Three Essays on Matching with Contracts

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ABSTRACT

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This dissertation consists of three theoretical essays. In all essays matching with contracts is a key factor. The first essay tries to explain effects of choosing primitives of the model and irrelevance of rejected contracts condition on some key existence theorems and results in matching with contracts literature. The second essay analyzes the properties of cumulative offer algorithm and presents an application of matching with contracts. It studies the achievability of responsive choices under a constrained setup. The last essay presents a new market design application of program-student matching where affirmative action policies are effective.

The first essay develops a hospital-doctor many-to-one matching with contracts model. Doctor preferences over contracts are part of primitive of the model. The other primitive of the model, our first essay suggests, hospital choice functions on sets of contracts. The first essay shows that if choice functions of hospitals are primitives of the model, then existence theorems used in many papers do not hold even when they satisfy strongest conditions. As a remedy, we introduced Irrelevance of Rejected Contracts (IRC) which guarantees stability if it is satisfied along with one substitutes condition.

Next, we show the relation between IRC and law of aggregate demand (LAD) conditions. Since LAD is satisfied by many application naturally, many models satisfying LAD and the strongest substitutes conditions are immune to our criticism.

On the other hand, many of the new and exiting applications satisfy only weakened substitutes condition. Therefore, assuming IRC explicitly does not only make their proofs accurate and also close the gap between theory and application.

The second chapter studies properties of cumulative offer algorithm under weakened substitutes condition. In this part we showed that in many-to-one matching with contracts problems order of proposals of COA does not change the outcome, under bilateral substitutes and IRC conditions. Also, bilateral substitutes and IRC conditions make COA equivalent to generalized deferred acceptance algorithm which produces the outcome in fewer steps.

This chapter also presents a new application area of matching with contracts. We used cadet-branch matching problem in USMA. In this application our main objective is, for a given branch, increasing cadet quality without giving up useful properties of allocation mechanism, such as stability and strategy-proofness.

The third essay studies a college admission with affirmative action problem. With this application, for the first time in the literature, we presented an affirmative action problem where students need to claim privilege if they want to be subject to affirmative action. We analyzed the current system and showed that current guideline is unfair and causes incentive compatibility issues. Also we showed that it fails to satisfy affirmative action requirements described in affirmative action law.

To solve these problems with the current system, we introduced a new choice function which is fair, respects affirmative action requirements and makes student optimal stable allocation stable and incentive compatible when used in conjunction with generalized deferred acceptance algorithm.

DEDICATION

I dedicate this dissertation to my wife Aysun Aygün, our twins Deniz Aygün and Uras Aygün, our parents Mehmet-Hatice Aygün and Ali Osman-Fatma Hiziroglu who are always with us whenever we need.

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CHAPTER 1.

MATCHING WITH CONTRACTS: THE CRITICAL ROLE OF IRRELEVANCE OF REJECTED CONTRACTS

1.1 Introduction

Formulation and analysis of *matching with contracts* model (Hatfield and Milgrom 2005) is widely considered as one of the most important developments of the last twenty years in theory of matching markets.[1] This powerful model embeds Gale and Shapley (1962) *two-sided matching* model and Kelso and Crawford (1982) *labor market model*,[2] among others, and it has given impetus to a flurry of theoretical research as well as to new practical applications of market design. Utilizing fixed-point techniques from lattice theory, Hatfield and Milgrom (2005) analyze the set of stable allocations in their rich framework. One of the main messages of their paper is that the set of stable allocations is non-empty under a substitutes condition. The substitutes condition that plays a key role in the analysis of Hatfield and Milgrom (2005), also induces a strong isomorphism between matching with contracts model and Kelso and Crawford (1982) labor market model (Echenique 2012). This isomorphism is considered to be a highly negative result since it reduces the scope of Hatfield and Milgrom (2005) to that of Kelso and Crawford (1982). Fortunately this restrictive "equivalence" between the two models breaks under two weaker conditions, *bilateral substitutes* and *unilateral substitutes*, introduced by Hatfield and Kojima (2010). The

[1] See also Adachi (2000), Fleiner (2003), and Echenique and Oviedo (2004).

[2] Kelso and Crawford (1982) builds on Crawford and Knoer (1981).

significance of these weaker substitutes conditions was further increased when Sönmez and Switzer (2011) - Sönmez (2011) introduced a brand new market design application of matching with contracts, *cadet-branch matching*, which satisfies the unilateral substitutes condition but not the substitutes condition.

In this paper we show that both Hatfield and Milgrom (2005) and Hatfield and Kojima (2010) implicitly assume an additional irrelevance of rejected contracts (IRC) condition throughout their analysis, and in the absence of IRC several of their results, including the guaranteed existence of a stable allocation, fail to hold. The implicit assumption of IRC is a result of choosing strict preferences as the primitive of their model. Unlike these two theoretical papers, many applications of matching with contracts do not assume strict preferences. That is the reason why choosing strict preferences as the primitive of the model and analyzing the model using IRC as an implicit assumption creates an ambiguity in results. There are two possible remedies to resolve this ambiguity. Of the two remedies, the first (and scientifically more sound) one results in the failure of many theorems in the absence of an additional *irrelevance of rejected contracts* (IRC) condition, whereas the second remedy eliminates the transparency of the results, reduces the scope of the model, and contradicts authors' interpretation of the role of the substitutes conditions. Fortunately all results are restored when IRC is explicitly assumed under the first remedy.

Since Hatfield and Milgrom (2005) and Hatfield and Kojima (2010) will likely play an important role in further applications of market design, it is important to remove the inconsistency in the model. Fortunately most market design applications of matching with contracts, including the above described cadet-branch matching, satisfy IRC, and as such they are shielded from our criticism.

The paper is structured as follows. The next section sets up the model. After introducing the model, background definitions and assumptions, we define stability notion and present a counter example. Having the example in hand, the paper proceeds with a remedy with the IRC and relation with the LAD condition. Then we analyze effect of choosing strict hospital preferences as primitives on results in the literature and on the scope of the model. It concludes with a brief discussion of the findings and provides the proofs for some of the results in the appendix part.

1.2 The Model

We mostly follow the notation of Hatfield and Milgrom (2005) and Hatfield and Kojima (2010). Since the purpose of this paper is presenting the implications of a major inconsistency in Hatfield and Milgrom (2005) and Hatfield and Kojima (2010), our presentation will also closely follow theirs.

There are finite sets D and H of doctors and hospitals, and a finite set X of contracts. Each contract $x \in X$ is associated with one doctor $x_D \in D$ and one hospital $x_H \in H$. Given a set of contracts $Y \subseteq X$, let Y_D denote the set of doctors who has contracts in Y. That is, $Y_D = \{d \in D | \exists y \in Y \text{ s.t. } y_D = d\}$. Each doctor $d \in D$ can sign at most one contract and his null contract where he signs no contract is denoted by $\emptyset_d$. A set of contracts $X' \subseteq X$ is an allocation if each doctor is associated with at most one contract under X'.

For each doctor $d \in D$, $\succ_d$ is a strict preference relation on his contracts $\big(\{x \in X|\ x_D = d\} \cup \{\emptyset_d\}\big)$. A contract is acceptable by doctor d if it is at least as good as the null contract $\emptyset_d$, and unacceptable by doctor d if it is worse than the null contract $\emptyset_d$. For each doctor $d \in D$ and a set of contracts $Y \subseteq X$, the chosen set $C_d(Y)$ of doctor d is defined as

$$C_d(Y) = \max_{\succ_d} \Big(\{x \in Y|\ x_D = d\} \cup \{\emptyset_d\}\Big).$$

For a given set of contracts $Y \subseteq X$, define $C_D(Y) = \bigcup_{d \in D} C_d(Y)$.

Given a hospital $h \in H$, define $X_h = \{x \in X|\ x_H = h\}$ to be the set of its non-empty contracts in X. Each hospital $h \in H$ can sign multiple contracts and has preferences $\succ_h$ on

$$\Big\{Y \subseteq X_h|\ y, y' \in Y \text{ and } y \neq y' \implies y_D \neq y'_D\Big\}.$$

Unlike doctor preferences, hospital preferences are not assumed to be strict in Hatfield and Milgrom (2005) and Hatfield and Kojima (2010). This point, which may initially seem to be a detail, will prove to be very important. For any $Y \subseteq X$, the **chosen set** $C_h(Y)$ of hospital h is defined in Hatfield and Kojima (2010) as

$$C_h(Y) = \max_{\succ_h} \Big\{Y' \subseteq Y \cap X_h|\ y, y' \in Y' \text{ and } y \neq y' \implies y_D \neq y'_D\Big\}.$$

Observe that the above definition of $C_h(Y)$ may include more than one set of contracts unless hospital preferences are also assumed to be strict. Since choice sets are assumed to be singletons throughout the analysis in Hatfield and Milgrom (2005) and Hatfield and Kojima (2010), this definition is not well-defined. There are two possible remedies for this inconsistency. Either hospital preferences shall be assumed to be strict (as in the case of the doctors), or $C_h(Y)$ shall be given as a *selection* from

$$\max_{\succ_h} \Big\{ Y' \subseteq Y \cap X_h |\; y, y' \in Y' \text{ and } y \neq y' \implies y_D \neq y'_D \Big\}.$$

As it turns out, each remedy introduces its own complications to the analysis of matching with contracts models. However, we will argue that the complications associated with the latter are considerably easier to resolve, and as such, we will for now allow indifferences in hospital preferences and assume that

$$C_h(Y) \in \max_{\succ_h} \Big\{ Y' \subseteq Y \cap X_h |\; y, y' \in Y' \text{ and } y \neq y' \implies y_D \neq y'_D \Big\}.$$

For a given hospital $h \in H$, we refer the function that maps each set of contracts to a chosen set as the choice function of hospital h. For a given set of contracts $Y \subseteq X$, define $C_H(Y) = \bigcup_{h \in H} C_h(Y)$.

An important advantage of this modeling choice is that, it introduces no *a priori* constraints on the structure of chosen sets. That is because, the preference relation where a hospital h is indifferent between all subsets of X_h is consistent with any selection of chosen sets. Thereby this modeling choice is equivalent to considering hospital choice functions to be primitives of the model. In contrast, if one adopts the first remedy assuming hospitals have strict preferences, that would introduce constraints on the structure of hospital choice functions including but not limited to a version of the *strong axiom of revealed preference*, and as such, the entire analysis would be superimposed on the implied structure, inconsistent with the authors' interpretation of the results in Hatfield and Kojima (2010). We will return to this important issue in Section 5, but for now, we adopt the first remedy and thereby assume that there is no a priori structure on hospital choice functions.

