Beyond Infinity: Georg Cantor and Leopold Kronecker's Dispute over Transfinite Numbers

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A Senior Honors Thesis

by

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Preface

In the past four years, I have devoted a significant amount of time to the study of mathematics and philosophy. Since I was quite young, toying with mathematical abstractions interested me greatly, and, after I was introduced to the abstract realities of philosophy four years ago, I could not avoid pursuing it as well. As my interest in these abstract fields strengthened, I noticed myself focusing on the 'big picture.' However, it was not until this past year that I discovered the natural apex of my studies.

While reading a book on David Hilbert[1], I found myself fascinated with one facet of Hilbert's life and works in particular: his interest and admiration for the concept of 'actual' infinity developed by Georg Cantor. After all, what 'bigger picture' is there than infinity? From there, and on the strong encouragement of my thesis advisor—Professor Patrick Byrne of the philosophy department at Boston College—the topic of my thesis formed naturally. The combination of my interest in the infinite and desire to write a philosophy thesis with a mathematical tilt led me inevitably to the incredibly significant philosophical dispute between two men with distinct views on the role of infinity in mathematics: Georg Cantor and Leopold Kronecker.

[1] Hilbert by Constance Reid. I recommend it

Introduction

Georg Cantor's theory of 'actual' infinity was revolutionary to mathematics. By producing an arithmetic of transfinite quantities that described an 'actual' infinity and thus infinities of *different sizes*, Cantor forced a revision of nearly all mathematical thought on the subject of infinity. Because of the methods he employed in defining his transfinite numbers and the claims he made about them, his work not only founded set theory as we know it, but also at once touched on the philosophical issues of mathematical existence, transcendence, and foundations.

Led by Leopold Kronecker, an entire school of thought opposed Cantor's abstractions. Because Cantor's ideas pushed the boundaries of mathematics and were often counter-intuitive, their mathematical importance and even validity came under intense scrutiny. Although strict at times, this opposing school of thought stressed a 'rigorous' basis for Cantor's work, and on the grounds of its 'lack of rigor' would harangue Cantor and his work for much of his life. Nevertheless, many of Cantor's contemporaries sided with him, and indeed it would be his ideas that would last. David Hilbert, for example, wholeheartedly supported Cantor's work, and once defended Cantor's work by claiming that it was nothing less than "the finest product of mathematical genius and one of the supreme achievements of purely intellectual human activity" (quoted in Reid 176).

In this work, I focus on exactly this philosophical dispute between Cantor and Kronecker and the broader philosophical implications that each man's ideas present. Because the ideas that motivate each of these men result in part from both their personal lives and mathematical work, I will start with separate descriptions of the upbringing, personal lives, education, influences, and works of these two men. The goal of these sections is to give the reader 'the facts' on each man, and thus they are relatively brief. I will attempt to mention only as much mathematics as is necessary to understand the following section, which describes philosophies of the infinite. Beginning with the major historical schools of thought on the infinite before Cantor, I will go on to describe the opposing philosophical ideas on the infinite espoused by Kronecker and Cantor. In this section, I must admit an unfortunate weakness of this work regarding information on Leopold Kronecker. In the absence of available sources to me about Kronecker's philosophical beliefs, I am forced to induce the major points of his philosophy from the specifics of his mathematics and third-party descriptions of him. Still, I will do my best to critique the two men in the final section, and finally relate my own thoughts on the subject.

In the end, I hope that the reader will gain a degree of appreciation for Cantor's work, the shock that it caused to the 19th century world of mathematics, and the importance of his philosophical dispute with Kronecker.

I. Cantor's Early Life and Work

Cantor's Upbringing and Personal Life

Georg Cantor was born on March 3, 1845 in St. Petersburg, Russia. While his mother, Maria Anna Böhm, was a baptized Roman Catholic, Cantor's father, Georg Woldemar Cantor, grew up in a Lutheran mission and raised his son a Lutheran. When Cantor was still a young child, the family moved from St. Petersburg, to Wiesdbaden, Germany, and finally to Frankfurt. Throughout these years, adolescent Cantor attended a boarding *Realschule* in Darmstadt and afterwards the *Höheren Gewerbeschule*[2] until 1862, consistently receiving strong evaluations across his courses.

Cantor knew by the spring of 1862 that he wanted to devote his life to mathematics. Feeling what he would consistently refer to as an "inner compulsion to study mathematics" (Dauben 277), Cantor followed his dreams ambitiously. The following August he took the *Reifeprüfung*, which qualified him to study the sciences at the university level, and within a few months Cantor began studying at the *Polytechnicum* in Zürich. Aside from his first semester in Zürich in 1863 and a summer in Göttingen in 1866, Cantor lived and studied in Berlin, where he spent the vast majority of his time studying "under some of the greatest mathematicians of the day: Kummer, Kronecker, and Weierstrass" (Dauben 31). He went on to receive his *Promovierung*[3] in 1866 from the University of Berlin and join the prestigious Schellbach Seminar for mathematics teachers in 1868. Within a year Cantor had accepted a teaching

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[2] "Higher Vocational School"
[3] Doctorate

position as a *Privatdozent*[4] at the University of Halle. Although he often expressed interest in a position at Berlin or Göttingen, Cantor's brilliant but often controversial work relegated him to a career at the second-tier University of Halle.

Although Georg Cantor led an extremely successful life, he was at times extremely self-critical and suffered from debilitating bouts of mental illness. While many scholars have proposed theories to explain these aspects of Cantor's personality, the lack of explicit information has resigned most to the realm of speculation. In general, however, they may be divided into two camps: those who blame pressure from Cantor's family (usually his father) for his breakdowns, and those who locate the source of his mental breakdowns as a combination of Cantor's firm belief in the truth of his controversial work and the antagonism of other intellectuals.

It is quite possible that Cantor felt extreme pressure from his father. E.T. Bell, for example, claimed in *Men of Mathematics* that "Georg Woldemar had a thoroughly deleterious and ruinous effect upon his son's psychological health" (Dauben 278). Throughout Cantor's childhood (and into adulthood), Georg Woldewar's religious fervor and high expectations for his son heavily influenced him. As a result, Cantor maintained a very self-critical mindset, possessing "the certainty that those who failed in life lacked a strong spirit and the belief that a truly religious spirit had to reside in the individual" (Dauben 276). He was the master of his own destiny, and failure was simply unacceptable. While this led to immeasurable success in the classroom for Cantor as a child and in the mathematical world as an adult, many experts accuse Georg Woldemar's strong beliefs and constant motivation of causing Cantor's mental illness.

Cantor certainly felt pressure from his father, but was that enough to drive him to mental illness? Many scholars would disagree entirely. Joseph Dauben, for example, describes

[4] Assistant Professor

Cantor's father as a "sensitive and gifted man, who loved his children deeply and wanted them to live happy, successful, and rewarding lives" (278). According to his argument, Cantor's breakdowns occurred as a result of overwork, frustration, and the criticism of such colleagues as Leopold Kronecker, a former professor (and mentor) of his in Berlin. This opinion has gained considerable favor among historians recently, as modern psychology has afforded us empirical studies of such conditions. In 1971, however, psychologist Ivor Grattan -Guinness concluded from the available historical data that Cantor's illness was "endogenous" (Dauben 285); that is, outside factors contributed little or nothing to his bouts of depression.

Despite these bouts of depression, Cantor had a tremendous mathematical career. Although these breakdowns debilitated him for a time, he would not enter into his first mental breakdown until after his *thirty-ninth* birthday, when he was already an established mathematical star. In fact, from the time he received his doctorate Cantor continued to prove himself an innovative and brilliant thinker. Before delving directly into the vast accomplishments of his early career, however, a brief treatment of the mathematical environment he entered into provides some necessary background information.

Early Influences

Throughout the 19^{th} century, the mathematical world exploded with new theories and developments. Often termed the 'century of rigor' by mathematical historians, mathematicians made tremendous strides in providing rigorous bases for many of the analytic theories of the past centuries. Since Cantor's own interests arose from topics in trigonometric series, the developments of the theory of trigonometric representations for functions provide a legitimate place to begin describing the mathematical developments of 19th century analysis.

In the early 19th century, Joseph Fourier initiated the study of trigonometric series in analysis. During the course of his studies on the conductivity of heat, Fourier formulated a method that allowed functions to be accurately represented by trigonometric series. Lead ing mathematicians such as Dirichlet, Riemann, Lipschitz, and Hankel then worked on expanding Fourier's approach to the representation of functions in order to determine a more exact definition of an analytic function. In 1829, Gustav Dirichlet published an article which defined a function as any correspondence between given domains, and determined that if a function is continuous (except possibly at its end points), then its Fourier series provides a complete representation of it. However, if a function is discontinuous, his result is not necessarily true. He was able to show that if a function has a finite number of discontinuities, then the corresponding Fourier series converges. Nevertheless, a pressing questioned remained: what happens when the function has an infinite number of discontinuities?

Picking up where Dirichlet left off, Georg Riemann devoted his *Habilitationsschrift*[5] to exactly this problem of Fourier-series representation for discontinuous functions. Redefining integration to include functions that are infinitely often discontinuous between any two limits, he expanded the number of functions that are representable by a Fourier series exponentially. With the strength of his new method of integration, he could approach functions of infinitely many oscillations on any finite interval. However, Riemann's treatment remained incomplete, since it could not be applied to functions with infinitely many maxima/minima, and made no attempt to determine whether or not a function may be uniquely represented by a trigonometric series.

