relationship in the bidder-optimality is weaker than strongly Pareto efficiency, we have $\overline{Wcore_\Gamma} \neq \emptyset$, and hence $\overline{BCcore_\Gamma} \neq \emptyset$ by Proposition 2.$\square$

The following theorem is the main result of this paper.

Theorem 1. In every menu auction game Γ, every truthful Nash equilibrium (TNE) is a coalition-proof Nash equilibrium (CPNE) and

$$\overline{Score_\Gamma} \subsetneq \overline{Wcore_\Gamma} = \overline{BCcore_\Gamma} = \mathcal{U}_\Gamma^{TNE} = \mathcal{U}_\Gamma^{CPNE}$$

where $\mathcal{U}_\Gamma^{TNE}$ and $\mathcal{U}_\Gamma^{CPNE}$ are the sets of TNE payoffs and CPNE payoffs in Γ.

The proof of this theorem is complex and we defer it to the appendix. It is interesting to compare this result with the existing literature. Dixit, Grossman, and Helpman (1997) show that under similar settings as this paper, the set of TNE payoffs under their definition is included in the set of strongly Pareto efficient allocations with respect to all bidders and the auctioneer.[25] Bernheim and Whinston (1986), under the assumptions of quasi-linear preferences and the absence of budget constraints, show that every TNE is a CPNE and

$$\overline{Score_\Gamma} = \overline{Wcore_\Gamma} = \mathcal{U}_\Gamma^{TNE} = \mathcal{U}_\Gamma^{CPNE}.^{26}$$

Therefore, Theorem 1 almost completely extends the results by Bernheim and Whinston (1986) to the generalized framework. The only difference is that in our framework $\overline{Score_\Gamma}$ does not coincide with $\overline{Wcore_\Gamma}$. This is unavoidable since $\overline{Wcore_\Gamma}$ is non-empty, whereas $\overline{Score_\Gamma}$ can be empty. However, it is possible to reconcile our result with Bernheim and Whinston (1986) with one of the following two additional assumptions. Bidders' preferences satisfy **indispensability of private good** if for all $i \in N$, $U_i(a, \omega_i(a)) = U_i(\tilde{a}, \omega_i(\tilde{a}))$ for all $a, \tilde{a} \in A$ (Mas-Colell 1977). Alternatively, bidders are deep-pocketed if for all $i \in N$, for all $a \in A$, $U_i(a, \omega_i(a)) < \min_{\tilde{a} \in A} U_i(\tilde{a}, 0)$. Either assumption implies that BC-blocking and weak blocking are the same, so we have $\overline{BCcore_\Gamma} = \overline{Score_\Gamma}$. Thus, Theorem 1 implies the following result.

Corollary 1. In every menu auction game Γ, if bidders' preferences satisfy indispensability of private good or bidders are deep-pocketed, then every TNE

[25] Note that condition (iii-b) is absent in their definition of Nash equilibrium.

[26] Though our bidder-optimality takes budget constraints into account, it is the same as the standard definition of the bidder-optimality when bidders have no budget constraints.

is a CPNE and

$$\overline{Score_\Gamma} = \overline{Wcore_\Gamma} = \mathcal{U}_\Gamma^{TNE} = \mathcal{U}_\Gamma^{CPNE}.$$

Corollary 1 implies the main result of Bernheim and Whinston (1986).

Corollary 2. In every menu auction game Γ, if the auctioneer and all bidders have quasi-linear preferences ($U_0(a) = V_0(a) + \sum_{i\in N} T_i(a)$ and $U_i(a) = V_i(a) - T_i(a)$ for all $i \in N$), and bidders have no budget constraint, then every TNE is a CPNE, the auctioneer chooses $a^* \in \max_{a\in A} V_0(a) + \sum_{i\in N} V_i(a)$ in every TNE/CPNE, and $\mathcal{U}_\Gamma^{TNE} = \mathcal{U}_\Gamma^{CPNE} = \overline{Score_\Gamma} = \overline{Wcore_\Gamma} = \{u \in \mathcal{U}_\Gamma(N_0) : \sum_{i\in S} u_i \leq W(N) - W(N\backslash S)$ for all $S \subseteq N\}$ where $W(S) \equiv \max_{a\in A} V_0(a) + \sum_{i\in S} V_i(a)$ for all $S \subseteq N$.

1.4 Concluding Remarks

In this paper, we generalize Bernheim and Whinston's (1986) menu auction game to the class of non-transferable utility game with budget constraints. This extension is useful since it allows more applications, as discussed in Section 1. However, there is another reason to study this extension. The efficiency result in menu auctions under a restricted domain has been used as a benchmark in general package/combinatorial auction designs. For example, Ausubel and Milgrom (2002) propose a generalized ascending package auction, which allows non-quasi-linear preferences and budget constraints. After bidders report their preferences, the auction mechanism uses an algorithm to determine an allocation that is shown to be in the weak core with respect to *reported* preferences.[27] This paper provides the theoretical basis for a comparison: *every*

[27]There might be equilibria where bidders do not report their actual preferences.

allocation in the *bidder-optimal* weak core with respect to *actual* preferences is implemented by a generalized menu auction game in CPNEs, irrespective of reported preferences.[28]

[28]Note that bidder-optimality is slightly modified for budget constraints as defined in section 3.

Appendix: Proof of Theorem 1

First, Proposition 5 shows every TNE is a CPNE. Second, we establish $\overline{BCcore_\Gamma} = \mathcal{U}_\Gamma^{TNE} = \mathcal{U}_\Gamma^{CPNE}$ by $\overline{BCcore_\Gamma} \subseteq \mathcal{U}_\Gamma^{TNE}$ (Proposition 4), $\mathcal{U}_\Gamma^{TNE} \subseteq \mathcal{U}_\Gamma^{CPNE}$ (Proposition 5) and $\mathcal{U}_\Gamma^{CPNE} \subseteq \overline{BCcore_\Gamma}$ (Proposition 6). By Proposition 2, $\overline{Wcore_\Gamma} = \overline{BCcore_\Gamma}$. By definition, we have $Score_\Gamma \subseteq Wcore_\Gamma$ but Example 3 shows it is possible to have $Score_\Gamma = \emptyset$ and $Wcore_\Gamma \neq \emptyset$, so in general $\overline{Score_\Gamma} \subsetneq \overline{Wcore_\Gamma}$, which completes the proof.

Proposition 4. In every menu auction game Γ, every allocation $u^* \in \overline{BCcore_\Gamma}$ can be supported by a truthful Nash equilibrium.

Proof. Consider $u^* \in \overline{BCcore_\Gamma}$ supported by $(a^*, T) \in A \times \mathcal{T}$. Suppose that for all $i \in N$, bidder i chooses the truthful bidding menu T_i^* relative to u_i^*, that is, $T_i^* = T_i^{u_i^*}$. It suffices to check (a^*, T^*) is a Nash equilibrium since the resulting allocation is u^* by construction. Clearly, $T^* \in \mathcal{T}$. Suppose $a^* \notin M(T^*)$. There exists $\tilde{a} \in A$ such that $U_0(\tilde{a}, T^*(\tilde{a})) > U_0(a^*, T^*(a^*))$. Let $K \equiv \{i \in N : U_i(\tilde{a}, 0) < u_i^*\}$ so that $U_0(\tilde{a}, (T_i^*(\tilde{a}))_{i \in N \backslash K}) > U_0(a^*, T^*(a^*))$. Truthful bidding menus imply $U_i(\tilde{a}, T_i^*(\tilde{a})) \geq U_i(a^*, T_i^*(a^*))$ for all $i \in N \backslash K$. Then u^* is BC-blocked by $N_0 \backslash K$, which contradicts $u^* \in \overline{BCcore_\Gamma}$. Hence, $a^* \in M(T^*)$.

Suppose for some $j \in N$, there exists $(\bar{a}, \bar{T}_j) \in A \times \mathcal{T}_j$ such that $\bar{a} \in M(\bar{T}_j, T_{-j}^*)$, $U_j(\bar{a}, \bar{T}_j(\bar{a})) > U_j(a^*, T_j^*(a^*))$ and $\bar{T}_j(\bar{a}) < \omega_j(\bar{a})$. There are two cases. (Case 1): $m(T^*) \leq m(\bar{T}_j, T_{-j}^*)$. Let $\bar{K} \equiv \{i \in N : U_i(\bar{a}, 0) < u_i^*\}$, $\bar{u}_0 = U_0(\bar{a}, (T_i^*(\bar{a}))_{i \in (N \backslash \bar{K}) \backslash \{j\}}, \bar{T}_j(\bar{a}))$, $\bar{u}_j = U_j\left(\bar{a}, \bar{T}_j(\bar{a})\right)$ and $\bar{u}_i = U_i(a^*, T_i^*(a^*))$ for all $i \in (N \backslash \bar{K}) \backslash \{j\}$. Then $\bar{u}_{N_0 \backslash \bar{K}}$ BC-blocks u^* by $N_0 \backslash \bar{K}$, which contradicts $u^* \in BCcore_\Gamma$. (Case 2): $m(T^*) > m(\bar{T}_j, T_{-j}^*)$. Note that $a^* \in M(T^*)$ implies $T_j^*(a^*) > 0$. There exists $\varepsilon_j > 0$ such that $\hat{T}_j(a^*) = T_j^*(a^*) - \varepsilon_j > 0$ and

$\hat{T}_j(a) = 0$ for all $a \in A \backslash \{a^*\}$ so that $a^* \in M(\hat{T}_j, T^*_{-j})$ and $U_j(a^*, \hat{T}_j(a^*)) > U_j(a^*, T^*_j(a^*))$. Denote $\hat{u}_0 = U_0(a^*, \hat{T}_j(a^*), T^*_{-j}(a^*))$, $\hat{u}_j = U_j(a^*, \hat{T}_j(a^*))$ and $\hat{u}_i = U_i(a^*, T^*_i(a^*))$ for all $i \in N \backslash \{j\}$. Note that $\hat{u}_i \geq u^*_i$ for all $i \in N$ and $\hat{u}_j > u^*_j$. If $\hat{u} \in BCcore_\Gamma$, then $u^* \notin \overline{BCcore_\Gamma}$, which is a contradiction. Hence, $\hat{u} \notin BCcore_\Gamma$ holds. Then for some $S \subseteq N$, there exists $\bar{\bar{u}}_{S_0} \in \mathcal{U}_\Gamma(S_0)$ such that $\bar{\bar{u}}_{S_0}$ BC-blocks $\hat{u}$ and $\bar{\bar{u}}_{S_0} \neq u^*_{S_0}$.[29] Since $\hat{u}$ and u^* are supported by $(a^*, \hat{T}_j, T^*_{-j})$ and (a^*, T^*), it must be $\bar{\bar{u}}_0 \geq u^*_0$.[30] However, this implies $\bar{\bar{u}}_{S_0}$ BC-blocks u^*, which contradicts $u^* \in BCcore_\Gamma$. $\square$

Before stating Proposition 5 and Proposition 6, it is useful to prove the following two lemmas.

Lemma 1. In every menu auction game Γ, if an allocation u^* is supported by a truthful Nash equilibrium (a^*, T^*) in Γ, then $u^*_{J_0} \in \overline{BCcore_{\Gamma \backslash (T^*_k)_{k \in N \backslash J}}}$ for all non-empty $J \subseteq N$ where $u^*_{J_0} \equiv (u^*_0, (u^*_i)_{i \in J})$.

Proof. First, we show $u^*_{J_0} \in BCcore_{\Gamma \backslash (T^*_k)_{k \in N \backslash J}}$ for all non-empty $J \subseteq N$. Suppose $u^*_{J_0} \notin BCcore_{\Gamma \backslash (T^*_k)_{k \in N \backslash J}}$ for some non-empty $J \subseteq N$. For some non-empty $S \subseteq J$, there exists $\tilde{u}_{S_0} \in \mathcal{U}_{\Gamma \backslash (T^*_k)_{k \in N \backslash J}}(S_0)$ supported by $(\tilde{a}, (\tilde{T}_i)_{i \in S}) \in A \times (\mathcal{T}_i)_{i \in S}$ such that $\tilde{u}_i \geq u^*_i$ for all $i \in S_0$ and $\tilde{u}_j > u^*_j$ with $\tilde{T}_j(\tilde{a}) < \omega_j(\tilde{a})$ for some $j \in S$,[31] where $\tilde{u}_0 = U_0(\tilde{a}, (\tilde{T}_i(\tilde{a}))_{i \in S}, (T^*_k(\tilde{a}))_{k \in N \backslash J})$, $u^*_0 = U_0(a^*, T^*(a^*))$, and for all $i \in S$, $\tilde{u}_i = U_i(\tilde{a}, \tilde{T}_i(\tilde{a}))$ and $u^*_i = U_i(a^*, T^*_i(a^*))$. There exists $\varepsilon_j > 0$ such that $\hat{T}_j(\tilde{a}) = \tilde{T}_j(\tilde{a}) + \varepsilon_j < \omega_j(\tilde{a})$ and $\hat{T}_j(a) = 0$ for all $a \in A \backslash \{\tilde{a}\}$ so that $U_j((\tilde{a}, \hat{T}_j(\tilde{a})) > U_j(a^*, T^*_j(a^*))$ and $U_0(\tilde{a}, \hat{T}_j(\tilde{a}), T^*_{-j}(\tilde{a})) > U_0(\tilde{a}, \tilde{T}_j(\tilde{a}), T^*_{-j}(\tilde{a}))$. For all $i \in S$, since T^*_i is a truthful bidding menu, we have $T^*_i(\tilde{a}) \geq \tilde{T}_i(\tilde{a})$ so

[29] If $\bar{\bar{u}}_{S_0} = u^*_{S_0}$, then $\hat{u}_i \geq \bar{\bar{u}}_i$ for all $i \in S_0$. This implies $\bar{\bar{u}}_{S_0}$ cannot BC-block $\hat{u}$.

[30] Otherwise, we can construct $\hat{u}$ such that it is not BC-blocked by any $\bar{\bar{u}}_{\tilde{S}_0}$ for all $\tilde{S} \subseteq N$.

[31] The auctioneer is not maximizing if there is no $j \in S$ such that $\tilde{u}_j > u^*_j$ with $\tilde{T}_j(\tilde{a}) < \omega_j(\tilde{a})$.

that $U_0(\tilde{a},\tilde{T}_j(\tilde{a}),T^*_{-j}(\tilde{a})) \geq U_0(\tilde{a},(\tilde{T}_i(\tilde{a}))_{i\in S},(T^*_k(\tilde{a}))_{k\in N\setminus J})$. Therefore, $U_0(\tilde{a},\hat{T}_j(\tilde{a}),T^*_{-j}(\tilde{a})) > U_0(a^*,T^*(a^*))$ implies $\tilde{a}\in M(\hat{T}_j,T^*_{-j})$, which contradicts (a^*,T^*) being a Nash equilibrium.

It remains to show $u^*_{J_0} \in \overline{BCcore_{\Gamma\setminus(T^*_k)_{k\in N\setminus J}}}$ for all non-empty $J\subseteq N$. Suppose not. There exists $\tilde{\tilde{u}}_{J_0} \in \overline{BCcore_{\Gamma\setminus(T^*_k)_{k\in N\setminus J}}}$ supported by $(\tilde{\tilde{a}},(\tilde{\tilde{T}}_i)_{i\in J}) \in A\times(\mathcal{T}_i)_{i\in J}$ such that for all $i\in J$, $\tilde{\tilde{u}}_i \geq u^*_i$ and for some $j'\in J$, $\tilde{\tilde{u}}_{j'} > u^*_{j'}$ and $\tilde{\tilde{T}}_{j'}(\tilde{\tilde{a}}) < \omega_{j'}(\tilde{\tilde{a}})$. By Proposition 4, $\tilde{\tilde{u}}_{J_0}$ can be supported by a TNE $(\tilde{\tilde{a}},(T_i^{\tilde{\tilde{u}}_i})_{i\in J})$ in $\Gamma\setminus(T^*_k)_{k\in N\setminus J}$ implying $m(T^*)\geq m((T_i^{\tilde{\tilde{u}}_i})_{i\in J},(T^*_k)_{k\in N\setminus J})$. There are two cases, both of which contradict (a^*,T^*) being a Nash equilibrium. (Case 1): $m(T^*) = m((T_i^{\tilde{\tilde{u}}_i})_{i\in J},(T^*_k)_{k\in N\setminus J})$. There exists $\varepsilon_{j'}>0$ such that $\bar{T}_{j'}(\tilde{\tilde{a}}) = T_{j'}^{\tilde{\tilde{u}}_{j'}}(\tilde{\tilde{a}})+\varepsilon_{j'} < \omega_{j'}(\tilde{\tilde{a}})$ and $\bar{T}_{j'}(a)=0$ for all $a\in A\setminus\{\tilde{\tilde{a}}\}$ so that $\tilde{\tilde{a}}\in M(\bar{T}_{j'},T^*_{-j'})$ and $U_{j'}(\tilde{\tilde{a}},\bar{T}_{j'}(\tilde{\tilde{a}})) > U_{j'}(a^*,T^*_{j'}(a^*))$. This contradicts (a^*,T^*) being a Nash equilibrium; (Case 2): $m(T^*) > m((T_i^{\tilde{\tilde{u}}_i})_{i\in J},(T^*_k)_{k\in N\setminus J})$. Then $m(T^*) > m(T_{j''}^{\tilde{\tilde{u}}_{j''}},T^*_{-j''})$ for some $j''\in J$. Note that $a^*\in M(T^*)$ implies $T^*_{j''}(a^*)>0$. There exists $\varepsilon_{j''}>0$ such that $\bar{\bar{T}}_{j''}(a^*) = T^*_{j''}(a^*) - \varepsilon_{j''} > 0$ and $\bar{\bar{T}}_{j''}(a)=0$ for all $a\in A\setminus\{a^*\}$ so that $a^*\in M(\bar{\bar{T}}_{j''},T^*_{-j''})$ and $U_{j''}(a^*,\bar{\bar{T}}_{j''}(a^*)) > U_{j''}(a^*,T^*_{j''}(a^*))$ and $\bar{\bar{T}}_{j''}(a^*) < \omega_{j''}(a^*)$. This contradicts (a^*,T^*) being a Nash equilibrium.□

Lemma 2. In every menu auction game Γ with $|N|\geq 2$, if an outcome (a^*,T^*) is self-enforcing in Γ, then for all non-empty $S\subsetneq N$, $u^*_{S_0}\in\overline{BCcore_{\Gamma\setminus(T^*_k)_{k\in N\setminus S}}}$ where $u^*_{S_0}\in\mathcal{U}_\Gamma(S_0)$ is supported by $(a^*,(T^*_i)_{i\in S})$ in $\Gamma\setminus(T^*_k)_{k\in N\setminus S}$.

Proof. Consider $|N|=2$. For all $i\in N$, (a^*,T^*) is self-enforcing in Γ if (a^*,T^*_i) is a CPNE in $\Gamma\setminus T^*_{-i}$. By definition, (a^*,T^*_i) is a CPNE in $\Gamma\setminus T^*_{-i}$ if and only if it is a Nash equilibrium in $\Gamma\setminus T^*_{-i}$. If $(u^*_0,u^*_i)\notin\overline{BCcore_{\Gamma\setminus T^*_{-i}}}$ for some $i\in N$, then (a^*,T^*) cannot be a Nash equilibrium in Γ. The rest of argument will be completed by induction.

Consider $|N| > 2$. By the induction assumption, $u^*_{S_0} \in \overline{BCcore_{\Gamma\backslash\left(T^*_k\right)_{k\in N\backslash S}}}$ for all $S \subseteq N$ with $|S| < |N| - 1$. Suppose $u^*_{\tilde{S}_0} \notin \overline{BCcore_{\Gamma\backslash\left(T^*_k\right)_{k\in N\backslash \tilde{S}}}}$ for some $\tilde{S} \subseteq N$ with $|\tilde{S}| = |N| - 1$. There exists $\tilde{u}_{\tilde{S}_0} \in \overline{BCcore_{\Gamma\backslash\left(T^*_k\right)_{k\in N\backslash \tilde{S}}}}$ supported by $(\tilde{a}, (\tilde{T}_i)_{i\in\tilde{S}}) \in A \times (\mathcal{T}_i)_{i\in\tilde{S}}$ in $\Gamma\backslash\left(T^*_k\right)_{k\in N\backslash \tilde{S}}$ such that $\tilde{u}_i \geq u^*_i$ for all $i \in \tilde{S}$, and $\tilde{u}_j > u^*_j$ and $\tilde{T}_j\left(\tilde{a}\right) < \omega_j\left(\tilde{a}\right)$ some $j \in \tilde{S}$. By Proposition 4, $(\tilde{a}, (T_i^{\tilde{u}_i})_{i\in\tilde{S}})$ is a TNE in $\Gamma\backslash\left(T^*_k\right)_{k\in N\backslash \tilde{S}}$. Then, by Lemma 1, for all non-empty $J \subseteq \tilde{S}$, $\tilde{u}_{J_0} \in \overline{BCcore_{\Gamma\backslash((T^*_k)_{k\in N\backslash\tilde{S}},(T_i^{\tilde{u}_i})_{i\in\tilde{S}\backslash J})}}$. Hence, $(\tilde{a}, (T_i^{\tilde{u}_i})_{i\in\tilde{S}})$ is self-enforcing in $\Gamma\backslash\left(T^*_k\right)_{k\in N\backslash \tilde{S}}$ as there is no credible deviation in $\Gamma\backslash((T^*_k)_{k\in N\backslash\tilde{S}}, (T_i^{\tilde{u}_i})_{i\in\tilde{S}\backslash\hat{S}})$ for all $\hat{S} \subseteq \tilde{S}$. Since $U_j(\tilde{a}, T_j^{\tilde{u}_j}\left(\tilde{a}\right)) > U_j(a^*, T^*_j\left(a^*\right))$ and $T_j^{\tilde{u}_j}\left(\tilde{a}\right) < \omega_j\left(\tilde{a}\right)$, the outcome (a^*, T^*) cannot be a CPNE in $\Gamma\backslash\left(T^*_k\right)_{k\in N\backslash \tilde{S}}$. This contradicts (a^*, T^*) being self-enforcing in Γ. Hence, $u^*_{\tilde{S}_0} \in \overline{BCcore_{\Gamma\backslash\left(T^*_k\right)_{k\in N\backslash \tilde{S}}}}$ for all $\tilde{S} \subsetneq N$. By induction, Lemma 2 is proved.□

Lemma 1 shows that every allocation supported by a TNE is also in the bidder-optimal BC-core of every component games, and Lemma 2 shows that every allocation supported by a self-enforcing outcome is also in the bidder-optimal BC-core of every component games. Since every CPNE is self-enforcing, Proposition 5 can be shown readily.

Proposition 5. In every menu auction game Γ, every truthful Nash equilibrium is a coalition-proof Nash equilibrium.

Proof. It is trivial when $|N| = 1$. Consider $|N| \geq 2$. Let (a^*, T^*) be a TNE in Γ. We proceed by induction on $S \subseteq N$. By induction assumption, for all non-empty $J \subseteq S$, $(a^*, (T^*_j)_{j\in J})$ is self-enforcing in $\Gamma\backslash\left(T^*_k\right)_{k\in N\backslash J}$. By Lemma 1, for all non-empty $J \subseteq S$, if $u^*_{J_0} \in \mathcal{U}_{\Gamma\backslash(T^*_k)_{k\in N\backslash J}}(J_0)$ is supported by a TNE $(a^*, (T^*_j)_{j\in J})$ in $\Gamma\backslash\left(T^*_k\right)_{k\in N\backslash J}$, then $u^*_{J_0} \in \overline{BCcore_{\Gamma\backslash(T^*_k)_{k\in N\backslash J}}}$. By Lemma 2, a self-enforcing allocation must be in the bidder-optimal BC-core in every component games. Therefore, for all non-empty $J \subseteq S$, there exists no self-enforcing allocation $\tilde{u}_{J_0} \in \mathcal{U}_{\Gamma \backslash (T_k^*)_{k \in N \backslash J}}(J_0)$ supported by $(\tilde{a}, (\tilde{T}_i)_{i \in J}) \in A \times (\mathcal{T}_i)_{i \in J}$ in $\Gamma \backslash (T_k^*)_{k \in N \backslash J}$ such that $\tilde{u}_i \geq u_i^*$ for all $i \in J$, and $\tilde{u}_{i'} > u_{i'}^*$ and $\tilde{T}_{i'}(\tilde{a}) < \omega_{i'}(\tilde{a})$ for some $i' \in J$. Hence, $(a^*, (T_j^*)_{j \in S})$ is a CPNE in $\Gamma \backslash (T_k^*)_{k \in N \backslash S}$. By induction, (a^*, T^*) is a CPNE in Γ.$\square$

Proposition 6. In every menu auction game Γ, every allocation u^* supported by a coalition-proof Nash equilibrium is in the bidder-optimal BC-core.

Proof. Suppose not. There exists $\tilde{u} \in \overline{BCcore_\Gamma}$ supported by $(\tilde{a}, \tilde{T}) \in A \times \mathcal{T}$ such that $\tilde{u}_i \geq u_i^*$ for all $i \in N$, and $\tilde{u}_j > u_j^*$ and $\tilde{T}_j(\tilde{a}) < \omega_j(\tilde{a})$ for some $j \in N$. From Proposition 4 and Proposition 5, $\tilde{u}$ is supported by a CPNE so $\tilde{u}$ is also supported by a self-enforcing outcome. However, the allocation u^* supported by a CPNE implies that there exists no self-enforcing allocation weakly improves all bidders and strictly improves some budget-unconstrained bidders. This is a contradiction.$\square$

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Chapter 2

Profit-Maximizing Matchmaker

2.1 Introduction

In their influential paper, Shapley and Shubik (1971) introduce an assignment problem that is a transferable utility (cooperative) game in a two-sided one-to-one matching problem. Kelso and Crawford (1982) generalize the assignment model to a many-to-one setting: they allow firms to choose how many workers to hire, and they analyze the resulting market equilibrium and the core. They consider a central planning authority that matches up firms and workers and propose a price adjustment mechanism by generalizing the Gale-Shapley deferred acceptance algorithm (Gale and Shapley 1962; Roth and Sotomayor 1990). Their algorithm finds the firm-optimal stable assignment that is a market equilibrium and a core allocation. As in many centralized market clearing mechanisms successfully used in the real world, such as entry-level medical markets and school choice problems, Kelso and Crawford (1982) assume that the matchmaker is a benevolent central planner who tries to achieve a desirable allocation—a market equilibrium.

By contrast, in this paper, we consider another matching mechanism that utilizes an auctioneer (matchmaker) who chooses a matching of firms and workers that maximizes profit in an environment of heterogeneous firms and workers. Specifically, we consider a two-stage noncooperative game in a many-to-one assignment problem with a matchmaker. In the first stage, each firm proposes how much it is willing to pay workers if they are matched, and each worker proposes what salary she is willing to accept from each firm if they are matched. These proposals are made simultaneously. Then, in the second stage, the matchmaker matches up firms and workers in order to maximize profits (the sum of the differences between the offering and asking salaries from each matched firm-worker(s)). This matchmaker game can be regarded as a resource allocation mechanism with an auctioneer in a two-sided matching problem.