1.3 Stability Under Substitutes and Weakened Substitutes Conditions: A Counter Example

Stability axiom plays a central role in analysis of two-sided matching models, and it is extended to matching with contracts as follows:

Definition 1 *A set of contracts $X' \subseteq X$ is a stable allocation (or a stable set of contracts) if*

1. $C_D(X') = C_H(X') = X'$, *and*

2. *there exists no hospital $h \in H$ and set of contracts $X'' \neq C_h(X')$ such that*

$$X'' = C_h(X' \cup X'') \subseteq C_D(X' \cup X'').$$

When the first condition fails, the allocation X' fails individual rationality and there is a blocking doctor or a hospital. When the second condition fails, there is a blocking coalition made of an hospital h and a subset of doctors $\{x_D\}_{x \in X''}$. In this case we say that X'' blocks X' .

Hatfield and Milgrom (2005) claim that the set of of stable allocations is always non-empty under the following condition:

Definition 2 *Contracts are substitutes for hospital h if there do not exist a set of contracts $Y \subset X$ and a pair of contracts $x, z \in X \setminus Y$ such that*

$$z \notin C_h(Y \cup \{z\}) \text{ and } z \in C_h(Y \cup \{x, z\}).$$

Loosely speaking, the substitutes condition captures the intuitive idea that a contract that is rejected from a set of contracts shall remain to be rejected when there is "increased competition". The following example shows that the set of stable allocations may be empty under the substitutes condition, in the absence of additional structure.

Example 1 *Consider a problem with one hospital, h, and two doctors d_1, d_2. Doctor d_1 has two contracts x, x' and doctor d_2 has one contract y. Preferences of the doctors and the choice function of the hospital are given as follows:*

$$\succ_{d_1}: \quad x \succ_{d_1} x' \succ_{d_1} \emptyset_{d_1}$$

$$\succ_{d_2}: \quad y \succ_{d_2} \emptyset_{d_2}$$

$C_h(\{x\}) = \{x\}$	$C_h(\{x, x'\}) = \{x\}$	$C_h(\{x, x', y\}) = \emptyset$
$C_h(\{x'\}) = \{x'\}$	$C_h(\{x, y\}) = \{y\}$	
$C_h(\{y\}) = \{y\}$	$C_h(\{x', y\}) = \{x'\}$	

It is easy to verify that C_h satisfies the substitutes condition. Moreover no allocation is stable in this example. Here is a list of blocking coalitions for each possible allocation:

Allocation	*Blocking Coalition*	*Allocation*	*Blocking Coalition*
$\{x\}$	$\{h, d_2\}$ *via* y	$\{x, y\}$	$\{h\}$ *via removing* x
$\{x'\}$	$\{h, d_1\}$ *via* x	$\{x', y\}$	$\{h\}$ *via removing* y
$\{y\}$	$\{h, d_1\}$ *via* x'	$\emptyset$	$\{h, d_1\}$ *via* x

The existence claim of Hatfield and Milgrom (2005) is not only key for several of their results, but also for a large number of follow-up papers on matching with contracts. Primary contributions of Hatfield and Kojima (2010) are (1) the intoduction of two weaker versions of the substitutes condition, and (2) the analysis of the structure of stable allocations under these weaker conditions. The weakest version of substitutes introduced in Hatfield and Kojima (2010) is the following:

Definition 3 *Contracts are bilateral substitutes for hospital h if for any set of contracts $Y \subset X$ and any pair of contracts $x, z \in X \setminus Y$,*

$$z \notin C_h(Y \cup \{z\}) \text{ and } z \in C_h(Y \cup \{x, z\}) \implies z_D \in Y_D \text{ or } x_D \in Y_D.$$

In Theorem 1 of Hatfield and Kojima (2010), the authors claim that bilateral substitutes is sufficient for the existence of a stable allocation. More specifically, they claim that the following cumulative offer algorithm (Hatfield and Kojima 2005) always produces a stable allocation:

Step 1: One of the doctors offers her first choice contract x_1. The hospital receiving the offer, $h_1 = (x_1)_H$, holds the contract if $x_1 \in C_{h_1}\big(\{x_1\}\big)$ and rejects it otherwise. Let $A_{h_1}(1) = \{x_1\}$, and $A_h(1) = \emptyset$ for all $H \setminus \{h_1\}$.

In general, at

Step t: One of the doctors with no contract on hold offers her most preferred contract x_t that has not been rejected in earlier steps. The hospital receiving the offer, $h_t = (x_t)_H$, holds the contracts in $C_{h_t}\big(A_{h_t}(t-1) \cup \{x_t\}\big)$ and rejects the rest. Let $A_{h_t}(t) = A_{h_t}(t-1) \cup \{x_t\}$, and $A_h(t) = A_h(t-1)$ for all $H \setminus \{h_t\}$.

The algorithm terminates when either every doctor is matched to at least one hospital or every unmatched doctor has had all acceptable contracts rejected. Since each contract is offered at most once, the algorithm terminates in some finite Step T. The outcome of the algorithm is, $\bigcup_{h\in H} C_h\big(A_h(T)\big)$.[3]

Given that the original (and stronger) substitutes condition is not sufficient for the existence of a stable allocation, it is clear that Theorem 1 of Hatfield and Kojima (2010) cannot hold in the absence of additional structure.

1.4 A Remedy with the Irrelevance of Rejected Contracts

A close look at the proof of Theorem 1 in Hatfield and Kojima (2010) reveals the source of the complication. The following additional condition on hospital choice functions is implicitly assumed throughout the paper.

Definition 4 *Contracts satisfy the irrelevance of rejected contracts (IRC) for hospital h if*

$$\forall Y \subset X, \forall z \in X \setminus Y \qquad z \notin C_h(Y \cup \{z\}) \implies C_h(Y) = C_h(Y \cup \{z\}).$$

This condition simply requires that, the removal of rejected contracts shall not affect chosen sets.[4] It turns out that, results of Hatfield and Milgrom (2005) and Hatfield and Kojima (2010) are restored once IRC is assumed throughout their analysis.

[3] Observe that while the algorithm necessarily terminates, in principle it may pick a set of contracts which is not an allocation. That is, multiple contracts of a given doctor may be chosen by the algorithm, in the absence of additional assumptions.

[4] This condition is earlier used by Blair (1988) in the context of many-to-many matching. In an extension of Blair's results, Alkan (2002) refers it as *consistency*. More recently Echenique (2007) refers this condition as *independence of irrelevant alternatives* in the context of combinatorial choice rules.

In the absence of IRC, of the seven theorems in Hatfield and Kojima (2010), six theorems do not hold. Likewise, in Hatfield and Milgrom (2005), several theorems including existence of stable allocation do not hold. Fortunately, in both cases, all results are recovered once IRC is assumed in addition to existing hypotheses. In Appendix, we provide proofs for the modified versions of Theorems 1, 4, and 5 of Hatfield and Kojima (2010) which follow the general flow of the original proofs and emphasizes the role of IRC. We omit the proof for the modified version of Theorem 3, since it is not directly related to the structure of stable allocations. We also omit the proofs for the modified versions of Theorems 6 and 7 of Hatfield and Kojima (2010) since their original proofs are valid, once Theorems 1, 4, and 5 are recovered, without additional need to invoke IRC.

Recall that the substitutes condition together with IRC guarantee the existence of a stable allocation. Indeed, the cumulative offer algorithm gives the same stable outcome as the celebrated agent-proposing deferred acceptance algorithm under these conditions. This is no longer the case when substitutes is replaced with bilateral substitutes since a hospital may hold a contract at Step t of the cumulative offer algorithm that was rejected at an earlier Step $t' < t$. Hatfield and Kojima (2010) refers this feature as renegotiation. In Theorem 4 of Hatfield and Kojima (2010), the authors claim that the renegotiation feature ceases to exist and the cumulative offer algorithm yields the same outcome as the agent-proposing deferred acceptance algorithm under the following version of substitutes, that is still weaker than Hatfield and Milgrom (2005) substitutes condition, but stronger than the bilateral substitutes:

Definition 5 *Contracts are unilateral substitutes for hospital h if for any set of contracts $Y \subset X$ and any pair of contracts $x, z \in X \setminus Y$,*

$$z \notin C_h(Y \cup \{z\}) \text{ and } z \in C_h(Y \cup \{x, z\}) \implies z_D \in Y_D.$$

Showing that the inconsistencies in this important research program can be eliminated with an easy fix is important because bilateral substitutes and unilateral substitutes have already established themselves not only as important conditions in theoretical analysis of matching with contracts but also for its practical applications.

While these conditions might initially appear to be minor technical deviations from the substitutes condition, a recent paper by Echenique (2012) makes it clear that they differ from it in one very significant way. In a surprising result Echenique (2012) shows that, Hatfield and Milgrom (2005) matching with contracts model is isomorphic to Kelso and Crawford (1982) labor market model under the substitutes condition. He has also shown that this isomorphism breaks under bilateral substitutes. Hence applications of matching with contracts that are outside the scope of Kelso and Crawford (1982) have to rely on conditions other than the substitutes condition. Sonmez and Switzer (2011) - Sonmez (2011) have recently introduced the first market design application of matching with contracts of that nature: Cadet-branch matching at U.S. Army programs. Both of these market design papers heavily utilize the unilateral substitutes condition, and as such it is important to emphasize that Hatfield and Kojima (2010) research program is not broken in a substantial way. We shall also emphasize that market design applications of matching with contracts, including cadet-branch matching, are shielded from our criticism, since these applications almost always satisfy IRC.

1.5 Relation with the Law of Aggregate Demand

Much of the literature on matching with contracts, including several results in Hatfield and Milgrom (2005), assumes the following condition in addition to the substitutes condition.

Definition 6 *Contracts satisfy the law of aggregate demand (LAD) for hospital h if*

$$\forall Y, Y' \subseteq X, \forall z \in X \setminus Y \qquad Y \subset Y' \implies |C_h(Y)| \leq |C_h(Y')|.$$

A bit of a good news is that substitutes along with LAD implies IRC, and hence results in the literature assuming LAD are immune to our criticism.

Proposition 1 *Suppose contracts satisfy the substitutes condition along with the LAD condition for hospital h. Then contracts also satisfy the IRC condition for hospital h.*

Proof. Suppose contracts satisfy the substitutes condition along with the LAD condition for hospital h. Let $Y \subset X$ and $z \in X \setminus Y$ be such that $z \notin C_h(Y \cup \{z\})$. We want to show that $C_h(Y) = C_h(Y \cup \{z\})$.