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[5] Dissertation

Rudolph Lipschitz and Hermann Hankel also made significant contributions to mathematical knowledge on Fourier series. Although Lipschitz introduced concepts "equivalent to both everywhere- and nowhere-dense sets" (Dauben 22) that would become important in set theory, he only considered the values of the domain with certain functional characteristics (e.g. continuity). Similarly, Hankel's important distinction between point-wise discontinuous and totally discontinuous functions became very important to later work. However, it was limited due to his focus on the behavior of functions at specific points of singularity, as opposed to the entire domain of singularities. Later mathematicians, led by Cantor, would focus on the structure of the set itself.

Fourier series thus provided enough material for **a** entire generation of mathematicians to work on and led to the development of many new mathematical concepts. From his work on Fourier series, Dirichlet determined that a more comprehensive definition of an analytic function would be necessary for further progress and posited his own. Riemann, recognizing the hole in Dirichlet's theory formed by his exclusion of infinitely discontinuous functions, redefined the integral to include more such functions, but failed to fully characterize them. Taking yet another step towards rigor, Lipschitz and Hankel refined the concept of discontinuity of functions, but were ultimately limited by their consideration of discontinuous *functions* alone. Furthermore, the question of uniqueness of representation by a trigonometric series remained open. The mathematical world, consistent with the century's trend towards rigor, seemed to shout out for a mathematician to come along and provide a convincing answer to the questions raised by Fourier series.

The man who would attempt this task was Georg Cantor.

Early Work

Soon after reaching Halle in 1870, the problem of representation of functions by trigonometric series was brought to Georg Cantor's attention. Although Cantor's dissertation dealt with number theory, Edward Heine—a colleague of his at Halle with an interest in problems of analysis—encouraged Cantor to take up the study of trigonometric series Cantor subsequently pursued the question of uniqueness of trigonometric representations of arbitrary functions, and within a few months, building on Heine's own work, proved the result for every function that can be represented by a convergent trigonometric series at all values of its domain. However, as Cantor sought to further generalize his result, he realized that in order to extend the uniqueness theorem to infinite domains, he needed to establish "a rigorous theory of real numbers" (Dauben 37).

In 1872, Cantor published a paper that studied the form of the real numbers at a previously unattained level of rigor. Focusing on the irrational numbers, Cantor rejected the generally accepted definition (which used infinite series to define the irrationals), calling it one of "those rare cases in which real errors can cause no significant harm in calculations" (Dauben 37). In order to redefine the irrationals, Cantor proposed a new system that would define the real numbers as the set of limits of sequences of rational numbers. That is, any real number could be represented as the limit of a sequence of rational numbers.

With his idea of building a new domain from the set of limits of another domain, Cantor introduced the method that would lead him to his revolutionary definition of limit points of sets in general. Defining a limit point of a point set P as "a point of the line for which any neighborhood of same, infinitely many points of P are found" (Dauben 41), Cantor then used the entire set of these limit points (denoted P') to define further domains P'', P''', etc. For example, if P = Q (the set of rational numbers), then P' = R (the real numbers). Any real number contains an infinite number of rational numbers around it, and thus may be constructed as the set of limit points of the rational numbers. As Cantor continued to generalize these ideas, he eventually introduced the distinction between sets P of the 'first species,' which would end up as a finite set after a finite number of derivations (i.e. there exists a finite number 'v' for which P^v was a finite set), and those of the 'second species' which after infinitely many derivations remained infinite sets.

Throughout his early work Cantor displayed his ability to respond to limitations in theory by abstracting from already existing concepts in order to make them more rigorous and general. Starting off studying trigonometric series, Cantor moved back to a more fundamental look at the real numbers in order to extend his uniqueness proof. However, in the process, Cantor not only defined the real numbers, but also described the structure of point sets as they never had been before. Attaining such a remarkable degree of generality, Cantor was in fact able to extend his uniqueness theorem to "countable infinite sets for which representation was given up" (Dauben 49). Although Cantor had begun his study in order to 'rigorize' the theory of the real numbers, "Cantor in 1872 was interested in more than just analysis of the real numbers system and rigorous construction of the irrationals" (Dauben 44). The further he abstracted from already existing concepts, the more he delved into what would become transfinite set theory.

Opponents of Early Work

As his proof of the uniqueness theorem for trigonometric representations of functions spread throughout the mathematical community in 1870, Cantor earned himself a degree of notoriety. While still on relatively good terms with Leopold Kronecker from his days as a student in Berlin, Cantor's relationship with the famous German mathematician became strained. Although Kronecker actually helped Cantor simplify his proof of the uniqueness theorem, he "became increasingly uneasy" (Dauben 34) as Cantor proved corollaries which extended the theorem to infinite domains of definition. Due to its inclusion of functions that do not admit representation and those whose series do not converge (both concepts which Kronecker disapproved of), Cantor's extension of the uniqueness theorem to infinite domains displeased Cantor's former mentor greatly.

Kronecker was not alone in his opposition to Cantor's work. Although he did not explicitly denounce Cantor's theories, Richard Dedekind did express the opinion that Cantor's ideas of derived sets would lead to nothing new. Finding the distinction between derived sets "quite unnecessary" (Dauben 44), Dedekind was far from extolling praise on Cantor for his revolutionary thought. While these famous mathematicians fretted over Cantor's proof, Cantor was busy creating a new, even more controversial, branch of mathematics.

Transition to Set Theoretic Work and Major Breakthroughs

In the period from 1873 to 1879, Cantor made astounding strides in the development of transfinite set theory. In a letter to Dedekind in December 1873, Cantor related his discovery that no one-to-one correspondence can exist between the real numbers and natural numbers (that is, the real numbers are *nondenumerable*) and published his proof in early 1874. Perhaps in an attempt not to stir up too much controversy, Cantor titled his paper "On a property of the Collection of All Real Algebraic Numbers" (Dauben 50) and inserted his proof as a corollary of another theorem.

In general, however, the most significant development in this text was Cantor's use of a one-to-one correspondence to evaluate the size of an infinite set and corresponding introduction of the denotation of 'equal power' to any sets between which a unique, reciprocal one-to-one correspondence could be mapped. In other words, sets are of equal power when, for every element of one set, one (and only one) member of the other set exists. Between finite sets of numbers, this amounts to asking whether or not the two sets have an equal number of elements (or equal *cardinalities*). For example, {1,2,3} and {4,5,6} are of the same power, because there exists at least one way to map each element of {1,2,3} to {4,5,6}. There are, in fact, six such mappings:

(1) {1, 2, 3} → {4, 5, 6}, (2) {1, 2, 3} → {4, 6, 5}, (3) {1, 2, 3} → {5, 4, 6}

(4) {1, 2, 3} → {5, 6, 4}, (5) {1, 2, 3} → {6, 5, 4}, (6) {1, 2, 3} → {6, 4, 5}.

Now, between infinite sets, the number of whose elements is impossible to determine, the concept of 'power' (*Mächitgkeit*) becomes much more significant. Using this definition, it becomes possible to group different infinite sets by their power. Interesting, often counter-intuitive results become possible, like, for example, the result that the set of natural numbers and even natural numbers are actually of equal power (since for every natural number n, there exists a number 2n). More broadly applied, this definition offers relatively simple explanations to many complex results.

Given his proof of the nondenumerability of the real numbers, Cantor was able to produce further revolutionary results. In 1874, he posited that a one-to-one correspondence could be constructed between the line of real numbers and the plane (i.e. from R to R^2), and proved it in 1877. In 1878 Cantor published this result in his *Beitrag*, along with further development of the concept of a set's 'power.' Importantly, he established the power of the natural numbers as the smallest infinite power and the equality of the respective powers of the set of irrational numbers and real numbers. Cantor immersed himself in the study of transfinite set theory, and would spend the next four years refining and expanding his theory of transfinite numbers.

The years from 1879 to 1883 changed the face of Cantor's transfinite set theory. In a paper he published in 1879, Cantor elaborated on the idea of sets being 'everywhere dense' and discussed sets of power equivalent to that of the natural or real numbers. In 1880, he described more fully his sets of the 'second species.' However, the most comprehensive treatment of the transfinite numbers yet came in 1883 with the publishing of Cantor's *Grundlagen einer allgemeinen Mannigfaltigkeitslehre*[6]. In it, he explained that the role of transfinite numbers was to provide a description not of *potential* infinities, but of *actual* infinities. He did not want to present an inductive definition of the extent of the set of natural numbers, but rather attempted to provide a method of defining higher powers; that is, he wanted to describe sets of higher power than the natural numbers. He expressed the order of the entire set of natural numbers as the Greek letter 'ω' (a denotation still used by mathematicians) and defined this number as "the first number following the sequence of natural numbers" (Dauben 97). In a sense, the number represents a limit that the natural numbers approach but never quite reach.

[6] "Foundations of a General Set Theory"

Cantor's most important pages in his *Grundlagen* introduced his two 'Principles of Generation,' two corresponding number classes, and consistent arithmetic operations for transfinite numbers. The First Principle of Generation may be simply defined as the successive addition of units. We use this first principle to define the natural numbers when we claim that for any arbitrarily large finite natural number n, there exists another finite natural number (n+1). However, this principle alone allows us only to define variably finite numbers, and prevents any method of transcending the finite to reach the infinite, so Cantor introduced his Second Principle of Generation. In order to surmount the finite, Cantor stated that, given a limitless set of numbers, another number may be introduced as the first number larger than the entire set. Thus, given the infinite set of natural numbers (produced by the First Principle of Generation), Cantor's Second Principle of Generation legitimates the existence of ω as the first number larger than the set of natural numbers. Using these two Principles of Generation, Cantor provided his first distinction between number classes, naming the first (I) as the natural numbers, and the second (II) as "the collection of all numbers…which can be formed by means of the two principles of generation" (Dauben 98). Finally, Cantor fully fleshed out his theory by defining arithmetic operations of transfinite arithmetic. In providing logical definitions of addition, subtraction, multiplication, division, and prime factorization, Cantor managed to prove that his transfinite numbers formed a *consistent* whole.