Recently, Milgrom (2010) proposes a framework that analyzes the effect on equilibria of restricting the message space of a game. He defines a certain "outcome closure property" on a simplification of message space, and shows that if the condition is satisfied, then every (ϵ)-Nash equilibrium in the simplified mechanism is an (ϵ)-Nash equilibrium of the original mechanism. Moreover, he illustrates the benefits of working with the simplified mechanism by noting that the set of Nash equilibria is intact by simplifying message space through adopting simple (individualized price) strategies in a combinatorial auction game, and also that the Gale-Shapley algorithm selects the same outcome even with individualized prices in the Kelso-Crawford assignment game under (gross)-substitute assumption. Thus, it is interesting to investigate the performance of using simple (individualized price) strategies in our matchmaker game, which combines a two-sided matching problem and a combinatorial auction game.

Our matchmaker game can be considered a two-sided version of a combinatorial auction game. It satisfies the outcome closure property, so a Nash equilibrium in simple (individualized price) strategies is a Nash equilibrium in general (package price) strategies. However, in contrast to Milgrom's observation on a combinatorial auction game and the Gale-Shapley algorithm, restricting the message space significantly reduces the set of Nash equilibria in our matchmaker game. In particular, all Nash equilibria in simple strategies generate zero profit for the matchmaker (Theorem 1), but Nash equilibria in general strategies may generate positive profits (Example 3). This result shows that while the simple strategy restriction excludes some of Nash equilibria in our matchmaker game, the performance of the mechanism improves with the restriction since profit for the matchmaker is a waste of resource. We also use a stronger equilibrium concept and investigate how the equilibrium outcomes differ under strategy restriction. A coalition-proof Nash equilibrium is a strategy profile that is immune to every credible coordinated change in strategies for any coalition (Bernheim, Peleg, and Whinston, 1987; Konishi, Le Breton and Weber 1999). In our matchmaker game, a coalition-proof Nash equilibrium in simple strategies is a coalition-proof Nash equilibrium in general strategies as well (Proposition 1). We show that every coalition-proof Nash equilibrium outcome in simple strategies is a stable assignment (a core allocation) (Theorem 3). However, perhaps somewhat surprisingly, in general strategies, even coalition-proof Nash equilibria may still yield positive profits to the matchmaker (Example 3). Thus, in our matchmaker game, restricting the message space to simple strategies not only reduces the information requirement but also enhances efficiency.

Applying the above theorems, we obtain results on the implementation of popular social choice correspondences in the Kelso-Crawford many-to-one assignment problem with monetary transfers. Alcalde et al. (1998) show that the stable correspondence (competitive equilibrium correspondence) is subgame-perfect-Nash-implementable by a simple two-stage game. Hayashi and Sakai (2009) characterize the stable correspondence by Nash implementation. Note that their results cannot treat the one-to-one problem or a many-to-one problem with quotas. By noting that the set of Nash equilibrium outcomes is equivalent to the set of acceptable assignments, we can show that the acceptable correspondence is Nash-implementable by our simple matchmaker game by applying Theorem 1 (Corollary 1). Theorem 2 directly shows that a stable correspondence is coalition-proof-Nash-implementable in a simple matchmaker game (Corollary 3). These results are not dependent on the presence of monetary transfers (Theorem 4) or quotas.[1]

Our matchmaker game is related to the *menu auction game* introduced by Bernheim and Whinston (1986), although the results in the literature of the menu auction game do not have much to do with ours except for the one-to-one problem. In a menu auction game, there are multiple principals (players) and an agent, and a set of actions. All players and the agent have preferences over actions, and each player offers a contribution schedule to the agent, which is a function from the action set to a monetary contribution. The agent sees the players' contribution schedules and chooses the action with the highest total payoff. We show that the class of our matchmaker games in general (package price) strategies can be embedded into that of the menu auction games (Proposition 2). In this sense, our game is related to the menu auction game. However, many important results in the literature of menu auction games have something to do with Nash equilibrium in restricted strategies: *truthful strategies* as defined in Bernheim and Whinston (1986). Unfortunately, in general,

[1]The effect of quota can be muted by setting quota equal to the size of the labor force.

truthful strategies and simple (individualized price) strategies are incompatible with each other except for a special domain of one-to-one assignment problems, and we cannot apply the results to our many-to-one assignment case. Still, our Theorem 1 implies that the one-to-one assignment problem is a new domain that satisfies the *no-rent property* introduced by Laussel and Le Breton (2001), under which many nice results hold.

The rest of the paper is organized as follows. In Section 2, the (many-to-one) Kelso-Crawford assignment problem and our matchmaker game are introduced with a few examples. Section 3 presents our main results. Section 4 provides applications of our main results to the implementation of acceptable and stable matchings and discusses the relationship of our results with menu auction games. Section 5 contains the proof of the main theorem.

2.2 The Model

2.2.1 A Many-to-One Matching Problem

We consider the Kelso-Crawford many-to-one assignment problem without imposing complementarity or substitutability of workers (Kelso and Crawford 1982). There are two disjoint finite sets of players: the set of firms F and the set of workers W. Let $N = F \cup W$. Each firm $f \in F$ has a finite **quota** q_f and each of q_f positions can hold one worker. Production technology is described by a function $Y : F \times 2^W \to \mathbb{R}_+$ such that $Y(f, W_f) \geq 0$ is the output that firm f can produce by hiring $W_f \subseteq W$ workers. We assume that $Y(f, W_f) = 0$ when $W_f = \varnothing$ or $|W_f| > q_f$ for all $f \in F$. Let $\mathcal{Y}$ be the set of all possible production technologies. Each worker $w \in W$ hired by firm f has some disutility from working d_{wf} independent of his or her position. If unemployed, then w receives zero disutility ($d_{w\phi} = 0$). We assume that $d_{wf} \geq 0$ for all $f \in F$ and all $w \in W$. Let $D = (d_{wf})_{w \in W, f \in F \cup \{\varnothing\}}$ be a disutility matrix, and let $\mathcal{D}$ be the set of all possible disutility matrices. A **many-to-one matching** $\mu : W \cup F \twoheadrightarrow W \cup F$ is a mapping such that (i) $\mu(f) \subseteq W$ and $\mu(w) \in F \cup \{\varnothing\}$ for all $f \in F$ and all $w \in W$; (ii) $|\mu(f)| \leq q_f$ for all $f \in F$; (iii) $w \in \mu(f)$ if $f = \mu(w)$; (iv) $\mu(w) = f$ for all $w \in \mu(f)$. Let $\mathcal{M}$ be the set of all matchings μ. An **efficient matching** is $\mu^* \in \arg\max_{\mu \in \mathcal{M}} \sum_{f \in F} \left[Y(f, \mu(f)) - \sum_{w \in \mu(f)} d_{wf} \right]$. We denote payoffs of firm f and worker w by v_f and u_w, respectively. Let $v = (v_f)_{f \in F}$ and $u = (u_w)_{w \in W}$ be firms' and workers' payoff vectors. A (nonwasteful) **allocation** is a list $(v, u, \mu) \in \mathbb{R}^F \times \mathbb{R}^W \times \mathcal{M}$ such that (i) $v_f = 0$ for all $f \in F$ with $\mu(f) = \varnothing$, (ii) $u_w = 0$ for all $w \in W$ with $\mu(w) = \varnothing$ and (iii) $v_f + \sum_{w \in \mu(f)} u_w = Y(f, \mu(f)) - \sum_{w \in \mu(f)} d_{wf}$ for all $f \in F$. An allocation (v, u, μ) is **efficient** if μ is an efficient matching. An allocation is **individually rational** if for all $f \in F$ and all $w \in W$, $v_f \geq 0$ and $u_w \geq 0$. An allocation is an **acceptable assignment** if (i) it is individually rational and (ii) $Y(f, W_f) - \sum_{w \in W_f} d_{wf} \leq v_f + \sum_{w \in W_f} u_w$ for all $f \in F$ and all $W_f \subset \mu(f)$. Condition (ii) of acceptability requires that firm f cannot be better off by firing some of its workers. Note that individual rationality is equivalent to acceptability in the one-to-one assignment problem, but not in the many-to-one problem. An allocation is a **stable assignment** if (i) it is individually rational and (ii) there is no pair $(f, W_f) \in F \times 2^W$ with $|W_f| \leq q_f$ such that $Y(f, W_f) - \sum_{w \in W_f} d_{wf} > v_f + \sum_{w \in W_f} u_w$. Clearly, stability requires acceptability.

2.2.2 The Matchmaker Game

Consider a mechanism by which a matchmaker matches up firms and workers under complete information. This matchmaker can be regarded as an auctioneer, or as a central planning authority who chooses a matching based on information submitted by firms and workers. In the first stage, a matchmaker asks each worker what salary she demands from each firm, and asks each firm how much it is willing to offer workers if they are matched. Thus, each worker $w \in W$ submits $s_w : F \to \mathbb{R}$ (or $s_w = (s_w(f))_{f \in F}$). However, the strategy for the firm has two possible formulations. One is a **simple strategy** (or an individualized price strategy) such that each firm $f \in F$ submits $\sigma_f : W \to \mathbb{R}$ (or $\sigma_f = (\sigma_f(w))_{w \in W}$). That is, irrespective of other workers assigned to firm f, f always pays $\sigma_f(w)$ to the matchmaker for getting worker w. The other is a **general strategy** (or a package price strategy) such that each firm $f \in F$ submits $\tilde{\sigma}_f : \mathcal{S}_f \to \mathbb{R}$ where $\mathcal{S}_f = \{W_f \subseteq W : |W_f| \leq q_f\}$. Clearly, simple strategies are special cases of general strategies. The matchmaker is allowed to take the difference between $\sigma_f(w)$ and $s_w(f)$ if she matches f and w in the case of simple strategies, and the matchmaker is allowed to take the difference between $\tilde{\sigma}_f(W_f)$ and $\sum_{w \in W_f} s_w(f)$ from matching up f and W_f in the case of general strategies. Needless to say, the matchmaker would not match a pair (f, w) if $\sigma_f(w) < s_w(f)$ in the case of a simple strategy, and would not match (f, W_f) if $\tilde{\sigma}_f(W_f) < \sum_{w \in W_f} s_w(f)$ in the case of a general strategy: the matchmaker would rather leave them unmatched.

In the second stage, using these submitted strategies, the matchmaker chooses a matching $\mu \in \mathcal{M}$. This game is called a **matchmaker game**, and the matching games with firms' simple and general strategies are called **simple** and **general matchmaker games**, respectively.

In a *simple matchmaker game*, the matchmaker has a payoff function $U : \mathbb{R}^{F\times W} \times \mathbb{R}^{W\times F} \times \mathcal{M} \to \mathbb{R}$ with $U(\sigma, s, \mu) = \sum_{f\in F} \sum_{w\in\mu(f)} (\sigma_f(w) - s_w(f))$. Let the set $M(\sigma, s) \subset \mathcal{M}$ be $M(\sigma, s) \equiv \text{argmax}_{\mu\in\mathcal{M}} U(\sigma, s, \mu)$. Each firm f, worker w, and the matchmaker obtain the following payoffs under $\mu \in M(\sigma, s)$: $v_f(\sigma, s, \mu) = Y(f, \mu(f)) - \sum_{w\in\mu(f)} \sigma_f(w)$, $u_w(\sigma, s, \mu) = s_w(\mu(w)) - d_{w\mu(w)}$, and $U(\sigma, s, \mu) = \sum_{f\in F} \sum_{w\in\mu(f)} (\sigma_f(w) - s_w(f))$, respectively.

In a *general matchmaker game*, the matchmaker has a payoff function $\tilde{U} : \mathbb{R}^{\times_{f\in F}\mathcal{S}_f} \times \mathbb{R}^{W\times F} \times \mathcal{M} \to \mathbb{R}$ with $\tilde{U}(\tilde{\sigma}, s, \mu) = \sum_{f\in F} \left(\tilde{\sigma}_f(\mu(f)) - \sum_{w\in\mu(f)} s_w(f)\right)$. Let the set $\tilde{M}(\tilde{\sigma}, s) \subset \mathcal{M}$ be $\tilde{M}(\tilde{\sigma}, s) \equiv \text{argmax}_{\mu\in\mathcal{M}} \tilde{U}(\tilde{\sigma}, s, \mu)$.Each firm f, worker w, and the matchmaker obtain the following payoffs under $\mu \in \tilde{M}(\tilde{\sigma}, s)$: $\tilde{v}_f(\tilde{\sigma}, s, \mu) = Y(f, \mu(f)) - \tilde{\sigma}_f(\mu(f))$, $u_w(\tilde{\sigma}, s, \mu) = s_w(\mu(w)) - d_{w\mu(w)}$, and $\tilde{U}(\tilde{\sigma}, s, \mu) = \sum_{f\in F} \left(\tilde{\sigma}_f(\mu(f)) - \sum_{w\in\mu(f)} s_w(f)\right)$, respectively. Note that each firm f cares only about $\mu(f)$. The rest of the matching is irrelevant. Similarly, each worker w cares only about $\mu(w)$.

A list (σ^*, s^*, μ^*) is a **Nash equilibrium** in a *simple matchmaker game* if (i) $\mu^* \in M(\sigma^*, s^*)$, (ii) there is no $f \in F$ such that $\sigma_f : W \to \mathbb{R}$ and $\mu \in M(\sigma_f, \sigma^*_{-f}, s^*)$ such that $v_f(\sigma_f, \sigma^*_{-f}, s^*, \mu) > v_f(\sigma^*, s^*, \mu^*)$, and (iii) there is no $w \in W$ such that $s_w : F \to \mathbb{R}$ and $\mu \in M(\sigma^*, s_w, s^*_{-w})$ such that $u_w(\sigma^*, s_w, s^*_{-w}, \mu) > u_w(\sigma^*, s^*, \mu^*)$.[2] While a Nash equilibrium is immune to any unilateral deviation such that the deviant is strictly improved, a **(strictly) strong Nash equilibrium (SNE)** is immune to any coalitional deviation such that all members weakly are improved and at least one member is strictly improved, and a (**strictly**) **coalition-proof Nash equilibrium** (**CPNE**) is immune to any credible coalitional deviation such that all members weakly are improved and at least one member is strictly improved, where credibility

[2]Although strictly speaking the game is a two-stage game, because the second stage is a mere maximization problem by the matchmaker, we can regard the game as static (see Bernheim and Whinston 1986; Laussel and Le Breton 2001).

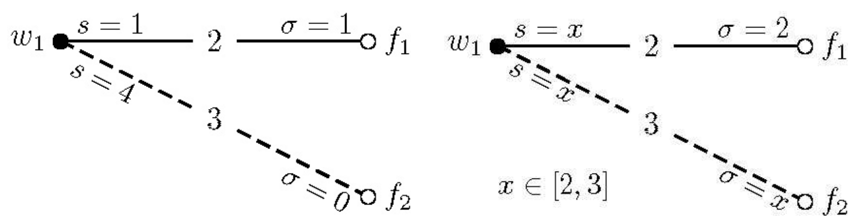
is defined recursively by fixing strategies of outsides of a coalition.[3] An **outcome** of a Nash equilibrium (σ^*, s^*, μ^*) in a *simple matchmaker game* is a list $(v, u, \mu) \in \mathbb{R}^F \times \mathbb{R}^W \times \mathcal{M}$ such that $v_f = Y(f, \mu^*(f)) - \sum_{w \in \mu^*(f)} \sigma_f^*(w)$ for all $f \in F$, $u_w = s_w^*(\mu^*(w)) - d_{w\mu^*(w)}$ for all $w \in W$ and $\mu = \mu^*$. An outcome of a strong Nash equilibrium and a coalition-proof Nash equilibrium in a *simple matchmaker game* are defined similarly. Corresponding definitions in a *general matchmaker game* are given in the same manner.

2.2.3 Examples

In this subsection, we illustrate what Nash equilibria and coalition-proof Nash equilibria look like. We start with a very simple one-to-one matching example.

Example 1. There are two firms $\{f_1, f_2\}$ and one worker $\{w_1\}$. Each firm has one position $q_{f_1} = q_{f_2} = 1$. Let $Y(f_1, \{w_1\}) = 2$, $Y(f_2, \{w_1\}) = 3$ and $d_{w_1 f_1} = d_{w_1 f_2} = 0$. Even in this simple example, there are multiple Nash equilibria with different matchings. Let $\sigma_{f_1}(w_1) = 1$ and $\sigma_{f_2}(w_1) = 0$, and $s_{w_1}(f_1) = 1$ and $s_{w_1}(f_2) = 4$. See Fig. 1 (a). Under this strategy profile, the matchmaker chooses $\mu(f_1) = w_1$ and $\mu(f_2) = \varnothing$, and makes no profit. This is a Nash equilibrium, but the resulting matching is inefficient. This inefficiency is due to a coordination failure. Firm f_2 has no incentive to hire w_1 by changing its strategy unilaterally since w_1 is asking an unreasonable salary, while worker w_1 has no incentive to try to be hired by changing her strategy unilaterally since f_2 is offering zero salary. However, if both firm f_2 and worker w_1 jointly

[3] Strictly speaking, we need to allow the matchmaker to have preferences over matchings. The standard definition of a CPNE requires that all reduced games (where the outsiders of a coalition keep their strategies fixed, and the members of the coalition play the game) belong to the same class of games. However, if outsiders make their salary offers and demands to coalition-members, then the matchmaker will have preferences over the matchings it chooses.



(a) NE can be inefficient. (b) CPNE achieves efficiency.

Figure 2.1: Illustration for Example 1.

change their strategies, then both can be better off by being matched up, thus achieving efficiency.

In contrast, let $\sigma_{f_2}(w_1) = x$ and $\sigma_{f_1}(w_1) = 2$, and $s_{w_1}(f_1) = x$ and $s_{w_1}(f_2) = x$, where $x \in [2, 3]$. See Fig. 1(b). If the matchmaker chooses $\mu'(f_2) = w_1$ and $\mu'(f_1) = \varnothing$ (indeed, unless $x = 2$, it must choose μ'), this is a coalition-proof Nash equilibrium, since there is no profitable deviation. Thus, any salary $x \in [2, 3]$ can be supported by a coalition-proof Nash equilibrium, and efficiency is achieved. Note that each of these allocations is a stable assignment.

Example 1 shows that Nash equilibria in matchmaker games can generate inefficient matchings. The matchmaker's profit is zero in all Nash equilibria. In the next example, we consider more general situations and show that the matchmaker's profit is still zero.

Example 2. There are two firms $\{f_1, f_2\}$ and two workers $\{w_1, w_2\}$. Each firm has one position $q_{f_1} = q_{f_2} = 1$. Let $Y(f_1, \{w_1\}) = Y(f_2, \{w_2\}) = 3$ and $Y(f_1, \{w_2\}) = Y(f_2, \{w_1\}) = 0$, and let $d_{w_j f_i} = 0$ for all $i, j = 1, 2$. Clearly, the efficient matching is $\mu(f_1) = w_1$, and $\mu(f_2) = w_2$. Suppose that the matchmaker is earning a positive profit in a Nash equilibrium at least from the pair $\{f_1, w_1\}$ by choosing μ, that is, $\sigma_{f_1}(w_1) > s_{w_1}(f_1)$. If $\sigma_{f_2}(w_2) = s_{w_2}(f_2)$, we have $\sigma_{f_1}(w_1) - s_{w_1}(f_1) = \sigma_{f_2}(w_1) - s_{w_1}(f_2)$ and $\sigma_{f_1}(w_1) - s_{w_1}(f_1) = \sigma_{f_1}(w_2) -$

$s_{w_2}(f_1)$ to prevent firm f_1 from offering less salary and worker w_1 from asking more salary. Then, a matching μ' with $\mu'(f_1) = w_2$, and $\mu'(f_2) = w_1$ generates a higher profit than μ. Hence, $\sigma_{f_2}(w_2) > s_{w_2}(f_2)$. Note that unless $\sigma_{f_1}(w_2) > s_{w_2}(f_1)$ or $\sigma_{f_2}(w_1) > s_{w_1}(f_2)$, f_1 can gain by reducing $\sigma_{f_1}(w_1)$ because the matchmaker would still choose μ. Without loss of generality, assume $\sigma_{f_1}(w_2) > s_{w_2}(f_1)$. Then f_1 can earn more by reducing $\sigma_{f_1}(w_1)$ and $\sigma_{f_1}(w_2)$ by the same amount without affecting the resulting matching. As a result, $\sigma_{f_1}(w_1) = s_{w_1}(f_1)$ must hold in every Nash equilibrium. Thus, in this example again, the matchmaker's profit must be zero in every Nash equilibrium. It is easy to see that the set of coalition-proof Nash equilibrium outcomes is equivalent to the set of stable assignments.

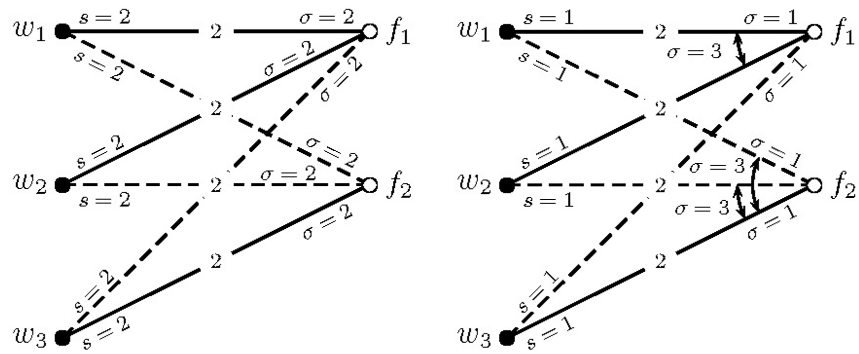
Now we consider a many-to-one problem. The following simple example illustrates a very important point: in general matchmaker games, a strong Nash equilibrium may yield a positive profit to the matchmaker.

Example 3. There are two firms $\{f_1, f_2\}$ and three workers $\{w_1, w_2, w_3\}$. All firms and workers are symmetric. Each firm has two positions $q_{f_1} = q_{f_2} = 2$. For all $i = 1, 2$ and all $j, k = 1, 2, 3$ $(j \neq k)$, $d_{w_j f_i} = 0$ and $Y(f_i, \{w_j\}) = 2$ and $Y(f_i, \{w_j, w_k\}) = 4$. In a simple matchmaker game, the wage offered to each worker is individualized, and similar arguments as above follow, since the matchmaker cares only about how much it can earn from each match of a firm with a worker. Thus, we can show that all Nash equilibria in this simple matchmaker game generate zero profit to the matchmaker. The unique coalition-proof Nash equilibrium (up to permutations) in the simple matchmaker game is (σ, s, μ) such that $\sigma_{f_i}(w_j) = 2$ and $s_{w_j}(f_i) = 2$ for all i and j, and $\mu(f_1) = \{w_1, w_2\}$ and $\mu(f_2) = \{w_3\}$. See Fig. 2(a). The salaries are pinned down owing to excess demand for workers. Note that this coalition-proof

Nash equilibrium generates a stable assignment. Clearly, there is no profit for the matchmaker in the coalition-proof equilibrium of the simple matchmaker game. From the above coalition-proof Nash equilibrium in a simple matchmaker game, let $\tilde{\sigma}_{f_i}(w_j) = 2$ and $\tilde{\sigma}_{f_i}(\{w_j, w_k\}) = 4$, and $s_{w_j}(f_i) = 2$ for all i, j, and k. This is indeed a coalition-proof Nash equilibria in this general matchmaker game. However, in the general matchmaker game, there are other coalition-proof Nash equilibria with positive profits. Consider the following strategy profile $(\tilde{\sigma}, s, \mu)$: $\tilde{\sigma}_{f_i}(\{w_j\}) = 1$ for all i and j, and $\tilde{\sigma}_{f_i}(\{w_j, w_k\}) = 3$ (if firm f_i is willing to pay 3 in total if it is matched with subset $\{w_j, w_k\}$) for all i, j, and k, and $s_{w_j}(f_i) = 1$ for all i and j. This results in $\mu(f_1) = \{w_1, w_2\}$ and $\mu(f_2) = \{w_3\}$ (up to permutations). See Fig. 2(b). This is a coalition-proof Nash equilibrium,[4] and firms are indifferent between hiring one or two workers. However, the matchmaker receives a profit of 1 from f_1. Note that firms are better off in this coalition-proof Nash equilibrium in the general matchmaker game: they obtain positive profits. Note also that Nash equilibrium may generate a positive profit in the general matchmaker game, since a coalition-proof Nash equilibrium is also a Nash equilibrium.

This example shows that unlike the one-to-one matching problem, restrictions on firms' strategy sets may affect the outcomes of a matchmaker game. A "simple" matchmaker game selects zero-profit Nash equilibria and coalition-proof Nash equilibrium equilibria from the larger sets of equilibria in the general matchmaker game.

[4]The set of CPNE would be $\tilde{\sigma}_{f_i}(w_j) = x \in [0, 2]$ for all i and j, and $\tilde{\sigma}_{f_i}(\{w_j, w_k\}) = 2 + x$ for all i and distinct j and k, and $s_{w_j}(f_i) = x$ for all i and j, and $(\mu(f_1), \mu(f_2)) \in \{(\{w_i, w_j\}, \{w_k\}), (\{w_i, w_j\}, \{w_k\})\}$ for all distinct i, j and k.



(a) Unique SNE with zero profit in simple matchmaker game.

(b) A CPNE with positive profit in general matchmaker game.

Figure 2.2: Illustration for Example 3.

In the next section, we will investigate whether the above observations hold in general.

2.3 The Results

2.3.1 Preliminaries

We first review Milgrom's recent contribution. Let (N, X, ω) be a normal-form mechanism where N is the set of players, $X = (X_i)_{i \in N}$ is the set of strategy profiles, Ω is the set of possible outcomes where Ω is endowed with a topology, and $\omega : X \to \Omega$ is an outcome function. A normal-form game can be constructed given utility functions $u = (u_i)_{i \in N}$ where $u_i : \Omega \to \mathbb{R}$. A normal-form mechanism $(N, \hat{X}, \omega|_{\hat{X}})$ is a **simplification** of (N, X, ω) if $\hat{X} \subseteq X$. A simplification $(N, \hat{X}, \omega|_{\hat{X}})$ of (N, X, ω) has the **outcome closure property** if, for every i, every $\hat{x}_{-i} \in \hat{X}_{-i}$, every $x_i \in X_i$, and every open neighborhood O of $\omega(x_i, \hat{x}_{-i})$, there exists $\hat{x}_i \in \hat{X}_i$ such that $\omega(\hat{x}) \in O$. The simplification $(N, \hat{X}, \omega|_{\hat{X}})$ of (N, X, ω) is **tight** if, for every continuous function u and every $\varepsilon \geq 0$, every pure strategy profile x that is an ε-Nash equilibrium of $(N, \hat{X}, \omega|_{\hat{X}})$ is also an ε-Nash equilibrium of (N, X, ω). Milgrom (2010) shows the following simplification theorem.