For any $x \in C_h(Y \cup \{z\})$, we have $x \neq z$ by assumption. This implies $x \in Y$ which in turn implies $x \in C_h(Y)$ by the substitutes condition. Therefore

$$C_h(Y \cup \{z\}) \subseteq C_h(Y).$$

Moreover we have$|C_h(Y)| \leq |C_h(Y \cup \{z\})|$ by the LAD and hence the above inclusion must hold with equality completing the proof. ■

Recall that results of Hatfield and Milgrom (2005) which assume LAD in addition to the substitutes condition are accurate. It turns out that, in contrast, Theorem 1 of Hatfield and Kojima (2010) fails to hold even if LAD is assumed. The following example shows that, not only the cumulative offer algorithm may produce an unstable allocation under unilateral substitutes, which is stronger than bilateral substitutes, and LAD, but also the set of stable allocations may be empty under these conditions:

Example 2 *Consider a problem with one hospital, h, and two doctors d_1, d_2. Doctor d_1 has three contracts x, x', x'' and doctor d_2 has two contracts y, y'. Preferences of the doctors and the choice function of the hospital are given as follows:*

$$\succ_{d_1}: \quad x \succ_{d_1} x' \succ_{d_1} x'' \succ_{d_1} \emptyset_{d_1}$$

$$\succ_{d_2}: \quad y' \succ_{d_2} y \succ_{d_2} \emptyset_{d_2}$$

$C_h(\{x\}) = \{x\}$	$C_h(\{x, x'\}) = \{x\}$	$C_h(\{x', y\}) = \{y\}$
$C_h(\{x'\}) = \{x'\}$	$C_h(\{x, x''\}) = \{x\}$	$C_h(\{x', y'\}) = \{x'\}$
$C_h(\{x''\}) = \{x''\}$	$C_h(\{x', x''\}) = \{x'\}$	$C_h(\{x'', y\}) = \{y\}$
$C_h(\{y\}) = \{y\}$	$C_h(\{x, y\}) = \{y\}$	$C_h(\{x'', y'\}) = \{x''\}$
$C_h(\{y'\}) = \{y'\}$	$C_h(\{x, y'\}) = \{x\}$	$C_h(\{y, y'\}) = \{y\}$

$C_h(\{x,x',x''\}) = \{x\}$	$C_h(\{x,y,y'\}) = \{y\}$	$C_h(\{x,x',x'',y\}) = \{y\}$
$C_h(\{x,x',y\}) = \{y\}$	$C_h(\{x',x'',y\}) = \{y\}$	$C_h(\{x,x',x'',y'\}) = \{x\}$
$C_h(\{x,x',y'\}) = \{x\}$	$C_h(\{x',x'',y'\}) = \{x'\}$	$C_h(\{x,x',y,y'\}) = \{y\}$
$C_h(\{x,x'',y\}) = \{y\}$	$C_h(\{x',y,y'\}) = \{y\}$	$C_h(\{x,x'',y,y'\}) = \{y\}$
$C_h(\{x,x'',y'\}) = \{x\}$	$C_h(\{x'',y,y'\}) = \{y\}$	$C_h(\{x',x'',y,y'\}) = \{y\}$
		$C_h(\{x,x',x'',y,y'\}) = \{x,y\}$

It is easy to verify that contracts satisfy unilateral substitutes as well as the LAD condition for hospital h.

Consider the cumulative offer algorithm and start the sequence of offers with doctor d_1. *Hospital* h *receives the following sequence of offers:* x, y', y, x', x''. *The cumulative offer algorithm terminates when all contracts are offered, and at this point* $C_h(\{x,x',x'',y,y'\}) = \{x,y\}$. *Hence the outcome is* $\{x,y\}$. *However the allocation* $\{x,y\}$ *is not stable, since hospital* h *blocks it:* $C_h(\{x,y\}) = \{y\}$. *This directly conflicts with the proof of Theorem 1 in Hatfield and Kojima (2010) where the authors argue that the cumulative offer algorithm always results in a stable allocation under bilateral substitutes. Indeed, not only the cumulative offer algorithm yields an unstable allocation in this example, but also the set of stable allocations is empty. Here is a list of blocking coalitions for every possible allocation in this example.*

Allocation	*Blocking Coalition*	*Allocation*	*Blocking Coalition*
$\{x\}$	$\{h, d_2\}$ *via* y	$\{x, y\}$	$\{h\}$ *via removing* x
$\{x'\}$	$\{h, d_2\}$ *via* y	$\{x', y\}$	$\{h\}$ *via removing* y
$\{x''\}$	$\{h, d_2\}$ *via* y	$\{x'', y\}$	$\{h\}$ *via removing* x
$\{y\}$	$\{h, d_2\}$ *via* y'	$\{x, y'\}$	$\{h\}$ *via removing* y'
$\{y'\}$	$\{h, d_1\}$ *via* x	$\{x', y'\}$	$\{h\}$ *via removing* x''
		$\{x'', y'\}$	$\{h\}$ *via removing* y'

1.6 Strict Hospital Preferences as Primitives

We have so far argued that the preferred way to recover the results of Hatfield and Kojima (2010) is

1. maintaining the original structure that allows for weak hospital preferences over sets of contracts that name them,

2. but adjusting the original results by imposing the IRC condition on hospital choice functions throughout the analysis.

This approach allows us to treat hospital choice functions as primitives of the model, consistent with the presentation of several of the results in Hatfield and Kojima (2010). One might be tempted instead to recover the results by assuming hospitals have strict preferences, since IRC is directly implied in this case. We will next present why this would be a poor modeling choice.

Loss of Transparency via a Priori Structure Imposed Hospital Choice Functions Let's suppose each hospital h has a strict preference relation $\succ_h$ over

$$\left\{Y \subseteq X_h |\ y, y' \in Y \text{ and } y \neq y' \implies y_D \neq y'_D\right\}, \text{ and}$$

for a given set of contracts $X' \subseteq X$ its chosen set $C_h(X')$ is derived as

$$C_h(X') = \max_{\succ_h} \left\{Y \subseteq X' \cap X_h |\ y, y' \in Y \text{ and } y \neq y' \implies y_D \neq y'_D\right\}.$$

Under this modeling choice, chosen sets are derivatives of strict hospital preferences, and as such, they must be consistent with these preferences. One potential appeal of this approach is, it assures that the resulting hospital choice functions automatically satisfy the IRC condition.[5] The IRC condition, however, is not the only condition that shall be satisfied by the resulting hospital choice functions. They shall also satisfy the following condition to assure that the underlying hospital preferences are transitive.

Definition 7 *Contracts satisfy the Strong Axiom of Revealed Preference (SARP) for hospital h, if there is no distinct $X^1, X^2, \ldots, X^k \subset X$ and no distinct $Y^1, Y^2, \ldots, Y^k \subset X$ with $k > 1$, such that*

$$\begin{aligned} \forall l &\in \{1, \ldots, k\} & Y^l &= C_h(X^l), \textit{ and} \\ \forall l &\in \{1, \ldots, k-1\} & Y^l &\subset X^l \cap X^{l+1} \textit{ and } Y^k \subset X^k \cap X^1 \end{aligned}$$

So before the analysis even starts, there is strong a priori structure imposed on hospital choice functions under this approach. This is especially troubling since the

[5] See, for example, Lemma 1 in Hatfield, Immorlica and Kominers (2012) for a short proof of this observation.

key results of Hatfield and Kojima (2010) concern the impact of particular properties of hospital choice functions on sets of stable allocations or mechanisms that select stable allocations. A loose analogy here would be, trying to appreciate a picture that is drawn on top of another picture. To illustrate how this affects the interpretation of their results, let's take Theorem 1 of Hatfield and Kojima (2010). This result reads:

"Result 1: Suppose that contracts are bilateral substitutes for every hospital. Then there exists a stable allocation."

The reader, however, is expected to interpret this statement as follows:

"Consider hospital choice functions that can be obtained from strict hospital preferences via derivation above. In addition, suppose that contracts are bilateral substitutes for every hospital. Then there exists a stable allocation."

As such, the exact role of bilateral substitutes in this existence result is not transparent. Perhaps the existence is "mostly" due to the underlying structure of feasible hospital choice functions which is already imposed before bilateral substitutes. Hence all results shall be interpreted in the context of an underlying structure which is not even discussed in the paper.

At the end, majority of results in Hatfield and Kojima (2010) rely on IRC no matter how the ambiguity is resolved. When hospital choice functions are treated as primitives (or alternatively when underlying hospital preferences allow for indifferences), this condition is explicitly stated in the results. When hospital preferences are strict, this condition is not only hidden in the results, but also accompanied by another implicit assumption, SARP, which has no role in any of the proofs. We next elaborate on how this redundancy reflects itself on applications of this important research program.

As far as we can see, SARP is not needed in any of the proofs of Hatfield and Milgrom (2005). Hence assuming that hospital choice functions are derivatives of the underlying strict hospital preferences not only imposes a strong structure on "feasible" choice functions on which the substitutes condition must be superimposed, but also potentially weakens the scope of their analysis. As such, assuming hospital choice functions to be primitives of the model and restating the results by explicitly assuming IRC might be the preferred approach. If, however, one takes strict hospital preference relations to be primitives, it is important to understand how SARP interacts with the substitutes condition. Here an important observation is, as in the case of the IRC condition, SARP might be violated even when contracts satisfy the substitutes condition. However, as we show next, the substitutes condition together with IRC implies SARP.