Soon after the publishing of his *Grundlagen*, Cantor suffered a debilitating bout of depression. He did not publish any further work on transfinite set theory until 1895, when he released part I of his *Beiträge zur Begründung der transfiniten Mengenlehre*[7]; he published part II in 1897.

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[7] "Articles on the Foundation of Transfinite Set Theory"

In part I and II of his *Beiträge*, Cantor focused on the representation of the transfinite cardinal numbers. Although Cantor developed the theory of transfinite *ordinal* number in his *Grundlagen* in 1883, he did not fully describe the transfinite *cardinal* numbers until the *Beiträge*. While the ordinal number of a set refers to the ordering of the elements within the set, the cardinal numbers represent the number of elements in any transfinite set. He used varying notations to represent them, eventually settling on the Hebrew letter "aleph" ($\aleph$) as a representation. The cardinality of the natural numbers, the smallest transfinite cardinal number, would be represented by $\aleph$-null (or $\aleph_0$). To this day, ω and $\aleph$-null remain standard notations for, respectively, the ordinal and cardinal numbers of the set of natural numbers N.

The level of abstraction and ingenuity in Cantor's later work speaks for itself. Beginning with his proof of nondenumerability of the real numbers, Cantor applied the method of using a one-to-one correspondence as a means of evaluating the size of infinite sets in relation to one another. Continuing with his work on relative sizes of infinite sets, he proved that a one-to-one correspondence could be constructed between the line of real numbers and the plane, and went on to define explicitly the concept of a set's 'power.' During his subsequent study of the structure of continuous domains, he defined 'everywhere dense' sets and provided a consistent foundation for the transfinite numbers, including arithmetic operations and notation for transfinite ordinal numbers.

There is no way to overestimate the impact Cantor's work would have on later generations. Not only did he strengthen already existing foundations for analysis and prove exciting analytic results, but, most significantly, he "succeeded in presenting the first outline of a new mathematical discipline, one that would eventually enrich and transform mathematics by forcing it to come to terms with his new theory of the infinite" (Dauben 118).

Opponents of Set Theoretic Work

Once again Leopold Kronecker did not hold back in his criticism of Cantor's work. By 1870 Kronecker objected to the use in rigorous mathematics of the Bolzano-Weierstrass theorem[8], upper and lower limits, and the irrational numbers. Cantor's proof of the nondenumerability of the real numbers, which made use of many of these concepts, bothered Kronecker intensely. In an attempt to counter what he saw as a sort of mathematical perversion, Kronecker tried to delay the publishing of Cantor's *Beitrag* due to the discussion of a one-to-one correspondence between the line and plane and infinite 'powers' in general; both were ideas he found absurd at best. In Kronecker's opinion, Cantor's proof of a mapping from R to R^2 "was meaningless, without any hope of salvation" (Dauben 69).

Cantor's transfinite set theory had created a division in mathematics. His theories and proofs often went against the grain of normal mathematical thought, surprising many but also causing a fair amount of discomfort among certain mathematicians. His proof of the mapping from the line from the plane was no exception. As Herbert Meschkowski so eloquently wrote, "Cantor's theorem is thus a beautiful example of a mathematical paradox, of a true statement which seems to be false to the uninformed" (qtd. in Dauben 69). With such controversial results, Cantor's theories would take some time to settle into the hearts and minds of European mathematician

[8] The Bolzano-Weierstrass Theorem states that every bounded infinite set has a limit point.

II. Who is Leopold Kronecker?

Leopold Kronecker had become Georg Cantor's arch nemesis, but why? Given his importance to Cantor's career and work, it warrants a moment to attempt to describe him. Now, it may be unfair to cast Kronecker as an old curmudgeon who resisted all mathematical progress, although he certainly had a reputation for stubbornness. His dispute with Cantor over infinite set theory, although his most famous quarrel, was merely one of the mathematical feuds he found himself in. Interestingly, while many mathematicians held negative opinions of him, it is not entirely obvious as to why they did. Harmless intellectual disagreements are common among academics, so how did Kronecker manage to involve himself in such intense disputes? Was he a disagreeable man with a Napoleon complex, or a man lacking social graces, or perhaps something else entirely? In order to fully understand this controversial figure in the history of mathematics, I will review the basics of his personality, the extent of his power influence, and the foci of his mathematical works.

Personality

Perhaps the item of greatest dispute among those who study Kronecker is the nature of his personality. Since every extant text from Kronecker's hand is strictly mathematical, one must rely on the accounts of others regarding his personality in order to pass any sort of judgment on what type of person he was.

Today's prevailing point of view portrays Kronecker quite negatively. Described by some as a "difficult and prejudiced man" (Edwards, "Kronecker's Place in History," 139), Kronecker is often portrayed as a vindictive and unfriendly man who leveled vendettas against many of his colleagues. Consistent with this image, his lectures in Berlin were reputedly so flighty and disorganized that they "did not attract great numbers of students" (Biermann 506). In sum, according those historians who share this point of view, he was far from a benevolent personality.

Other historians, in an attempt to explain his personality and revise the evaluation of his reputation, claim that he suffered from nothing but an unusually abstract mathematical capability coupled with extremely strong convictions. Although he disagreed completely with Cantor's set theoretic notions, there is "little evidence of his alleged hostility and aggressiveness" (Edwards 140) towards Cantor. Perhaps he simply disagreed with Cantor's ideas fundamentally. Kronecker certainly was "a man who had strong opinions" (Edwards, "Kronecker's Place," 140), and one who thus needed extremely strong evidence to be convinced of something. Such a favorable claim seems plausible, especially when one remembers that this feud was not his only mathematical disagreement. For example, his views significantly "put him in opposition not only to Cantor…, but also to his colleague Weierstrass, who likewise had undertaken the 'arithmetization of analysis" (141). Furthermore, Kronecker apparently wished to avoid bad feelings between the two men, insisting "a disagreement over mathematical questions should not affect their personal relations" (142). It is at least possible that Kronecker was not as vindictive a man as he is often made out to be.

Such a favorable portrayal of Kronecker's personality begins to decrease in plausibility as one examines his relationships more closely. After all, most mathematicians hold strong convictions about their beliefs, but few engage in such heated disputes. If Kronecker were merely a man with interests only in mathematics and strong convictions, why did Cantor suffer bouts of depression and claim to feel targeted by him? Historian H.M. Edwards poses two options:

> “in the first place because he was in a much weaker position at a provincial university, and in the second place because Cantor’s personality contained a strain of paranoia that deeply disturbed at one time or another his relations with many contemporaries other than Kronecker” (142).

Notable among those other contemporaries were Weierstrass, H.A. Schwarz, and G. Mittag-Leffler. In other words, Edwards argues (perhaps apologetically) that Cantor himself inflated the significance of their disagreement, as he did with other mathematicians in his day.

It is at least possible that Cantor’s lack of mathematical clout and own mental instability caused his relationship with Kronecker to deteriorate, but this is not necessarily the case. Cantor did suffer from occasional bouts of mental illness. However, they did not *precede* his feud with Kronecker, but rather *followed* it. Nonetheless, such psychological questions extend beyond the scope of this paper.

Both of the portrayals of Kronecker’s personality contain plausible arguments. The number of strained relationships that Kronecker took part in seems to point to him as a disagreeable person. However, as most (if not all) mathematics students would probably attest, the opposing theory is possible as well. After all, one must admit that math professors are not normally renowned for their ‘people-skills.’ Perhaps it is due to their focus on the otherworldly abstraction of mathematics, although I do not propose to know what the exact reasons for such reputations are. Thus, it could happen that an incredibly nice professor seems, in initial conversation, obscure, stubborn, and generally disagreeable. Certainly, given my own experience of interacting with math professors, the opposing theory of Kronecker's personal traits seems quite plausible.

Although a full depiction of Kronecker's personality traits may never be obtained, drawing out the common strands from both schools of thought presents a rough outline. First, Kronecker had a reputation as a stubborn and disagreeable man (regardless of whether it was true or not). His thinking was highly abstract, allowing few to follow his train of thought, and his convictions were strong, especially his opposition to Cantor's infinite set theory and Weierstrass' ideas on analysis. Thus, although not much can be definitively stated about Kronecker's personality, his reputation, level of abstraction in teaching, and convictions cannot be denied.

Power and Influence

Throughout his mathematical career, Leopold Kronecker was a man to be both venerated and feared. As a professor in Berlin, he rose quickly, becoming a member of the prestigious Berlin Academy within 4 years. Realizing the potential for prestige inherent in any position at the University of Berlin, Kronecker even turned down an offer for the chair at Göttingen. Furthermore, he became quite active in his position at the Academy, aiding fifteen like-minded mathematicians in becoming members throughout his career as a member (Biermann 506). Kronecker had thus attained a tremendous degree of power and prestige among German mathematicians of his time. However, due to the obvious favoritism involved in his nominations and their importance in the mathematical world, his presence loomed as a regulatory force to those mathematicians that disagreed with him.