Theorem 0. (Milgrom 2010) Any simplification $(N, \hat{X}, \omega|_{\hat{X}})$ of (N, X, ω) that has the outcome closure property is tight.

In a matchmaker game, the set of players is the set of firms and workers, $N = F \cup W$. For player $w \in W$, a strategy is $s_w : F \to \mathbb{R}$ and X_w is a collection of all strategies for w. For player $f \in F$, a (general) strategy is $\tilde{\sigma}_f : \mathcal{S}_f \to \mathbb{R}$, and X_f is the collection of all possible general strategies for f. The restriction $\hat{X}_f$ is the set of all general strategies that can be created from simple strategies.[5] The set of possible outcomes is denoted by $\Omega = \mathbb{R}^F \times \mathbb{R}^W \times \mathcal{M}$, where $\omega = (v, u, \mu) \in \Omega$, and an outcome function is $\omega : X \to \Omega$ such that $v_f = Y(f, \mu(f)) - \tilde{\sigma}_f(\mu(f))$ for all $f \in F$, $u_w = s_w(\mu(w)) - d_{w\mu(w)}$ for all $w \in W$, and $\mu \in \tilde{M}(\tilde{\sigma}, s)$. A simple matchmaker game is a simplification of a general matchmaker game, and the simplification satisfies the outcome closure property. Then the following observation immediately emerges by selecting the appropriate outcome function to support each Nash equilibrium:[6]

Observation. Every Nash equilibrium in a simple matchmaker game is a Nash equilibrium in the general matchmaker game.

[5] Setting $\tilde{\sigma}_f(S) = \sum_{w \in S} \sigma_f(w)$ for all $S \subseteq W$ with $S \neq \emptyset$, we can create a general strategy $\tilde{\sigma}_f : \mathcal{S}_f \to \mathbb{R}$ from a simple strategy $\sigma_f : W \to \mathbb{R}$.

[6] In a Nash equilibrium of a (simple and general) matchmaker game, the matchmaker is indifferent among at least two actions.

2.3.2 Main Result

Given the above Observation, it makes sense to analyze the Nash equilibrium in the simple matchmaker game. The first and most important result of this paper is as follows.

Theorem 1. In every simple matchmaker game, the matchmaker's profit is zero in every Nash equilibrium.

The proof of this theorem is complicated, and we defer it to the last section of the paper. If there is only one firm, it is not surprising that the firm can reduce wages without changing the matching if the matchmaker is getting a positive profit as in Example 1. However, if multiple firms are competing for workers, a firm's reducing its wage offers may not improve the firm's payoff, since the matchmaker may match other firms with workers whom the firm could have had if it had not reduced wages. Thus the result of Theorem 1 is more subtle than the argument that leaving the profit margin to the matchmaker is never a best response. To provide some intuition behind this result, we briefly describe the proof for a special case of a one-to-one assignment problem: $q_f = 1$ for all $f \in F$ (the formal proof is postponed to Section 5). Suppose that there is a Nash equilibrium with a positive profit, and let (σ, s, μ) be a Nash equilibrium with the highest profit. Pick a firm-worker pair f and w such that $\mu(f) = w$ and $\sigma_f(w) > s_w(f)$. Since μ is the outcome of a Nash equilibrium, firm f and worker w do not deviate for the fear of μ not being chosen. Since the matchmaker is profit-maximizing, if f deviates, the matchmaker chooses a matching $\mu' \neq \mu$ with $\mu'(w) \neq f$ that generates exactly the same profit as μ does (see Corollary 4 in Section 5 for the formal statement). Similarly, if w deviates, the matchmaker chooses matching $\mu'' \neq \mu$ with $\mu''(f) \neq w$ that generates exactly the same profit as μ does. By combining μ' and μ'' with some adjustments we can create a new matching without a match between f and w, which generates an even higher profit than μ. Then the matchmaker can improve its profit by choosing the new matching, which contradicts that (σ, s, μ) is a Nash equilibrium. Thus, even with interactions among firm-worker pairs, leaving the profit margin to the matchmaker cannot be supported by a Nash equilibrium of a simple matchmaker game.

The result of Theorem 1 provides a stark contrast with Nash equilibria in the general matchmaker game. Example 3 in the previous section showed that there might be Nash equilibria that give a positive profit to the matchmaker. Thus, unlike the Nash equilibrium in a (one-sided) combinatorial auction game and the Gale-Shapley algorithm in the two-sided matching problem, restricting the message space to simple strategies has a real impact on the set of Nash equilibria. Is this result bad news for simple strategies? We think that it is actually good news. In a resource allocation problem, a positive profit for the matchmaker (or the auctioneer) is a waste of resources. If a restriction in message space eliminates profit made by the matchmaker, thus achieving a nonwasteful allocation, then it should be considered a desirable property.

Although this result is somewhat surprising by itself, it also turns out to be quite useful when we consider a refinement of Nash equilibrium. With the zero profit result for Nash equilibrium, we will have coalition-proof Nash versions of Observation.

Proposition 1. Every coalition-proof Nash equilibrium in a simple matchmaker game is a coalition-proof Nash equilibrium in the general matchmaker game.

Proof. Suppose that a coalition-proof Nash equilibrium in a simple matchmaker game is not immune to a credible coalitional deviation with general strategies. Then, at least one player improves by the credible deviation. Suppose that firm f is such a player. Then, after the deviation, f is matched with a subset of workers W_f. Clearly, all $w \in W_f$ cannot be made worse off by the deviation. That is, $Y(f, W_f) - \sum_{w \in W_f} d_{wf}$ must achieve a higher value than the sum of their Nash equilibrium payoffs. However, by Theorem 1, every Nash (and thus coalition-proof Nash) equilibrium leaves zero profit to the matchmaker. Thus, all output is divided up by firms and workers, and the coalition-proof Nash equilibrium outcome is a nonwasteful allocation. Since $Y(f, W_f)$ would improve over the allocation, the original matching is not a stable assignment. This is a contradiction (see Theorem 3 below). The same logic applies to the case where no firm is strictly better off (but there is a worker who is better off).□

That is, "simple" strategies refine the Nash equilibrium and the coalition-proof Nash equilibrium in a general matchmaker game. From previous examples, it is easy to observe that every Nash equilibrium outcome is an acceptable assignment.

Theorem 2. In every many-to-one assignment problem, the set of Nash equilibrium outcomes in the simple matchmaker game is equivalent to the set of acceptable assignments.

Proof. Let (v, u, μ) be the outcome of a Nash equilibrium (σ, s, μ). It is clearly individually rational, as negative payoffs can be avoided. Suppose for firm f there exists some $C \subset \mu(f)$ such that $Y(f, C) - \sum_{w \in C} d_{wf} > v_f + \sum_{w \in C} u_w$. From Theorem 1, that the matchmaker earns zero profit implies $v_f + \sum_{w \in \mu(f)} u_w = Y(f, \mu(f)) - \sum_{w \in \mu(f)} d_{wf}$ and $\sigma_f(w) = s_w(f) = u_w + d_{wf}$ for all $w \in \mu(f)$. Consider $\sigma'_f(w) = \sigma_f(w) + \varepsilon$ for all $w \in C$ and $\sigma'_f(w) = 0$ for all $w \notin C$, where $\varepsilon > 0$ satisfies $\varepsilon < \frac{1}{|C|}\Big[Y(f, C) - Y(f, \mu(f)) + \sum_{w \in \mu(f) \setminus C}(u_w + d_{wf})\Big]$. The matchmaker can make a positive profit by matching f and C. Hence, (σ, s, μ) cannot be a Nash equilibrium. Thus, a Nash equilibrium outcome is an acceptable assignment.

Consider an acceptable assignment (v, u, μ). For every matched firm f, consider for all $w \in \mu(f)$, $\sigma_f(w) = s_w(f) = u_w + d_{wf}$, and for all $w' \notin \mu(f)$, $\sigma_f(w') = 0$ and $s_{w'}(f)$ is prohibitively high. For each single firm, let its salary offer be zero for all workers, and for each single worker, let her salary demand be at a prohibitively high level. It is easy to see (σ, s, μ) is a Nash equilibrium.$\square$

We notice in Example 3 that if a Nash equilibrium is refined by a coalition-proof Nash equilibrium, then a stable assignment is achieved. The next theorem shows that this is not a coincidence. Using Theorem 1, we obtain the following.

Theorem 3. In every many-to-one assignment problem, the set of coalition-proof Nash equilibrium in the simple matchmaker game is equivalent to the set of stable assignments.

Proof. From Theorem 1, the matchmaker earns zero profit in every Nash equilibria, hence earns zero profit in every coalition-proof Nash equilibrium. Let (v, u, μ) be a coalition-proof Nash equilibrium outcome, and suppose that it is not a stable assignment. Then, there is a pair $(f, W_f) \in F \times 2^W$ with

$|W_f| \le q_f$ such that $Y(f, W_f) - \sum_{w \in W_f} d_{wf} > v_f + \sum_{w \in W_f} u_w$. Consider $\sigma'_f(w) = u_w + d_{wf} + Y(f, W_f) - \sum_{w \in W_f} d_{wf} - \left(v_f + \sum_{w \in W_f} u_w\right)$ for all $w \in W_f$ and $\sigma'_f(w') = 0$ for all $w' \notin W_f$, and $s'_w(f) = u_w + d_{wf}$ for all $w \in W_f$ and $s'_{w'}(f)$ is prohibitively high for all $w' \notin W_f$. If this deviation is credible, then (v, u, μ) cannot be a coalition-proof Nash equilibrium outcome. Hence, this deviation is not credible because there exists $C \subset \{f\} \cup W_f$ can credibly further deviate. However, this deviation by C is feasible at the original game so that (v, u, μ) cannot be a coalition-proof Nash equilibrium outcome. Thus, a coalition-proof equilibrium outcome is a stable assignment.

Now, let (v, u, μ) be a stable assignment. Consider the following strategy. For all matched firms $f \in F$ and all $w \in \mu(f)$, $\sigma_f(w) = s_w(f) = u_w + d_{wf}$ and $\sigma_f(w') = 0$ and $s_{w'}(f)$ is prohibitively high for $w' \notin \mu(f)$. For each single firm, let its salary offer be zero for all workers, and for each single worker, let her salary demand be at a prohibitively high level. The matchmaker chooses μ and gets zero profit. Given the strategy (σ, s), the matchmaker would create a new match only when a pair $(f', W_{f'}) \in F \times 2^W$ with $|W_{f'}| \le q_{f'}$ provides a positive profit. However, by the definition of a stable assignment, there is no pair $(f'', W_{f''}) \in F \times 2^W$ with $|W_{f''}| \le q_{f''}$ such that $Y(f'', W_{f''}) - \sum_{w \in W_{f''}} d_{wf''} > v_{f''} + \sum_{w \in W_{f''}} u_w$. Thus, there is no subset of players who agree to offer a positive profit to the matchmaker to create a new matching. Therefore, a stable assignment is supportable by a coalition-proof Nash equilibrium. $\square$

From Example 3 in the previous section, we know that some coalition-proof Nash equilibria in a general matchmaker game leave positive profits to the matchmaker, which implies that some coalition-proof Nash outcomes are not nonwasteful allocations. This implies that in a general matchmaker game, the outcomes of coalition-proof Nash equilibria are not necessarily stable assignments.[7] Thus, using simple strategies refines the set of coalition-proof Nash equilibria, and this refinement selects socially desirable allocations: non-wasteful allocations. We conclude that in matchmaker games, restricting the strategy space to simple ones is socially beneficial.

2.4 Discussion

In this section, we discuss the issues of implementation in matching problems. We then discuss the relationship between our matchmaker games and the menu auction games in Bernheim and Whinston (1986).

2.4.1 Implementation

Here we discuss the implementation of popular social choice correspondences by using our matchmaker games. We then show how our results can be connected with the literature on matching problems without money. Let us first introduce some notation. A mapping $\varphi : \mathcal{Y} \times \mathcal{D} \twoheadrightarrow \mathbb{R}^{F \cup W} \times \mathcal{M}$ is a **social choice correspondence** if $\varphi(Y, D) \neq \varnothing$ for all $(Y, D) \in \mathcal{Y} \times \mathcal{D}$. An **individually rational correspondence** $\varphi^{IR} : \mathcal{Y} \times \mathcal{D} \twoheadrightarrow \mathbb{R}^{F \cup W} \times \mathcal{M}$ is a social choice correspondence such that $\varphi^{IR}(Y, D) \subset \mathbb{R}^{F \cup W} \times \mathcal{M}$ is the set of all individually rational allocations (v, u, μ) for (Y, D). An **acceptable correspondence** $\varphi^{A} : \mathcal{Y} \times \mathcal{D} \twoheadrightarrow \mathbb{R}^{F \cup W} \times \mathcal{M}$ is a social choice correspondence such that $\varphi^{A}(Y, D) \subset \mathbb{R}^{F \cup W} \times \mathcal{M}$ is the set of all acceptable allocations (v, u, μ) for (Y, D). A **stable correspondence** $\varphi^{S} : \mathcal{Y} \times \mathcal{D} \twoheadrightarrow \mathbb{R}^{F \cup W} \times \mathcal{M}$ is a social

[7] However, we can say that the resulting matching of a coalition-proof Nash equilibrium is always efficient even in a general matchmaker game by using results from Bernheim and Whinston (1986: see Section 4.2).

choice correspondence such that $\varphi^S(Y, D) \subset \mathbb{R}^{F \cup W} \times \mathcal{M}$ is the set of all stable assignments (v, u, μ) for (Y, D).

By Theorem 2, we know that the set of Nash equilibrium outcomes and the set of acceptable assignments are equivalent. Thus, we have the following implementation result.

Corollary 1. In every many-to-one assignment problem, the acceptable correspondence $\varphi^A : \mathcal{Y} \times \mathcal{D} \twoheadrightarrow \mathbb{R}^{F \cup W} \times \mathcal{M}$ is implemented by the simple matchmaker game in Nash equilibria.

In the one-to-one matching problem, the acceptable allocations and individual rational allocations are the same, and there is no difference between simple and general strategies. Thus, the above corollary implies the following.

Corollary 2. In every one-to-one assignment problem, the individually rational correspondence $\varphi^{IR} : \mathcal{Y} \times \mathcal{D} \twoheadrightarrow \mathbb{R}^{F \cup W} \times \mathcal{M}$ is implemented by the matchmaker game in Nash equilibria.

Theorem 3 directly implies the following.

Corollary 3. In every many-to-one assignment problem, if workers are gross substitutes for each firm, then the stable correspondence $\varphi^S : \mathcal{Y} \times \mathcal{D} \twoheadrightarrow \mathbb{R}^{F \cup W} \times \mathcal{M}$ is implemented by the simple matchmaker game in coalition-proof Nash equilibria.

Without the gross substitutability assumption, φ^S may be empty valued. This is why we require the assumption. Note that Corollaries 1 and 2 are not affected by the presence of quotas. Hayashi and Sakai (2009) characterize the stable correspondence by Nash implementation. Note that their results cannot treat the one-to-one problem or a many-to-one problem with quotas.

Finally, we connect our results with the implementation literature in a matching problem without money: a many-to-one assignment problem when salaries between each firm and worker are fixed exogenously.[8] Roth (1985) and Shin and Suh (1996) show that under any stable mechanism, the individually rational (acceptable matching, in our definition) correspondence and the stable correspondence are implemented in Nash and strong Nash equilibria, respectively.[9]

Our simple matchmaker game can generate similar results. Suppose for each firm f and each worker w, the salary has been fixed at x_{fw}. Then if firm f hires $W_f \subseteq W$ workers, the payoff for f is $Y(f, W_f) - \sum_{w \in W_f} x_{fw}$. Similarly, if worker w works for firm f, the payoff for w would be $x_{fw} - d_{wf}$. Firms without any workers pay no salary, and unemployed workers receive no salary, so that being unmatched would still result in a payoff of 0. Under this setting, it is easy to see that the definitions in Section 2.1 can be expressed in similar fashion in models of matching without money. Since salaries are fixed here, a firm's offer and a worker's demand are considered as an additional monetary transfer. The matchmaker takes the difference between these two bids. For simplicity, we assume preference orderings are *strict*. Firm f's preference $\succ_f$ is a linear ordering over subsets of workers $\mathcal{S}_f$, while worker w's preference $\succ_w$ is a linear ordering over firms F. An **NTU matching problem** is a list $\{F, W, (\succ_f)_{f \in F}, (\succ_w)_{w \in W}\}$. A **many-to-one matching** $\mu : W \cup F \twoheadrightarrow W \cup F$ is a mapping such that (i) $\mu(f) \subseteq W$ and $\mu(w) \in F \cup \{\varnothing\}$ for all $f \in F$ and

[8]See, say, Chapters 5 and 6.1 in Roth and Sotomayor (1990).

[9]Sonmez (1997) generalizes these results to the class of all efficient and individually rational mechanisms. The results by Suh and Shin (1996) and Sonmez (1997) are on one-to-one matching problems.

for all $w \in W$; (ii) $|\mu(f)| \leq q_f$ for all $f \in F$; (iii) $w \in \mu(f)$ if $f = \mu(w)$; (iv) $\mu(w) = f$ for all $w \in \mu(f)$. A matching μ is **individually rational** if $\mu(f) \succeq_f \varnothing$ for all $f \in F$ and $\mu(w) \succeq_w \varnothing$ for all $w \in W$. A matching μ is **acceptable** if it is individually rational and $\mu(f) \succ_f C$ for all $C \subsetneqq \mu(f)$. A matching μ is **stable** if there is no pair $(f, W_f) \in F \times 2^W$ with $|W_f| \leq q_f$ such that $W_f \succ_f \mu(f)$ and $f \succeq_w \mu(w)$ for all $w \in W_f$.[10] Let $Ch_f : 2^W \to \mathcal{S}_f$ be firm f's choice function such that $Ch_f(C) = \{S \subseteq C : S \in \mathcal{S}_f$ and $S \succeq_f S'$ for all $S' \subseteq C$ with $S' \in \mathcal{S}_f\}$. Firms' preferences are **substitutable** if for all $f \in F$, all $C \in 2^W$, and all $w \in Ch_f(C)$, $Ch_f(C)\backslash\{w\} \subseteq Ch_f(C\backslash\{w\})$ holds.

We restrict available monetary transfers by firms and workers to the set $\{-L, 0, K\}$, where $L > \max_{w \in W} \max_{f \in F} (x_{fw} - d_{wf})$ and $K > \max_{f \in F} \max_{W_f \subseteq W}(Y(f, W_f) - \sum_{w \in W_f} x_{fw})$. Each firm's offer will be chosen from the set $\{-L, 0\}$, since for any firm K is an amount of money that is not worthwhile to pay to any worker. Similarly, each worker's request will be chosen from $\{0, K\}$, since for any worker $-L$ is an amount of money that is not worthwhile to request from any firm. We assume the following **tie-breaking rule**: the matchmaker matches up a pair of a firm and a worker if she is indifferent between matching them up or not.[11] What remains is exactly the same as a simple matchmaker game. Call this game a **simple NTU matchmaker game**. We can show the following result.

Theorem 4. In every many-to-one matching problem without transfer, if firms' preferences are substitutable, then the set of Nash equilibrium matchings in the simple NTU matchmaker game is equivalent to the set of acceptable

[10] With strict preferences, this definition is the same as requiring no (f, W_f) such that firm f and all workers in W_f are weakly better off and at least one of them is strictly better off.

[11] This tie-breaking rule is sufficient to pin down the Nash equilibrium under strict preferences. However, if indifference in preferences is allowed, more careful treatment is needed in the NTU setting. See Ko (2011) for further discussion.

matchings, and the set of coalition-proof Nash equilibrium matchings in the simple NTU matchmaker game is equivalent to the set of stable matchings.

Proof. First, we show that a Nash equilibrium matching is individually rational and acceptable. A Nash equilibrium matching is individually rational because, by construction, every worker will not be matched up with a firm if she requests $K > 0$ from it, and every firm will not be matched with a worker if it offers $-L < 0$ to her. This implies that in every Nash equilibrium the matchmaker earns zero profit. We can show that for every Nash equilibrium $((\sigma, s), \mu)$, the matching μ is an acceptable matching. Suppose not. Then, there exist a firm f and a subset of workers $C \subsetneq \mu(f)$ such that $C \succ_f \mu(f)$. However, firm f can improve its payoff by switching its strategy to σ'_f such that $\sigma'_f(w) = 0$ if $w \in C$ and $\sigma'_f(w) = -L$ if $w \notin C$, which is a contradiction.

Second, an acceptable matching can be implemented by a Nash equilibrium. Consider an acceptable matching μ. For each matched firm f, consider for all $w \in \mu(f)$, $\sigma_f(w) = s_w(f) = 0$, and for all $w' \notin \mu(f)$, $\sigma_f(w') = -L$ and $s_{w'}(f) = K$. Given the tie-breaking rule by the matchmaker, the matching μ is chosen given the strategy profile (σ, s). This is a Nash equilibrium because all unmatched pairs would never be matched up by choosing other strategies.

Third, we show that a coalition-proof Nash equilibrium matching is stable. Let (σ, s, μ) be a coalition-proof Nash equilibrium. Suppose it is not a stable matching. Then there is a pair $(f, W_f) \in F \times 2^W$ with $|W_f| \leq q_f$ such that $W_f \succ_f \mu(f)$ and $f \succ_w \mu(w)$ for all $w \in W_f$ (strict preference). Consider a deviation by (f, W_f) such that (i) $\sigma'_f(w) = 0$ for all $w \in W_f$ and $\sigma'_f(w) = -L$, otherwise, and (ii) for all $w \in W_f$, $s'_w(f) = 0$ and $s'_w(f') = K$ for $f' \neq f$. Since the matchmaker would still make zero profit by matching (f, W_f) and no player in $\{f\} \cup W_f$ can be matched with outsiders, the matchmaker matches them up by the tie-breaking rule. If it is credible deviation, then $((\sigma, s), \mu)$ cannot be a coalition-proof Nash equilibrium. Hence, it is not a credible deviation because there is a coalition $C \subset \{f\} \cup W_f$ that can credibly further deviate. However, this deviation by C is feasible at the original game so that (σ, s, μ) cannot be a coalition-proof Nash equilibrium.

Finally, we show that a stable matching can be implemented by a coalition-proof Nash equilibrium. Let μ be a stable matching. Consider the following strategy. For each matched firm f, consider a strategy profile (σ, s) such that $\sigma_f(w) = s_w(f) = 0$ if and only if $w \in \mu(f)$. Given the tie-breaking rule, matching μ is chosen by the matchmaker. Since this is a stable matching, it is immune to coalitional deviations, which implies that (σ, s, μ) is a coalition-proof Nash equilibrium.$\square$

2.4.2 Relationship with Menu Auction Games

A menu auction game is a complete information multi-principal-one-agent game, introduced by Bernheim and Whinston (1986). The agent is going to choose an action, which will affect her own payoff as well as the payoffs to principals. Principals can affect the agent's decision by offering a menu of side payments: a side payment schedule for each possible action. The agent maximizes the sum of her own utility and side payments from the principals when choosing an action. We can consider our matchmaker's problem as a menu auction game by interpreting a matching μ as an action, and letting the matchmaker be intrinsically indifferent over μ (except for side payments).

A **menu auction problem** $\Gamma \equiv \left\{A, (V_k)_{k \in N \cup \{0\}}\right\}$ is described by $(N+2)$ tuples where A is the set of actions, $V_k : A \to \mathbb{R}$ is k's (quasi-linear) payoff function, 0 denotes the agent, and N is the set of principals. In the extensive form of the game, the principals simultaneously offer contingent payment schedules to the agent, who subsequently chooses an action that maximizes her total payoff. A strategy for each principal $k \in N$ is a function $T_k : A \to [b_k, \infty)$, which is a monetary reward (or punishment) of $T_k(a)$ to the agent for selecting a, where b_k is the lower bound for payment from principal k. For each action a, principal k receives a net payoff: $U_k(a, T) = V_k(a) - T_k(a)$, where $T = (T_{k'})_{k' \in N}$ is a strategy profile. The set of all possible strategies for principal k is denoted by $\mathcal{T}_k$. The agent chooses an action that maximizes her total payoff: the agent selects an action in the set $M(T)$, where $M(T) \equiv \underset{a \in A}{\operatorname{argmax}} \left[V_0(a) + \sum_{k \in N} T_k(a) \right]$.

A **menu auction game** $(\Gamma, \mathcal{T})$ is a pair consisting of a menu auction problem Γ and a set of strategies for all principals $\mathcal{T} = (\mathcal{T}_k)_{k \in N}$. This menu auction game is merely a game among principals, although, strictly speaking, a tie-breaking rule among $M(T)$ needs to be specified for the agent.

Let $\mathcal{T}_k^I \equiv \{T_k \in \mathcal{T}_k : T_k(a) = T_k(a') \text{ for all } a, a' \in A \text{ with } V_k(a) = V_k(a')\}$ be the restricted domain of strategies that requires principal k must bid the same amount for all actions among which principal k is indifferent. If all principals' strategy spaces belong to this domain, then we say that the principals' strategy spaces belong to the set of strategy spaces $\mathcal{T}^I = \left(\mathcal{T}_k^I\right)_{k \in N}$.[12]

An outcome of a menu auction game $(\Gamma, \mathcal{T})$ is (a, T). An outcome (a^*, T^*) is a **Nash equilibrium** if $a^* \in M(T^*)$ and there is no $k \in N$ such that $T_k : A \to [b_k, \infty)$ and $a \in M\left(T_k, T_{-k}^*\right)$ such that $U_k\left(a, T_k, T_{-k}^*\right) > U_k(a^*, T^*)$. However, the set of Nash equilibria in a menu auction game is quite large owing to coordination problems. So, Bernheim and Whinston (1986) propose

[12] Although this restriction is needed for the formal statement of Proposition 1, the set of Nash equilibrium payoffs with the restriction is the same as the set of Nash equilibrium payoffs without the restriction.

a refinement of Nash equilibrium by using what they call "truthful strategies." A strategy T_k is **truthful relative to** $\bar{a}$ if and only if for all $a \in A$ either (i) $U_k(a, T) = U_k(\bar{a}, T)$ or (ii) $U_k(a, T) < U_k(\bar{a}, T)$ and $T_k(a) = b_k$. Clearly, truthful strategies belong to the domain $\mathcal{T}^I$. An outcome (a^*, T^*) is a **truthful Nash equilibrium (TNE)** if and only if it is a Nash equilibrium, and T_k^* is truthful relative to a^* for all $k \in N$.