Proposition 2 *Suppose contracts satisfy the substitutes condition along with the IRC condition for hospital h. Then contracts also satisfy SARP for hospital h.*

Proof. Suppose contracts satisfy the substitutes condition along with the IRC condition for hospital h. Towards a contradiction, suppose SARP is violated. Then there exists distinct $X^1, X^2, \ldots, X^k \subset X$ and distinct $Y^1, Y^2, \ldots, Y^k \subset X$ with $k > 1$, such that

$$\begin{aligned} \forall l &\in \{1, \ldots, k\} && Y^l = C_h(X^l), \text{ and} \\ \forall l &\in \{1, \ldots, k-1\} && Y^l \subset X^l \cap X^{l+1} \text{ and } Y^k \subset X^k \cap X^1 \end{aligned}$$

Define $\overline{X} = \bigcup_{l \le k} X^l$, $\overline{Y} = \bigcup_{l \le k} Y^l$, and $\underline{Y} = \bigcap_{l \le k} Y^l$. Also define $Y^{k+1} \equiv Y^1$ and $X^{k+1} \equiv X^1$ for notational convenience. For any $x \in X$,

$$x \in \overline{X} \setminus \overline{Y} \implies \exists l \le k \text{ s.t. } x \in X^l \setminus Y^l \implies x \notin C_h(\overline{X}) \text{ (1)}$$

where the last implication holds by the substitutes condition. Moreover for any $x \in X$,

$$x \in \overline{Y} \setminus \underline{Y} \implies \exists l \le k \text{ s.t. } x \in Y^l \setminus Y^{l+1} \implies x \in X^{l+1} \setminus Y^{l+1} \implies x \notin C_h(\overline{X}) \text{ (2)}$$

where the second implication holds by the relation $Y^l \subset X^l \cap X^{l+1} \subset X^{l+1}$ and the third implication holds by the substitutes condition. Therefore by (1) and (2) we have, for any $x \in X$,

$$x \in \overline{X} \setminus \underline{Y} \implies x \notin C_h(\overline{X}) \implies C_h(\overline{X}) \subseteq \underline{Y} \text{ (3)}$$

Pick any $l \le k$. We have $\overline{X} \supseteq X^l \supseteq Y^l \supseteq \underline{Y} \supseteq C_h(\overline{X})$ where the last inclusion holds by (3). Therefore for any $l \le k$, we must have $Y^l = C_h(X^l) = C_h(\overline{X})$ by IRC contradicting the distinct choice of sets $Y^1, Y^2, \ldots, Y^k$ and completing the proof. ■

An immediate corollary of Propositions 1 and 2 is the following.

Corollary 1 *Suppose contracts satisfy the substitutes condition along with the LAD condition for hospital h. Then contracts also satisfy SARP for hospital h.*

Reduced Scope of the Analysis None of the results in Hatfield and Kojima (2010) rely on the SARP condition as discussed here, and its sole purpose is assuring the existence of underlying strict preferences for the hospitals. While it is certainly important to cover choice functions that are derivatives of strict hospital preferences, imposing such a structure significantly reduces the scope of the analysis without any clear benefit. Indeed, Sonmez and Switzer (2011) and Sonmez (2011) have recently presented the first practical application of the unilateral substitutes condition in a brand new application of market design, cadet-branch matching, and this first application builds on choice functions that are derivatives of branch priorities that capture Army policies, and they are not derivatives of branch preferences. This and similar potential applications of Hatfield and Kojima (2010) might be left outside the scope of their paper, if hospital choice functions are required to be derivatives of underlying strict preferences.

Adverse Impact on Interpretation of the Results While the substitutes condition and SARP are logically independent in the absence of other conditions, the substitutes condition together with IRC (or alternatively together with LAD) imply SARP (Aygun and Sonmez 2012). What that means is, once IRC is assured, the substitutes condition guarantee the existence of underlying strict hospital preferences. It turns out that, this result has no counterpart for bilateral substitutes or even for the stronger unilateral substitutes. In other words relaxing the substitutes condition to its weaker versions may not be "free." This is in sharp contrast with the authors' interpretation of their results, and promotion of their weaker substitutes conditions. To illustrate this point, consider the following statement in page 1715:

"We have seen that the bilateral substitutes condition is a useful notion in matching with contracts in the sense that it is the weakest condition guaranteeing the existence of a stable allocation known to date."

Since bilateral substitutes guarantee existence of a stable allocation only in the presence of an underlying structure a priori imposed on hospital choice functions, the interaction of bilateral substitutes with the underlying structure is important. In the presence of IRC, the substitutes condition guarantee compatibility with the underlying structure, whereas bilateral substitutes or unilateral substitutes do not. As such, bilateral substitutes (or unilateral substitutes) can no longer be considered to be a "costless" relaxation of the substitutes condition. Observe that this issue is entirely caused by compatibility with SARP, which was never needed in entire analysis. Therefore taking strict hospital preferences as primitives of the model introduces an artificial difficulty in interpretation of the role of the weaker substitutes conditions.

1.7 Concluding Remarks

We presented two remedies to resolve a critical inconsistency in Hatfield and Milgrom (2005) and Hatfield and Kojima (2010). We believe the first one which essentially treats hospital choice functions as the primitives of the model is the scientifically sound remedy since it maintains the transparency of the results, increases the scope of the paper by embracing applications such as cadet-branch matching, and allows for more transparent comparisons between the roles of various substitutes conditions. It is important to emphasize that market design applications of matching with contracts almost always satisfy the IRC condition, and therefore shielded from our criticism.

Our observations have potentially adverse implications on a large number of follow-up papers on matching with contracts. However two strands of the literature are mostly shielded from our criticism. A significant portion of the literature assume

LAD in addition to the substitutes condition. Substitutes along with LAD implies IRC and hence our criticism has no bite under LAD. In addition, market design applications of matching with contracts, including the earlier mentioned applications on school choice with soft caps and cadet-branch matching, typically construct choice sets based on other primitives including but not limited to preferences, thereby automatically satisfy the IRC condition. Therefore most of the results in market design applications are likely correct, even though their proofs might be slightly inaccurate.

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1.9 Appendix

Proofs for Modified Versions of Theorems 1, 4 and 5

We will mostly follow the general outline of the original proofs in Hatfield and Kojima (2010), so that the extensive role of IRC can be clearly observed.

Theorem 1 (Theorem 1 of Hatfield and Kojima (2010)) *Suppose that contracts are bilateral substitutes for every hospital and they satisfy IRC. Then there exists a stable allocation.*

Proof. Suppose that contracts are bilateral substitutes for every hospital and they satisfy IRC. We will show that the cumulative offer algorithm yields a stable allocation under these conditions. Cumulative offer algorithm always terminates in finite steps and produces a set of contracts since there are finite number of contracts, and no contract can be offered more than once. Let the algorithm terminate at Step T producing the set of contracts X'. We want to show that X' is a stable allocation.

We first show that X' is an allocation. To do so, we will show that no doctor can have multiple contracts in his name under X'. This is a direct implication of the following Claim which states that a hospital cannot hold at any step a contract it rejected in the previous step unless in the previous step it holds another contract of the same doctor. This does not rule out the possibility that a previously rejected contract to be held later on, but it rules out the possibility that multiple contracts of the same doctor to be on hold at any given step across all hospitals.

Claim 1 *For any $h \in H$, $z \in X$ with $z_H = h$, and $t \geq 2$,*

$$z \in A_h(t-1) \setminus C_h(A_h(t-1)) \text{ and } z_D \notin [C_h(A_h(t-1))]_D \implies z \notin C_h(A_h(t)).$$

Proof. Proof of the Claim: We have three cases to consider.

Case 1 : Hospital h receives no offers at Step t. This case immediately follows since $A_h(t-1) = A_h(t)$.

Case 2 : Hospital h receives an offer z' from doctor z_D at Step t. Since $z \in A_h(t-1)$, we have $z \neq z'$, and thus $A_h(t) = A_h(t-1) \cup \{z'\}$. Towards a contradiction suppose $z \in C_h(A_h(t))$. Then $z' \notin C_h(A_h(t))$ and hence by IRC we have $C_h(A_h(t)) = C_h(A_h(t-1))$ contradicting $z \in A_h(t-1) \setminus C_h(A_h(t-1))$ and completing Case 2.

Case 3 : Hospital h receives an offer x from doctor $x_D \neq z_D$ at Step t. Let $Y = A_h(t-1) \setminus \{y \in X | y_D \in \{x_D, z_D\}\}$. Observe that $x_D, z_D \notin Y_D$. Since doctor x_D makes an offer at Step t, we have $x_D \notin [C_h(A_h(t-1))]_D$; furthermore by assumption $z_D \notin [C_h(A_h(t-1))]_D$. Finally by IRC, $C_h(A_h(t-1)) = C_h(Y \cup \{z\})$, and therefore $z \notin C_h(Y \cup \{z\})$, which in turn implies $z \notin C_h(Y \cup \{x, z\})$ by bilateral substitutes. Towards a contradiction suppose $z \in C_h(A_h(t))$. Since $z \notin C_h(A_h(t-1))$, that means $C_h(A_h(t-1)) \neq C_h(A_h(t))$, which in turn implies $x \in C_h(A_h(t))$ by IRC and $A_h(t) = A_h(t-1) \cup \{x\}$. Thus $x, z \in C_h(A_h(t))$ which means neither doctor

x_D nor doctor z_D can have another contacts in $C_h(A_h(t))$. Therefore IRC implies $x, z \in C_h(Y \cup \{x, z\})$ contradicting relation and completing Case 3. This completes the proof of the Claim. ■

We will next show that allocation X' is stable. First observe that no doctor can block X' since a doctor never offers an unacceptable contract. Hence $C_D(X') = X'$. Next suppose $C_H(X') \neq X'$, and observe that $C_H(X') = \bigcup_{h \in H} C_h(A_h(T))$ under IRC. Therefore there exists a hospital h and a contract x such that $x \in C_h(A_h(T))$ but $x \notin C_h(C_h(A_h(T)))$. This is ruled out by IRC and hence $C_H(X') = X'$.

Finally, towards a contradiction, suppose there exists a hospital h and a set of contracts $X'' \neq C_h(X')$ such that

$$X'' = C_h(X' \cup X'') \subseteq C_D(X' \cup X'').$$

Let $X'_h = \{x \in X' | x_H = h\}$. That is, X'_h is the subset of X' that pertains to hospital h. Observe that $X'_h = C_h(A_h(T))$ by the mechanics of the cumulative offer algorithm. Also recall that, we have already shown $C_h(X') = X'_h$ by the above individual rationality argument. Hence

$$X'_h = C_h(X') = C_h(A_h(T)).$$

Since $X'' = C_h(X' \cup X'')$, we have $x_H = h$ for all $x \in X''$. Moreover since $X'' \subseteq C_D(X' \cup X'')$,

$$\forall x \in X'', \qquad x \succeq_{x_D} x'_{x_D}$$

Therefore each contract in X'' is offered to hospital h by step T by the mechanics of the cumulative offer algorithm. Hence $X'' \subseteq A_h(T)$. This in turn implies

$$X'' = C_h(X' \cup X'') = C_h(X'_h \cup X'') = C_h(C_h(X') \cup X'') = C_h(A_h(T)) = C_h(X')$$

contradicting $X'' \neq C_h(X')$. This shows that X' is stable completing the proof. ■

The next theorem by Hatfield and Kojima (2010) states that a contract that is rejected at any step is rejected for good under unilateral substitutes and IRC.