In his prime, Kronecker's approval was synonymous with success. If a young mathematician were to gain Kronecker's favor, he could be sure of some degree of membership in the Berlin Academy, which would permit him to offer a series of lectures in Berlin, his articles would be published in the *Journal für die reine und angewandte Mathematik*[9] (which Kronecker edited), as well as a respectable professorship. However, if a mathematician produced a theory not in accord with Kronecker's line of thinking (e.g. Cantor), he could expect a steep uphill battle to gain support for it (507).

In time, Kronecker's influence extended beyond the borders of Germany. Within five years of his 1863 nomination to the Berlin Academy, Kronecker became a member of the Paris Academy and within 21 years was named a foreign associate to Royal Society of London (507). Kronecker, a small man in height, truly lived up to his reputation as "small in stature, but large in power" (Edwards, "Kronecker's Place," 139).

Through his subjective favoritism and increasing international renown, Kronecker promoted his own ideology and muffled all challenges to it.

Capability as a Mathematician and Major Work

His ideology and favoritism aside, Leopold Kronecker was an extremely capable mathematician. There must have been, after all, a reason that he became "the uncrowned king of the German mathematical world" (Edwards, "Kronecker's Place," 139). Kronecker's ideas received considerable praise, and his work was quite significant in its day. In Berlin he lectured on the theory of numbers, the theory of determinants, and the theory of simple and multiple integrals. He produced respectable works on number theory, algebra, the theory of elliptical

[9] Journal for pure and applied mathematics

functions, and in particular “the interdependence of these mathematical disciplines” with regard to natural numbers (Biermann 506). The rigor and value of his work was so undeniable that “even Kronecker’s enemies admit he was a superb mathematician” (Edwards, “Kronecker’s Place,” 140).

Kronecker’s greatest mathematical achievements can be found in his effort to unify arithmetic, algebra, and analysis, especially in his work on elliptical functions. He often “attempted to simplify and refine existing theories and present them from new perspectives” (Biermann 506), creating new proofs of old theorems whose original proofs utilized concepts he did not approve of. For example, he would remove mention of irrational numbers from a proof and refer only to natural numbers, while retaining logical validity. Also, he revised and simplified many already existing theories in algebra and number theory (508).

At the height of his career, Kronecker focused the vast majority of his creative energy towards attempting to reduce all mathematical operations to those dealing in positive whole numbers. He became “preeminent in uniting the separate mathematical disciplines” (508). These efforts are among the most significant parts of his mathematical legacy for mathematicians today.

In fact, Kronecker’s legacy is not as remote as many outside the mathematical world (and some inside) would think. Although a student of mathematics today is more likely to hear the names of such contemporaries of his as Dedekind, Cantor, or Weierstrass in a mathematics lecture, Kronecker’s name lives on in concepts like Kronecker’s Lemma, Kronecker Delta, and Kronecker’s Polynomial Theorem, which deal with series approaching infinity, discrete delta functions in analysis, and algebra, respectively. Other mathematicians such as Hecke, Siegel, and Weil have used his work as a foundation for their own research, especially his boundary formulas, theorem on the convergence of infinite series, and theorems of number theory, algebra, and cyclotomy (Biermann 508; Edwards, "Kronecker's Place," 140). In fact, Kronecker's work in elliptical functions may have contributed at least indirectly to Andrew Wiles' 1993 proof of Fermat's Last Theorem (Van der Poorten 153, 156).

Yet, despite his work's importance among a few mathematicians, Kronecker's work remains relatively obscure in today's mathematical world for several reasons. First, although his work was "worthy of the highest respect," it was notoriously difficult to read (Edwards, "Kronecker's Place," 141), which strains the ability of a modern mathematician to reach back and build upon his work. Also, his theory of basing mathematics on purely natural numbers is simply not popular in today's mathematical culture. Thus, because much of his work reinterpreted previous theorems in terms of natural numbers, these proofs, insofar as they are his work, are most likely much more complicated than the original and thus not considered the most elegant proof of a theorem. Finally, in today's computer age, the idea of using algorithms to solve problems is highly valued, and while Kronecker valued the algorithm as a useful tool, he viewed them as something not to be relied on too heavily. They were to him a means to an end, and far from an end-in-themselves (Biermann 508). Thus, due to its difficulty, reliance on natural numbers, and avoidance of algorithms, Kronecker's work carries little weight today.

As H.M. Edwards states in "Kronecker's Place in History," Kronecker has been "undervalued and caricatured by historians because they have been following the lead of the philosophers of mathematics and of mathematicians themselves" (142), but is there perhaps another reason? Many of Kronecker's works and letters (in fact his entire *Nachlass*) seem to have been destroyed sometime after the Second World War. Sometime around 1943 it was decided that valuable books, papers and other items from the University of Göttingen should be stored at the bottom of a mineshaft that was already used to store explosives and field rations. Eventually, something exploded. A fire "with a flame spouting 100 meters out of the mine shaft" (Edwards, "On the Kronecker Nachlass," 422) broke out, and nearly everything inside was destroyed. Had some of the items of Kronecker's *Nachlass* been retrieved, it is at least possible that greater clarity might have been shed upon his work and philosophy.

Conclusion

Leopold Kronecker was a preeminent figure in late 19th century German mathematics. Whether he was a good or bad person, his imposing personality did not mesh well with those of other German mathematicians, most notably Cantor, thus resulting in a series of harsh, discriminatory, and long-lasting feuds. Due to his positions in various mathematical societies in Berlin, London, and Paris, as well as his pro-active involvement in nominating mathematicians to professorships, membership in mathematical societies, and publishing in mathematical journals, Kronecker's power and influence was unparalleled. However, despite acclaim for his work in his own time, Kronecker's work in number theory, algebra, and cyclotomy is usually considered dated due to its rejection of the irrational numbers and reliance on the natural numbers

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III. Philosophical Positions on the Infinite

Historical Positions on the Infinite

Cantor's transfinite set theory aroused the attention of more than just mathematicians in Europe. While it is certainly the case that Cantor's work on transfinite set theory impacted generations of mathematicians, "transfinite numbers were to prove no less revolutionary for philosophers and theologians who were concerned with the problem of infinity" (Dauben 118). The idea of describing numbers that are not finite using mathematical terms seemed to challenge existing opinions on the infinite and caused a great degree of controversy in certain philosophical circles.

Historically, the philosophical treatment of the infinite is to define infinity reductively; that is, to define it recursively with regard to something else. For example, Aristotle defined infinity in terms of a 'variable finite.' In mathematical terms, this amounts to saying thatthe natural numbers, for example, are infinitebecause for an y natural number 1, 2, …, n, (n+1),…, (n+1) > n is true, and since (n+1) is a natural number, one may conclude that there is an infinite progression of finite natural numbers. In practical terms, the variable finite is the finite thatcan always be pushed a stage further. According to this school of thought the infinite exists as a *potentiality*, not as an *actuality*

The definition of the infinite as a *potentiality* has gained more recent support as well. Kant, for example, supported such a definition of the infinite, holding that "while there may very well be a *regressus in indefinitum*…there cannot ever really be a *regressus in infinitum* which covers all the conditions or all the presuppositions of some actual stage of affairs" (Findlay 149-150).

Closer to Cantor's time, Carl Friedrich Gauss wrote that that the infinite was in mathematics "only a *façon de parler*, in which one properly speaks of limits" (Dauben 120). Other philosophers, such as Spinoza, and Leibniz rejected the idea on the basis of man's ignorance. Because we, as finite beings, are unable to comprehend any absolute thing (including, notably, God) "any attempt to assign a basis for determining magnitudes other than merely potential ones was predestined to fail" (Dauben 123). Finally, in Cantor's own time his former mentor Leopold Kronecker led an intellectual assault on Cantor's work, doing his best to halt the spreading of Cantor's ideas. The philosophical grounds for his objections are the topic of a later section.

Many Christian philosophers also supported the infinite as a potentiality, although the reasoning behind their argument differs slightly. Thomas Aquinas, for example, argued vehemently against the actual infinite, regarding "the idea as a direct challenge to the unique and absolutely infinite nature of God" (Dauben 120). That is,because the infinite belongs exclusively to God, admitting a description of the actual infinite is equivalent to accepting the existence of some infinite presence *other than God*. Furthermore, the Absolute, in its nature, is uniquely predicated and thus beyond determination. Since the very act of describing infinity determines it as something other than absolute, it becomes "necessarily finite by definition" (Dauben 123). Cardinal Johannes Franzelin, a leading Jesuit philosopher of the 19th century, warned Cantor that believing in the existence of the transfinite numbers in any form "could not be defended and in a certain sense would involve the error of Pantheism" (Dauben 145). Pantheism, a hot-button issue for the Roman Catholic Church in the 19th century (it was formally condemned in 1861), equates God with the forces and laws of the universe and had been growing in popularity due partly to the works of Spinoza. Since "any attempt to correlate God's infinity with a concrete, temporal infinity suggested Pantheism" (Dauben 145), Cantor seemed to be aiding the cause of the Pantheists. Clearly, the devoutly religious Cantor would not agree.

Cantor's Defense of the Infinite and Mathematical Philosophy

Such historical opposition to the absolute infinite was difficult to challenge, much less disprove. Conscious of this, mathematicians typically steered clear of any attempt to describe the infinite as an actuality rather than a potentiality. Georg Cantor, however, was far from typical. Driven by the belief that his was a divinely inspired theory that "had been revealed to him" (Dauben 147), Cantor felt the responsibility to defend it with his heart and soul. He studied the various positions of philosophers on the infinite and prepared counterarguments to each of their claims, as well as formed his own mathematical philosophy.