It is clear that if workers are objects (with no preferences), and if firms are bidding on workers, then we can easily formulate a combinatorial auction game by this menu auction.[13] In the following, we show that our *general* matchmaker game can also be embedded in the class of menu auction games by reinterpreting players' strategies. In a general matchmaker game, firm f's strategy $\tilde{\sigma}_f : \mathcal{S}_f \to \mathbb{R}_+$ is **truthful relative to** W_f if and only if for all $S \in \mathcal{S}_f$ either (i) $Y(f, S) - \tilde{\sigma}_f(S) = Y(f, W_f) - \tilde{\sigma}_f(W_f)$ or (ii) $Y(f, S) - \tilde{\sigma}_f(S) < Y(f, W_f) - \tilde{\sigma}_f(W_f)$ and $\tilde{\sigma}_f(S) = 0$.

Proposition 2. A general matchmaker game can be embedded in the class of menu auction games with strategy space $\mathcal{T}^I$. A strategy in a general matchmaker game is truthful if and only if the corresponding strategy is a truthful strategy in the corresponding menu auction game.

Proof. Let the matchmaker be the agent, and firms and workers be principals. Let $\mathcal{M}$ be the set of actions A. Firm f receives monetary payoff $V_f : \mathcal{M} \to \mathbb{R}$ where $V_f(\mu) \equiv Y(f, \mu(f))$, worker w receives monetary payoff $V_w : \mathcal{M} \to \mathbb{R}$ with $V_w(\mu) \equiv -d_{w\mu(w)}$, and the matchmaker's (denoted by 0) monetary payoff is $V_0(\mu) = 0$ for all $\mu \in \mathcal{M}$.[14]

[13] Milgrom (2004) discusses menu auction games in the context of a combinatorial auction problem.

[14] We normalize $V_0(\mu) = 0$ because the matchmaker has no preferences over the matchings themselves.

Under $\mathcal{T}^I$, principals are able to choose any contribution menu over potential partners but not over the entire matching. A strategy for firm f that is generated from σ_f is a function $T_f : \mathcal{M} \to \mathbb{R}_+$, where $T_f(\mu) \equiv \sigma_f(\mu(f))$. A strategy for worker w that is generated from s_w is a function $T_w : \mathcal{M} \to \mathbb{R}_-$, where $T_w(\mu) \equiv -s_w(\mu(w))$. We can set a lower bound for the value for $T_w(\mu)$ without losing anything, since worker w would not be matched anyway, if $T_w(\mu) < -Y(f, \mu(f))$ holds. Thus, we assume that for each $k \in N = W \cup F$, there is a lower bound b_k: $T_k(\mu) \geq b_k$ that must be satisfied for all $k \in N$. Thus, a matchmaker game can be represented as a menu auction game. Clearly, a truthful strategy $\tilde{\sigma}_f$ or s_w trivially can be extended to a truthful strategy T_k, and vice versa. This completes the proof.□

Remark. Note that in a one-to-one assignment problem, the general strategy and the simple strategy are equivalent. Thus, Proposition 2 together with Theorem 1 implies that the agent earns zero rent in every Nash equilibrium in a menu auction game that is generated from a matchmaker game in a one-to-one assignment problem.

Laussel and Le Breton (2001) define a menu auction game as possessing the **no-rent property** if and only if all truthful Nash equilibrium (TNE) outcomes leave no profit to the agent. They prove that if a cooperative game from a menu auction game Γ is convex,[15] then Γ possesses the no-rent property. However, although convexity is satisfied in interesting classes of menu auction games such as the public good provision game, in our assignment problem convexity is clearly not satisfied.[16] Moreover, the following example shows that even if

[15] A system $(v(S))_{S \subseteq N}$ is **convex** if and only if for all $S, T \subseteq N$, $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ holds.

[16] For example, imagine $N = \{f_1, w_1, w_2\}$ with $y_{11} = y_{12} = 1$. Letting $S = \{f_1, w_1\}$ and $T = \{f_1, w_2\}$, we can see a violation of convexity.

convexity holds, there exists a Nash equilibrium such that the agent earns a positive profit.

Example 4 (discrete public good provision). Consider a public good provision problem with two principals (consumers) $N = \{1, 2\}$ and an agent (the government) with two actions $A = \{a_1, a_2\}$. Actions a_2 and a_1 are regarded as provision and no provision of a discrete public good. Consumers prefer a_2 to a_1 but a_2 is more costly for the government: $V_i(a_1) = 0$ and $V_i(a_2) = 5$ for $i = 1, 2$ and $V_0(a_1) = 0$ and $V_0(a_2) = -1$ (public good provision cost is 1). This creates a transferrable utility cooperative game (N, v) such that $v(\{1, 2\}) = 9$, $v(\{1\}) = v(\{2\}) = 4$, and $v(\varnothing) = 0$, where $v(S)$ is the value of coalition $S \subseteq N$. This is a convex game, and Le Breton-Laussel's no-rent property holds. Consider $T_1(a_1) = 2$, $T_1(a_2) = T_2(a_1) = 0$, and $T_2(a_2) = 3$. Then (a_2, T) is a Nash equilibrium where the agent earns a positive profit. However, the set of truthful Nash equilibria is $\{(a_2, \tilde{T}) : \tilde{T}_1(a_1) = \tilde{T}_2(a_1) = 0$ and $\tilde{T}_1(a_2) + \tilde{T}_2(a_2) = 1\}$ since the game satisfies the no-rent property.

In contrast, in our one-to-one matchmaker game, the matchmaker always earns zero profit not only in all truthful Nash equilibria but also in all Nash equilibria. Since the simple strategy and the general strategy are the same in the one-to-one matchmaker game, Theorem 1 provides another interesting class of menu auction games that possess the no-rent property. Thus, we can extend Theorem 3 in the domain of the one-to-one matching problem.[17] However, we cannot obtain the same result by Proposition 2 (and Example 3).

[17]Bernheim and Whinston (1986) show that TNE and CPNE are equivalent in a utility space. Under the no-rent property, Laussel and Le Breton (2001) and Konishi, Le Breton, and Weber (1999) show the equivalences of CPNE and the core of underlying TU game, and CPNE and SNE, respectively.

Theorem 3'. Suppose that the matchmaker is allowed to have preferences over matchings. Then, in every one-to-one assignment problem, the sets of truthful Nash equilibrium outcomes, strong Nash equilibrium outcomes, and coalition-proof Nash equilibrium outcomes in the matchmaker game, and the set of stable assignments (the core) are all equivalent.

2.5 Proof of Theorem 1

In this section, we prove Theorem 1. First, we introduce some notation. For all $S \subseteq N$, let $C(S,\mu) \equiv \{k \in S : \mu(k) \in S \text{ and } \mu(k) \neq \varnothing\}$. That is, $C(S,\mu)$ is the set of members of S who have partners in S under matching μ (coupled). Given a strategy profile $(\sigma, s) \in \mathbb{R}^{F\times W} \times \mathbb{R}^{W\times F}$, let $R(S,\sigma,s,\mu) \equiv \sum_{f\in C(S,\mu)\cap F}(\sigma_f(\mu(f)) - s_{\mu(f)}(f))$ be the profit (rent) generated in S under μ. Let $R^*(S,\sigma,s) \equiv \max_{\mu\in\mathcal{M}} R(S,\sigma,s,\mu)$ and $A^*(S,\sigma,s) \equiv \text{argmax}_{\mu\in\mathcal{M}} R(S,\sigma,s,\mu)$ be the maximum profit generated in coalition S given firms' strategies σ and workers' strategies s, and its associated matching μ, respectively. We can characterize Nash equilibrium in an interesting way.

Proposition 3. In every simple matchmaker game, in every Nash equilibrium (σ, s, μ), (1A) for all $f \in F$ with $\mu(f) = \varnothing$, $R^*(N,\sigma,s) = R(N,\sigma,s,\mu) = R^*(N\backslash\{f\},\sigma,s)$; (1B) for all $f \in F$ with $\mu(f) \neq \emptyset$, and all $w \in \mu(f)$, there exists μ' such that (i) $\mu'(f) \subsetneqq \mu(f)\backslash\{w\}$, and (ii) $R^*(N,\sigma,s) = R(N,\sigma,s,\mu')$; and (2) for all $w \in W$, there exists μ'' such that (i) $\mu''(w) = \varnothing$, and (ii) $R^*(N,\sigma,s) = R(N,\sigma,s,\mu'') = R^*(N\backslash\{w\},\sigma,s)$.

Proof. Since (2) is a special case of (1), we focus on case (1).

Case (1A) is trivial, since we can use the same matching μ to achieve the same profit. Thus, we will work on case (1B). Clearly, if $\sigma_f(w) = s_w(f)$ for all $w \in \mu(f)$, then we can find a μ' that satisfies all three conditions: the matchmaker makes no money by matching f with workers, so she might as well cancel the matching (let $\mu'(f) = \varnothing$). Thus, let us focus on $\mu(f) \in W$ and $\sigma_f(w) > s_w(f)$ for some $w \in \mu(f)$ for the rest of the proof.

Consider $\sigma'_f(w) = \sigma_f(w) - \epsilon$, $\sigma'_f(w') = \max\{\sigma_f(w') - \epsilon, 0\}$ for all $w' \notin \mu(f)$ and $\sigma'_f(w'') = \sigma_f(w'')$ for all $w'' \in \mu(f) \setminus \{w\}$. Let $\mu' \in A^*(N, (\sigma'_f, \sigma_{-f}), s)$. By construction, $R(N, (\sigma'_f, \sigma_{-f}), s, \mu') = R(N, \sigma, s, \mu') - \epsilon |\mu'(f) \setminus \mu(f)|$ and $R(N, (\sigma'_f, \sigma_{-f}), s, \mu) = R(N, \sigma, s, \mu) - \epsilon$. By optimalities of μ and μ', we have $R(N, \sigma, s, \mu) \geq R(N, \sigma, s, \mu')$ and $R(N, \sigma'_f, \sigma_{-f}, \mu') \geq R(N, (\sigma'_f, \sigma_{-f}), \mu)$. Since $|\mu'(f) \setminus \mu(f)| > 1$ leads to a contradiction, either $|\mu'(f) \setminus \mu(f)| = 1$ or $|\mu'(f) \setminus \mu(f)| = 0$. Suppose $|\mu'(f) \setminus \mu(f)| = 1$. This implies $R(N, (\sigma'_f, \sigma_{-f}), s, \mu') = R(N, (\sigma'_f, \sigma_{-f}), s, \mu)$. However, if this is the case, then firm f can improve its payoff by $\epsilon > 0$ by choosing σ''_f such that $\sigma''_f(w) = \sigma_f(w) - \epsilon$, $\sigma''_f(w') = 0$ for all $w' \notin \mu(f)$ and $\sigma''_f(w'') = \sigma_f(w'')$ for all $w'' \notin \mu(f) \setminus \{w\}$ as the matchmaker is forced to choose μ. This is a contradiction. Hence, we have $|\mu'(f) \setminus \mu(f)| = 0$ or $\mu'(f) \subseteqq \mu(f)$. Hence, $R(N, \sigma, s, \mu') = R(N, (\sigma'_f, \sigma_{-f}), s, \mu')$.

(i) Suppose $w \in \mu'(f)$. By construction, $R(\{f, \mu'(f)\}, \sigma, s, \mu') > R(\{f, \mu'(f)\}, (\sigma'_f, \sigma_{-f}), s, \mu')$ and $R(N \setminus \{f, \mu'(f)\}, \sigma, s, \mu') = R(N \setminus \{f, \mu'(f)\}, \sigma'_f, \sigma_{-f}, s, \mu')$. Since $R(N, \sigma, s, \mu') = R(\{f, \mu'(f)\}, \sigma, s, \mu') + R(N \setminus \{f, \mu'(f)\}, \sigma, s, \mu')$ and $R(N, (\sigma'_f, \sigma_{-f}), s, \mu') = R(\{f, \mu'(f)\}, (\sigma'_f, \sigma_{-f}), s, \mu') + R(N \setminus \{f, \mu'(f)\}, (\sigma'_f, \sigma_{-f}), s, \mu')$, we have $R(N, \sigma, s, \mu') > R(N, (\sigma'_f, \sigma_{-f}), s, \mu')$. This is a contradiction. Thus, $\mu'(f) \subseteqq \mu(f) \setminus \{w\}$.

(ii) Suppose not. Then $R^*(N,\sigma,s) > R(N,\sigma,s,\mu')$. Consider $\delta \equiv R^*(N,\sigma,s) - R(N,\sigma,s,\mu') > 0$. Since $R(N,\sigma,s,\mu') = R(N,(\sigma'_f,\sigma_{-f}),s,\mu')$, firm f can improve its payoff by $\epsilon < \delta$ by choosing σ'''_f such that $\sigma'''_f(w) = \sigma_f(w) - \epsilon$, $\sigma'''_f(w') = 0$ for all $w' \notin \mu(f)$ and $\sigma'''_f(w'') = \sigma_f(w'')$ for all $w'' \notin \mu(f) \setminus \{w\}$. This is a contradiction.□

Although Theorem 1 deals with a simple matchmaker game in a many-to-one matching problem, it is more convenient to start with a one-to-one matching problem, since the result of a one-to-one matching problem can be extended to the case of a many-to-one matching problem. Let $q_f = 1$ for all $f \in F$. In the one-to-one matching problem, Proposition 3 becomes the following simple statement.

Corollary 4. In every one-to-one matchmaker game, in every Nash equilibrium (σ,s,μ), $R^*(N,\sigma,s) = R^*(N\backslash\{k\},\sigma,s)$ for all $k \in N$.

Let $S_k = \{k' \in N\backslash\{k\} : \mu(k') \neq \varnothing\}$. This implies that $R^*(S_k,\sigma,s) = R(S_k,\sigma,s,\mu) = R^*(N\backslash\{k\},\sigma,s)$. Then, Corollary 4 says that in every Nash equilibrium (σ,s,μ), for all $k \in N$, there exists $S_k \subseteq N\backslash\{k\}$ such that the following equation holds:

$$R^*(N,\sigma,s) = R^*(S_k,\sigma,s).$$

This **system of Nash equations** characterizes a Nash equilibrium (σ, s, μ) of the one-to-one matchmaker game.[18] The following is the first main result of this section.

Proposition 4. In every one-to-one matchmaker game, the matchmaker's profit is zero in every Nash equilibrium.

Proof. We will prove the theorem by contradiction. Assume that there is a Nash equilibrium allocation (σ, s, μ) with a positive profit $(R(N, \sigma, s, \mu) = R^*(N, \sigma, s) > 0)$, and we will reach a contradiction.

First, note that $R(N, \sigma, s, \mu) = \sum_{f \in C(N,\mu) \cap F} R(\{f, \mu(f)\}, \sigma, s, \mu)$. Pick up a pair $(f_1, w_1) \subset N$ that generates the *highest* positive profit under (σ, s) and μ:

$$R(\{f_1, w_1\}, \sigma, s, \mu) > 0. \tag{$*$}$$

The relevant Nash equations for f_1 and w_1 can be written as

$$\sum_{f \in C(S_{f_1}, \mu') \cap F} R(\{f, \mu'(f)\}, \sigma, s, \mu') = \sum_{f \in C(N,\mu) \cap F} R(\{f, \mu(f)\}, \sigma, s, \mu),$$

$$\sum_{f \in C(S_{w_1}, \mu'') \cap F} R(\{f, \mu''(f)\}, \sigma, s, \mu'') = \sum_{f \in C(N,\mu) \cap F} R(\{f, \mu(f)\}, \sigma, s, \mu)$$

where $\mu' \in A^*(S_{f_1}, \sigma, s)$ and $\mu'' \in A^*(S_{w_1}, \sigma, s)$.

Our first lemma is the following.

[18] Our system of Nash equations is inspired by the system of fundamental equations given by Laussel and Le Breton (2001). However, these two systems of equations are very different from each other. Laussel and Le Breton's (2001) system of fundamental equations is constructed from each coalition's value (the maximal value of the sum of the payoffs of the agent and the principals in the coalition), and all truthful equilibrium payoff vectors satisfy the same system of equations. In contrast, our system of Nash equations is constructed from the matchmaker's (the agent's) total profit for each coalition when a Nash strategy profile is picked.

Lemma 1. We have $w_1 \in S_{f_1}$ and $R(\{\mu'(w_1), w_1\}, \sigma, s, \mu') > 0$. Similarly, $f_1 \in S_{w_1}$ and $R(\{f_1, \mu''(f_1)\}, \sigma, s, \mu'') > 0$.

Proof of Lemma 1. We will prove the first half (the second half follows by a symmetric argument). Suppose $w_1 \notin S_{f_1}$ or $\mu'(w_1) = \varnothing$. Then, we can construct a new matching μ^* such that $\mu^*(k) = \mu'(k)$ for all $k \in S_{f_1}$, $\mu^*(f_1) = w_1$, and $\mu^*(k) = \varnothing$ for all $k \in N \backslash (S_{f_1} \cup \{w_1, f_1\})$. Then, we have $R(N, \sigma, s, \mu^*) = R(S_{f_1}, \sigma, s, \mu') + R(\{w_1, f_1\}, \sigma, s, \mu) > R(S_{f_1}, \sigma, s, \mu') = R(N, \sigma, s, \mu)$. Note that the last equality comes from the Nash equation. This is in contradiction with $\mu \in A^*(N, \sigma, s)$.

Now, suppose $R(\{w_1, \mu'(w_1)\}, \sigma, s, \mu') = 0$ (if profit is negative, the matchmaker would rather leave them unmatched). Then, we have $R(S_{f_1}, \sigma, s, \mu') = R(S_{f_1} \backslash \{w_1, \mu'(w_1)\}, \sigma, s, \mu') + R(\{w_1, \mu'(w_1)\}, \sigma, s, \mu') = R(S_{f_1} \backslash \{w_1, \mu'(w_1)\}, \sigma, s, \mu')$. Then we could construct μ^* such that $\mu^*(k) = \mu'(k)$ for all $k \in S_{f_1}$, $\mu^*(f_1) = w_1$, and $\mu^*(k) = \varnothing$ for all $k \in N \backslash (S_{f_1} \cup \{w_1, \mu'(w_1)\})$. Then we have $R(N, \sigma, s, \mu^*) = R(S_{f_1} \backslash \{w_1, \mu'(w_1)\}, \sigma, s, \mu') + R(\{w_1, f_1\}, \sigma, s, \mu') = R(S_{f_1}, \sigma, s, \mu') + R(\{w_1, f_1\}, \sigma, s, \mu) > R(N, \sigma, s, \mu)$. This violates $\mu \in A^*(N, \sigma, s)$.$\square$

Recall μ' and μ'' are matchings that achieve values $R^*(S_{f_1}, \sigma, s)$ and $R^*(S_{w_1}, \sigma, s)$, respectively. By using Lemma 1, we will construct chains of pairs from matchings μ, μ', and μ''. Let $f_{\ell+1} \equiv \mu'(w_\ell)$ and $w_{\ell+1} = \mu(f_{\ell+1})$ for $\ell = 1, 2, ...L$, where L is such that $\mu(f_\ell) \in C(N, \mu) \cap W$ and $\mu'(w_\ell) \in C(N, \mu) \cap F$ for all $\ell < L$ and $\mu(f_L) \notin C(N, \mu) \cap W$. Similarly, let $\tilde{w}_{\ell+1} \equiv \mu''(\tilde{f}_\ell)$ and $\tilde{f}_{\ell+1} = \mu(\tilde{w}_{\ell+1})$ for $\ell = 1, 2, ..., \tilde{L}$, where $\tilde{L}$ is such that $\mu(\tilde{w}_\ell) \in C(N, \mu) \cap F$ and $\mu'(\tilde{f}_\ell) \in C(N, \mu) \cap W$ for all $\ell < \tilde{L}$ and $\mu(\tilde{w}_{\tilde{L}}) \notin C(N, \mu) \cap F$. The following is our key lemma.

Lemma 2. Either $\sum_{\ell=1}^{L} R\left(\{w_\ell, f_\ell\}, \sigma, s, \mu\right) > \sum_{\ell=1}^{L-1} R\left(\{w_\ell, f_{\ell+1}\}, \sigma, s, \mu'\right)$ or $\sum_{\ell=1}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, \sigma, s, \mu) > \sum_{\ell=1}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, \sigma, s, \mu'')$ holds.

Proof of Lemma 2. Optimality of μ implies: $\sum_{\ell=1}^{L} R\left(\{w_\ell, f_\ell\}, \sigma, s, \mu\right) \geq \sum_{\ell=1}^{L-1} R\left(\{w_\ell, f_{\ell+1}\}, \sigma, s, \mu'\right)$ and $\sum_{\ell=1}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, \sigma, s, \mu) \geq \sum_{\ell=1}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, \sigma, s, \mu'')$. Thus, suppose to the contrary that

$$\begin{aligned}
\sum_{\ell=1}^{L} R\left(\{w_\ell, f_\ell\}, \sigma, s, \mu\right) &= \sum_{\ell=1}^{L-1} R\left(\{w_\ell, f_{\ell+1}\}, \sigma, s, \mu'\right), \\
\sum_{\ell=1}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, \sigma, s, \mu) &= \sum_{\ell=1}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, \sigma, s, \mu'').
\end{aligned} \tag{$**$}$$

There are two cases: (Case 1) $\left(\cup_{\ell=1}^{L-1} \{w_\ell, f_{\ell+1}\}\right) \cap (\cup_{\ell=1}^{\tilde{L}-1}\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}) = \varnothing$, and (Case 2) $\left(\cup_{\ell=1}^{L-1} \{w_\ell, f_{\ell+1}\}\right) \cap (\cup_{\ell=1}^{\tilde{L}-1}\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}) \neq \varnothing$. We will analyze the two cases by noting $\{w_1, f_1\} = \{\tilde{w}_1, \tilde{f}_1\}$. Let us start with the simpler case.

(Case 1) Suppose $\left(\cup_{\ell=1}^{L-1} \{w_\ell, f_{\ell+1}\}\right) \cap (\cup_{\ell=1}^{\tilde{L}-1}\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}) = \varnothing$. See Figure 3. Summing the two equations in ($**$), we have

$$\begin{aligned}
&\sum_{\ell=1}^{L-1} R\left(\{w_\ell, f_{\ell+1}\}, \sigma, s, \mu'\right) + \sum_{\ell=1}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, \sigma, s, \mu'') \\
&= \sum_{\ell=1}^{L} R\left(\{w_\ell, f_\ell\}, \sigma, s, \mu\right) + \sum_{\ell=1}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, \sigma, s, \mu) \\
&= \left(\sum_{\ell=1}^{L} R\left(\{w_\ell, f_\ell\}, \sigma, s, \mu\right) + \sum_{\ell=2}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, \sigma, s, \mu)\right) \\
&+ R(\{w_1, f_1\}, \sigma, s, \mu)
\end{aligned}$$

where the last equality comes from $\{\tilde{w}_1, \tilde{f}_1\} = \{w_1, f_1\}$. Let $A \equiv \left(\cup_{\ell=1}^{L-1} \{w_\ell, f_{\ell+1}\}\right) \cap (\cup_{\ell=1}^{\tilde{L}-1}\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\})$. There is no double counting of players

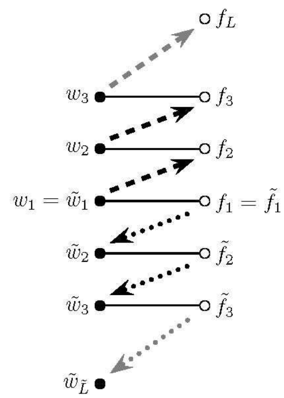
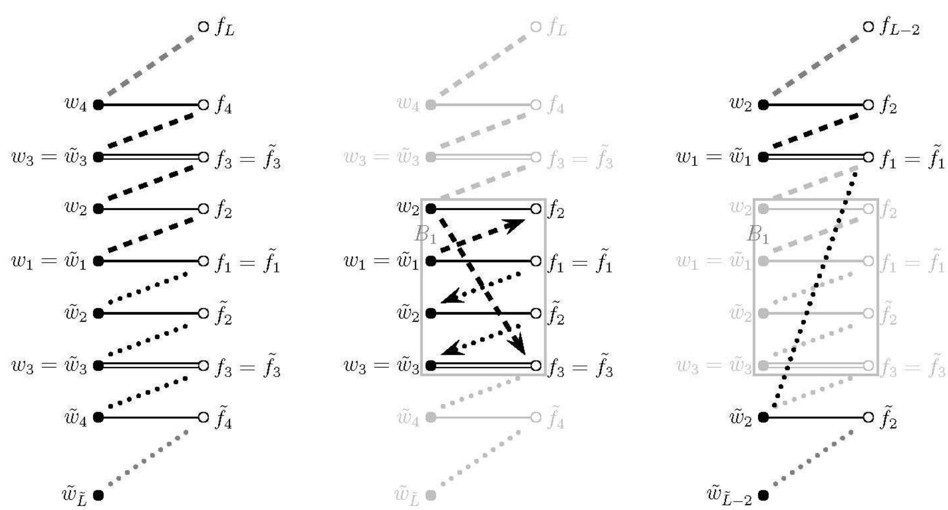


Figure 2.3: Illustration of (Case 1). Solid, dashed, and dotted lines represent matchings μ, μ', and μ'', respectively. Arrows represent μ^*.



(a) $w_3 = \tilde{w}_3, f_3 = \tilde{f}_3$. **(b)** Construct B_1. **(c)** Remove B_1.

Figure 2.4: Illustration of (Case 2) when $\ell' = \bar{\ell} = 3$. Solid, dashed, and dotted lines represent matchings μ, μ', and μ'', respectively. Arrows represent μ^*.

in A. Let $\mu^* \in \mathcal{M}$ be such that $\mu^*(w_\ell) = f_{\ell+1}$ for $\ell = 1, ..., L-1$ and $\mu^*(\tilde{f}_\ell) = \tilde{w}_{\ell+1}$ for $\ell = 1, ..., \tilde{L}-1$. Replacing μ by μ^*, the total value in A increases by $R(\{w_1, f_1\}, \sigma, s, \mu)$. By the prevailing assumption $(*)$, $R(\{w_1, f_1\}, \sigma, s, \mu) > 0$. This contradicts the optimality of μ.