Theorem 2 (Theorem 4 of Hatfield and Kojima (2010)) *Suppose that contracts are unilateral substitutes for every hospital and they satisfy IRC. A contract z that is rejected by a hospital h at any step of the cumulative offer algorithm cannot be held by hospital h in any subsequent step.*

Proof. Towards a contradiction let t' be the first step a hospital h holds a contract z it previously rejected at Step $t < t'$. Since z is rejected by hospital h at Step t, either it was on hold by hospital h at Step $(t-1)$ or it was offered to hospital h at Step t. In either case no other contract of doctor z_D could be on hold by hospital h at Step $(t-1)$. But then, since z is the first contract to be held after an earlier rejection, hospital h cannot have held another contract by doctor z_D at Step t. That is,

$$z_D \notin [C_h(A_h(t))]_D$$

Then by IRC $z \in A_h(t) \setminus C_h(A_h(t))$ implies

$$z_D \notin C_h\left(C_h(A_h(t)) \cup \{z\}\right)$$

and yet

$$z \in C_h(A_h(t'))$$

Since $(C_h(A_h(t)) \cup \{z\}) \subseteq A_h(t')$, three relations above contradict unilateral substitutes completing the proof. ■

Theorem 3 (Theorem 5 of Hatfield and Kojima (2010)) *Suppose that contracts are unilateral substitutes for every hospital and they satisfy IRC. Then there exists a doctor-optimal stable allocation each doctor weakly prefers to any other stable allocation. The allocation that is produced by the cumulative offer algorithm is the doctor-optimal stable allocation.*

Proof. Since unilateral substitutes implies bilateral substitutes, there exists a stable allocation by Theorem 1 of Hatfield and Kojima (2010). To prove the theorem, it suffices to show that for any stable allocation $X' \subseteq X$ and any contract $z \in X'$, contract z is not rejected by the cumulative offer algorithm. To obtain the desired contradiction, suppose not. Let t be the first step where a hospital $h = z_H$ rejects such a contract z, and let $Y = C_h(A_h(t))$. Then by IRC, $z \notin C_h(Y \cup \{z\})$. By Theorem 4 of Hatfield and Kojima (2010), $z_D \notin Y_D$. Since t is the first step a contract in any stable allocation is rejected, every doctor in Y_D weakly prefers their contract in Y to their contract in X' which is stable by assumption. We complete the proof via two cases each of which yields the desired contradiction:

Case 1 : $z \notin C_h(Y \cup X')$. In this case hospital h blocks allocation X' together with doctors in Y_D (unless $Y_D = \emptyset$ in which case hospital h blocks X' by itself). That is, Y blocks X' contradicting its stability.

Case 2 : $z \in C_h(Y \cup X')$. This case immediately gives a contradiction by unilateral substitutes since $(Y \cup \{z\}) \subseteq (Y \cup X')$, $z \notin C_h(Y \cup \{z\})$, and $z_D \notin Y_D$. ■

Hatfield and Kojima (2010) observe that, the cumulative offer algorithm overlaps with the doctor proposing deferred acceptance algorithm (for any sequence of offers). This observation is directly implied by Theorem 5.

CHAPTER 2.

ORDER INDEPENDENCE OF CUMULATIVE OFFER ALGORITHM AND CADET QUALITY IN USMA

2.1 Introduction

Matching with contract literature is young and very exciting literature with its brand new applications. In most of the applications, as in matching literature, achieving a stable allocation is one of the main goals. In the matching with contracs literature, the weakest set of conditions known that guarantees existence of stablity is shown in Aygün and Sönmez (2012) paper. These conditions are bilateral substitutes condition and Irrelevance of Rejected Contracts (IRC) condition. In their paper, Aygün and Sönmez (2012) showed that under bilateral substitutes and IRC condition assumptions, existence of stable allocation can be guaranteed. These conditions are satisfied by most of the applications of matching with contract literature. To achieve stable allocation, Hatfield and Kojima (2010) described Cumulative Offer Algorithm (COA) in the paper and Hatfield and Kojima (2010) and Aygün and Sönmez (2012) showed that under bilateral substitutes and IRC assumptions, Doctor Proposing COA produces a stable allocation in any doctor hospital matching problem. In this paper, we study cadet-branch matching problem and we analyze Cadet Proposing Cumulative Offer Algorithm. Cadet Proposing version of this algorithm is a generalized version of the Student Proposing Deferred Acceptance Algorithm (DAA) described by Gale and Shapley (1962). There are two main differences between COA and DAA. The first one is that COA keeps rejected contracts considered after rejection. The second difference is that under COA cadets propose their contracts one by one but not altogether. Cadets make their offer one at a time and in each step cadet is chosen randomly among the ones whom there is no contract held by a branch.

In the first part of this paper, we analyze COA and show that the selection of cadets in each step of COA does not affect the outcome under bilateral substitutes and IRC condition assumptions. Also under these assumptions, COA eqiuvalent to Generalized Deferred Acceptance Algorithm (GDAA) described in Hatfield and Milgrom (2005) which allows cadets make their offer altogether. In the second part, we study cadet-branch matching problem where the new choice function we designed only satisfies bilateral substitutes and IRC conditions, yet GDAA produces stable allocation.

The United States Military Academy, to increase manpower in the army, changed its matching mechanism to match cadets and branches. In the new choice function used by branches in the allocation process, cadets can get priority for the %25 of the seats of any branch if they apply with longer term contracts. Therefore, USMA gives incentive to cadets to serve more and increases manpower in the US Army. First, we showed that for any given branch, we can increase cadet quality by changing choice function and to increase cadet quality, we designed a new choice function for branches. Next, we showed that the new choice function proposed satisfies bilateral substitutes and IRC which makes cadet-branch matching problem perferct application for GDAA if we use new alternative choice function.

Next, we study incentive compatibility property of Cadet Optimal Stable Mechanism induced by GDAA. Prior to this paper, Hatfield and Kojima (2010) showed that bilateral substitutes and IRC are not sufficient for strategy-proofness of COA. However, the lexicographic choice functions described in Kominers and Sonmez (2012) satisfy bilateral substitutes and IRC and guarantee strategy-proofness. The choice function we suggest is not in the class of lexicographic choice functions. But, it makes Cadet Optimal Stable Mechanism strategy-proof. Therefore, the application we present here is the second application in the matching with contracts literature such that choice functions only satisfies the weakest substitutes condition and IRC, yet makes COA, equivalently GDAA, strategy-proof.

The paper is structured as follows. The next section sets up the model. After introducing the model, we analyze order independence property of COA. Then, we analyze cadet-branch matching problem in USMA and propose an alternative choice function. It concludes with a brief discussion of the findings and provides the proofs for some of the results in the appendix part.

2.2 The Model

There are finite sets I and B of cadets and branches, and a finite set X of contracts. Each contract $x \in X$ is associated with one cadet $x_I \in I$ and one branch $x_B \in B$. Each cadet $i \in I$ can sign at most one contract or his null contract which is denoted by $\emptyset_i$. A set of contracts $X' \subseteq X$ is an allocation if each cadet is associated with at most one contract under X'. For any set of contracts $Y \subseteq X$, Y_I and Y_B are the set of cadets and set of branches that has at least one contract in Y respectively. Also, for any set of contracts $Y \subseteq X$, $Y(j)$ is the subset of Y that includes all contracts of $j \in I \cup B$.

For each cadet $i \in I$, P_i is a strict preference relation on his contracts $X(i) \cup \{\emptyset_i\}$. A contract is acceptable by cadet i if it is at least as good as the null contract $\emptyset_i$, and unacceptable by cadet i if it is worse that the null contract $\emptyset_i$. For each cadet $i \in I$ and a set of contracts $Y \subseteq X$, the chosen set $C_i(Y)$ of cadet i is defined as

$$C_i(Y) = \max_{P_i}(Y(i) \cup \{\emptyset_i\}).$$

For a given set of contracts $Y \subseteq X$, define $C_I(Y) \equiv \bigcup_{i \in I} C_i(Y)$.

Given a branch $b \in B$ and any set of contracts $Y \subseteq X$, the chosen set $C_b(Y)$ of an branch b is a subset of the contracts associated with branch b. That is, $C_b(Y) \subseteq Y(b)$. Moreover, a branch can sign only one contract with any given cadet:

$$\forall b \in B, \forall Y \subseteq X, \forall x, x' \in C_b(Y) \qquad x \neq x' \Longrightarrow x_I \neq x'_I.$$

For a given set of contracts $Y \subseteq X$, define $C_B(Y) \equiv \bigcup_{b \in B} C_b(Y)$. For a given branch $b \in B$, we refer the function that maps each set of contracts to a chosen set as the choice function of branch b.

In a given cadet-branch matching problem stable allocation can be described as the situation where no cadet or branch would be better off by either walking away of forming bilateral arrangements outside of the allocation.

Definition 8 *A set of contracts $X' \subseteq X$ is a stable allocation if*

1. $C_I(X') = C_B(X') = X'$, *and*

2. *there exists no branch $b \in B$ and set of contracts $X'' \neq C_b(X')$ such that*

$$X'' = C_b(X' \cup X'') \subseteq C_I(X' \cup X'').$$

As in the matching literature, first condition of stable allocation is individual rationality and the second condition is no blocking pair of cadets and branches. By Hatfield and Milgrom (2005) and Aygun and Sonmez (2013) papers, we know that if choice functions of branches satisfy substitutes condition along with IRC condition, set of stable allocations is not empty and Cadet Proposing GDAA produces a stable allocation which is Pareto Efficient among stable allocations and strategy-proof if cadets are considered as the only strategic agents in the problem.

Definition 9 *A choice function satisfies substitutes condition if for any set of contracts $Y \subseteq Y' \subseteq X$ and a contract $x \in Y$,*

$$x \notin C(Y) \implies x \notin C(Y')$$

Definition 10 *A choice function satisfies Irrelevance of Rejected Contracts condition if for any set of contracts $Y, Y' \subseteq X$,*

$$C(Y') \subseteq Y \subseteq Y' \implies C(Y) = C(Y')$$

In Hatfield and Kojima (2010), it is shown that although substitutes condition is sufficient to guarantee existence of stable allocation, it is not necessary. To widen the scope of the model they defined weaker versions of substitutes condition, unilateral substitutes and bilateral substitutes conditions. The weakest version, bilateral substitutes condition, is defined below.