Cantor's reaction to those who believed in the infinite as merely a potentiality involved a common fallacy that he came across in their arguments. Almost every philosopher wrote under the implicit assumption that finite properties such as addition and subtraction would necessarily apply to transfinite numbers exactly as they do to finite ones. For example, many scholastics used the Aristotelian argument of the 'annihilation of number' against the infinite. They argued that during, for example, the addition of a finite and a transfinite number, the finite number "would be swallowed up by any infinite number of magnitude" (Dauben 122). Indeed, such would be the case, were the operations equal in the finite and transfinite arithmetic. However, such a statement held no bearing on Cantor's theory because he had defined an entirely unique and consistent transfinite arithmetic. In a certain respect, it is true that a finite number may be "swallowed up" by a transfinite number, such as in the following addition of 1 and ω (the smallest transfinite ordinal number):

$$1 + \omega = 1, 1, 2, 3,\ldots = \omega$$

This follows because the addition of 1 from the left side does not alter the ordering of the natural numbers (ω) in any way. However, Cantor's definition does not allow the reverse case to be true. That is,

$$(\omega) + 1 = 1, 2, 3, \ldots, 1 = (\omega + 1) \neq \omega$$

In other words, when a finite number is added to a transfinite on the right side, it alters the ordering of the set of natural numbers. Thus, it may not be 'annihilated,'but rather it modifies the transfinite ordinal number ω. By fleshing out his definition of transfinite arithmetic for subtraction, multiplication, and division in a similarly consistent manner, Cantor laid all such objections to rest.

Rebutting those who attacked his transfinite numbers on religious grounds, Cantor argued that in fact his theory displays God's glory. As a matter of fact, religion was so important to Cantor that, in a letter to Hermite, he calls it his "first love" [*erste Flamme*] (Meschkowski 124). Although he found errors in Thomas Aquinas' writings, Cantor expressed himself as an enthusiastic neo-Thomist. Hoping to fully to explain his mathematical concepts as a wonder of God, Cantor held that his theory of the infinite could remedy any failings. As such, he aimed at not just the correction but "the *perfection* of Christian philosophy" (Dauben 296). Oddly, in what Herbert Meschkowski calls an "astonishing" (124) phenomenon, Cantor made great attempts to reconcile his ideas with Church fathers and Catholic theology, although he was, in fact, a Lutheran. While some historians suggest that his deference to the Catholic Church arise from his love for his Catholic mother, Meschkowski disagrees. He appealed to the Church, he argues, not as much due to his Catholic mother, but rather because he received a great deal of support for his theory of the actual infinite from Catholic philosophers and theologians.

Constantin Gutberlet was among Cantor's greatest contemporary philosophical allies. A leading philosopher of the Church who supported Cantor's absolute infinite (and avowed neo-Thomist), Gutberlet claimed that "instead of diminishing the extent of God's nature and dominion, the transfinite numbers actually made it all the greater" (Dauben 143). Indeed, Gutberlet argued that God's infinite greatness guaranteed the existence of Cantor's transfinite numbers. Such an argument appealed to the devout neo-Thomist in Cantor greatly. Disinclined to underestimate the significance of his works, Cantor "viewed his series of alephs as 'something holy', as 'the steps that lead up to God's throne'" (Meschkowski 124). His work not only was significant to mathematics, but its awe-inspiring capability could even be useful to the Church in attracting converts within the scientific community. Eventually, he "hoped his efforts would help to promote the same spirit Pope Leo XIII seemed to encourage in urging a revival of neo-Thomism" (Dauben 296).

Clearly, Cantor believed wholeheartedly in the truth and existence of his transfinite numbers. But how did he determine that they existed, and how exactly did their existence attest to the glory of God? In order to explain the reasoning behind his beliefs, I will need to take a step back for a moment and consider some of Cantor's basic philosophical beliefs.

Cantor believed that a theory's internal consistency proved the mathematical possibility of a concept. That is, if one can define the parameters of the theory such that no contradictions may be produced as a result of mathematical manipulations (i.e. it is consistent), it is mathematically possible. Importantly, for Cantor "consistency alone was the determining factor in any question of mathematical existence, since God could realize any 'possibility'" (Dauben 229). In other words, internal *consistency* was sufficient to prove the *possibility* of a mathematical object, and this mathematical *possibility* in turn was sufficient to prove mathematical *existence* of the object, because God would necessarily bring any possibility to existence. But what sort of existence? To Cantor, phenomenological existence held little weight on the absolute existence of an object, and thus never claimed that his ideas had any existence in the physical world. As Joseph Dauben writes:

> "In the phenomenological world, one might be used to thinking of numbers as linked in succession, from a given n proceeding to (n+1), but Cantor thought the numbers used in mathematics had an entirely separate existence" (230).

These mathematical numbers existed immanently were thusindependent of any form of sequencing and totally infinite. In fact, Cantor would argue, it would contradict the fact of God's overarching power to claim that they did *not* exist, since that would imply that God was not all-powerful. To sum, "if ideas were consistent, then they were possibilities, and as possibilities they had to exist in the mind of God as eternally true ideas" (229); claiming that God was incapable of realizing this possibility seemed to Cantor a great sacrilege.

As perhaps the reader may have already noticed, Cantor's words on mathematical objects draw strict parallels to the world of Platonic forms. This was no accident. Cantor espoused a "strong form of Platonism…to which [he] returned repeatedly for support" (Dauben 229). Cantor's works are highly Platonic, and some have claimed that he was "truly the last great messenger of Platonic thought in mathematics" (Meschkowski, *Vorwort*). Even in his early works on well-defined sets, for example, one cannot help but notice the "sense of Platonism underlying his position" (Dauben 83) on their ontological existence. In time, Cantor argued not only the existence of his transfinite numbers, but also proposed their higher level of existence.

Cantor wrote in a letter to Hermite that "the (whole) numbers seem to me to be constituted as a world of realities which exist outside of us" (228), and compared the level of their existence to that of Nature. However, he went on to explain that the reality and absolute existence of the natural numbers was in fact much greater than that of Nature because their existence is not based on knowledge that we receive from our senses. In a sense, one may interpret him as saying that mathematical objects are *more real* than physical objects, due to their eternal and immanent nature. Thus the natural numbers, for example, "exist in the highest degree of reality as eternal [*ewige*] ideas in the *Intellectus Divinus*" (228). Clearly, then, the transfinite numbers were also real, because they had the same level of internal consistency (and thus immanent reality) as the finite numbers. Without a doubt, Cantor believed that the 'Transfinitum' (all the transfinite numbers) existed as eternal ideas outside the physical world.

Cantor's Platonist ideas led him to famously announce that "the essence of mathematics is its freedom" (Dauben 132). Whether or not mathematical objects had physical representations in the world, their immanent existence, ensured by their internal consistency, allowed them to be considered eternal. This unchanging and transcendent aspect of mathematics allowed the mathematician a degree of creative license unavailable to any physical scientists (since their work depends on the empirical world). In fact, the physical world, i.e. space and time, "could contribute nothing" (Dauben 108) to mathematical inquiry. According to these ideas, his transfinite numbers become virtually impervious to criticism. Given their internal consistency, they existed immanently as eternal and true concepts. Thus all attempts to deny the validity of the transfinite numbers due to their lack of an 'image' in the physical world or mere doubt about their existence become dubious accusations of men who do not understand the nature of mathematics.

Cantor's reliance on internal consistency of mathematical concepts would influence an entire mathematical movement. His work "cleared the path for modern Formalism" (Meschkowski, *Vorwort*), a movement popularized by David Hilbert that relied solely on the consistent manipulation of mathematical signs. Although the movement lacked Cantor's religiousness and thus his Platonism, his influence was marked. Even in his own time, Cantor realized that many would not agree with his religious beliefs, and thus did not insist that other mathematicians accept his arguments for the existence of transfinite numbers. Whether or not they accepted what he saw as their eternal truth, they were nonetheless mathematically valid due to their internal *consistency*. Like the rational, irrational, or complex numbers, as long as any new numbers or concepts are defined via a relation from older ones so that "in given cases they can definitely be distinguished from one another…[they] must be regarded as existent and real in mathematics" (Dauben 129). In the end, formalists would inherit from Cantor his emphasis on the internal consistency of, for example, his transfinite numbers as the only litmus test mathematicians need consider before accepting them as valid.

An interesting application of Cantor's formalism may be found in his vehement opposition to the existence of infinitesimal quantities. Significantly referred to as "the clearest expression of his formalist justification for ideas in mathematics" (Dauben 131-132), Cantor objected to their existence as a result of their incompatibility with his definition of linear numbers. He viciously attacked mathematicians who entertained the notion of infinitesimals,a ccusing them of trying to "infect mathematics with the Cholera-Bacillus of infinitesimals" (131), and mademany attempts to refute them. Throughout his various attempts, however, his method remained more or less constant. Using the Archimedean Axiom[10] to discount infinitesimals, Cantor argued that these

[10] The Archimedean Axiom states that, given any two real numbers a and b such that $a<b$, there exists a natural number n such that $na>b$.

infinitesimals cannot exist because they are not *linear*, i.e. that no finite or infinite number of infinitesimals can be added together to produce a finite magnitude. However, his 'proofs' quickly fell apart as it was noticed that it depended on the *petition principii* that all numbers must be linear (130-131). Thus, try as he might, Cantor was never able to convincingly disprove the existence of infinitesimals.