(Case 2) $\left(\cup_{\ell=1}^{L-1}\{w_\ell, f_{\ell+1}\}\right) \cap (\cup_{\ell=1}^{\tilde{L}-1}\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}) \neq \varnothing$. Let $\bar{\ell}$ be such that for all $1 \leq \ell < \bar{\ell}$, $\tilde{w}_\ell, \tilde{f}_\ell \notin \cup_{\ell=1}^{L}\{w_\ell, f_\ell\}$, and $\tilde{w}_{\bar{\ell}}, \tilde{f}_{\bar{\ell}} \in \cup_{\ell=1}^{L}\{w_\ell, f_\ell\}$. Hence, $\{\tilde{w}_{\bar{\ell}}, \tilde{f}_{\bar{\ell}}\} = \{w_{\ell'}, f_{\ell'}\}$ for some $\ell' \in \{2, ..., L\}$. See Figure 4(a) for the case when $\ell' = \bar{\ell} = 3$. Denote the set of players $B_1 \equiv (\cup_{\ell=1}^{\ell'}\{w_\ell, f_\ell\}) \cup (\cup_{\ell=2}^{\bar{\ell}-1}\{\tilde{w}_\ell, \tilde{f}_\ell\})$ as in Figure 4(b). There is no double counting in B_1. Now, consider two matchings in B_1: μ and μ^* such that $\mu^*(w_\ell) = \mu'(w_\ell)$ for $\ell = 1, ..., \ell' - 1$, and $\mu^*(\tilde{f}_\ell) = \mu''(\tilde{f}_\ell)$ for $\ell = 1, ..., \bar{\ell} - 1$ (note $\tilde{f}_1 = f_1$ and $\tilde{w}_{\bar{\ell}} = w_{\ell'}$). We now compare the values of these two. First,

$$
\begin{aligned}
&R(B_1, \sigma, s, \mu^*) \\
&= \sum_{\ell=1}^{\ell'-1} R(\{w_\ell, f_{\ell+1}\}, \sigma, s, \mu') + \sum_{\ell=1}^{\bar{\ell}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, \sigma, s, \mu'') \\
&= \left[\sum_{\ell=1}^{L-1} R(\{w_\ell, f_{\ell+1}\}, \sigma, s, \mu') + \sum_{\ell=1}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, \sigma, s, \mu'')\right] \\
&- \left[\sum_{\ell=\ell'}^{L-1} R(\{w_\ell, f_{\ell+1}\}, \sigma, s, \mu') + \sum_{\ell=\bar{\ell}}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, \sigma, s, \mu'')\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&R(B_1, \sigma, s, \mu) \\
&= \sum_{\ell=1}^{\ell'} R(\{w_\ell, f_\ell\}, \sigma, s, \mu) + \sum_{\ell=2}^{\bar{\ell}-1} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, \sigma, s, \mu) \\
&= \left[\sum_{\ell=1}^{L} R(\{w_\ell, f_\ell\}, \sigma, s, \mu) + \sum_{\ell=1}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, \sigma, s, \mu)\right] \\
&- \left[\sum_{\ell=\ell'}^{L} R(\{w_\ell, f_\ell\}, \sigma, s, \mu) + \sum_{\ell=\bar{\ell}}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, \sigma, s, \mu)\right] \\
&- [R(\{w_1, f_1\}, \sigma, s, \mu) - R(\{w_{\ell'}, f_{\ell'}\}, \sigma, s, \mu)]
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\sum_{\ell=1}^{L-1} R\left(\left\{w_\ell, f_{\ell+1}\right\}, \sigma, s, \mu'\right) + \sum_{\ell=1}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, \sigma, s, \mu'')\right] \\
&- \left[\sum_{\ell=\ell'}^{L} R\left(\left\{w_\ell, f_\ell\right\}, \sigma, s, \mu\right) + \sum_{\ell=\bar{\ell}}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, \sigma, s, \mu)\right] \\
&- \left[R(\{w_1, f_1\}, (\sigma, s), \mu) - R\left(\left\{w_{\ell'}, f_{\ell'}\right\}, (\sigma, s), \mu\right)\right]
\end{aligned}
$$

where the last equality follows from $(**)$. Thus, we have

$$
\begin{aligned}
& R(B_1, \sigma, s, \mu^*) - R(B_1, \sigma, s, \mu) \\
&= R(\{w_1, f_1\}, \sigma, s, \mu) - R\left(\left\{w_{\ell'}, f_{\ell'}\right\}, \sigma, s, \mu\right) \\
&+ \left[\sum_{\ell=\ell'}^{L} R\left(\left\{w_\ell, f_\ell\right\}, \sigma, s, \mu\right) - \sum_{\ell=\ell'}^{L-1} R\left(\left\{w_\ell, f_{\ell+1}\right\}, \sigma, s, \mu'\right)\right] \\
&+ \left[\sum_{\ell=\bar{\ell}}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, \sigma, s, \mu) - \sum_{\ell=\bar{\ell}}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, \sigma, s, \mu'')\right].
\end{aligned}
$$

Note that the contents in both brackets must be nonnegative since μ maximizes the total value in N. Since $\{w_1, f_1\}$ generates the highest profit under (σ, s) and μ, $R(\{w_1, f_1\}, \sigma, s, \mu) \geq R\left(\left\{w_{\ell'}, f_{\ell'}\right\}, \sigma, s, \mu\right)$ must hold. Thus, $R(B_1, \sigma, s, \mu^*) \geq R(B_1, \sigma, s, \mu)$ must hold. If $R(B_1, \sigma, s, \mu^*) > R(B_1, \sigma, s, \mu)$, we have a contradiction, so assume that $R(B_1, \sigma, s, \mu^*) = R(B_1, \sigma, s, \mu)$. For this to happen, the following three conditions must hold:

(i) $R(\{w_1, f_1\}, \sigma, s, \mu) = R\left(\left\{w_{\ell'}, f_{\ell'}\right\}, \sigma, s, \mu\right)$.

(ii) $\sum_{\ell=\ell'}^{L} R\left(\left\{w_\ell, f_\ell\right\}, \sigma, s, \mu\right) = \sum_{\ell=\ell'}^{L-1} R\left(\left\{w_\ell, f_{\ell+1}\right\}, \sigma, s, \mu'\right)$.

(iii) $\sum_{\ell=\bar{\ell}}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, \sigma, s, \mu) = \sum_{\ell=\bar{\ell}}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, \sigma, s, \mu'')$.

Recall that $\{w_{\ell'}, f_{\ell'}\} = \{\tilde{w}_{\bar{\ell}}, \tilde{f}_{\bar{\ell}}\}$. Rename w_ℓ, f_ℓ, $\tilde{w}_\ell$, $\tilde{f}_\ell$, L, and $\tilde{L}$ as $w_{\ell-\ell'+1}$, $f_{\ell-\ell'+1}$, $\tilde{w}_{\ell-\bar{\ell}+1}$, $\tilde{f}_{\ell-\bar{\ell}+1}$, $L - \ell' + 1$, and $\tilde{L} - \bar{\ell} + 1$, respectively. Then, we again have exactly the same problem as before: $\sum_{\ell=1}^{L} R\left(\{w_\ell, f_\ell\}, \sigma, s, \mu\right) = \sum_{\ell=1}^{L-1} R\left(\{w_\ell, f_{\ell+1}\}, \sigma, s, \mu'\right)$ and $\sum_{\ell=1}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, \sigma, s, \mu) = \sum_{\ell=1}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, \sigma, s, \mu'')$ as in Fig. 4(c).

If (Case 1) applies, then we have a contradiction. If (Case 2) applies, then we again find $\{w_{\ell'}, f_{\ell'}\} = \{\tilde{w}_\ell, \tilde{f}_\ell\}$, and we can again find a cycle set B_2. If the cycle achieves a strict improvement, we reach a contradiction. So, assuming equalities, firms and workers that remain after taking B_2 out still satisfy the above three conditions. Applying this procedure repeatedly, eventually, (Case 1) applies (by a finite number of players). Hence, we conclude that $\sum_{\ell=1}^{L} R\left(\{w_\ell, f_\ell\}, \sigma, s, \mu\right) > \sum_{\ell=1}^{L-1} R\left(\{w_\ell, f_{\ell+1}\}, \sigma, s, \mu'\right)$ or $\sum_{\ell=1}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, \sigma, s, \mu) > \sum_{\ell=1}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, \sigma, s, \mu'')$ holds.■

The last part of the proof of Proposition 4. Now we will complete the proof of Proposition 4. Suppose, without loss of generality, that $\sum_{\ell=1}^{L} R(\{w_\ell, f_\ell\}, \sigma, s, \mu) > \sum_{\ell=1}^{L-1} R(\{w_\ell, f_{\ell+1}\}, \sigma, s, \mu')$ holds. There are two possibilities: (1) $S_{f_1} = \cup_{\ell=1}^{L-1}\{w_\ell, f_{\ell+1}\}$, or (2) $S_{f_1} \supsetneqq \cup_{\ell=1}^{L-1}\{w_\ell, f_{\ell+1}\}$. In the first case, $R(S_{f_1}, \sigma, s, \mu') < R(N, \sigma, s, \mu)$. This contradicts the Nash equation. In the second case, the new matching created from μ and μ' is broken in the middle. There are two subcases: (i) $\mu(f_L) = \varnothing$, and (ii) $\mu'(w_L) = \varnothing$. In either subcase, $R(N \backslash \sum_{\ell=1}^{L}\{w_\ell, f_\ell\}, \sigma, s, \mu) = R(N \backslash \sum_{\ell=1}^{L}\{w_\ell, f_\ell\}, \sigma, s, \mu')$.[19] This again implies $R(S_{f_1}, \sigma, s, \mu') < R(N, \sigma, s, \mu)$. Hence, assumption $(*)$ cannot be true. Thus, no pair can generate a positive profit.□

The proof of Proposition 4 utilizes only Corollary 4 and the matchmaker's profit-maximizing behavior given the system of profit on each pair of firms and workers (generated from σ and s). As the Nash equations apply to each position instead of each firm, we can extend our Proposition 3 to the simple

[19] This is a slight abuse of notation: in subcase (i), w_L does not exist, since f_L is single.

matchmaker game in the many-to-one assignment problem. Let us separate firm f into q_f **positions** $f' = \left\{f'_1, \ldots, f'_{q_f}\right\}$ where each position offers the same wages. Denote $F' \equiv \bigcup_{f \in F} \left\{f'_1, \ldots, f'_{q_f}\right\}$ as the set of positions (decomposed firms). Then, we can generate a one-to-one matching of positions and workers. Let μ-**decomposed matching** $\bar{\mu} : W \cup F' \to W \cup F'$ be a bijection such that (i) $\bar{\mu}\left(f'_i\right) = w$ if there exists $f' \ni f'_i$ such that $w \in \mu\left(f\right)$; (ii) $\bar{\mu}\left(w\right) = f'_i$ if $\mu\left(w\right) = f$; (iii) $\bar{\mu}\left(f\right) \in F'$ implies $\bar{\mu}\left(f'_i\right) = f'_i$ for all $f'_i \in f'$ and $\bar{\mu}\left(w\right) \in W$ implies $\bar{\mu}\left(w\right) = \varnothing$. Since Proposition 3 implies that Corollary 4 applies to μ-decomposed matching in the artificial one-to-one assignment problem, Proposition 4 directly implies that the zero-profit result for the simple matchmaker game will hold in the many-to-one assignment problem. This completes the proof of Theorem 1.

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Chapter 3

Choosing a Licensee from Heterogeneous Rivals

3.1 Introduction

We examine a firm licensing its production technology to a rival in a product market, but relax the standard assumption that the rivals are homogeneous in their production technologies. Specifically, the firms compete in Cournot competition differing in their constant marginal cost of production and a technology transfer reduces a licensee's marginal cost to the level of the licensor. This implies that the size of the technology transfer varies with the licensee's efficiency (a less efficient rival receives a larger transfer). As is standard in the licensing literature, the production decisions of the firms remain independent with any transfer agreement. That is, we are interested in the direct effects from the licensing and so abstract from any possible effects from collusion. As we allow for heterogeneous rivals and for corner solutions (some firms may choose to shut down: zero production), we focus on the case where a single licensor chooses an exclusive licensing partner and assume complete information, though we allow negative externalities of licensing to third party firms.[1] We first analyze the gains in joint profit for the licensor and a licensee from licensing, and then social welfare gains. Then, we consider two auction games to determine the licensee, and investigate how efficient the resulting licensee is.

We begin, following the seminal work by Katz and Shapiro (1985), by analyzing whether such a transfer is always jointly profitable in a Cournot model.[2] Katz and Shapiro (1985) have shown that a complete technology transfer (so that the licensee has the same cost as the innovator) could reduce joint profit in a duopoly if the licensor has a near-monopoly position because then the transfer would reduce the licensor's near-monopoly profit. Further, in our setting a partial technology transfer (so that the licensee does not have the same cost as the innovator) can reduce joint profit. Despite this we are able to show that a complete technology transfer is always jointly profitable so long as the demand curve is weakly concave and there are at least three firms in the market after the transfer (Theorem 1). That is, a complete transfer is always jointly profitable no matter its absolute size. The licensor does not have to be the most efficient firm for this result to hold.

We then focus on which partner would maximize joint profit. One might at first glance expect then that this would be the most inefficient rival. We find that for weakly concave demand, it is neither a very inefficient nor a very efficient rival that maximizes joint profit (Observation 1). With heterogeneous firms, the less efficient the licensee, the greater is the technology transfer. Thus,

[1] For example, Jehiel et al (1996) and Jehiel and Moldovanu (2000) in examining a single transfer allow for the presence of private information in auction stage. In this paper, we analyze auction methods in licensing, but we concentrate on the effects of having (negative) externalities on auction outcomes.

[2] This is also equivalent to fixed fee licensing examined in Kamien and Tauman (1986).

a technology transfer to a nearly equally efficient rival is very small and has little benefit on the rival's profit, although such a transfer does not reduce the market price much and the licensor's profit. On the other hand, a technology transfer to a very inefficient firm benefits the licensee greatly, while reduces the licensor's output and profit through a large reduction in the market price. Given that profit is convex in output, the licensor's profit reduction is large if a technology transfer is made to a very inefficient firm. Hence, the licensor is better off by choosing a partner who is neither very efficient nor very inefficient.

Turning to the welfare effects of a technology transfer, it is known that making an inefficient firm slightly more efficient can reduce welfare (Lahiri and Ono 1988). This implies that, as a corollary of Theorem 1, a jointly profitable transfer can reduce social welfare if there are more than two firms and if both the licensor and licensee are sufficiently similar and inefficient (Observation 2). This is in contrast to Katz and Shapiro (1985) who found that profitable transfers are never welfare reducing in a duopoly, hence, the importance of considering non-duopoly markets. This is also in contrast to Katz and Shapiro (1986) and Sen and Tauman (2007) who find that with homogeneous firms, licensing always raises welfare and so heterogeneity is also important in evaluating the welfare implications of licensing. Despite the fact that a transfer from a sufficiently inefficient licensor can reduce welfare, we show that if the most efficient firm makes a complete transfer then social welfare always increases under general demand (Theorem 2). However, the joint-profit maximizing licensee is not necessarily the social welfare maximizing licensee because the joint-profit maximizing selection does not take into account the negative externality imposed on other firms. Or, to put it differently, because technology transfers affect the rival firms' production decisions, including efficient rivals, total costs can be lower with a *more* efficient licensee. The conclusion for a policy maker that maximizes social welfare is that most efficient firm should not be discouraged from licensing their technology to rivals, but technology transfers between marginal firms could bear some scrutiny.

Analyzing the joint-profit maximizing licensee is a natural benchmark, allows comparison to Katz and Shapiro (1985) as well as work that examines fixed-fee setting licensing (e.g., Kamien and Tauman 1986), and, as we will see, is useful for later analysis.[3] However, with more than one rival, the above mechanism does not exploit the entire possible gains for the licensor if it can credibly threaten to change partners while negotiating with a potential licensee. That is, letting potential licensees compete over the technology transfer should be more profitable in the presence of externalities. Katz and Shapiro (1986) and others since,[4] take this into account when they examined an auction game in a homogeneous licensee environment by endogenizing the number of licenses. We follow their approach, but in a setting with heterogeneous firms and ask which firm would win the right to use the technology and how much would the licensor collect from licensing. Specifically, we examine what happens when the most efficient firm (the natural analogy to when rivals are homogeneous) uses the first-price auction mechanisms to sell the right to use its technology. In the first-price auction method (a simple auction game), which is a modification of the method by Katz and Shapiro (1986) so as to take into account the heterogeneous firm environment, each potential licensee submits a bid and only the winner pays for the bid. Since there are many Nash equilibria and most of them are less plausible, we refine the set of Nash equilibrium by requesting

[3] This can also be justified by noting that often a licensee is selected and then the two parties negotiate the contract. Since negotiating a technology transfer is not trivial, at that point it may be too costly for the licensor to credibly threaten to use a different firm in the bargaining and so the fee should be determined as a function of the increase in their joint profit. In this case, the joint-profit-maximizing licensee should be selected by the licensor as the recipient of the technology.

[4] For a review of auctions in licensing see Giebe and Wolfstetter (2007)

that non-licensees would not be worse-off if the licensor happens to choose it: truthful Nash equilibrium (TNE in simple auction). Roughly speaking this is akin to a trembling-hand refinement. In this refined set of Nash equilibrium, licensing fee can be pinned down and the licensee is the partner that maximizes the joint profit of the licensee, the licensor and any other potential rival.

Given the complex negative externalities created by a technology transfer, even if a firm is not willing to bid enough to win the license it might find it profitable to bribe the licensor so as to affect which of its rivals does obtain the license.[5] For this reason, we also consider a menu auction (Bernheim and Whinston 1986), in which each potential licensee submits a menu that offers a payment to the licensor depending on which rival is the licensee (i.e., specifies a payment for each possible licensee the licensor might select). Similar to the simple auction, we refine the set of Nash equilibrium by truthful Nash equilibrium (TNE in menu auction).[6] We show that a simple auction licensee is at least as efficient as the joint-profit-maximizing partner, and a menu auction licensee is at least as efficient as simple auction licensee (Theorem 3). Furthermore, if only the menu auction licensee pays in a menu auction game, then the same licensee also win the license in simple auction, and in particular, if the number of firms is three then these two auction mechanisms generate the same outcome (Proposition 6).

In the next section we introduce the basic modeling assumptions. Section 3 examines the effect the amount of technology transferred has on profit while section 4 examines the effect of the type of partner. Section 5 identifies which

[5]For example, recently, when it looked that Google would acquire bankrupt Nortel's patents a coalition of Apple, EMC, Ericsson, Microsoft, Research In Motion, and Sony out-bid Google (Claburn, 2011).

[6]Truthful Nash equilibria in simple auction and in menu auction appear to be similar in their definitions, but their implications are somewhat different. In simple auction, TNE is a rather innocuous refinement of Nash equilibrium, while in menu auction, TNE has an implication for communication-based refinement (Bernheim and Whinston 1986).

firm will get the right to use technology in license auction games. Section 6 contains the welfare analysis and section 7 concludes.

3.2 The Model

We consider the basic Cournot market structure. There is a commodity besides a numeraire good, and its inverse demand is a continuous function $P(Q)$ in $[0, \bar{Q}]$ that is twice continuously differentiable with $P'(Q) < 0$ for all $Q \in (0, \bar{Q})$ and $P(\bar{Q}) = 0$. There are K firms in the market with no fixed cost of production.

Firms are indexed as $i \in \{1, ..., K\}$ and differ in their constant marginal cost c_k. We order firms by their degrees of efficiency: $c_1 \leq c_2 \leq ... \leq c_K$. With a little abuse of notation, let the set $\{1, 2, \ldots, K\}$ be denoted by K as well.

Each firm i's production level is denoted by q_i. Firm i's profit function is written as

$$\pi_i(q_i, q_{-i}) = (P(Q) - c_i)\, q_i,$$

where $Q = \sum_{i \in K} q_i$. The first order condition for profit maximization (assuming interior solution) is

$$P'(Q)q_i + P(Q) - c_i = 0.$$

This implies

$$q_i = \frac{(P(Q) - c_i)}{-P'(Q)},$$

and firm i's profit is written as

$$\pi_i(q_i, q_{-i}) = \frac{(P(Q) - c_i)^2}{-P'(Q)}.$$

We assume the **strategic substitutability** condition throughout the paper: for all $i \in K$:

$$P''(Q)q_i + P'(Q) \leq 0.$$

Note that the second order condition for profit maximization ($P''(Q)q_i + 2P'(Q) \leq 0$) is guaranteed by the strategic substitutability. The strategic substitutability is weaker than requiring that the inverse demand is **weakly concave** $P''(Q) \leq 0$.[7] In proving some of our main results, we strengthen the strategic substitutability by the weak concavity of inverse demand.

The strategic substitutability condition guarantees the uniqueness of equilibrium of this game. Let $C = \sum_{i \in K} c_i$ denote the aggregate marginal cost. With this we can establish a standard result, whose derivation will be useful for later analysis.

Lemma 1. Under the strategic substitute condition, equilibrium is unique. Moreover, keeping other firms' marginal costs intact, an increase in c_j decreases equilibrium total output level Q if $c_j < P(Q)$, and has no effect, otherwise.

Proof. First note that equilibrium output of firm i, q_i, is expressed by equation

$$q_i = \frac{(P(Q) - c_i)}{-P'(Q)},$$

[7]That is, the weak concavity of inverse demand implies the second order condition for profit maximization.

if $P(Q) > c_i$, and $q_i = 0$ if $P(Q) \leq c_i$. Recall that $c_1 \leq c_2 \leq ... \leq c_K$. Summing up the first order conditions for profit maximization over firms in subset $L \subset K$, and assuming these firms produce positive outputs, we obtain

$$P'(Q)Q + LP(Q) = \sum_{\ell \in L} c_\ell,$$

where L also denotes the number of firms in set L. If the solution of the above equation Q satisfies $P(Q) \geq c_\ell$ for all $\ell \in L$, and $P(Q) < c_k$ for all $k \in K \backslash L$, then Q is the aggregate equilibrium output. Rewriting the above equation, we obtain

$$P'(Q)Q + \sum_{\ell \in L} (P(Q) - c_\ell) = 0,$$

or

$$P'(Q)Q + \sum_{k \in K} \max \{0, P(Q) - c_k\} = 0.$$

The LHS of the above equation is continuous in Q, although it is not continuously differentiable since firms stop producing in order as Q increases. However, for each $L \subset K$, the LHS is differentiable for Q satisfying $P(Q) \geq c_\ell$ for all $\ell \in L$, and $P(Q) < c_k$ for all $k \in K \backslash L$, and the derivative is

$$\frac{d\,(LHS)}{dQ} = P''(Q)Q + (L+1)\,P'(Q)$$

Summing the strategic substitutability conditions up over firms $\ell \in L$, we obtain

$$P''(Q)Q + LP'(Q) \leq 0.$$

This implies that the LHS of the aggregated first order condition is decreasing in Q since $P'(Q) < 0$. This implies that equilibrium aggregate output Q is uniquely determined for every marginal cost profile $(c_1, ..., c_K)$.

Now, we conduct a comparative static analysis with respect to c_j. By the above analysis, it is easy to see that Q decreases as c_j increases if $P(Q) > c_j$, and Q is intact otherwise.$\square$

3.3 Production Technologies and Transfers

Each firm i has its own technology of producing the commodity (the marginal cost of production is c_i), and it has the property right to its own technology (e.g., it holds a patent). We focus on a firm that has a single unit of technology to transfer and assume that the output decisions remain independent after any transfer as the independence of production decisions is usually a condition imposed by competition authorities as well as being the standard assumption in the literature. Firm i can license its technology with an exclusive usage agreement to another firm . As is standard in the literature (Katz and Shapiro 1986, etc.), we assume **complete** technology transfer throughout the paper: the obtaining firm j (*licensee*) reduces its marginal cost to that of firm i. That is, if firms i and j have technologies with marginal costs c_i and c_j with $c_i < c_j$, respectively, then firm j can reduce its marginal cost of production to c_i by adopting firm i's technology.

The following simple lemma plays a key role in the subsequent analysis. As the proof is straightforward, it is left to the appendix.

Lemma 2. Suppose that there are initially K firms engaging in production. Pick three firms i, j, and j' with $c_i < c_j < c_{j'}$, and consider two scenarios: (i) firm i transfers its technology to firm j, and (ii) firm i transfers its technology to firm j'. Then, equilibrium aggregate output $\hat{Q}$ in scenario (i) is not more than equilibrium aggregate output $\tilde{Q}$ in scenario (ii), resulting in $P(\hat{Q}) \geq P(\tilde{Q})$.

3.3.1 Jointly Profitable Transfers

Katz and Shapiro (1985) show that complete transfers could reduce joint profits in a duopoly and we examine if this result can extend to markets with more than two firms. We can show that under weakly concave demand (which includes linear demand) a complete technology transfer is always profitable as long as there is a third firm. This result is somewhat surprising not only because Katz and Shapiro (1985) found such transfers could be unprofitable but also because Creane and Konishi (2009b) show that partial transfers (i.e., the licensee's cost is not completely reduced to the licensor's cost) when firms are heterogeneous could reduce joint profit. Due to this fact that a small transfer may reduce joint profit, we cannot simply rely on comparative statics on technology transfers: we need to utilize an artificial economy to prove the theorem. The proof is involved, and found in the appendix.

Theorem 1. Pick firms $i, j \in K$ with $c_i < c_j$. Assume that firm i is in operation originally, and that even after firm i transfers technology to firm j, there is still another firm k in operation ($q_k > 0$) with $c_k \neq c_i$. If demand is weakly concave ($P''(Q) \leq 0$), then a complete technology transfer from firm i to firm j is joint profit improving.

Notice that we assume that at least three firms remain in the market after the technology transfer. Although Katz and Shapiro (1985) obtain conditions for a complete technology transfer to reduce joint profit, they examine a duopoly case. The existence of at least a third firm drives the theorem as part of the gain to the licensee comes from lost profits of the non-licensor firm(s). Thus, while the licensor's profits decrease from the transfer, the licensee's gain is sufficient to offset the loss to the licensor. However, since a partial technology transfer could reduce joint profit, one may wonder how it can be guaranteed that a complete transfer does not reduce joint profit. To intuitively see the reason, consider what happens when a partial technology transfer would reduce joint profit. In this case, consider what happens if, instead, the licensee's cost is *increased* (thereby raising joint profit) until the licensee is driven out of the market. Joint profit has now increased. At this point we note from the divisionalization literature (Baye, et al 1996) that if the licensee could create a second, identical division then its profits increase.

3.3.2 The Joint-Profit-Maximizing Partner

While in the previous section we considered the profitability of technology transfers, in this section we consider which partner would maximize joint profit. That is, for firm i, which firm j would create the greatest increase in joint profit from a technology transfer? Recall that the licensee choosing a less efficient partner leads to a larger technology transfer.