Definition 11 *A choice function satisfies bilateral substitutes condition if for any set of contracts $Y \subseteq X$ and a pair of contracts $x, y \in X \backslash Y$,*

$$x \notin C(Y \cup \{x\}) \textit{ and } x \in C(Y \cup \{x, y\}) \implies x_I \in Y_I \textit{ or } y_I \in Y_I$$

So, by Hatfield and Kojima (2010) and Aygun and Sonmez (2012b) papers, we know that if choice functions of branches satisfy bilateral substitutes condition along with IRC condition, set of stable allocations is not empty and Cadet Proposing GDAA produces a stable allocation. Next, it is shown that substitutes or unilateral substitutes conditions along with the IRC condition make DAA, COA and GDAA equivalent. On the other hand, there is no equivalence result for these three algorithms under bilateral substitutes condition. In this paper, we show that although DAA and COA are different due to possible renegotiations, COA and GDAA are equivalent under bilateral substitutes and IRC conditions.

2.3 Order Independence of Cumulative Offer Algorithm

In this section, we formally describe GDAA and COA and show the relationship between these two algorithms.

Generalized Deferred Acceptance Algorithm The generalized deferred acceptance algorithm description we use here, was previously introduced by Hatfield and Milgrom (2005). For any given many to one matching with contracts problem, cadet proposing GDAA works as the following:

Step 1: All cadets offer their first choice contracts. Call the set of contracts offered in this step X_1. Let $A'(0) = \emptyset$, and $A'(1) = X_1$. Each branch b, holds contracts in set $C_b(X_1)$ and rejects the rest of the contracts.

In general,

Step $l \geq 2$: All the cadets for whom no contract is currently held by a branch offer their most preferred contracts that has not been rejected in previous steps. Call the set of contracts offered in this step X_l. Let $A'(l) = A'(l-1) \cup X_l$. Each branch b, holds contracts in set $C_b(A'(l))$ and rejects the rest of the contracts.

The algorithm terminates when either every cadet has a contract that is held by a branch or every unmatched cadet has had every acceptable contract rejected. As there are a finite number of contracts, the algorithm terminates in some finite number L of steps. At that point, the algorithm produces $X' = \bigcup_{b \in B} C_b(A'(L))$, i.e., the set of contracts that are held by some branch at the terminal step L.

As we mentioned above, in general, keeping previously rejected contracts available to branches is the main difference between DAA and GDAA. However, under substitutes or unilateral substitutes conditions, no rejected contracts have chance to be chosen in the GDAA. Therefore, DAA and GDAA are equivalent.

Cadet Proposing Cumulative Offer Algorithm The cumulative offer algorithm description we use here, was previously introduced by Hatfield and Kojima (2010). For any given many to one matching with contracts problem, cadet proposing COA works as the following:

Step 1: One randomly selected cadet i_1 offers her first choice contract x^1, according to her preferences P_{i_1}. The branch that receives the offer $b_1 = x^1_B$ holds $C_{b_1}(\{x^1\})$. Let $A_{b_1}(1) = x^1$, and $A_b(1) = \emptyset$ for all $b \neq b_1$.

In general,

Step $k \geq 2$: One of the cadets for whom no contract is currently held by a branch, say i_k, offers the most preferred contract, based on her preferences P_{i_k}, that has not been rejected in previous steps. Call the new offered contract, x^k. Branch $b_k = x^k_B$ holds $C_{b_k}(A_{b_k}(k-1) \cup \{x^k\})$ and rejects all other contracts. Let $A_{b_k}(k) = A_{b_k}(k-1) \cup \{x^k\}$, and $A_b(k) = A_b(k-1)$ for all $b \neq b_k$.

The algorithm terminates when either every cadet is matched to a branch or every unmatched cadet has no contract left to offer. The algorithm terminates in some finite number K of steps due to a finite number of contracts. At that point, the algorithm produces $X' = \bigcup_{b \in B} C_b(A_b(K))$, i.e., the set of contracts that are held by some branch at the terminal step K.

<u>Order Independence of Cumulative Offer Algorithm</u> Assume that we use cadet proposing cumulative offer algorithm described above. In each step, cadet who is proposing her contract, i_k, is randomly selected among cadets for whom no contract is currently held by a branch. Therefore, in general, outcome of the algorithm depends on the selection of cadet who makes offer in each step. On the other hand, in many applications we observe that at least bilateral substitutes and IRC are satisfied and our main theorem states that under bilateral substitutes and IRC conditions assumptions outcome of cumulative offer algorithm is independent of the selection of cadets.

An order θ, is a function that gives the name of the cadet that offers a contract for each step, $\theta : \mathbb{N} \longrightarrow I$. Set of all possible orders is denoted as Θ. For a fixed problem and a cumulative offer algoritm, an order θ is feasible if each cadet offers a new contract when their turn comes.

Definition 12 *An order $\theta \in \Theta$ is feasible if* $\forall k, \theta(k) \in \{i \in I : \nexists x \in C_B(A(k-1))$ *and* $x_I = i\}$.

By the definition above, one can say that any cadet proposing cumulative offer process can be described by a feasible order. To prove order independency of COA, first we show that under bilateral substitutes and IRC conditions, if all of the proposed contracts of a cadet are rejected in some step, no other cadet makes the rejected contracts desirable.

Lemma 1 *Assume that all choice functions satisfy bilateral substitutes condition along with the IRC condition. If θ is feasible, a cadet i has no chosen contract at k_1 and offers his next contract at k_2, then $k_1 < k < k_2 \implies \nexists x \in C_B(A(k))$ s.t. $x_I = i$.*

Unlike unilateral substitutes and substitutes conditions, bilateral substitutes condition allows renegotiation. Therefore, in general, COA is not equivalent to DAA. Our first lemma, Lemma 1, states that renegotiation is possible for chosen cadets only.

Next, we show that swapping orders of two cadets does not change the outcome. Assume that we have a feasible order θ such that $\theta(k^*) = i', \theta(k^*+1) = i$ and $i \neq i'$ for a given k^*. Now, we construct a new order θ' by swapping i and i''s turns.

$$\theta'(k) = \begin{cases} \theta'(k) = \theta(k^*+1) & \text{if } k = k^* \\ \theta'(k) = \theta(k^*) & \text{if } k = k^*+1 \\ \theta'(k) = \theta(k) & \text{otherwise} \end{cases}$$

Lemma 2 *Assume that all choice functions satisfy bilateral substitutes condition along with the IRC condition. For any feasible order θ, if there is no chosen contract* *of cadet i at step $k^* - 1$, then the outcome of cumulative algorithm remains the same if one changes ordering from θ to θ'.*

The lemma above is the main part of the proof ot the main theorem. By swapping orders of cadets one at a time will let us achieve any feasible order without changing the outcome. Now, think about the following class of order, θ^*:

In the first K_1 period each cadet offers her best contract one at a time. Between K_1 and K_2, each cadet for whom no contract is currently held by a branch at K_1 ($\{i \in I : \nexists x \in C_B(A(K_1))$ and $x_I = i\}$) offers her next best contract among contracts that are not offered yet. Between K_{l-1} and K_l, each cadet for whom no contract is held by a branch at K_{l-1} ($\{i \in I : \nexists x \in C(A(K_{l-1}))$ and $x_I = i\}$) offers her next best contract among contracts that are not offered yet. Algorithm terminates when either every cadet has a contract held by a branch or every unmatched cadet has no contract left to offer.

Lemma 3 *Assume that all choice functions satisfy bilateral substitutes condition along with the IRC condition. The order θ^* given above is feasible and the outcomes of all orders in class θ^* are identical.*

We are going to use the lemma above to show order independence of cumulative offer algorithm by transforming any feasible order to an order in class θ^*. The transformation can be done changing order of cadets one at a time by making all cadets offer their top contracts in the first K_1 steps, making rejected cadets offer their next best contracts in the next $K_2 - K_1$ steps etc. The first main theorem is stated below.

Theorem 4 *Assume that all choice functions satisfy bilateral substitutes condition along with the IRC condition. Cumulative offer algorithm induced by any feasible order gives unique, stable allocation.*[1]

As a corollary, we can state the equivalence of GDAA and the COA induced by any feasible order under bilateral substitutable choice functions that satisfy IRC condition. For any step l of GDAA, set of contracts offered $A'(l)$ is identical to the set of contracts $A(K_l)$ that is constructed under COA induced by any order in class θ^*. Therefore, under bilateral substitutes and IRC conditions assumption, GDAA and COA are equivalent. Also, GDAA produces its outcome much faster since it allows all the cadets offer together. The result we mentioned here unifies algorithms used generally in matching with contracts applications. In the rest of the paper, we present an application of this result in the cadet-branch matching problem in USMA.

2.4 Cadet-Branch Matching Problem in USMA

Prior to 2006, cadet-branch assignment in USMA was a typical application of Balinski and Sonmez (1999). Assignment used to be done according to cadets' preferences over branches and their unique priority ranking. These priorities are known as order of merit list (OML) that is based on order of merit score which is a weighted average of cadet's academic performance, physical fitness test score and military performance. Prior to 2006, USMA used serial dictatorship as a matching mechanism to assign cadets to branches due to the unique priority structure. At

[1] This theorem is independently studied and proved by Hirata and Kasuya (2014).

the end of the assignment process, cadets used to serve 5 years at the army branch assigned.

As of 2006, USMA changed its policy on cadet-branch assignment to increase manpower in the US Army. The new policy introduced two new aspect to this assignment problem. The first one is introducing 8-year contract. By introducing the 8-year contracts USMA aimed to increase manpower. The second one is giving cadets incentive to sign 8-year contracts by giving higher priority for 25% of the seats in all branches.

There are finite sets $I = \{i_1, i_2, \ldots, i_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$ of cadets and branches. Each branch b has its own capacity q_b and $\sum_{b \in B} q_b \geq n$. There are two possible serving times for each cadet (t_0, t_+). In our setup, t_0 represents five years serving time, or "low cost", and t_+ represents eight years serving time, or "high cost". Set T is the potential contract terms. Order of merit list for each cadet is determined by a priority ranking function $\pi : I \rightarrow \{1, 2, \ldots, n\}$.

A contract $x \in X$ is a tuple (i, b, t), where $i \in I$, $b \in B$ and $t \in T$. For a contract x; x_I, x_B and x_T represent cadet, branch and term of the contract x respectively. Let X be the set of all contracts. For any cadet $i \in I$, P_i is the preference of cadet i over her possible contracts and being unassigned $X(i) \cup \{\emptyset\}$. Let $\mathcal{P}_i$ be the set of all possible preferences over contracts for cadet i. An allocation is a set of contracts $X' \subseteq X$, such that for every $i \in I$ and every $b \in B$, $|X'(i)| \leq 1$ and $|X'(b)| \leq q_b$. Let χ be set of all possible allocations.