His first statements concerning the existence of infinitesimals, however, provide us with an insight to another idea that Cantor would confront in his work: therole of intuition in mathematics. As he approached the question of the difference between discrete and continuous domains in his paper of 1882 on multi-dimensional domains, Cantor inserted a footnote to clarify his opposition to infinitesimals. Objecting to geometric bases for infinitesimals, Cantor claimed to strive further for atheory of number wholly independent of intuition. He argued for arithmetic treatments of number and sets and rejected treatments that admitted infinitesimals due to the purely intuitive nature of their conception. In general, he wrote that such intuition was "a poor guide in rigorous mathematics" (Dauben 87). As of 1882, though, Cantor did not explain this belief further than mention in as a note in his paper. Later, however, Cantor would engage in highly intuitive mathematical abstractions. He appealed, for example, to "our active powers of thought" (Dauben 171) when defining the concepts of 'set', 'power', and 'cardinality' in his *Beiträge*, and would receive a great amount of criticism for the highly intuitive nature of his work from such esteemed intellectuals as Frege. As such, it is difficult to understand Cantor's opinion on the role of intuition in mathematical creativity.

Throughout his lifetime, Cantor's works made use of a philosophically revolutionary method of transcendence to provide a direct (i.e. non-reductive) description of the infinite. In the past, theories of the infinite used the inexhaustibility of any infinite quantity as a means of proving that infinity itself may never be determined. Cantor would not disagree, although he would append a further notion of transcendence to their theory. While it is not possible to exhaust an infinite series by successively naming terms until the last one is reached, it may be exhausted in a different sense; to use different terms, while it is not possible to exhaust an infinite series by applying Cantor's First Principle of Generation, the Second Principle allows us to do just that. By regarding infinite sets as complete unities, Cantor uses his Second Principle of Generation to name the first number that is greater than of the infinite set. By redefining the idea of 'exhausting' a series, Cantor creates a situation in which "the paradoxes attending our former conceptions of the infinite become the truisms stating its essential properties" (Findlay 151). By merely appending a new principle of exhaustion to the exact same principle of succession that created so many headaches for philosophers, Cantor created a consistent system that described the actual infinite.

Cantor's work on and philosophy of the infinite stood in direct opposition to many of the great philosophical and mathematical minds in history. And yet, he defended his mathematical work against them masterfully. To his Aristotelian detractors, he exposed the fallacy of the 'annihilation of number' argument, stressing the consistency of his transfinite numbers, and to his religious detractors, Cantor appealed to God's ability to realize any possibility as a means of reinforcing Thomism. In doing so, he exposed his reliance on mathematical consistency and Platonist ideas. As a result, he believed in absolute freedom of mathematical objects that could be shown to be internally consistent. Unfortunately, intuitive methods did not seem to suffice for Cantor, leading him to reject the existence of infinitesimals. Furthermore, his works, particularly his 'Principles of Generation' altered the concepts of transcendence and abstraction forever. By successfully describing infinities of various powers directly, he contributed to a re-ordering of the way we think about the infinite in general, bringing “into our ken a whole family of infinites, each living above the next on an entirely different floor of the family mansion” (Findlay 151).

However, Cantor’s philosophy was not readily accepted in his time.

Kronecker’s Philosophical Position

Leopold Kronecker’s philosophical views of mathematics are not easy to list. In much the same way that it is difficult to describe his personality, the difficulty of describing Kronecker’s philosophy lies with the fact that he devoted his works to purely mathematical themes and “published nothing on his constructive program” (Edwards , “Kronecker’s Place,” 141). However, despite the lack of a published statement of his views, he exhibited them in other ways. In order to explain his mathematical philosophy and attempt to summarize it in the absence of any explicit philosophical work, I will inspect these other means by which his opinions became known. By examining the extent of his philosophical background, his relationships with contemporary mathematicians, the philosophical undertones of his works and lectures, and a later mathematical movement he influenced, a fuller picture of his philosophy should arise.

Kronecker was far from ignorant of philosophy. As a student at the University of Berlin in the latter half of the 19th century, he studied a wide spectrum of subjects in addition to mathematics. Kronecker devoted a significant amount of his time to the study of philosophy, and even selected the history of philosophy as one of the topics to be questioned on during his final oral examination. He attended Schelling’s philosophy lectures and made “a thorough study of the works of Descartes, Spinoza, Leibniz, Kant, and Hegel, as well as those of Schopenhauer, whose ideas he rejected" (Biermann 505). Thus, one may be certain that Kronecker certainly was aware of the philosophical implications of his work, and was well-enough versed in philosophy to have reasoned out his own views.

In an admittedly speculative way, one may conjecture as to how these philosophers may have influenced him. The three major Rationalists—Descartes, Leibniz and Spinoza—valued reason over experience in their philosophical undertakings, and Schelling's romantic philosophy valued intuition over reason (Markie 740-741). Schelling's ideas would imply that the only valid mathematics would be one that reaches back to intuitive concepts. Kant's discussions of potential versus "actual, non-constructive infinity" (Körner 31) and Hegel's rejection of the absolute infinite (Findlay 149) would have confronted Kronecker with distinctions regarding infinity, which he would later bring to bear in his career.

Given his educational background, Kronecker's feuds with other mathematicians ought to be regarded from a philosophical standpoint. Perhaps his earliest feud with great philosophical significance was his (above-mentioned) falling-out with Karl Weierstrass, a colleague of his in Berlin. In their early careers, the two were such great friends that Weierstrass seconded Kronecker's nomination to the Berlin Academy in 1861. However, as early as the 1870s their friendship began to deteriorate due to "very different temperaments of the two men…, and their professional and scientific differences" (Biermann 507). More specifically, the two disagreed fundamentally on the appropriate role of the natural numbers in mathematics. Although Weierstrass admitted their importance, he did not see the natural numbers as all-encompassing, and used conceptual constructions like irrational numbers in his works. By 1885, Weierstrass concluded that their disagreement stemmed from the fact that "for Kronecker it was an axiom that equations could exist only between whole numbers," while he (Weierstrass) "granted irrational numbers the same validity as any other concepts" (507). Kronecker's firm belief in the sole validity of the natural numbers played a major role in his objection to the famous Bolzano-Weierstrass Theorem[11], whose proof made use of irrational numbers. Kronecker's faith in a mathematics based purely on the natural numbers would become a hallmark of his mathematical philosophy.

Although Kronecker's views had become known among many mathematicians in the course of time, it was not until a lecture he gave in Berlin in 1886 that Kronecker revealed a great many of his mathematical convictions to a greater public. In it, he "argued against the theory of irrational numbers used by Dedekind, Cantor and Heine giving the arguments by which he opposed…the 'irrationals' in general [and]…the concept of an infinite series" (Calkins). Specifically, he held that the essence of mathematics lies not in its freedom (as Cantor would have it), but rather "in its truth…its power to convince us of its correctness" (Edwards, "Kronecker's Place," 141). This truth, he claimed, lay in the natural numbers, which are the only mathematical objects that exist beyond doubt. As he famously stated, he believed that "*Gott schuf die natürlichen Zahlen, alles andere ist Menschenwerk*" (Biermann 507)[12]. In other words, because they exist innately in us, the natural numbers must serve as the basis for every other mathematical notion. Thus, the standard of rigor (i.e. truth) in a mathematical concept or proof boils quite simply down to whether or not its included concepts can be restated in terms of the natural numbers. In contrast, those notions *not* based on the natural numbers cannot be considered mathematical (due to their lack of rigor). As a result, consistently constructed concepts such as imaginary numbers, transcendental numbers, and irrational numbers have no place in mathematics, due to their lack of reference to the natural numbers. Furthermore, as a

[11] The Bolzano-Weierstrass Theorem states that every bounded infinite set has a limit point.
[12] "God created the natural numbers, all else is the work of man"

corollary to his opposition of infinite series, he rejected proofs that utilized an infinite number of steps, since infinity stands in contrast to finite whole numbers. After his 1886 lecture, Kronecker's emphasis on the mathematical 'truth' of natural numbers and rejection of infinity became known throughout the academic world.

Still, Kronecker never fleshed out his philosophical views by publishing them. But had he done so, how could one describe them? Since Kronecker's ideas "may be compared to that of intuitionists" (Biermann 508), a group of philosophers of mathematics, we gain an example of his philosophy in its fully 'fleshed-out' form by looking at their work. Led by such "proponents" of Kronecker's views as Henri Poincaré and L.E.J. Brouwer (Edwards, "Kronecker's Place," 140), the Intuitionists held that mathematics must be based on the intuitive process of counting the natural numbers. As one would expect of proponents of Kronecker's views, they rejected absolute infinity, favoring the view of the natural numbers as a potentially infinite sequence (140). Furthermore, they held that a mathematical concept must be defined in a finite number of steps, rejected proofs of existential claims that do not provide a method of determining an instance of those claims, opposed Cantor's actual infinities, and in general sought a "wholesale reconstruction of mathematics" (McCarty 402-403). Differing from Kronecker only in the extent to which they pursued their constructivist ideals, the Intuitionists share his faith in the sole truth of the natural numbers and rejection of Cantor's absolute infinity.