Since we need to compare the point profits when a different partner has been chosen, for heuristic reasons it is more convenient for us to use linear demand with explicit solutions and assume that all firms are in operation: $q_k > 0$ for all $k \in K$. Let $P(Q) = a - bQ$. With this demand curve we have $Q = \frac{(aK-C)}{(K+1)b}$, $P = \frac{a+C}{(K+1)}$, and $q_i = \frac{\frac{a+C}{(K+1)} - c_i}{b}$. Then the change in the joint profit by firms i and j from the technology transfer is

$$\begin{aligned}
&(\text{joint profit after transfer}) - (\text{joint profit before transfer}) \\
&= \frac{2}{b}\left(\frac{a+C-(c_j-c_i)}{K+1} - c_i\right)^2 - \frac{1}{b}\left(\frac{a+C}{K+1} - c_i\right)^2 - \frac{1}{b}\left(\frac{a+C}{K+1} - c_j\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= \underbrace{\frac{1}{b}\left(\frac{a+C}{K+1}-c_i-\frac{c_j-c_i}{K+1}\right)^2-\frac{1}{b}\left(\frac{a+C}{K+1}-c_i-(c_j-c_i)\right)^2}_{\text{increase in firm } j\text{'s profit}} \\
&\underbrace{+\frac{1}{b}\left(\frac{a+C}{K+1}-c_i-\frac{c_j-c_i}{K+1}\right)^2-\frac{1}{b}\left(\frac{a+C}{K+1}-c_i\right)^2}_{\text{decrease in firm } i\text{'s profit}} \\
&= \frac{1}{b}\left(2\frac{a+C}{K+1}-2c_i-\frac{(K+2)(c_j-c_i)}{K+1}\right)\left(\frac{K}{K+1}(c_j-c_i)\right) \\
&-\frac{1}{b}\left(2\frac{a+C}{K+1}-2c_i-\frac{c_j-c_i}{K+1}\right)\left(\frac{c_j-c_i}{K+1}\right) \\
&= \frac{2}{b}\left(\frac{a+C}{K+1}-c_i\right)(c_j-c_i)-\frac{(K(K+2)-1)(c_j-c_i)^2}{b(K+1)^2}.
\end{aligned}
$$

This is a quadratic function in the difference in marginal costs $c_j - c_i$. The first positive term increases if firm j is a less efficient partner, while the second negative term gains its magnitude as firm j is a less efficient partner. Hence, this implies that the gain is highest when c_j is neither too big nor too small. Firm i should choose some firm in the middle. Although the above analysis is based on linear demand assumption, a quantitatively similar result applies for general demand (see Creane and Konishi 2009b).

Observation 1. With a complete transfer, the joint-profit maximizing partner for a firm is neither too efficient nor too inefficient relative to the firm under weakly concave demand.

This condition is intuitive: you cannot make a rival who is efficient that much more efficient. Thus, there is a benefit from picking a less efficient rival as there is a greater transfer and so increase in profit of the licensee from the transfer. However, you do not want to pick too inefficient of a rival. The reason is that as you pick a more inefficient rival the technology transfer causes the price to fall more, harming you as well as the rival. At the same time, when considering sufficiently inefficient firms, a slightly more inefficient firm does not have that much less profit (since its output is approaching zero, i.e., marginal cost is approaching the price) and the gain from selecting a slightly more inefficient rival approaches zero.

The following simple example illustrates this observation. We denote by $\pi_k(i,j)$ firm k's (equilibrium) profit when firms i and j are the licensor and licensee for all $j,k = 1,...,K$. Notation $\pi_k(i,i)$ means firm k's profit when the licensor i does not license its technology to any firm.

Example 1. Consider a market with four firms with marginal costs $c_1 = 0$, $c_2 = .05$, $c_3 = .15$ and $c_4 = .25$, and the licensor is the most efficient firm: $i = 1$. Demand function is linear $P(Q) = 1 - Q$. In the following table are the resulting profits for each firm from a transfer to firm j (with j =1 implying no transfer has occurred).

$i = 1$	P	π_1	π_2	π_3	π_4
$j = 1$	0.29	0.0841	0.0576	0.0196	0.0016
$j = 2$	0.28	0.0784	0.0784	0.0169	0.0009
$j = 3$	0.26	0.0676	0.0441	0.0676	0.0001
$j = 4$	0.24	0.0576	0.0361	0.0081	0.0576

As is easily seen, if $j = 2$ then the gains in joint profit is $2 \times 0.0784 - (0.0841 + 0.0576) = 0.0151$; if $j = 3$ then it is $2 \times 0.0676 - (0.0841 + 0.0196) = 0.0315$; and if $j = 4$ then it is $2 \times 0.0576 - (0.0841 + 0.0081) = 0.023$. Thus, firm 3 maximizes joint profits.

3.4 Welfare Effects

We now investigate the effect of technology transfers on social welfare, which we define as the sum of the firms' profit and consumer surplus. Since technology transfers reduce production cost, social welfare tends to increase in the amount of technology transferred. Indeed, Katz and Shapiro (1985) show that with a duopoly, licensing that increases joint profit always increases welfare (and welfare decreasing licensing always decreases joint profit). Likewise Sen and Tauman (2007) find licensing to be welfare improving under general licensing schemes.

Despite previous results, profitable licensing could reduce welfare when firms are heterogeneous. This possibility arises because if a very inefficient firm obtains a technology transfer that reduces its cost only slightly, then its resulting increase production will displace the production of more efficient firms, thereby reducing social welfare. This result has already been observed by Lahiri and Ono (1988). The question then is whether this implies that jointly profitable licensing can reduce welfare contrary to previous results. By the use of Theorem 1 combined with Lahiri and Ono's result we are able to state that the previous results do not generalize to when there are more than two firms and firms are heterogeneous: profitable licensing can be welfare reducing.

Given this result one may wonder if there are conditions that guarantee that a technology transfer raises welfare. We then show that if the most efficient firm makes a complete technology transfer, then welfare increases. The policy implications of these results appear straightforward: competition authorities should be scrutinous of technology transfers (through licensing, joint venture, or merger) between marginal firms (in the technological efficiency sense) in an industry. On the other hand, the most efficient firm within an industry should not be discouraged from making a technology transfer to a rival.

3.4.1 Welfare-reducing profitable licensing

We begin by presenting Lahiri and Ono's condition for when an improvement in the marginal cost of an inefficient firm reduces social welfare.

Observation 2. (Lahiri and Ono 1988): When firm j's marginal cost (c_j) decreases, social welfare decreases if c_j is sufficiently high, though consumer welfare (surplus) increases.

From this observation there is an immediate corollary to Theorem 1 that yields a result contrary to previous ones in the literature: there are profitable technology transfers that reduce total welfare though benefiting consumers.

Corollary 1. Suppose that demand is weakly concave and that there are more than two firms. Then, if firm j has sufficiently high marginal cost (c_j) and firm i's marginal cost is sufficiently close to firm j's, then firm i licensing its technology to firm j is jointly profitable and welfare reducing though consumer welfare (surplus) increases.

The following example shows that the social welfare can indeed decrease by a jointly profitable technology transfer.

Example 2. Consider a market with five firms with marginal costs $c_1 = 0$, $c_2 = .075$, $c_3 = .15$, $c_4 = .225$ and $c_5 = .29$. Demand function is linear $P(Q) = 1 - Q$. Consider two cases: (i) firm 3 licenses its technology to firm 5

($i = 3$ and $j = 5$), and (ii) firm 4 licenses its technology to firm 5 ($i = 4$ and $j = 5$). Numbers in the table below are rounded to two decimal places.

	P	π_1	π_2	π_3	π_4	π_5	CS	SW
none	0.29	0.084	0.046	0.020	0.004	0	0.252	0.406
$i = 3$, $j = 5$	0.267	0.071	0.037	0.014	0.002	0.013	0.269	0.406
$i = 4$, $j = 5$	0.279	0.078	0.041	0.017	0.003	0.003	0.260	0.402

In both cases, although the joint profit improves (case (i): from 0.020 to 2×0.014; case (ii): from 0.004 to 2×0.003), the social welfare goes down. Note that consumer surplus (CS) improves by technology transfer. The social welfare goes down by technology licensing because relatively high marginal cost firms crowd out more efficient firms' production. (Although before the technology transfer, firm 5 is producing zero since its marginal cost and the market price are the same, this is just for convenience of calculations. We can make π_5 positive by setting c_5 a slightly lower than 0.29, and still have welfare decreasing in both cases.)

There are previous results in the literature that may at first glance appear to be similar even though they are quite distinct. First, Katz and Shapiro (1985) have shown that in a duopoly a technology transfer can reduce welfare, but only when it reduces joint profit. Hence, such transfers would never actually occur. In contrast, here there can be technology transfers that reduce welfare, but increase joint profit. Second, Faulí-Oller and Sandonís (2002) have shown that in a duopoly profitable licensing can reduce welfare, but this requires the use of a royalty (raising the recipient's marginal cost) and only occurs in price competition. As they note, the royalty works as a collusive device" and so reduces welfare. More generally, licensing contracts can reduce welfare through their collusive effects (Shapiro 1985 and others), which do not exist here.

3.4.2 Welfare-improving profitable licensing

Since technology transfers between inefficient firms can reduce welfare, the next question is whether there are conditions for transfers to increase welfare. Since the social welfare reduction occurs only because relatively inefficient firms' production crowd out more efficient firms' production, we can naturally guess that if the licensor is the most efficient firm then the social welfare should improve. Indeed, we can show that it is the case. For this result, we need no condition on demand function (see the appendix for the proof and all following proofs).

Theorem 2. Suppose that the most efficient firm (firm 1) makes a complete transfer to any firm j ($c_1 \leq c_2 \leq ... \leq c_j \leq ... \leq c_K$ and $c_1 < c_j$). Then, the social welfare improves.

Somewhat interestingly, the social welfare maximizing partner is not necessarily the least efficient firm. Although it is true that aggregate output and consumer surplus are maximized by choosing the least efficient firm as the partner, industry profit is also part of social welfare. The following example illustrates how the harm to industry profit means that welfare is not maximized by licensing to the least efficient firm.

Example 3. Consider a market with five firms with marginal costs $c_1 = 0$, $c_2 = 0.05$, $c_3 = 0.1$, $c_4 = 0.14$, and $c_5 = 0.2$. Demand function is linear $P(Q) = 1 - Q$. Firm 1 is the unique licensor ($i = 1$). Numbers in the table below are rounded to two decimal places.

$i=1$	P	π_1	π_2	π_3	π_4	π_5	$\sum \pi$	CS	SW
$j=1$	0.248	0.061	0.039	0.022	0.011	0.002	0.137	0.282	0.419
$j=2$	0.24	0.058	0.058	0.020	0.01	0.001	0.146	0.289	0.435
$j=3$	0.232	0.05	0.033	0.0537	0.008	0.001	0.150	0.295	0.445
$j=4$	0.225	0.051	0.031	0.016	0.051	0.001	0.148	0.300	0.448
$j=5$	0.215	0.046	0.027	0.013	0.006	0.046	0.138	0.308	0.446

The social welfare maximizing partner is firm 4, the consumer surplus maximizing partner is firm 5, and the industry-profit maximizing partner is firm 3. A different way to see why firm 4 maximizes social welfare is to note that social welfare is consumers' total benefit less the cost of production (which appears in industry profit). Licensing to the least efficient firm (firm 5) does result in the greatest cost reduction, however, there are two countervailing effects that result in lower total costs when firm 4 is licensed. First, total production by the lowest cost firms is greater when firm 4 is licensed (the price is higher). Second, when firm 4 is licensed, the least efficient firm still produces (firm 5), but its production is quite small so its contribution to total cost is negligible while when firm 5 is licensed firm 4's production is several times larger. As a result, in the above example while licensing instead firm 5 increases output by .01, total cost increase by .004 giving a "marginal cost" of .4 well above the price.

3.5 Choosing a Licensee Through Auctions

In this section we study i) which firm wins the right to use the technology and ii) how much is paid when the technology is licensed by auction. When there is a duopoly, as in Katz and Shapiro (1985), there is only one potential licensee and so a fixed-price licensing fee (calculated as the difference between the licensee's post-transfer profit minus its original profit) is optimal. However, if there are multiple potential licensees, then they compete over the exclusive license. As noted by Katz and Shapiro (1986), if one firm obtains the license, its rivals suffer from the market price reduction caused by the licensing. A firm's auction bid, then, must take this externality into consideration. However, unlike in Katz and Shapiro (1986), here the potential licensees are heterogeneous and so the non-licensees differ in the harm from a given firm winning the license and so in their willingness-to-pay.

We consider two types of auctions: The first is that each potential licensee bids for the right to use the technology and when a winner is selected, only the winner pays the license fee according to its bid (simple auction). The second is that each potential licensee offers a menu that describe how much it will pay the licensor depending on which of the potential licensees gets the technology; when a winner is selected all potential licensees pay the licensor according to their bids for that particular winner (menu auction). These two license auctions have advantages and disadvantages. A simple auction can be considered as a natural auction, since only the winner of the license auction pays for the license. However, the externalities created by potential licensees are not identical. If a firm is harmed more by rival j getting the license than rival k, it obviously would prefer firm k instead of j to obtain the license and so might be willing to pay to have firm k instead of j obtain the license. Given this, a menu auction also makes sense in a licensing market, although it is less natural at first glance (and for this reason may be viewed disfavorably by competition authorities).

*For the rest of the paper we will assume that the licensor is the most efficient firm, **firm 1**: i.e.,* $c_1 < c_2 \leq c_3 \leq ... \leq c_K$. This is a natural setup for the licensing problem as it is (trivially) the structure in the literature when the licensees as homogeneous, and as Theorem 2 assures, such licensing will certainly improve the welfare. Given that the licensor i is firm 1, we simplify our notation $\pi_k(1,j)$ by $\pi_k(j)$ which means the profit of firm k when firm j obtains the license from firm 1 for $k, j = 1, .., K$.

3.5.1 Simple Auction

A **simple auction** is a version of the first-price auction played by firms $2, 3, ..., K$, in which each firm $k \in \{2, .., K\}$ simultaneously offers $T_k \geq 0$ to be the unique licensee to the licensor who chooses as a licensee the firm (say, firm j) that maximizes the sum of firm 1's profit and T_j: i.e., $j \in M(T) \equiv \arg\max_{k \in \{1,...,K\}} (\pi_1(k) + T_k)$, where $T = (T_1, T_2, ..., T_K)$ (Recall that when $j = 1$, firm 1's technology is not transferred to any firm.). Knowing this, each firm in $\{2, ..., K\}$ chooses its bid T_k. In a simple auction, an outcome (j^*, T^*) is a **Nash equilibrium** if $j^* \in M(T^*)$ and there is no $k \in \{2, ..., K\}$ such that $k \in M\left(T_k, T^*_{-k}\right)$ and $U_k(k, T) > U_k(j^*, T)$, where $U_k(j, T) = \pi_k(k) - T_k$ if $j = k$.

Note that Nash equilibrium in simple auction is different from **competitive equilibrium** outcome (j^c, T^c), which is also the joint-profit maximizing outcome, that is described by the list of each potential licensee's offer to pay for the technology (comparing with the status quo) and the winning licensee whose offer is most attractive to firm 1: formally, it is defined by $j^c \in M(T^c) \equiv \arg\max_{k\in\{1,...,K\}} (\pi_1(k) + T_k^c)$ where $T_k^c = \pi_k(k) - \pi_k(1)$ for all $k = 1, ..., K$. In order to see the difference between competitive price and Nash equilibria, the following simple example would be helpful.

Example 4. There are $K = 3$ firms with marginal costs $c_1 = 0$ and $c_2 = c_3 = 0.2$. Note that this example corresponds to the standard assumption in the literature: the potential licensees are homogeneous. Demand function is $P(Q) = 1 - Q$. Without technology transfer ($j = 1$), equilibrium allocation is described by $P(1) = 0.35$, $q_1(1) = 0.35$, $q_2(1) = q_3(1) = 0.15$, and $\pi_1(1) = 0.1225$ and $\pi_2(1) = \pi_3(1) = 0.0225$. With firm 2 (symmetrically, firm 3 with permutation) being chosen as the licensor, the equilibrium allocation becomes $P(2) = 0.3$, $q_1(2) = q_2(2) = 0.3$, $q_3(2) = 0.1$, and $\pi_1(2) = \pi_2(2) = 0.09$ and $\pi_3(2) = 0.01$.

	P	π_1	π_2	π_3
$j = 1$	0.35	0.1225	0.0225	0.0225
$j = 2$	0.3	0.09	0.09	0.001
$j = 3$	0.3	0.09	0.01	0.09

If licensing is done through competitive pricing, both firms 2 and 3 are willing to pay $T_2^c = \pi_2(2) - \pi_2(1) = 0.0675$ which is the amount of additional profits generated by acquiring license. Firm 1 improves its profit from $\pi_1(1)$ to $\pi_1(2) + T_2^c$ by 0.035. Is T_c a Nash equilibrium licensee fee? It is not. To see this, assume that firm 2 got the licensee with fee T_c. However, firm 3's profit after licensing is $\pi_3(2) = 0.01$, but if it offers a fee more than 0.0675, it can earn $\pi_3(3) = \pi_2(2) = 0.09$. Firm 3 is willing to pay up to $T_3^* = 0.09 - 0.01 = 0.08$. Indeed, if $T_2^* = T_3^* = 0.08$, the rival firm will not challenge its bid; they form a Nash equilibrium. Conversely, if the bid by a firm is lower than 0.08, the other firm has an incentive to lower its bid, though both firms are willing to pay 0.08. Thus, this is unique Nash equilibrium of this simple auction (with permutation: both firms 2 and 3 can be selected as the licensee). The reason that $T_2^* > T_2^c$ is the negative externality effect from licensing on the rival firm.

We now turn to characterizing the Nash Equilibrium. The first question is whether there is a unique equilibrium in the simple auction. In the above example there was, but the potential licensees (firms 2 and 3) were symmetric (homogeneous) as is the standard assumption in the literature. However, once potential licensees are heterogeneous, then the uniqueness of Nash equilibrium may be lost as is shown in the following example.

Example 5. There are $K = 3$ firms with marginal costs $c_1 = 0$, $c_2 = 0.16$, and $c_3 = 0.24$. Demand function is $P(Q) = 1-Q$. Then, we have the following table.

	P	π_1	π_2	π_3
$j = 1$	0.35	0.1225	0.0361	0.0121
$j = 2$	0.31	0.0961	0.0961	0.0049
$j = 3$	0.29	0.0841	0.0169	0.0841

Suppose that firm 2 receives the license: $j = 2$. Then, firm 3 receives 0.0049, although it earns 0.841 if it gets the license. Thus, firm 3 is willing to pay $0.0841 - 0.0049 = 0.0792$. Now suppose that firm 3 receives the license: $j = 3$. Then, firm 2 receives 0.0169, although it earns 0.961 if it gets the license. Thus, firm 2 is also willing to pay $0.0961-0.0169 = 0.0792$ (the values are the same: it is just a coincidence). However, firm 1 prefers to give the license to firm 2 if the license fee is the same since the equilibrium price is higher with firm 2 instead of firm 3 as the licensee (a smaller transfer). Given the situation, one would expect firm 2 to get the license, and firm 3 to offer $T_3^* = 0.0792$, i.e., firm 3's bid less the gain to firm 1 from having firm 2 obtain the license. Then, in order to get the license, firm 2 can offer a little more than $T_3^* - (\pi_1(2) - \pi_1(3)) = 0.0672$. Since firm 3 cannot match such T_2^*, and it is a Nash equilibrium.

Is this the only Nash equilibrium? Unfortunately, it is not the only equilibrium without any additional refinements. Recall firm 2's gain from winning the license over firm 3 is 0.0792. If firm 3 somehow bid $T_3^* = 0.0792 + (\pi_1(2) - \pi_1(3)) = 0.0912$, firm 2 is willing to offer $T_2^* = 0.0792$. Of course, firm 3 does not want to pay that much, but as long as firm 2 matches the offer, and firm 1 selects firm 2, firm 3 does not lose anything. Thus, this allocation is also a Nash equilibrium. In fact, there is a continuum of Nash equilibria in this example.

Notice that, among all Nash equilibria (a continuum of equilibria), firm 3 will suffer if the license were somehow granted to it by accident (so it has to pay its bid) except for the equilibrium with $T_3^* = 0.0672$. In all other equilibria, the only role of firm 3's offer is to induce firm 2 pay more to the licensor. In the light of this, we consider a reasonable refinement of Nash equilibrium that is a version of truthful equilibrium. The idea is vaguely related to trembling-hand argument for the licensor. The licensor may make a slight mistake in choosing a potential licensee. Hence, each firm would be better-off by making a weakly dominant offer relative to the equilibrium outcome. For firm $j \in K \backslash \{1\}$, a strategy T_j is said to be **truthful relative to $\bar{j}$** if and only if either (i) $U_j(j, T) = U_j(\bar{j}, T)$ or (ii) $U_j(j, T) < U_j(\bar{j}, T)$ and $T_j(j) = 0$. A **truthful Nash equilibrium (TNE)** is a Nash equilibrium (j^*, T^*) such that each firm chooses a truthful strategy relative to j^*. With this refinement we can pin down and characterize the unique equilibrium in the above example.

Proposition 1. No licensing is a TNE of the simple auction game if and only if $\pi_1(1) + \pi_j(1) \geq \pi_1(j) + \pi_j(j)$ for all $k = 2, ..., K$. Suppose that no licensing is not a TNE. Then, a profile (j^*, T^*) is a TNE with licensing $(j^* > 1)$, if and only if $T^*_{j^*} = \max_{j \in K \setminus \{1\}} \{\pi_1(j) - \pi_1(j^*) + \pi_j(j) - \pi_j(j^*)\}$, $T^*_j = \pi_j(j) - \pi_j(j^*)$ for all $j \neq j^*$, and $\pi_1(j^*) + \pi_{j^*}(j^*) + \pi_j(j^*) \geq \pi_1(j) + \pi_j(j) + \pi_{j^*}(j)$ for all $j \neq 1$.

The last condition means that firm j^* is willing to challenge firm j by paying more if firm j gets the license. Suppose firm j receives the license. Then firm j^*'s payoff is $\pi_{j^*}(j)$, and firms 1 and j are jointly earning $\pi_1(j) + \pi_j(j)$. That is, the sum of these three firms's payoffs is $\pi_1(j) + \pi_j(j) + \pi_{j^*}(j)$. Now, if firm j^* receives the license then the total profit of these three firms is $\pi_1(j^*) + \pi_{j^*}(j^*) + \pi_j(j^*)$. If this value exceeds $\pi_1(j) + \pi_j(j) + \pi_{j^*}(j)$, firm j^* can beat firm j. If firm j^* can beat all other potential licensees, firm j^* wins the licensing auction. As a corollary of the first part of Proposition 1 and Theorem 1 (joint profit increases), we can state the following.

Corollary 2. Under weakly concave demand, no-licensing is not a TNE of the simple auction game if at least three firms remain in operation after licensing.

We call the licensee in a truthful Nash equilibrium outcome a **simple auction licensee**. Without negative externalities, the joint-profit-maximizing partner is the simple auction licensee. Taking externalities into account, a firm is the simple auction licensee if and only if such a transfer maximizes the joint profit of the licensor, the licensee and any one potential licensee firm. That is, if the licensor's profit plus the licensee's gain from not another potential licensee getting the license is greater than the licensor's profit and the gain of this other potential licensee. Comparing a simple auction licensee and the joint-profit-maximizing partner, it turns out that the simple auction licensee, if exists, is at least as efficient as the joint-profit-maximizing partner.

Proposition 2. Under weakly concave demand, the simple auction licensee (if it exist) is at least as efficient as the joint-profit-maximizing partner.

From the characterization of TNE (Proposition 1), it is easy to see that a Nash equilibrium in pure strategy must satisfy many inequalities. If there are only two potential licensees ($K = 3$), then it is easy to show the existence of TNE and to characterize it. However, if there are more than two potential licensees, finding Nash equilibrium (in pure strategies) is hard. Although we are unable to show the existence of a TNE in a simple auction game under weakly concave demand, we can show it always exists under the linear demand assumption.

Proposition 3. (1) Suppose that $K = 3$. Then, a profile (j^*, T^*) is a TNE with licensing ($j^* > 1$), if and only if (i) $j^* \in \arg\max_{j\in\{2,3\}} (\pi_1(j) + \pi_2(j) + \pi_3(j))$, and (ii) $T^*_{j^*} = \pi_1(j) + \pi_j(j) - \pi_1(j^*) - \pi_j(j^*)$ where $j \neq 1, j^*$. (2) Under the linear demand, there exists a TNE in a simple auction game.

Note that if $K = 3$, the TNE licensee in simple auction maximizes the industry's aggregate profit, and equilibrium is unique unless potential licensees are identical.

3.5.2 Menu Auction

Consider the effect firm 1 (the most efficient firm) has on the licensees when it chooses to license its technology to a firm (licensee) $j \in N = K\backslash\{1\} = \{2, \ldots, K\}$. Since there is a negative externality from the technology transfer, the other firms not receiving the transfer (non-licensees $N \backslash \{1, j\}$) would like to influence the licensing decision and may be willing to offer firm 1 not to license to firm j. We try to capture such strategic interaction using the menu auction framework proposed by Bernheim and Whinston (1986).

A **menu auction game** Γ is described by $(N+2)$ tuples:

$$\Gamma \equiv \left\{ A, (V_k)_{k \in N \cup \{1\}} \right\},$$

where A is the set of actions, $V_k : A \to \mathbb{R}$ is k's (quasi-linear) payoff function, 0 denotes the agent, and N is the set of principals. In the extensive form of the game the principals simultaneously offer contingent payments to the agent who subsequently chooses an action that maximizes her total payoff. A strategy for each principal $k \in N$ is a function $T_k : A \to [b_k, \infty)$, which is a monetary reward (or punishment) of $T_k(a)$ to the agent for selecting a, where b_k is the lower bound for payment from principal k. For each action a, principal k receives a net payoff:

$$U_k(a, T) = V_k(a) - T_k(a),$$

where $T = (T_{k'})_{k' \in N}$ is a strategy profile. The agent chooses an action that maximizes her total payoff: the agent selects an action in the set $M(T)$ with:

$$M(T) \equiv \arg\max_{a \in A} \left[V_0(a) + \sum_{k \in N} T_k(a) \right].$$

The menu auction game is merely a game among principals, although, strictly speaking, a tie-breaking rule among $M(T)$ needs to be specified for the agent. An outcome of a menu auction game Γ is (T, a). An outcome (a^*, T^*) is a **Nash**

equilibrium if $a^* \in M(T^*)$ and there is no $k \in N$ such that $T_k : A \to \mathbb{R}_+$ and $a \in M\left(T_k, T^*_{-k}\right)$ such that $U_k(a, T) > U_k(a^*, T)$. Unfortunately, with many coordination problems amongst the many players (principals), there are too many Nash equilibria in menu auction game.