USMA Priorities In general, the chosen set of branch b from a set of contacts Y depends on the policy on who has higher claims for slots in branch b. Here our focus is the current USMA policy, where cadets with higher OML ranking have higher claims for the top $(1-\lambda)q_b$ slots and the priority for the last λq_b slots is adjusted to favor cadets who are willing to pay the increased service cost. We are ready to define the USMA chosen set of branch b from a set of contracts Y.

For a given order-of-merit priority ranking π, $\lambda \in [0,1]$ and Y, chosen set $C_b(Y)$ is constructed as follows:

1. Branch chooses contracts of cadets based on their OML rankings, one contract at a time. If there are two contracts of a cadet, branch chooses the one with lower cost, t_0 and rejects the other. Either all contracts will be considered or $(1-\lambda)q_b$ contracts are chosen. If the former happens branch terminates the procedure and rejects all the remaining contracts if the latter happens then branch continues with the next step.

2. Branch only considers contracts with high cost, t_+. Branch chooses contracts of cadets based on their OML rankings, one contract at a time. If there are two contracts in Y of a cadet whose contract is chosen in this step, branch chooses the one with high cost, t_+ and rejects the other. Either q_b contracts are chosen or all contracts with high cost, t_+, will be considered. If the former happens branch terminates the procedure and rejects all the remaining contracts if the latter happens then branch continues with the next step.

3. All the remaining contracts are low cost contracts by construction. Branch

chooses contracts of cadets based on their OML rankings, one contract at a time. Branch terminates the procedure when either all the remaining contracts are chosen or q_b contracts chosen.

In this paper, our objective is choosing "better" sets of cadets for any given branch. The idea of choosing better set is based on responsive preferences in college admission problem. As we mentioned, before 2006, branches were used to choose their cadets based on single OML rankings. Given any set of applications branches chose top cadets in the application pool up to their capacities. In that sense, prior to 2006, branches had responsive preferences. Like the college admission with affirmative action problems, due to the new policy shift in 2006, responsive choices are not available for branches anymore. However, by replacing lower ranked cadets with higher ranked cadets without hurting incentive policy introduced by USMA, we can choose sets of cadets closer to a responsive choice function chooses. Next, example shows that if there exists a cadet that prefers a contract with the higher cost to a contract with the base cost, choice function derived from USMA priorities may not choose best set of cadets among the available contract set.

Example 3 *Let* $I = (i_1, i_2, i_3)$, $B = \{b\}$ *with* $q_b = 2$. *Let the merit ranking be* $\pi(i_1) < \pi(i_2) < \pi(i_3)$ *and* $\lambda = 0.5$. *Let* $Y = \{(i_1, b, t_+), (i_2, b, t_0), (i_3, b, t_+)\}$. *If we use choice function derived from USMA priorities we get* $C_b^{USMA}(Y) = \{(i_1, b, t_+), (i_3, b, t_+)\}$. *Therefore, cadets* i_1, i_3 *will be chosen by branch* b. *However, if we use choice function described above we get* $C'_b(Y) = \{(i_1, b, t_+), (i_2, b, t_0)\}$. *Here cadets* i_1, i_2 *will be chosen by branch* b. *which improves cadet quality of branch* b *without violating reserve for* *contracts with the higher cost t_+.*

Definition 13 *A choice function $C()$ is q-acceptant if for any set of contracts $Y \subseteq X$,*

$$|C(Y)| = \min\{|Y_I|, q\}$$

For any given set of cadets $I' = (i_1, \ldots, i_{|I'|}) \subseteq I$ function $\mathcal{O}(j, I')$ is internal order of cadets and defined as the following:

$$\begin{aligned} \mathcal{O}(1, I') &= \arg\min_{i \in I'} \pi(i) \\ \mathcal{O}(\eta, I') &= \arg\min_{i \in I' \setminus \cup_{j<\eta} O(j, I')} \pi(i) \end{aligned}$$

Definition 14 *Among two q-acceptant choice functions $C()$ and $C'()$, $C()$ **dominates** $C'()$ if for any set of contracts $Y \subseteq X$ such that $|Y_I| > q$ and any $\eta \in \{1, \ldots, q\}$, $\pi(\mathcal{O}(\eta, C'(Y)_I)) \geq \pi(\mathcal{O}(\eta, C(Y)_I))$.*

Alternative Choice Function For a given order-of-merit priority ranking, $\lambda \in [0, 1]$ and Y chosen set $C_b(Y)$ is constructed as follows:

1. Branch chooses contracts of cadets based on their OML rankings, one contract at a time. If there are two contracts of a cadet, branch chooses the one with lower cost, t_0 and rejects the other. During the process if the number of chosen contracts with low cost, t_0, reaches $(1 - \lambda)q_b$, branch tentatively rejects all remaining contracts with the with low cost and continue with the contracts with the high cost, t_+. If q_b contracts are chosen, branch terminates the procedure and rejects all remaining and tentatively rejected contracts. If all contracts are considered and $|C_b(Y)|, q_b$, branch continues with the next step.

2. Branch rejects tentatively rejected contracts of cadets that have a chosen contract in the first step.

3. For the remaining potential elements of $C_b(Y)$, branch only considers contracts tentatively rejected contracts in Phase 1 and chooses contracts of cadets based on their OML rankings, one contract at a time. Branch terminates the procedure when either all the tentatively rejected contracts are chosen or q_b contracts chosen.

Proposition 3 *The alternative choice function, $C'()$, defined above dominates choice function induced by USMA priorities, $C^{USMA}()$.*

Following example shows that contracts are not necessarily unilateral substitutes under the choice function described above.

Example 4 *Let $I = (i_1, i_2, i_3), B = \{b\}$ with $q_b = 2$. Let the merit ranking be $\pi(i_1) < \pi(i_2) < \pi(i_3)$ and $\lambda = 0.5$. Let $Y = \{(i_1, b, t_+), (i_2, b, t_0), (i_3, b, t_+)\}$ and $Y' = \{(i_1, b, t_0), (i_1, b, t_+),\ (i_2, b, t_0), (i_3, b, t_+)\}$. We have $C'_b(Y) = \{(i_1, b_1, t_+), (i_2, b_1, t_0)\}$ and $C'_b(Y') = \{(i_1, b, t_0), (i_3, b, t_+)\}$. Hence, even though contract (i_3, b, t_+), the only contract of cadet i_3 in Y, is rejected from Y, it is not rejected from $Y' \supset Y$.*

Proposition 4 *Elements of X are bilateral substitutes for each branch b under the choice function defined above.*

Proposition 5 *The alternative choice function satisfies IRC condition.*

A mechanism is a strategy space Δ_i for each cadet i along with an outcome function $\varphi : (\Delta_{i_1}, \Delta_{i_2}, \ldots, \Delta_{i_n}) \to \chi$ that selects an allocation for each strategy vector

$(\delta_{i_1}, \delta_{i_2}, \ldots, \delta_{i_n}) \in (\Delta_{i_1}, \Delta_{i_2}, \ldots, \Delta_{i_n})$. Given a cadet i and a strategy profile $\delta_i \in \Delta_i$, let δ_{-i} denote the strategy of all cadets except cadet i. A direct mechanism is a mechanism where strategies are preferences over contracts. Hence a direct mechanism is simply a function $\psi : (\mathcal{P}_i)^n \to \chi$ that selects an allocation for each preference profile.

Cadet Optimal Stable Mechanism is a direct mechanism where cadets submit their preferences over contracts and central authority runs cadet proposing generalized deferred acceptance algorithm with submitted preferences and some choice functions $(C_b())_{b \in B}$. By the help of propositions above, one can guarantee existence of stability. Next proposition states that stable allocation can be achieved by Cadet Optimal Stable Mechanism.

Proposition 6 *Cadet Optimal Stable Mechanism induced by alternative choice function $C'()$ is a stable machanism.*

Cadet Sub-Branch Matching Problem In this section, we are going to study incentive properties of cadet optimal stable mechanism induced by alternative choice function. It is well known by Hatfield and Kojima (2010) that bilateral substitutes property, even along with LAD, is not sufficient for existence of strategy-proof and stable mechansims. Although, Kominers and Sonmez (2013) provides a class of choice functions satisfying only bilateral substitutes and making cadet proposing COA strategy-proof, the alternative choice function proposed here is not in that class. In order to show strategy-proofness of Cadet Optimal Stable Mechanism, in this section, we are going to construct a new problem which is parallel to our cadet branch matching problem.

To construct a cadet sub-branch matching problem we extend the contract set X to the set $\tilde{X}$ defined by

$$\tilde{X} \equiv \{\langle x, s\rangle = \langle (i,b,t), s\rangle : x \in X \text{ and } s = 1,2\}$$

Consider a parallel problem where each branch b is divided into two sub-branches, i.e. b_1 and b_2, where $q_{b_1} = q_b(1-\lambda)$ and $q_{b_2} = q_b\lambda$. In this setup, for any contract $\langle x, s\rangle$, s denotes the sub-branch associated with the contract and preference of cadet i, P_i^*,over the contracts in set $\tilde{X}$ are as the following:

$$\langle (i,b,t_0),1\rangle \; P_i^* \; \langle (i,b,t_0),2\rangle \; P_i^* \; \langle (i,b',t'),s\rangle \text{ , for } s = 1,2 \quad \text{if } (i,b,t_0)P_i(i,b',t')$$

$$\langle (i,b,t_+),2\rangle \; P_i^* \; \langle (i,b,t_+),1\rangle \; P_i^* \; \langle (i,b',t'),s\rangle \text{ , for } s = 1,2 \quad \text{if } (i,b,t_+)P_i(i,b',t')$$

For any branch b, choice functions of b_1 and b_2 are examples of lexicographic choice functions described in Kominers and Sonmez (2013). The priorities of slots and the precedence orders are the following:

$C_{b_1}(Y)$:

All the slots give priority based on OML ranking of cadets and among the contracts of same cadet priority is given to contracts with low cost. Since all priorities are the same, any precedence order produces the same choice choice function.

$C_{b_2}(Y)$:

All the slots prefer high cost contracts to low cost contracts and among the contracts of same cost, priority is given based on merit list of cadet. Since all priorities are the same, any precedence order produces the same choice choice function.