I recently managed to obtain a copy of Kronecker's 1887 work on the conceptof number (*Über den Zahlbegriff)*, and found many of the above-mentioned ideas reflected. Unfortunately, he does not explicitly write of his objection to Cantor's transfinite numbers or infinity in general, but his preference for natural numbers is made perfectly clear to the reader. Although the title of the essay refers merely to numbers, his writing discusses and redefines both natural numbers and arithmetic. He claims that these two concepts reflect the mathematical word- and number-representations that arise from the *Menschengeist*; the proof of their innate veracity, he claims, lies in their existence in separate cultures throughout the world. Out of these innate word- and number-representations come not only arithmetic, but also the laws of cosmology and many of the laws that govern our daily lives, such as those for trade (Kronecker 274). Given the fundamental nature of the natural numbers and arithmetic, he strives in his work to 'arithmetize' all of mathematics; that is, to base all of mathematics—with the possible exception of geometry and mechanics—on the natural numbers and the arithmetical operations of addition and multiplication. In order to define numbers, he begins with the concept of a set [*eine Schaar*], then goes on to describe their ordinal and cardinal numbers, explaining that the numbers we use in everyday life are actually the latter. He defines counting as the result of the process involved when one applies the ordinal numbers to the objects of a set in a certain order until the cardinality (*Anzahl*) of the set is determined. Afterwards, any further counting is achieved by adding elements to the set (268-270). That is, when we count to the number 3, we are referring to a set of three objects, and were we to want to count further to the number 5, we would merely have to add two elements to the first set. The natural numbers, then, are the objects of this counting process. Consistent with his agenda of arithmetization, Kronecker goes on to define normal equivalence relations (<, >, =) as well as addition and multiplication between these numbers. In order to avoid subtraction, he introduces variable calculations (*Buchstabenrechnung*), and avoids division similarly by using a modular system of equations. For example, instead of writing '-1' he would refer to 'that x for which x + 1 = 0.' Notably, all of his definitions extend to some indeterminate *finite* number n, and never continue to infinity (263-273). Throughout the essay, his emphasis on the innate truth of the natural numbers, reliance on processes of counting, rejection of extensions of arithmetic beyond addition and multiplication, and limitation of sets to finite quantities is quite striking to a modern mathematics student.

IV. Who is right?

The feud between Cantor and Kronecker remained unresolved until Kronecker's death. Starting in 1870 with Cantor's extension of the uniqueness theorem from trigonometrically-represented functions to those functions with infinite domains[13], the two men became more and more firmly entrenched in their positions as time went on. Over the course of 21 years, their famous mathematical feud divided many mathematicians and philosophers into opposing camps.

Indeed, their ideas were often completely opposite. Cantor and Kronecker both believed in the mathematical *truth* of their own ideas about number, but themanner in which their numbers *existed* differed. To Cantor, mathematical ideas lived immanently as creations of God and needed no phenomenological corollary. Kronecker, on the other hand, admitted the existence of numbers only when he could see physical evidence of their existence. As a result, Cantor judged the existence of mathematical objects on their internal consistency, while Kronecker used the intuitive experience of counting (i.e. using the natural numbers). As Cantor stressed the freedom of his mathematical theories, Kronecker sought "to keep mathematics free of uncertain philosophical speculations" (Meschkowski 136). Cantor went on to define new modes of transcendence via his 'Principles of Generation,' while Kronecker restricted his mathematics to those numbers and operations thatwould produce results that were physically available (e.g. avoiding division because one cannot find an example of '4/5' in the phenomenological world). Thus Cantor could construct new abstract notions with his transfinite numbers, whereas Kronecker reverted back to the natural numbers and the operations of addition and

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[13] see page 6

multiplication; Cantor's abstractions, although consistent, on transfinite numbers could not be more meaningless to him.

But who was right? Certainly, each of their arguments had their inconsistencies.

Kronecker's (Un)Natural Numbers

Kronecker's reliance on the natural numbers due to their sole 'existence' in the world makes little sense. After all, what is so 'natural' about these numbers? Although we are accustomed to using numbers and counting as a means of describing the world around us, the set N = {1,2,3,4…} is not an entirely accurate devise. For example, when we refer to 3 apples, it is clear that we are referring to three individual apples, but clearly 3 apples are not the same as 3 pears. So what is exactly this'3' ? Does it refer to three objects in general? What if, for example, we were to cut one of these apples in half, would we then count 4 objects, or still 3, or 2 of one type and 2 of another? All of these numbers describe the same assembly, so does 4 = 3 = 2 + 2? According to Kronecker, an example of '3' ought to exist in the phenomenological world, but searching for one becomes quite difficult. So if the natural numbers are not quite useful as a means to describe nature, why not choose another set of numbers? The real numbers, for example, exist on the same ontological level as the natural numbers, since examples of rational and irrational numbers could be said to exist just as much as the aforementioned '3.' After all, splitting an apple in half could be said to create two quantities of size 1/2, and the diagonal of a square with sides of length 1 is an irrational number. So do the real numbers—that is, the union of the rational and irrational numbers—exist? Although they meet Kronecker's standards, he denied that they were mathematically true.

In general, the effect of Kronecker's choice of the natural numbers is not only to impose "an ordering, by the operation of adding, upon otherwise non-ordered units" (Byrne 21), but also to limit mathematics. Since the world is not neatly divided into concrete blocks of size 1, 2, 3, etc, Kronecker uses the natural numbers as a means of ordering it, most likely due to their relative simplicity. However, due to the limitations of these numbers, they fail to describe all possibilities. For this reason, further sets of numbers—which Kronecker openly rejected—were introduced that in fact facilitate descriptions of the world. Excluding these sets of numbers from mathematics on the basis of the phenomenological 'existence' of the natural numbers seems an incredible arbitrary choice. In short, "there is no reason to only allow this [the set of natural numbers] to be a set in mathematics" (Meschkowski 137).

Aside from arbitrariness and imprecision in Kronecker's methods, he contradicts his objection to Cantor's abstract mathematics in his use of abstraction in his own work As discussed above, Kronecker bases his mathematics on the natural numbers because their existence is exhibited in the act of counting. In other words, since we can literally point your finger at objects and count them, the natural numbers can be shown to exist without any difficulty. They are thus not extrapolated from other mathematical concepts, but simply exist. However, this reliance on counting 'existing' mathematical numbers causes contradictions because it actually *requires abstraction*. To return to the example above, we extrapolate this '3' from the aforementioned three apples, and the abstraction of the number '3' from '3 apples', although not entirely consistent, is relatively easy to understand since we can count: 1 apple, 2 apples, 3 apples. However, were we to increase this quantity, to say 10^{1000}, how could we be sure that such a number exists? It certainly fits Kronecker's definition of numbers, since after a finite number of additions it can be produced from other natural numbers, and yet, on the basic level of counting, it fails miserably. Due to certain limiting factors inherent in our status as humans—such as fatigue, boredom, and mortality—we could not possibly count to 10^{1000}, and yet, almost any mathematician, philosophical anomalies aside, would agree that this is a natural number. So if he cannot count to all natural numbers, how can Kronecker be *sure* that they exist? And yet, Kronecker's work is full of such abstractions.

In his *Über den Zahlbegriff*, Kronecker uses the whole numbers (i.e. the natural numbers) as a single unit repeatedly as a means of describing his ideas. For example, when he first makes reference to variable equations, he speaks of placing letters "in the place of the numbers 1, 2, 3…" (271). This sentence is quite significant. First, it implies an infinitely proceeding sequence and thereby denotes the natural numbers as a unity (or *set*). In general, when we define the natural numbers by a definition such as N = {1, 2, 3, 4,…}, we understand that the "…" means that this sequence produces an indefinitely large magnitude, and indeed cannot define the natural numbers without it. This sort of intellectual extrapolation is what the philosopher Bernard Lonergan would call an 'insight,' because it allows us to define meaningfully the natural numbers without writing all (infinitely many) of them out. Furthermore, as Patrick Byrne writes, it implies "simultaneously a principle of limitation and transcendence in an infinite series or sequence" (26), because it both limits our possible orderings of the elements to one and transcends any possible finite listing of the elements in a meaningful way. That is, Kronecker's 1, 2, 3,… is only meaningful to the reader of his essay in the context of an abstraction which allows us to understand its infinite nature and limitations of ordering.

Cantor used this exact principle as the basis for his First Principle of Generation, and indeed noted the inconsistency in Kronecker's work. As Joseph Dauben sums Cantor's argument,

> "finitists, who only allowed arguments of the sort: 'For any arbitrarily large number N there exists a number n > N,' necessarily presupposed (said Cantor) the existence of *all* such numbers n > N, taken as an entire, completed collection which he called the *Transfinitum*" (127).

That is, Kronecker, who vehemently rejected talk of infinite sets, based his entire work on the infinite set of natural numbers. So if we may make the insight necessary to realize that the sequence of natural numbers proceeds infinitely and refer to this as N, why is it contradictory to speak of Cantor's $\omega + 1$ as the next greatest number after the greatest ordinal number in N?

Later in Kroneceker's work, he uses his definitions of addition and multiplication to e*xtrapolate* further definitions of division and subtraction—and thus rational numbers, including negative numbers, and zero—from the natural numbers. Again, if we may construct not only the natural numbers, but also rational numbers, why are Cantor's transfinite numbers so dubious?

For whatever reason, none of these arguments ever changed Kronecker's mind on the nature of mathematics. Although his own work made use of the concepts thatCantor applied in the construction of his transfinite numbers, Kronecker stubbornly saw them as a perversion of rigorous mathematics rather than a progressive result. He "was simply living in the past" (Clegg 191) and could not accept the validity and ultimate importance of Cantor's work.

Cantor's Contradictions: Intuition, Infinitesimals, and Arrogance

It may already be obvious to the reader that I support Cantor's philosophical stance. However, my support remains qualified due to his stubbornly fallacious stances on the role of intuition and infinitesimal quantities in mathematics.