In order to get plausible predictions among the many allocations supported by Nash equilibrium, Bernheim and Whinston (1986) consider a reasonable refinement on the set of Nash equilibria and they argue that truthful strategies are quite crucial in menu auction. A strategy T_k is said to be **truthful relative to** $\bar{a}$ if and only if for all $a \in A$ either (i) $U_k(a, T) = U_k(\bar{a}, T)$ or (ii) $U_k(a, T) < U_k(\bar{a}, T)$ and $T_k(a) = b_k$. An outcome (a^*, T^*) is a **truthful Nash equilibrium (TNE)** if and only if it is a Nash equilibrium, and T^*_k is truthful relative to a^* for all $k \in N$. They show that in menu auction games, the set of truthful Nash equilibria (TNE) and the set of coalition-proof Nash equilibria (CPNE) are equivalent in utility space. Bernheim and Whinston (1986) show that efficient action (the industry-profit-maximizing licensee in our context) is chosen by the agent in every TNE outcome in a menu auction: if (a^*, T^*) is a TNE, then we have $a^* \in \operatorname{argmax}_{a \in A}[\sum_{i \in N} V_i(a) + V_0(a)]$.

We let the set of principals be the set of potential licensees, $N = K \backslash \{1\}$, and the set of actions be the set of potential licensees as well, $A = K \backslash \{1\}$. The agent is the licensor firm 1. As any potential licensee cannot extract payment from the licensor, we have $b_k = 0$ for all k. We call the licensee in a TNE of menu auction as a **menu auction licensee**. We can show the following result.

Proposition 4. A menu auction licensee is at least as efficient as a simple auction licensee.

The underlying intuition of this proposition is that as the menu auction licensee is the industry-profit-maximizing partner and simple auction licensee is the three-firm-profit-maximizing partner. Thus, the negative externality of the technology transfer would make the industry-profit-maximizing firm more efficient than the three-firm-profit-maximizing partner to counteract the effect of the greater negative externality. Propositions 2 and 4 can be summarized as the licensing partners' efficiency ranking among different regimes in the following Theorem.

Theorem 3. Suppose that firm 1 is licensing technology to another firm. Under weakly concave demand, the licensing partner that maximizes the gains in their joint profit is weakly less efficient than the partner determined in a simple auction, and the latter is weakly less efficient than the partner determined by a menu auction: i.e.,

$$\text{menu auction licensee} \leq \text{simple auction licensee} \leq \text{joint-profit-maximizing partner},$$

where firms are ordered by its efficiency in a descending manner.

The following example illustrates that joint-profit-maximizing (competitive equilibrium) partner, simple auction licensee and menu auction licensee can be different.

Example 3 (revisited). Consider a market with five firms with marginal costs $c_1 = 0$, $c_2 = 0.05$, $c_3 = 0.1$, $c_4 = 0.14$, and $c_5 = 0.2$. Demand function is linear $P(Q) = 1 - Q$.

	P	π_1	π_2	π_3	π_4	π_5
$k = 1$	0.24833	0.06167	0.03934	0.022	0.01174	0.00234
$k = 2$	0.24	0.0576	0.0576	0.0196	0.01	0.0016
$k = 3$	0.23167	0.05367	0.033	0.05367	0.0084	0.001
$k = 4$	0.225	0.05063	0.03063	0.01563	0.05063	0.00063
$k = 5$	0.215	0.04623	0.02723	0.01323	0.00563	0.04623

It is easy to see that firm 3 is the menu auction licensee, since it is the industry-profit-maximizing partner as we have seen before. With the characterization in Proposition 1, we can check that firm 4 is the simple auction licensee. It is also easy to see that firm 5 is the joint-profit-maximizing partner (maximizes $\pi_1(k) + \pi_k(k) - \pi_1(1) - \pi_k(k)$ over $k = 2, ..., 5$). Finally, for comparison, recall that firm 4 maximized social welfare.

We conclude this section by providing additional results on menu auction licensees under more restrictive demand functions. Proposition 5 shows that under linear demand the menu auction results in the technology being licensed, even though no licensing can be a Nash equilibrium in general. Proposition 6 provides a sufficient condition for the licensees in simple and menu auctions to coincide. Since in a menu auction, non-licensing firms may be paying to have a particular licensee so as to prevent a more inefficient firm from obtaining the license, it is natural to conjecture that when the licensee is the only paying firm, the licensee is also the licensee under the simple auction.

Proposition 5. If demand is linear and $K \geq 3$, then licensing must occur in a truthful Nash equilibrium in menu auction.

Proposition 6. If firm j^* is a menu auction licensee and only j^* is paying for the license, then j^* is a simple auction licensee. If $K = 3$, the menu auction allocation coincides with the simple auction outcome.

3.6 Conclusion

In this paper we have analyzed which rival a licensor would choose as a partner when rivals are heterogeneous. We assume that the technology transfer is complete when licensing is made, and first show that a technology transfer between any pair of firms would improve joint profit (thus licensing is profitable) as long as there are more than two firms in the industry. However, jointly profitable licensing can be welfare reducing. These results are in contrast with the ones in the duopoly case examined by Katz and Shapiro (1985) as well as the welfare results others have found with homogeneous rivals (Katz and Shapiro 1986 and Sen and Tauman 2007). However, we also show that licensing the most efficient firm's technology is always social welfare-improving, though the licensor may not select the welfare maximizing licensee. We also analyze the licensee when either a simple or a menu auction is used and find that the licensor may not select the welfare maximizing licensee under these schemes too. However, we can order efficiency of the licensee by the licensing method: the joint profit maximizing licensee is less efficient than the simple auction licensee which in turn is less efficient than the menu auction licensee.

Note that in our welfare results we implicitly assumed that firms do not pay any recoverable fixed costs so that if a firm shuts down it does not save any fixed cost. If instead a fixed cost are recoverable, then the firm can avoid paying it by exiting the market. This would change the characterization because then the market price jumps up if a firm exits the market since the firm produces positive output at the time of exit. Although Theorems 1 and 2 will not be affected by the presence of annual fixed costs (as long as the fixed cost is common across firms), the results on licensing can be affected, since there will be a predation effect on the inefficient firms by licensing superior technology to other firms. This idea has been analyzed by Creane and Konishi (2009a) in the case where the technology transfer is made without monetary transfer. With a licensing fee, this motivation may be strengthened since potential licensees compete over technology more vigorously since their survival in the industry is at stake. Moreover, the licensor (and surviving firms) might prefer licensing technology to a very inefficient firm to push the market price sufficiently low to predate many other firms. Once firms exit, the market price jumps up again, increasing the pie to share. Although the idea is interesting, this extension brings further complications. For one thing, we assumed complete technology transfer in this paper,[8] but it is possible that partial technology transfer is more beneficial if predation by licensing is the licensor's motivation (especially if the licensor and the licensee are the only firms that can survive). Thus, the analysis will be significantly more complicated by allowing for predations. This extension will be left for future research.

[8] Complete transfer is a very reasonable assumption in the current paper's framework. It is not only a standard assumption in the literature (Katz and Shapiro 1986), but also is justifiable by the following observation. If demand is linear, there are at least three firms, and solutions are interior, complete technology transfer always dominates partial technology transfer as long as partial technology transfer is profitable. It is also possible that a partial transfer is harmful while complete transfer is profitable. Thus, under the above assumptions, we do not lose anything by considering complete transfer only.

Appendix A

Lemma 2. Suppose that there are initially K firms engaging in production. Pick three firms i, j, and j' with $c_i < c_j < c_{j'}$, and consider two scenarios: (i) firm i transfers its technology to firm j, and (ii) firm i transfers its technology to firm j'. Then, equilibrium aggregate output $\hat{Q}$ in scenario (i) is not more than equilibrium aggregate output $\tilde{Q}$ in scenario (ii), resulting in $P(\hat{Q}) \geq P(\tilde{Q})$.

Proof. In scenario (i) c_j goes down to c_i, while in scenario (ii) $c_{j'}$ goes down to c_i. Suppose that in scenario (i), firms $\ell \in L$ remain in operation: $q_\ell > 0$ (and firms $k \in K \backslash L$ chooses $q_k = 0$). Clearly, firms i and j will be in operation after technology transfer: $i, j \in L$. First consider the case where $j' \in L$ in scenario (i). Then, the aggregate output $\hat{Q}$ in scenario (i) is described by (recall that firm j's cost is c_i)

$$
\begin{aligned}
0 &= P'(\hat{Q})\hat{Q} + \sum_{\ell \in L} \left(P(\hat{Q}) - c_\ell\right) \\
&= P'(\hat{Q})\hat{Q} + \sum_{\ell \in L \backslash \{j, j'\}} \left(P(\hat{Q}) - c_\ell\right) + \left(P(\hat{Q}) - c_i\right) + \left(P(\hat{Q}) - c_{j'}\right) \\
&< P'(\hat{Q})\hat{Q} + \sum_{\ell \in L \backslash \{j, j'\}} \left(P(\hat{Q}) - c_\ell\right) + \left(P(\hat{Q}) - c_i\right) + \left(P(\hat{Q}) - c_j\right).
\end{aligned}
$$

Since $P'(Q)Q + \sum_{k \in K} \max\{0, P(Q) - c_k\}$ is decreasing function in Q, the equilibrium aggregate output $\tilde{Q}$ in scenario (ii) satisfies $\hat{Q} < \tilde{Q}$.

Second, consider the case where $j' \notin L$ in scenario (i). Since $c_j < c_{j'}$ and $P(\hat{Q}) < c_{j'}$, the aggregate output $\hat{Q}$ in scenario (i) is described by

$$
\begin{aligned}
0 &= P'(\hat{Q})\hat{Q} + \sum_{\ell \in L} \left(P(\hat{Q}) - c_\ell\right) \\
&= P'(\hat{Q})\hat{Q} + \sum_{\ell \in L \backslash \{j\}} \left(P(\hat{Q}) - c_\ell\right) + \left(P(\hat{Q}) - c_i\right)
\end{aligned}
$$

$$\leq P'(\hat{Q})\hat{Q} + \sum_{\ell \in L \setminus \{j,j'\}} \left(P(\hat{Q}) - c_\ell\right) + \max\left\{0, P(\hat{Q}) - c_j\right\} + \left(P(\hat{Q}) - c_i\right).$$

Thus, as before, the equilibrium aggregate output $\hat{Q}$ in scenario (ii) satisfies $\hat{Q} < \tilde{Q}$ if $P(\hat{Q}) - c_j > 0$, and $\hat{Q} = \tilde{Q}$, otherwise.$\square$

Theorem 1. Pick firms $i, j \in K$ with $c_i < c_j$. Assume that firm i is in operation originally, and that even after firm i transfers technology to firm j, there is still another firm k in operation ($q_k > 0$) with $c_k \neq c_i$. If demand is weakly concave ($P''(Q) \leq 0$), then a complete technology transfer from firm i to firm j is joint profit improving.

Proof. The proof utilizes an artificial market. This device is useful by observing the fact that transferring technology partially can reduce the joint profit. Instead, we replace firm j with an artificial (public: not profit-maximizing) firm i' with marginal cost c_i, but we control its output level so that the joint profit between firms i and i' increases monotonically. After that, we go back to the original economy. This is the strategy to prove the theorem.

Consider an *artificial* market parametrized by $\alpha \in [0,1]$, in which firm j ($c_i < c_j$) is replaced by an artificial firm i' that satisfies (i) $c_{i'} = c_i$, (ii) $q_{i'}(\alpha) = \alpha q_i(\alpha)$, and (iii) $(q_k(\alpha))_{k \neq i'}$ is a solution of the system of equations, $q_k(\alpha) = \max\left\{0, \frac{P(Q(\alpha)) - c_k}{-P'(Q(\alpha))}\right\}$ for all $k \neq i'$ and $Q(\alpha) = \sum_{k \neq i'} q_k(\alpha) + \alpha q_i(\alpha)$. That is, although the output decision by firm i' is linked with that of firm i, firms $k \neq i'$ do not use this information by choosing the best response to $Q_{-k}(\alpha) = \sum_{\ell \neq k} q_\ell(\alpha)$ (the standard Cournot behavior: not the Stackelberg one). Note that when $\alpha = 1$, $Q(1)$ is the aggregate Cournot equilibrium output after the complete technology transfer from firm i to firm j, since the best response by firm i' is identical to the one by firm i when $\alpha = 1$.

In the following, we will show that in this artificial market, the joint profit of firms i and i', $\Pi^J(\alpha) = (1+\alpha)\pi_i(\alpha) = \frac{(1+\alpha)(P(Q(\alpha))-c_i)^2}{-P'(Q(\alpha))}$, increases monotonically as α goes up (step 1). Then, we connect this artificial economy with the original economy before technology transfer (step 2).

(Step 1) The best response by firm $k \neq i'$ is described by

$$q_k(\alpha) = \min\left\{0, \frac{P(Q(\alpha)) - c_k}{-P'(Q(\alpha))}\right\}.$$

Since firm i will be in operation after technology transfer, we have

$$q_i(\alpha) = \frac{P(Q(\alpha)) - c_i}{-P'(Q(\alpha))},$$

thus we can write

$$q_{i'}(\alpha) = \alpha \times \frac{P(Q(\alpha)) - c_i}{-P'(Q(\alpha))}.$$

Let $L(\alpha) \equiv \{k \in K : q_k(\alpha) > 0\}$. As before, we denote the cardinality of $L(\alpha)$ by $L(\alpha)$ as well. Summing up these equations, we have

$$\sum_{\ell \in L(\alpha)} q_\ell(\alpha) = (1+\alpha)\frac{P(Q(\alpha)) - c_i}{-P'(Q(\alpha))} + \sum_{\ell \in L(\alpha)\setminus\{i,i'\}} \frac{P(Q(\alpha)) - c_\ell}{-P'(Q(\alpha))},$$

or

$$P'(Q(\alpha))Q(\alpha) + (L(\alpha) - 1 + \alpha)\,P(Q(\alpha)) - \left(\sum_{\ell \in L(\alpha)} c_\ell - (1-\alpha)c_i\right) = 0.$$

Totally differentiating the above, we have

$$(P''Q + P' + (L(\alpha) - 1 + \alpha)\,P')\,dQ + (P - c_i)d\alpha = 0$$

$$\frac{dQ}{d\alpha} = \frac{P(Q(\alpha)) - c_i}{-P''(Q(\alpha))Q(\alpha) - (L(\alpha)+\alpha)\,P'(Q(\alpha))}.$$

Since $Q(\alpha) = \sum_{\ell \in L(\alpha)} q_\ell(\alpha)$ and $P''(Q)q_\ell + P'(Q) \leq 0$ holds for all $\ell \in L(\alpha)\backslash\{i'\}$, we have

$$\begin{aligned}
&- P''(Q(\alpha))Q(\alpha) - (L(\alpha) + \alpha)\, P'(Q(\alpha)) \\
&= - \sum_{\ell \in L(\alpha)\backslash\{i,i'\}} (P''(Q(\alpha))q_\ell(\alpha) + P'(Q(\alpha))) - (1+\alpha)\,(P''(Q(\alpha))q_i(\alpha) + P'(Q(\alpha))) \\
&> 0.
\end{aligned}$$

The inequality is strict as long as there is at least a firm with a different marginal cost from others' (i.e., if $P''(Q)q_k + P'(Q) = 0$ holds then $P''(Q)q_\ell + P'(Q) < 0$ must hold due to the strategic substitute assumption). That is, for each $L \subset K$ with $L = L(\alpha)$ for some range of $\alpha \in [0,1]$, $\frac{dQ}{d\alpha} > 0$ holds for the range of α. This implies that $Q(\alpha)$ monotonically increases as α increases, resulting in monotonic reduction of $P(Q(\alpha))$. Since firms shut down their production in order from higher marginal cost ones (if any firm does it), the set of active firms $L(\alpha)$ shrinks in nested manner: $L(\alpha') \subseteq L(\alpha)$ for all $\alpha' > \alpha$.

Now, we will show $\Pi^J(\alpha) = \frac{(1+\alpha)(P(Q(\alpha))-c_i)^2}{-P'(Q(\alpha))}$ changes as α increases. We consider

$$\begin{aligned}
&\frac{d\Pi^J}{d\alpha} \\
&= \frac{(P-c_i)^2}{-P'} + (1+\alpha) \times \frac{2(P-c_i)P'(-P') + P''\,(P-c_i)^2}{(-P')^2} \times \frac{P-c_i}{-P''Q - (L(\alpha)+\alpha)\,P'} \\
&= A \times \left[(-P')(-P''Q - (L(\alpha)+\alpha)\,P') + (1+\alpha)\left\{-2\,(-P')^2 + P''\,(P-c_i)\right\}\right] \\
&= A \times \left[(-P')(-P''Q - (L(\alpha)+\alpha)\,P') + (1+\alpha)\left\{-2\,(-P')^2 + P''\,(P-c_i)\right\}\right] \\
&= A \times \left[\{(L(\alpha)+\alpha) - 2(1+\alpha)\}\,(-P')^2 + (-P'')\,\{-P'Q - (1+\alpha)\,(P-c_i)\}\right] \\
&= A \times \left[(L(\alpha) - 2 - \alpha)\,(-P')^2 + (-P'')\,\{(-P')\,(Q - (1+\alpha)q_i) - (1+\alpha)\,(P'q_i + P - c_i)\}\right]
\end{aligned}$$

where $A = \frac{(P-c_i)^2}{(-P')^2(-P''Q-(L(\alpha)+\alpha)P')} > 0$. We can determine the sign of $\frac{d\Pi^J}{d\alpha}$. Note that $P' < 0$ and $P'' \leq 0$. Since $L(\alpha) \geq 3$, $L(\alpha) - 2 - \alpha \geq 0$ must follows, and the first term in the bracket of the last line is nonnegative for all $\alpha \in [0,1]$. Since $L(\alpha) \geq 3$ with interior solution, we have $Q > (1+\alpha)q_i$, and $P'q_i + P - c_i = 0$ holds by firm i's first order condition. This implies that the second term is positive. Thus, we can conclude that $\frac{d\Pi^J}{d\alpha} > 0$ holds for all $\alpha \in (0,1)$.[9]

(Step 2) Now, we show that the equilibrium allocation with firm j is mimicked by an equilibrium allocation in our artificial market at a certain $\hat{\alpha} \in (0,1)$. Let $(\hat{P}, (\hat{q}_k)_{k=1}^K)$ be the Cournot equilibrium allocation before firm j received a complete technology transfer. Let $\hat{\alpha} = \frac{\hat{q}_j}{\hat{q}_i}$. Since $c_j > c_i$, we have $\hat{q}_i > \hat{q}_j \geq 0$ and $0 < \hat{\alpha} < 1$. Thus, $(\hat{P}, (\hat{q}_k)_{k=1}^K) = (P(\hat{\alpha}), (q_k(\hat{\alpha}))_{k=1}^K)$ holds, and the initial equilibrium allocation is mimicked by the equilibrium in an artificial market with $\alpha = \hat{\alpha}$. Since $\hat{q}_j = \hat{\alpha}\hat{q}_i = \hat{\alpha}q_i(\hat{\alpha})$, we have

$$
\begin{aligned}
\hat{\pi}_i + \hat{\pi}_j &= \left(\hat{P} - c_i\right)\hat{q}_i + \left(\hat{P} - c_j\right)\hat{q}_j \\
&= \left(P(\hat{\alpha}) - c_i\right) q_i(\hat{\alpha}) + \left(P(\hat{\alpha}) - c_j\right) \hat{\alpha} q_i(\hat{\alpha}) \\
&< \left(P(\hat{\alpha}) - c_i\right) q_i(\hat{\alpha}) + \left(P(\hat{\alpha}) - c_i\right) \hat{\alpha} q_i(\hat{\alpha}) \\
&= \Pi^J(\hat{\alpha}).
\end{aligned}
$$

Since $\Pi^J(\alpha)$ is monotonically increasing in α, we have $\Pi^J(\hat{\alpha}) < \Pi^J(1)$. Since $\Pi^J(1)$ is the same as the joint profit by firms i and j after the complete technology transfer from firm i to firm j, we can conclude that the joint profit by firms i and j must increase after the complete technology transfer.$\square$

[9] At αs with $P(Q(\alpha)) = c_k$ for some $k \in K$, Π^J is not continuously differentiable (the right and left derivatives are different), though it is a continuous function. However, it is clear that Π^J is monotonically increasing in α.

Theorem 2. Suppose that the most efficient firm (firm 1) makes a complete transfer to firm j ($c_1 \leq c_2 \leq ... \leq c_j \leq ... \leq c_K$ and $c_1 < c_j$). Then, the social welfare improves.

Proof. By Lemma 1, we know that if a technology transfer is made from a technologically superior firm to a technologically inferior firm, the equilibrium total output Q increases. Now, consider firm k. If C decreases keeping c_k constant, Q increases while q_k shrinks. We can write the relationship between Q and q_k (through changes in C behind) as follows:

$$q_k(Q) = \frac{P(Q) - c_k}{-P'(Q)}.$$

Let us denote the original (before transfer) equilibrium by "hat," and the new equilibrium by "tilde." Since firm j's marginal cost c_j only goes down from $\hat{c}_j = c_j$ to $\tilde{c}_j = c_i$ keeping all other marginal costs constant, we have $\hat{Q} < \tilde{Q}$ and $\hat{q}_k > \tilde{q}_k$ for all $k \neq j$. Then, we necessarily have $\hat{q}_j < \tilde{q}_j$ and $\tilde{q}_j - \hat{q}_j > \tilde{Q} - \hat{Q}$.

Since the social welfare is written as

$$SW = (\text{total benefit}) - (\text{total cost}) = \int_0^Q P(Q')dQ' - \sum_{k=1}^{K} c_k q_k,$$

we have

$$\begin{aligned}
\widetilde{SW} &= \int_0^{\tilde{Q}} P(Q')dQ' - \sum_{k=1}^{K} c_k \tilde{q}_k \\
&= \int_0^{\hat{Q}} P(Q')dQ' + \int_{\hat{Q}}^{\tilde{Q}} P(Q')dQ' - \sum_{k \neq j} c_k \tilde{q}_k - c_1 \tilde{q}_j \\
&= \int_0^{\hat{Q}} P(Q')dQ' + \int_{\hat{Q}}^{\tilde{Q}} P(Q')dQ' - \sum_{k \neq j} c_k \tilde{q}_k - c_1(\tilde{Q} - \hat{Q}) - c_1 \left(\tilde{q}_j - (\tilde{Q} - \hat{Q}) \right)
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\hat{Q}} P(Q')dQ' - \sum_{k\neq j} c_k \tilde{q}_k - c_1 \left(\tilde{q}_j - (\tilde{Q}-\hat{Q})\right) \\
&+ \int_{\hat{Q}}^{\tilde{Q}} P(Q')dQ' - c_1(\tilde{Q}-\hat{Q}).
\end{aligned}
$$

The last two terms are obviously positive since $P(\tilde{Q}) > c_1$. Thus, we have

$$
\begin{aligned}
\widetilde{SW} - \widehat{SW} &> \int_0^{\hat{Q}} P(Q')dQ' - \sum_{k\neq j} c_k \tilde{q}_k - c_1 \left(\tilde{q}_j - (\tilde{Q}-\hat{Q})\right) - \widehat{SW} \\
&= \int_0^{\hat{Q}} P(Q')dQ' - \sum_{k\neq j} c_k \tilde{q}_k - c_1 \left(\tilde{q}_j - (\tilde{Q}-\hat{Q})\right) - \int_0^{\hat{Q}} P(Q')dQ' + \sum_{k=1}^{K} c_k \hat{q}_k \\
&= \sum_{k=1}^{K} c_k \hat{q}_k - \sum_{k\neq j} c_k \tilde{q}_k - c_1 \left(\tilde{q}_j - (\tilde{Q}-\hat{Q})\right) \\
&= \sum_{k\neq j} c_k \left(\hat{q}_k - \tilde{q}_k\right) + c_j \hat{q}_j - c_1 \left(\tilde{q}_j - \sum_{k=1}^{K} (\tilde{q}_k - \hat{q}_k)\right) \\
&= \sum_{k\neq j} \left(c_k - c_1\right)\left(\hat{q}_k - \tilde{q}_k\right) + \left(c_j - c_1\right)\hat{q}_j > 0.
\end{aligned}
$$

Hence, we conclude $\widetilde{SW} > \widehat{SW}$.$\square$

Appendix B: Licensing Equilibria

We first characterize the set of Nash equilibria and then prove the existence of a Nash equilibrium. The proof is relegated to the appendix. Although firm 1 is not a bidder, we let $T_1 \equiv 0$ for notational convenience.

Lemma B1. In a simple auction, an outcome (j^*, T^*) is a Nash equilibrium in a simple auction if and only if

(a) $\pi_1(j^*) + T^*_{j^*} \geq \pi_1(j) + T^*_j$ for all j.

(b) If $j^* > 1$, then $\pi_1(j^*) + \pi_j(j^*) + T^*_{j^*} \geq \pi_1(j) + \pi_j(j)$ for all $j \neq j^*$.

(c) If $j^* > 1$ and $T^*_{j^*} > 0$, then $\pi_1(j^*) + T^*_{j^*} = \pi_1(j) + T^*_j$ for some $j \neq j^*$ and $\pi_1(j^*) + T^*_{j^*} = \pi_1(\tilde{j}) + T^*_{\tilde{j}}$ implies $\pi_{j^*}(j^*) - T^*_{j^*} \geq \pi_{j^*}(\tilde{j})$.

Proof. We first characterize the set of Nash equilibria and then we show the existence.

Consider (j^*, T^*) is a Nash equilibrium outcome. Condition (a) is obvious from the structure of the game. For (b), suppose we have some $j \neq j^*$ such that $\pi_j(j^*) < \pi_j(j) - \left[\left(\pi_1(j^*) + T^*_{j^*}\right) - \pi_1(j)\right]$, then firm j can offer $\tilde{T}_j = \pi_1(j^*) + T^*_{j^*} - \pi_1(j) + \varepsilon_j$ for some $\varepsilon_j > 0$ so that $U_j\left(j, \tilde{T}_j, T^*_{-j}\right) \geq U_j(j^*, T^*)$ and $U_1\left(j, \tilde{T}_j, T^*_{-j}\right) \geq U_1\left(k, \tilde{T}_j, T^*_{-j}\right)$ for all k. What remains is condition (c). If there is no j such that $\pi_1(j^*) + T^*_{j^*} = \pi_1(j) + T^*_j$, then from condition (a), we have $\pi_1(j^*) + T^*_{j^*} > \pi_1(j) + T^*_j$ for all j. Then firm j^* can offer $\tilde{T}_{j^*} = T^*_{j^*} - \varepsilon_{j^*}$ for some $\varepsilon_{j^*} > 0$ so that $U_{j^*}\left(j^*, \tilde{T}_{j^*}, T^*_{-j^*}\right) \geq U_j(j^*, T^*)$ and $U_1\left(j^*, \tilde{T}_{j^*}, T^*_{-j^*}\right) \geq U_1\left(k, \tilde{T}_{j^*}, T^*_{-j^*}\right)$ for all k. If there is some $\tilde{j}$ with $\pi_1(j^*) + T^*_{j^*} = \pi_1(\tilde{j}) + T^*_{\tilde{j}}$ such that $\pi_{j^*}(j^*) - T^*_{j^*} < \pi_{j^*}(\tilde{j})$, then firm j^* can deviate to $\tilde{T}_{j^*} = 0$ so that $U_{j^*}\left(\tilde{j}, \tilde{T}_{j^*}, T^*_{-j^*}\right) \geq U_j(j^*, T^*)$ and $U_1\left(\tilde{j}, \tilde{T}_{j^*}, T^*_{-j^*}\right) \geq U_1\left(k, \tilde{T}_{j^*}, T^*_{-j^*}\right)$ for all k. Therefore, any Nash equilibrium satisfies all three conditions.

Suppose to the contrary that an outcome satisfies the three conditions but (j^*, T^*) is not a Nash equilibrium. Condition (a) implies that firm 1 selects firm j^*. First, consider the case that j^* has incentive to deviate from $T^*_{j^*}$ to $\tilde{T}_{j^*}$. It is clear that $\tilde{T}_{j^*} < T^*_{j^*}$ because $\tilde{T}_{j^*} \geq T^*_{j^*}$ would still make firm j^* be the licensee with no less payment. However, condition (c) implies that when j^* reduces payment, there exists $\tilde{j} \neq j^*$, $\pi_1(\tilde{j}) + T^*_{\tilde{j}} = \pi_1(j^*) + T^*_{j^*}$ with $\pi_{j^*}(j^*) - T_{j^*}(j^*) \geq \pi_{j^*}(\tilde{j})$ to be chosen as the licensee, which violates the condition that j^* will deviate. Now consider $j \neq j^*$ deviates from T^*_j to $\tilde{T}_j$. Then we have $\tilde{T}_j \geq 0$ such that $\pi_1(j) + \tilde{T}_j \geq \pi_1(k) + T^*_k$ for all $k \neq j$ and $\pi_j(j) - \tilde{T}_j > \pi_j(j^*)$.

From condition (b), we have $\pi_1(j^*) + \pi^*_{j^*}(j^*) + T^*_{j^*} \geq \pi_1(j) + \pi_j(j^*)$. Hence, we have $\pi_1(j^*) + T^*_{j^*} - \tilde{T}_j > \pi_1(j)$. From condition (c), we have for some $\tilde{j} \neq j^*$, $\pi_1(j^*) + T^*_{j^*} = \pi_1\left(\tilde{j}\right) + T^*_{\tilde{j}}$, so that $\pi_1\left(\tilde{j}\right) + T^*_{\tilde{j}} - \tilde{T}_j > \pi_1(j)$ which contradicts the conditions that j deviates.$\square$

We now move on to characterize TNEs.

Proposition 1. No licensing is a TNE of the simple auction game if and only if $\pi_1(1) + \pi_j(1) \geq \pi_1(j) + \pi_j(j)$ for all $k = 2, ..., K$. Suppose that no licensing is not a TNE. Then, a profile (j^*, T^*) is a TNE with licensing ($j^* > 1$), if and only if such that $T^*_{j^*} = \max_{j \in K\setminus\{1\}}\{\pi_1(j) - \pi_1(j^*) + \pi_j(j) - \pi_j(j^*)\}$, $T^*_j = \pi_j(j) - \pi_j(j^*)$ for all $j \neq j^*$, and $\pi_1(j^*) + \pi_{j^*}(j^*) + \pi_j(j^*) \geq \pi_1(j) + \pi_j(j) + \pi_{j^*}(j)$ for all $j \neq 1$.

Proof. First suppose that no licensing is a TNE. Then, $\pi_1(j) + T^*_j \leq \pi_1(1)$ and $\pi_j(j) = \pi_j(1) + T^*_j$ holds for all $j \neq 1$. Thus, $\pi_1(1) + \pi_j(1) \geq \pi_1(j) + \pi_j(j)$ holds. Conversely, if $\pi_1(1) + \pi_j(1) \geq \pi_1(j) + \pi_j(j)$ holds for all $j \neq 1$, then $\pi_1(j) + T^*_j \leq \pi_1(1)$ and $\pi_j(j) = \pi_j(1) + T^*_j$ holds.

Second, we consider the case with licensing. Let (j^*, T^*) be a TNE. In a TNE, we have $T^*_j = \pi_j(j) - \pi_j(j^*)$ for all $j \neq j^*$. From condition (a) of Lemma B1, we have $\pi_1(j^*) + T^*_{j^*} \geq \pi_1(j) + T^*_j$ for all $j \neq 1$ so that $T^*_{j^*} = \max_{j \in K\setminus\{1, j^*\}}\{\pi_1(j) + \pi_j(j) - \pi_j(j^*) - \pi_1(j^*)\}$. This implies condition (b) of Lemma B1. By condition (c) of Lemma B1, we have $\tilde{j} \neq j^*$ such that $\pi_1(j^*) + T^*_j = \pi_1\left(\tilde{j}\right) + T^*_{\tilde{j}}$ and $\pi_{j^*}(j^*) - T^*_{j^*} \geq \pi_{j^*}\left(\tilde{j}\right)$. Hence, we have $\pi_1(j^*) + \pi_{j^*}(j^*) - \pi_{j^*}\left(\tilde{j}\right) \geq \pi_1\left(\tilde{j}\right) + \pi_{\tilde{j}}\left(\tilde{j}\right) - \pi_{\tilde{j}}(j^*)$. Since $\pi_1\left(\tilde{j}\right) + \pi_{\tilde{j}}\left(\tilde{j}\right) - \pi_{\tilde{j}}(j^*) \geq \pi_1(j) + \pi_j(j) - \pi_j(j^*)$ for all $j \neq j^*$, we have $\pi_1(j^*) + \pi_{j^*}(j^*) + \pi_j(j^*) \geq \pi_1(j) + \pi_{j^*}(j) + \pi_j(j)$ for all $j \neq 1$.

Consider $\pi_1(j^*) + \pi_{j^*}(j^*) + \pi_j(j^*) \geq \pi_1(j) + \pi_{j^*}(j) + \pi_j(j)$ for all $j \neq 1$. Define $T_j^* = \pi_j(j) - \pi_j(j^*)$ for all $j \neq j^*$ and $T_{j^*}^* = \max_{j \in K \setminus \{1\}} \{\pi_1(j) + \pi_j(j) - \pi_j(j^*)\} - \pi_1(j^*)$. It is easy to check all conditions in a Nash equilibrium are satisfied.$\square$

Proposition 2. Under weakly concave demand, the simple auction licensee (if exist) is at least as efficient as the joint-profit-maximizing partner.

Proof. Let $j^* \in \arg\max_{j \in K} [(\pi_1(j) + \pi_j(j)) - (\pi_1(1) + \pi_j(1))]$ be the joint-profit-maximizing partner. Suppose that there exists $k > j^*$ (i.e., $c_k > c_{j^*}$) such that $\pi_1(k) + \pi_k(k) + \pi_{j^*}(k) > \pi_1(j^*) + \pi_k(j^*) + \pi_{j^*}(j^*)$. Since we have $\pi_1(j^*) + \pi_{j^*}(j^*) - \pi_{j^*}(1) > \pi_1(k) + \pi_k(k) - \pi_k(1)$, it is easy to see $\pi_{j^*}(k) - \pi_k(j^*) > \pi_{j^*}(1) - \pi_k(1)$. However, we have

$$\begin{aligned}
\pi_{j^*}(k) - \pi_k(j^*) &= \frac{(P(Q_k) - c_{j^*})^2}{-P'(Q_k)} - \frac{(P(Q_{j^*}) - c_k)^2}{-P'(Q_{j^*})} \\
&< \frac{(P(Q_k) - c_{j^*})^2}{-P'(Q_{j^*})} - \frac{(P(Q_{j^*}) - c_k)^2}{-P'(Q_{j^*})}
\end{aligned}$$

since $-P'(Q_{j^*}) < -P'(Q_k)$ by weak concavity of P and Lemma 2. Then we have

$$\begin{aligned}
&\frac{(P(Q_k) - c_{j^*})^2}{-P'(Q_{j^*})} - \frac{(P(Q_{j^*}) - c_k)^2}{-P'(Q_{j^*})} \\
&= \frac{(P(Q_k) + P(Q_{j^*}) - c_{j^*} - c_k)(P(Q_k) - P(Q_{j^*}) + c_k - c_{j^*})}{-P'(Q_{j^*})} \\
&< \frac{(2P(Q_1) - c_{j^*} - c_k)(c_k - c_{j^*})}{-P'(Q_1)} = \pi_{j^*}(1) - \pi_k(1)
\end{aligned}$$

since $2P(Q_1) \geq P(Q_k) + P(Q_{j^*})$, $-P'(Q_1) < -P'(Q_{j^*})$ and from equilibrium conditions we have $c_k - c_{j^*} = [-P'(Q_k)]Q_k - [-P'(Q_{j^*})]Q_{j^*} +$

$K\left[P\left(Q_{j^*}\right)-P\left(Q_k\right)\right]$ so that $0 \leq P\left(Q_k\right)-P\left(Q_{j^*}\right)+c_k-c_{j^*} \leq c_k-c_{j^*}$. Hence, we have $\pi_{j^*}(k)-\pi_k(j^*)<\pi_{j^*}(1)-\pi_k(1)$, which is a contradiction.□

Lemma B2. For any distinct i, j and k, define $\Pi(i,j,k)=\left[\pi_i(j)-\pi_k(j)\right]-\left[\pi_j(i)-\pi_k(i)\right]-\left[\pi_i(k)-\pi_j(k)\right]$. Under linear demand, we have $\Pi(i,j,k)=0$ for $i<j<k$.

Proof. Define Q_k be the industry equilibrium output if the licensee is firm k. First, we have

$$\begin{aligned}
\Pi(i,j,k) &= \left[\frac{\left(P\left(Q_j\right)-c_i\right)^2}{-P'\left(Q_j\right)}-\frac{\left(P\left(Q_j\right)-c_k\right)^2}{-P'\left(Q_j\right)}\right]-\left[\frac{\left(P\left(Q_i\right)-c_j\right)^2}{-P'\left(Q_i\right)}-\frac{\left(P\left(Q_i\right)-c_k\right)^2}{-P'\left(Q_i\right)}\right] \\
&\quad -\left[\frac{\left(P\left(Q_k\right)-c_i\right)^2}{-P'\left(Q_k\right)}-\frac{\left(P\left(Q_k\right)-c_j\right)^2}{-P'\left(Q_k\right)}\right] \\
&= \frac{\left(2P\left(Q_j\right)-c_k-c_i\right)\left(c_k-c_i\right)}{-P'\left(Q_j\right)}-\frac{\left(2P\left(Q_i\right)-c_k-c_j\right)\left(c_k-c_j\right)}{-P'\left(Q_i\right)} \\
&\quad -\frac{\left(2P\left(Q_k\right)-c_i-c_j\right)\left(c_j-c_i\right)}{-P'\left(Q_k\right)}
\end{aligned}$$

Under the linear (inverse) demand function be $P(Q)=a-bQ$ where $a,b>0$. Thus, we have $-P'\left(Q_h\right)=-P'\left(Q_j\right)=-P'\left(Q_k\right)=b$ and $P\left(Q_h\right)=\left(a-C-c_1+c_h\right)/\left(1+K\right)$ for all $h\neq 1$. Hence, we have

$$\begin{aligned}
\Pi(i,j,k) &= \left(2\frac{a-C-c_1+c_j}{1+K}-c_k-c_i\right)\frac{c_k-c_i}{b}-\left(2\frac{a-C-c_1+c_i}{1+K}-c_k-c_j\right)\frac{c_k-c_j}{b} \\
&\quad -\left(2\frac{a-C-c_1+c_k}{1+K}-c_j-c_i\right)\frac{c_j-c_i}{b} \\
&= \frac{2}{b(1+K)}\left[-c_j\left(c_k-c_i\right)+c_i\left(c_k-c_j\right)+c_k\left(c_j-c_i\right)\right]=0.\square
\end{aligned}$$

Proposition 3. (1) Suppose that $K = 3$. Then, a profile (j^*, T^*) is a TNE with licensing $(j^* \neq 1)$, if and only if (i) $j^* \in \arg\max_{j \in \{2,3\}} (\pi_1(j) + \pi_2(j) + \pi_3(j))$, and (ii) $T^*_{j^*} = \pi_1(j) + \pi_j(j) - \pi_1(j^*) - \pi_j(j^*)$ where $j \neq 1, j^*$. (2) Under the linear demand, there exists a TNE in a simple auction game.

Proof. First (1). From the statement of Proposition 1, it is straightforward to show this.

Second, (2). Let $j_1 = \arg\max_{j \in K} [\pi_1(j) + \pi_j(j)] - [\pi_1(1) + \pi_j(1)]$ be the joint-profit-maximizing partner. If $j_1 < K$, then Proposition 2 has already shown that $\pi_1(j_1) + \pi_k(j_1) + \pi_{j_1}(j_1) > \pi_1(k) + \pi_k(k) + \pi_{j_1}(k)$ for all $j_1 < k$. If we have $\pi_1(j_1) + \pi_k(j_1) + \pi_{j_1}(j_1) > \pi_1(k) + \pi_k(k) + \pi_{j_1}(k)$ for all $k < j_1$, then we are done. Suppose not. Define $j_2 = \max\{j < j_2 : \pi_1(j) + \pi_j(j) + \pi_{j_2}(j) > \pi_1(j_1) + \pi_j(j_1) + \pi_{j_2}(j_2)\}$. We are going to show we have $\pi_1(j_2) + \pi_k(j_2) + \pi_{j_2}(j_2) > \pi_1(k) + \pi_k(k) + \pi_{j_2}(k)$ for all $k > j_2$: First, we will show $\pi_1(j_2) + \pi_k(j_2) + \pi_{j_2}(j_2) > \pi_1(k) + \pi_k(k) + \pi_{j_2}(k)$ for all $k > j_1$. It is trivial if $j_1 = K$. Consider $j_1 < K$. Using Lemma B2, we have

$$
\begin{aligned}
&\pi_1(j_2) + \pi_{j_2}(j_2) + \pi_k(j_2) \\
&= \pi_1(j_2) + \pi_{j_2}(j_2) + \pi_k(j_2) - \Pi(j_2, j_1, k) \\
&= \pi_1(j_2) + \pi_{j_2}(j_2) + \pi_k(j_2) - [\pi_{j_2}(j_1) - \pi_k(j_1) - \pi_{j_1}(j_2) + \pi_k(j_2) - \pi_i(k) + \pi_{j_2}(k)] \\
&= \pi_1(j_2) + \pi_{j_2}(j_2) + \pi_{j_1}(j_2) - \pi_{j_2}(j_1) + \pi_k(j_1) + \pi_{j_2}(k) - \pi_{j_1}(k) \\
&> \pi_1(j_1) + \pi_{j_2}(j_1) + \pi_{j_1}(j_1) - \pi_{j_2}(j_1) + \pi_k(j_1) + \pi_{j_2}(k) - \pi_{j_1}(k) \\
&= \pi_1(j_1) + \pi_{j_1}(j_1) + \pi_k(j_1) + \pi_{j_2}(k) - \pi_{j_1}(k) \\
&\geq \pi_1(k) + \pi_{j_1}(k) + \pi_k(k) + \pi_{j_2}(k) - \pi_{j_1}(k) \\
&= \pi_1(k) + \pi_{j_2}(k) + \pi_k(k)
\end{aligned}
$$

What remains is to show we have $\pi_1(j_2)+\pi_h(j_2)+\pi_{j_2}(j_2) > \pi_1(h)+\pi_h(h)+\pi_{j_2}(h)$ for all h such that $j_2 < h < j_1$. (This step is trivial if $j_2 = j_1 - 1$.) By construction, we have $\pi_1(j_1)+\pi_{j_1}(j_1)+\pi_h(j_1) \geq \pi_1(h)+\pi_{j_1}(h)+\pi_h(h)$. Hence, we have

$$
\begin{aligned}
&\pi_1(j_2)+\pi_h(j_2)+\pi_{j_2}(j_2) \\
&= \pi_1(j_2)+\pi_h(j_2)+\pi_{j_2}(j_2)+\Pi(j_2,h,j_1) \\
&= \pi_1(j_2)+\pi_h(j_2)+\pi_{j_2}(j_2)+[\pi_{j_2}(h)-\pi_{j_1}(h)-\pi_h(j_2)+\pi_{j_1}(j_2)-\pi_{j_2}(j_1)+\pi_h(j_1)] \\
&= \pi_1(j_2)+\pi_{j_1}(j_2)+\pi_{j_2}(j_2)+\pi_{j_2}(h)-\pi_{j_1}(h)-\pi_{j_2}(j_1)+\pi_h(j_1) \\
&\geq \pi_1(j_1)+\pi_{j_1}(j_1)+\pi_{j_2}(j_1)+\pi_{j_2}(h)-\pi_{j_1}(h)-\pi_{j_2}(j_1)+\pi_h(j_1) \\
&= \pi_1(j_1)+\pi_{j_1}(j_1)+\pi_{j_1}(j_1)+\pi_{j_2}(h)-\pi_{j_1}(h) \\
&\geq \pi_1(h)+\pi_h(h)+\pi_{j_1}(h)+\pi_{j_2}(h)-\pi_{j_1}(h) \\
&= \pi_1(h)+\pi_h(h)+\pi_{j_2}(h)
\end{aligned}
$$

Therefore, we have $\pi_1(j_2)+\pi_k(j_2)+\pi_{j_2}(j_2) > \pi_1(k)+\pi_k(k)+\pi_{j_2}(k)$ for all $k > j_2$. If we have $\pi_1(j_2)+\pi_k(j_2)+\pi_{j_2}(j_2) > \pi_1(k)+\pi_k(k)+\pi_{j_2}(k)$ for all $k < j_2$, then we are done. Otherwise, we can inductively define $j_{n+1} = \max\{j < j_n : \pi_1(j)+\pi_j(j)+\pi_{j_n}(j) > \pi_1(j_1)+\pi_j(j_n)+\pi_{j_n}(j_n)\}$ and repeat the argument to show $\pi_1(j_{n+1})+\pi_k(j_{n+1})+\pi_{j_{n+1}}(j_{n+1}) > \pi_1(k)+\pi_k(k)+\pi_{j_{n+1}}(k)$ for all $k > j_{n+1}$. Since j_n is strictly decreasing and $j_n \geq 2$, the process must end in finite steps. Then, there exists some firm j^* with $2 \leq j^* \leq j_1$ such that $\pi_1(j^*)+\pi_{j^*}(j^*)+\pi_{j^*}(k) > \pi_1(k)+\pi_{j^*}(k)+\pi_k(k)$ for all $k \neq j^*$.$\square$

Bernheim and Whinston (1986) characterize the set of TNE payoffs and action of a menu auction game. We utilize their efficiency (industry profit maximization) result of TNEs:

Proposition 4. A menu auction licensee is at least as efficient as a simple auction licensee.

Proof. Denote j^S and j^M simple auction licensee and menu auction licensee. By the property of a TNE in menu auction, we have $\sum_{h\in K}\pi_h\left(j^M\right)\geq\sum_{h\in K}\pi_h\left(j^S\right)$. By Proposition 1, we have $\pi_1\left(j^S\right)+\pi_{j^M}\left(j^S\right)+\pi_{j^S}\left(j^S\right)\geq\pi_1\left(j^M\right)+\pi_{j^S}\left(j^M\right)+\pi_{j^M}\left(j^M\right)$. Hence, we have $\sum_{h\in K\backslash\{1,j^S,j^M\}}\pi_h\left(j^M\right)\geq\sum_{h\in K\backslash\{1,j^S,j^M\}}\pi_h\left(j^S\right)$.This implies that $j^M\leq j^S$.□

Proposition 5. If demand is linear and $K\geq 3$, then licensing must occur in a truthful Nash equilibrium in menu auction.

Proof. Since a TNE in menu auction always achieves the most efficient action, it suffices to show that transfer to firm 2 always leads to higher total industry profit.

We are going to show the linear case first. Let the inverse demand function be $P\left(Q\right)=a-bQ$. The industry total profit would be

$$\Pi=\sum_{i\in K}\frac{1}{b}\left[\frac{a+\sum_{j\in K}c_j}{(1+K)}-c_i\right]^2$$

So that if transfer from firm 1 to firm 2, then the industry total profit is

$$\hat{\Pi}=\sum_{i\in K\backslash\{2\}}\frac{1}{b}\left[\frac{a+\sum_{j\in K}c_j+c_1-c_2}{(1+K)}-c_i\right]^2+\frac{1}{b}\left[\frac{a+\sum_{j\in K}c_j+c_1-c_2}{(1+K)}-c_1\right]^2$$

the change in profit would then be

$$
\begin{aligned}
&\hat{\Pi} - \Pi \\
&= \frac{1}{b} \sum_{i \in K \setminus \{2\}} \left[\left(\frac{a + \sum_{j \in K} c_j + c_1 - c_2}{1+K} - c_i \right)^2 - \left(\frac{a + \sum_{j \in K} c_j}{1+K} - c_i \right)^2 \right] \\
&+ \frac{1}{b} \left[\left(\frac{a + \sum_{j \in K} c_j + c_1 - c_2}{(1+K)} - c_1 \right)^2 - \left(\frac{a + \sum_{j \in K} c_j}{(1+K)} - c_2 \right)^2 \right] \\
&= \frac{1}{b} \sum_{i \in K \setminus \{2\}} \left(\frac{2a + 2\sum_{j \in K} c_j + c_1 - c_2}{1+K} - 2c_i \right) \left(\frac{c_1 - c_2}{1+K} \right) \\
&+ \frac{1}{b} \left(\frac{2a + 2\sum_{j \in K} c_j + c_1 - c_2}{1+K} - c_1 - c_2 \right) \left(\frac{c_1 - c_2}{1+K} - c_1 + c_2 \right) \\
&= \frac{1}{b} \frac{c_1 - c_2}{1+K} \left((K-1) \frac{2a + 2\sum_{j \in K} c_j + c_1 - c_2}{1+K} - 2 \sum_{i \in K \setminus \{2\}} c_i \right) \\
&+ \frac{K}{b} \frac{c_2 - c_1}{1+K} \left(\frac{2a + 2\sum_{j \in K} c_j + c_1 - c_2}{1+K} - c_1 - c_2 \right) \\
&= \frac{1}{b} \frac{c_2 - c_1}{1+K} \left(\frac{2a + 2\sum_{j \in K} c_j + c_1 - c_2}{1+K} + 2 \sum_{i \in K \setminus \{2\}} c_i - Kc_1 - Kc_2 \right) \\
&= \frac{1}{b} \frac{c_2 - c_1}{(1+K)^2} \left(2a + 2\sum_{j \in K} c_j + c_1 - c_2 + 2(1+K) \sum_{i \in K \setminus \{2\}} c_i - K(1+K)c_1 - K(1+K)c_2 \right) \\
&= \frac{1}{b} \frac{c_2 - c_1}{(1+K)^2} \left(2a + 2(2+K) \sum_{j \in K} c_j - \left(K^2 + K - 1\right) c_1 - \left(K^2 + 3K + 3\right) c_2 \right)
\end{aligned}
$$

Since each firm is having positive output after transfer, price must be higher than c_K. Hence, we have $\frac{1}{1+K}\left(1 + \sum_{j \in K} c_j - c_2\right) - c_3 > 0$ which implies $a + \sum_{j \in K} c_j > (K+2) c_2$ if $K \geq 3$. Then, we have

$$
\begin{aligned}
&\hat{\Pi} - \Pi \\
&> \frac{1}{b} \frac{c_2 - c_1}{(1+K)^2} \left(2(K+2) c_2 + 2(K+1) \sum_{j \in K} c_j - \left(K^2 + K - 1\right) c_1 - \left(K^2 + 3K + 3\right) c_2 \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{b}\frac{c_2-c_1}{(1+K)^2}\left(2(K+1)\sum_{j\in K}c_j-\left(K^2+K-1\right)(c_1+c_2)\right)\\
&= \frac{1}{b}\frac{c_2-c_1}{(1+K)^2}\left(2(K+1)\sum_{i>2}c_i-\left(K^2-K-3\right)(c_1+c_2)\right)\\
&\geq \frac{1}{b}\frac{c_2-c_1}{(1+K)^2}\left(2(K+1)(K-2)c_2-\left(K^2-K-3\right)(c_1+c_2)\right)\geq 0
\end{aligned}
$$

Hence, transfer to firm 2 always generate strictly higher industry total profit than no licensing and as TNE in menu auction always achieves the most efficient action, no licensing cannot be a TNE outcome of menu auction game.□

Proposition 6. If firm j^* is a menu auction licensee and only j^* is paying for the license, then j^* is a simple auction licensee.

Proof. Clearly $j^*=1$, then $\pi_1(1)\geq\pi_j(j)+\sum_{h\in K}T_h(j)$ for all $j\in K$. By truthful strategies, we have $T_j(j)\geq\pi_j(j)-\pi_h(1)$ for all $j\in K$. Therefore, we have, $\pi_1(1)+\pi_j(1)\geq\pi_1(j)+\pi_j(j)+\sum_{h\in K}T_h(j)\geq\pi_1(j)+\pi_j(j)$ for all $j\neq 1$. Hence, by Lemma 5, it is a TNE in simple auction.

Consider $j^*>1$. Define $T_h=\sum_{k\in K}T_h^*(k)$. It is easy to check that (j^*,T) is a NE. What remains is to show that (j^*,T) is a TNE. Since (j^*,T^*) is a TNE in menu auction, we have $\pi_1(j^*)+T_{j^*}^*(j^*)\geq\pi_1(k)+\sum_{h\in K}T_h^*(k)$ for all k. Bernheim and Whinston (1986) shows that there exists a TNE $(j^*,\tilde{T})$ such that $\pi_j(j^*)-\tilde{T}_j(j^*)=\pi_j(j)-\tilde{T}_j(j)$ if $\pi_{j^*}(j^*)-\tilde{T}_{j^*}(j^*)\geq\pi_j(j)$ and $\tilde{T}_j(j)=0$ if $\pi_{j^*}(j^*)-\tilde{T}_{j^*}(j^*)<\pi_j(j)$. Hence, $\pi_1(j^*)+\pi_{j^*}(j^*)-\pi_{j^*}(j)\geq\pi_1(j)+\sum_{h\in P}[\pi_h(j)-\pi_h(j^*)]$ for all j where $P=\{j\in K\backslash\{j^*\}:\pi_j(j^*)\leq\pi_j(j)\}$. By rearranging, we have $\pi_1(j^*)+\pi_{j^*}(j^*)+\sum_{h\in P}\pi_h(j^*)\geq\pi_1(j)+\pi_{j^*}(j)+\sum_{h\in P}\pi_h(j)$. For all $h\in P$, we have $\pi_h(h)\geq\pi_h(j^*)$, which implies $\pi_1(j^*)+\pi_{j^*}(j^*)+\pi_k(j^*)\geq\pi_1(k)+\pi_{j^*}(k)+\pi_k(k)$. By Proposition 3, we know (j^*,T) is a TNE in simple auction.□

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