Cantor's occasional opposition to intuitive methods in mathematics is a glaring point of confusion in his philosophy. As mentioned above, he vehemently opposed intuition as it came into play regarding geometry and proofs of the existence of infinitesimals, rejecting such arguments "as subjective *a priori* forms of intuition" (Dauben 108). However, such statements seem somehow unfair. Perhaps by the word 'intuition' Cantor means something other than I interpret it, but it seems to me that 'intuition' is the very insight that guided Cantor to his greatest mathematical discoveries. Indeed, when defining the laws of arithmetic for his transfinite numbers, he explained that his definitions were far from arbitrary. Rather, they derived "immediately from the mind's inner intuition with absolute certainty" (104). In truth, Cantor "believed in his intuition, and… had always relied upon the strength and sharpness of his intuition " (Dauben 223-224) in his work. In effect, Cantor denied the claims of some because their work was based on intuition, and yet stood by his own works of intuition. Such subjective claims on the value of intuition represent an unfortunate weakness in Cantor's ideas.

Given Cantor's ambiguous stance on intuition in mathematics, it seems awfully hard to take his criticism of infinitesimals seriously. As Joseph Dauben writes,

> "there is even some irony in Cantor's position. To many mathematicians, his theory of actually infinite transfinite numbers seemed to justify intrinsically the infinitely small as well as the infinitely large" (129).

After all, Cantor's transfinite numbers, taken as reciprocals, that is,

$$\frac{1}{\omega}, \frac{1}{\omega+1}, \frac{1}{\omega+2}, \ldots,$$

seem to consistently define actual infinitesimals. If we replace ∞ with different classes of infinity, why do we consider the $\lim_{n\to\infty} \frac{1}{n}$ to be 0? Should it not be some infinitely small number, say, $\frac{1}{\omega}$? Yet to Cantor "such a step was irresponsible and under no circumstances could it be rigorously justified" (130). But, if they are consistent and thus mathematically possible why should Cantor reject them? He certainly never was able to disprove their existence. Could not God realize any possibility? Even Leibniz, one of Cantor's great philosophical allies in support of the absolute infinite, supported the existence of infinitesimals (124).

The problems in Cantor's argument against infinitesimals arose from their apparent applicability to his own theory and simultaneous incompatibility with his own definition of a linear quantity. Because they seemed to fit in so naturally as a corollary of his transfinite numbers, his own criticism of them applied "as effectively against the transfinite numbers as against the infinitesimals" (Dauben 131). And yet, any support for infinitesimals was thus a direct challenge to his own work, since "any acceptance of infinitesimals necessarily meant that Cantor's own theory of number was incomplete" (233). Because he never succeeded in proving the impossibility of their existence, he made the bold statement that they could not exist. Specifically, he argued that his transfinite numbers followed from *real* ideas produced directly from sets, and challenged mathematicians "to show any *real ideas* corresponding to the supposed infinitesimals" (236). According to Cantor, there were no sets from which infinitesimals could be directly abstracted as could ω and $\aleph_0$. But what about these two numbers themselves as a basis? As shown above, it seems that they could suffice. However, Cantor remained opposed to them in both principle and practice.

Cantor's inconsistent stances on the role of intuition in mathematics and infinitesimal quantities reflect a quality of Cantor's personality that both aided him and limited him throughout his career: rigidity. While firmness in belief is usually a good characteristic, Cantor was unable to rid himself of some of his prejudgments. For example, why should his intuition be more accurate than another's when both produce consistent mathematical theories? He was so firm in his conviction that "his characterization of the infinite was the *only* characterization possible" (Dauben 233) that it comes across as arrogance at times. For a man who espoused the 'freedom' of mathematics, he was entirely too dogmatic about his own theory

The Final Analysis

Despite his shortcomings, Georg Cantor's philosophical position remains both highly important and largely accurate for us today. In particular, his convictions regarding the freedom of mathematics and revolutionary ideas on transcendence and limitation are his most convincing ideas.

Although consistency in a mathematical system has been proven to be unprovable[14], mathematical freedom is as important today as ever. Without it, many important mathematical results would have never occurred. Cantor knew this then, as he claimed that:

> "Without the freedom to construct new ideas and connections in mathematics, Gauss, Cauchy, Abel, Jacobi, Dirichlet, Weierstrass, Hermite, and Riemann would never have made the significant advances they did….and consequently the world would be in no position, Cantor added with a note of cumin, to appreciate the work of Kronecker and Dedekind" (Dauben 133).

If such free inquiry has produced such tremendous mathematical breakthroughs in the past, for what possible reason should it be constrained?

[14] In 1930, Kurt Gödel proved in his Second Incompleteness theorem that no mathematical system could be proven to be internally consistent.

Mathematicians who operate under such constraints seem hopelessly limited. As Herbert Meschkowski wrote, the natural response to those who limit mathematics to the natural numbers is: “is that really all of mathematics? Are constructed concepts not acceptable and even necessary to define structures of a different type?” (Meschkowski 136). As mentioned above, even the natural numbers are to some extent constructed, so why reject other constructions? Admittedly, some of Cantor’s original ideas were quite vague, but is that not always the case when a theory is being developed? Early calculus, for example, became famous for its lack of rigor in the 19th century, but then gained new rigor due to work in analysis. Although it still requires the use of irrational numbers, its power to accurately describe the world remains essential to many scientists and engineers. So, if such a constructed concept can produce groundbreaking results that even apply to the phenomenological world, why should it be denounced? As with calculus, it is generally the case that “‘applied’ mathematics, such as analytical mechanics and physics, is *metaphysical* both in its foundations and in its ends. If it seeks to free itself of this…it degenerates into a ‘describing of nature,’ which must lack both the fresh breeze of free mathematical thought and the power of *explanation* and *grounding* of natural appearances” (Jourdain 69). In other words, if mathematicians could not venture to the realm of the metaphysical and create new work regardless of its apparent level of rigor, many radical shifts in the world of mathematics—both pure and applied—would never have happened.

Most importantly, Cantor’s abstractions of set theory opened up both mathematics and philosophy to new concepts of limitation and transcendence. Traditionally, philosophers and mathematicians held that transcendence in mathematics merely involved the addition of some units to others and that infinity itself could not be surmounted due to its endless nature. Kronecker’s ‘counting’ method and attempt of ‘arithmetization’ fit neatly in this mold.

However, Cantor's principles of generation allowed mathematical infinities to become not only surmountable but differentiable to one another. Thus the symbol 'N' came to denote the natural numbers as all of the numbers exhausted by the first principle of generation. However, what became revolutionary was his second principle, because it was is "a matter of recognizing that a series has a principle of limitation and that this recognition is itself a transcending of the limitation" (Byrne 22). Approaching the limitation that infinity presented in an entirely new light, Cantor proposed numbers that transcend the finite—are trans-finite—with the help of his second principle of generation. Especially in light of the idea of mathematical freedom, the veracity and significance of Cantor's work on limitation and transcendence is remarkable.

Conclusion

Today the dispute between Leopold Kronecker and Georg Cantor exists in history as little more than a footnote in the occasional book on the history of mathematics. Although Georg Cantor's work maintains consistent mention in the classrooms of students of mathematics around the world, the philosophical motivations for his work receive no mention nor do Kronecker's philosophical commitments. This sort of forgetfulness is unfortunate, because both the feud between these men and the individual views they expressed remain relevant.

Their feud stands out most significantly as a kind of focal point of the numerous disputes among mathematicians of that time. Kronecker objected not only to C antor and Weierstrass' work, but also to many others—including Dedekind—and these same men in turn objected to his work. Likewise, Cantor opposed vehemently the works of all who supported infinitesimals, and in turn his detractors included not just Kronecker, but other mathematicians as renowned as Henri Poincaré, who believed that "Cantor's ideas were a grave disease that seemed to infect all mathematics" (Dauben 266). However, the meaningfulness of their feud rests in the strong oppositions produced by it. That is, in the feud itself two very opposing views of mathematics present themselves.

Cantor and Kronecker stood on opposite ends of the spectrum of 19^{th} century mathematical philosophy. On the more avant-garde end of the spectrum stood Cantor's neo-Thomist and Platonist ideas, and on the more conservative end stood Kronecker's emphasis on 'rigor' and restrictive intuitionist beliefs. Cantor, so religious that "an evaluation of the works of Georg Cantor cannot overlook his religiousness" (Meschkowski 124), held that God himself guaranteed the existence of his transfinite numbers, due mainly to their high level of consistency.

Furthermore, his transcendence of infinity and creation—or discovery, as he might perhaps argue—of the infinite ordinal and cardinal numbers redefined concepts of transcendence altogether. In contrast to Cantor, Kronecker—also a religious man—based his views on the belief that we may use only the natural numbers in our calculations, since they are the only numbers made by God and thus universally true. To him, transcendence of infinity is as absurd as transcending God. The differences lie furthermore in beliefs about the existence of mathematical objects and the ability of mathematics to expand and redefine itself. While both argue from religious perspectives, they come out on completely different ends of the philosophical spectrum. Cantor's belief in God gives him a degree of flexibility, requiring that mathematical objects merely be *possible* in order for them to be *real*. Kronecker's religiousness, on the other hand, causes him to restrict mathematics on the basis that God made only one number system, and thus any deviation from it generates false conclusions.

The full application of each view on mathematics seems to spell the difference between the mindsets of these two men the clearest. Cantor's view is far more progressive, allowing mathematics to be freer than Kronecker's restrictive mindset. While Kronecker's opinion attempts to suffocate mathematics with arbitrary standards of rigor, Cantor's is a comparative breath of fresh air. David Hilbert once claimed that "no one shall expel us from the paradise that Cantor has created for us" (Reid 177). Cantor's mathematics is indeed a paradise in comparison with Kronecker's.

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