It is clear that any allocation in the parallel cadet-branch matching problem, say $\tilde{X}' \subset \tilde{X}$, corresponds to an allocation in original cadet-branch matching problem,

$X' \subset X$. This correspondence can be shown by the following projection, ϖ:

$$\varpi(\tilde{X}') \equiv \{x \in X : \langle x, s\rangle \in \tilde{X}' \text{ for some } s \in \{1, 2\}\}$$

We start with existence of stability in the cadet sub-branch matching problem. In order to guarantee existence of stable allocation and strategy-proofness of cadet proposing COA in cadet sub-branch problem, we are going to use following properties mentioned in the following three lemmas.

Lemma 4 *For any branch b, choice functions C_{b_1} and C_{b_2} satisfies substitutes condition.*

Lemma 5 *For any branch b, choice functions C_{b_1} and C_{b_2} satisfies IRC condition.*

Lemma 6 *For any branch b, choice functions C_{b_1} and C_{b_2} satisfies LAD condition.*

Three lemmas above help us to utilize theorem 1 and theorem 11 in Hatfield and Milgrom (2005) and theorem 1 in Aygun and Sonmez (2013) to show that cadet proposing COA gives stable allocation and is strategy-proof in cadet sub-branch matching problem. Next, we are going to show that the allocation chosen by cadet proposing COA in cadet sub-branch matching problem coincides with the allocation chosen by cadet proposing COA in original cadet branch matching problem.

Proposition 7 *For any cadet-branch matching problem with preference profile $(P_i)_{i\in I}$, outcome of cumulative offer process, $\tilde{X}'$, for cadet-branch matching problem with preference profile $(P_i^*)_{i\in I}$, is identical to outcome of cumulative offer process, X', for cadet-branch matching problem under projection ϖ:*

In the final step, we are going to define strategy-proofness formally and show this property on cadet optimal stable mechanism.

Definition 15 *A mechanism is strategy-proof if*

$$\nexists i \in I, \nexists \delta_{-i} \in \prod_{j \in I \setminus \{i\}} \Delta_j \text{ and } \nexists P_i, P_i' \in \Delta_i, \text{ such that } \psi(P_i', \delta_{-i}) P_i \psi(P_i, \delta_{-i}).$$

The definition above states that for any cadet, no matter what preference a cadet has and no matter what kind of strategies other cadets submit, submitting actual preferences over contracts should be weakly dominant strategy.

Theorem 5 *Cadet Optimal Stable Mechanism induced by alternative choice function $C'()$ is strategy-proof.*

By the teorem above, we showed that by a small modification branches can choose better set of cadets without sacrificing useful properties of allocation mechanism such as stability and strategy-proofness.

2.5 Concluding Remarks

In this paper, we first studied properties of cumulative offer algorithm which is designed to produce a stable allocation for matching with contracts problems. Next, we presented a new market design application where our result help us to design a mechanism which gives identical outcome with the COA and process ends in fewer steps. The application we introduce is particularly interesting in the sense that the choice function we designed is the second in the literature that satisfies only weakest substitutes condition, yet strategy proofness of the mechanism is preserved. Also, first time in literature, we dealt with constrained responsive preferences due to incentive objectives of USMA. On the other hand, this issue is not unique to this problem, since in any affirmative action model schools, firms etc. cannot use their responsive preferences. Therefore, the choice function suggested here may serve as a second best.

This paper shows that for a given set of applications, the current USMA priorities may not choose top cadets if there exists a cadet preferring long term contracts to short term contracts.

We proposed a new choice function that can also be used together with the cadet optimal stable mechanism to generate assignments. The choice function chooses a "better" set of cadets than current USMA priorities for any given set of contracts. Moreover, the mechanism we suggest is strategy-proof and yields a stable allocation for any problem.

2.6 References

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2.7 Appendix

Proof of Lemma 1. Assume that is not true. Therefore, there is a contract y such that $y_I = i$ and a step k such that $\nexists x \in C_B(A(k-1))$ and $x_I = i$, and

$y \in C_B(A(k))$. Let $\theta(k) = i'$ and y' be the contract offered by i' at step k . We can construct two sets Y and Y' in the following way:

$$\begin{aligned} Y &= A(k-1) \setminus \{x \in \chi : x_D \in \{i, i'\}\} \\ Y' &= Y \cup \{y, y'\} \end{aligned}$$

In words, to costruct Y we remove all rejected contracts of i and i' at $k-1$ and for Y' we just add contracts y and y' to set Y. By feasibility of the order θ and our assumption, no contract that belongs to cadets i and i' are chosen in step $k-1$. Next, by IRC condition, $C_B(Y) = C_B(A(k-1))$ since there is no chosen contract of i or i' at $k-1$ and $C_B(Y') = C_B(A(k))$ since all contracts we removed are some rejected contracts of i or i' at k. Also, by IRC condition $y \notin C(Y \cup \{y\})$, since $Y \cup \{y\}$ is a subset of $A(k-1)$ and includes all chosen contracts in $A(k-1)$.

Since choice functions $C()$ satisfies bilateral substitution condition and there is no contract of i and i' in Y, following must be true:

$$y \notin C(Y \cup \{y\}) \implies y \notin C(Y \cup \{y, y'\})$$

However, by our assumption we have $y \in C(A(k)) = C(Y') = C(Y \cup \{y, y'\})$, a contradiction. Hence, for any step $k_1 < k < k_2$ we have $\nexists x \in C(A(k))$ s.t. $x_I = i$.

■

Proof of Lemma 2. First, one can observe that up to step $k^* - 1$, $A(k)$ and outcomes of $C_B(A(k))$ under both orders θ and θ' are identical. By lemma 1, we know that there is no chosen contract of cadet i at step k^*, so cadet i can offer her next best contract without violating feasibility of the order. Also, under θ', we can say that there is no chosen contract of cadet i' at step k^*-1, because $A(k^*-1)$'s are identical under both orders and we know that i' offers a new contract at k^* under θ, that means there is no chosen contract of cadet i' at step k^*-1 under θ'. Then, by the help of lemma 1, we can say that cadet i's offer at k^* does not make any contract of i' chosen, so i' can make her new offer at step k^*+1 without violating feasibility of the order. Therefore, $A(k^*+1)$ and $C(A(k^*+1))$ under both orders are still identical. Since order of cadets are the same for the rest of the steps under θ and θ', $A(k)$ and $C(A(k))$ under both orders are the same for the rest. Hence, outcome of cumulative offer algorithm remains unchanged. ∎

Proof of Lemma 3. First of all, we are going to show that θ^* is feasible. For $0 < k \leq K_1$, each cadet offers her first contracts. So feasibility is not violated in the first K_1 step.

For $K_1 < k \leq K_2$, each cadet $\theta^*(k)$ for whom no contract is held by a branch at K_1 offers her next contract at time k. By lemma 1, we know that first offer of $\theta^*(k)$ is not accepted before time K_1. So, for $K_1 < k \leq K_2$, $\theta^*(k) \notin \{i \in I : \exists x \in C(A(k-1))$ and $x_I = i\}$. Hence, feasibility is not violated in the first K_2 steps.

For steps between K_{l-1} and K_l, each cadet $\theta^*(k)$ for whom no contract is currently held by a branch at K_{l-1} offers her next contract at time k. By lemma 1, we know that any offer of $\theta^*(k)$ is not accepted before time K_{l-1}. So, for $K_{l-1} < k \leq K_l$, $\theta^*(k) \notin \{i \in I : \exists x \in C(A(k-1))$ and $x_I = i\}$. Hence, feasibility is satisfied in the first K_l steps. Since l is arbitrary, one can say that $\forall k, \theta^*(l) \in I \setminus \{i \in I : \exists x \in C(A(k-1))$ s.t. $x_I = i\}$. Hence, θ^* is feasible.

Next, we are going to show that any order in the class θ^* gives identical outcome. For any cumulative offer algorithm induced by an order in class θ^*, no matter what is the order between 0 and K_1, one gets $A(K_1)$ at the end of step K_1. Since $A(K_1)$ and set of cadets giving new offers between K_1 and K_2 are the same for any order, one gets $A(K_2)$ at the end of step K_2. By induction, $A(K_l)$ for any l should be the same for any order in class θ^*. Hence, outcomes for any order in class θ^* must be identical. ■

Proof of Theorem 4. Stability of outcome comes directly from Hatfield and Kojima (2010) and Aygun and Sonmez (2012). We are going to prove that outcomes for any feasible order of offering are identical. To prove this theorem, we are going to convert any feasible order to some order in class θ^* without changing outcome.

In the first step, consider cadets offering their first contracts. By lemma 1, we know that we can swap their turn with the ones come before them. So, one can change their first offering times one at a time and make them offer their first contract in the first K_1 steps without changing outcome. Let the new order be θ^1.

In the second step, consider set of cadets for whom no contract is currently held by a branch at K_1. Since θ^1 is feasible, $\theta^1(K_1 + 1)$ should offer her next best contract. Take the second cadet making her next offer. Since that doctor is rejected at step K_1, by lemma 1, we can say that she has no accepted contract two steps before her next offer and we can swap her turn with the one who offered before her. By doing this procedure one at a time, we can change her turn to $K_1 + 2$. Then we can use the same technique to change offering orders of other cadets in the set and make them offer their next best contracts in steps between K_1 and K_2 without changing outcome. Let the new order be θ^2.

In step l, consider set of cadets for whom no contract is currently held by a branch at K_{l-1}. Since θ^{l-1} is feasible, $\theta^{l-1}(K_{l-1}+1)$ should offer her next best contract. Take the second cadet making her next offer. Since cadet is rejected at step K_{l-1}, by lemma 1, we can say that she has no accepted contract two steps before her next offer and we can swap her turn with the one who offered before her. By doing this procedure one at a time, we can change her turn to $K_{l-1}+2$. Then we can use the same technique to change offering orders of other cadets in the set and make them offer their next best contracts in steps between K_{l-1} and K_l without changing outcome. Let the new order of offering be θ^l.

If we continue untill we cover all steps in θ and construct some order of offering θ^L, it is easy to verify that θ^L belongs to the class θ^* and since we keep outcome unchanged during the procedure, one can say that feasible order of offering θ gives the same outcome as any order in class θ^*. Since we choose θ arbitrarily, this result is true for any feasible order. Hence, cumulative offer algorithm with any feasible order gives one identical, stable allocation. ■

Proof of Proposition 3. For any given set of contracts Y such that $|Y_I| > q$, top $q(1-\lambda)$ cadets will be accepted by both choice functions. Therefore, $\pi(\mathcal{O}(j, C'(Y)_I)) = \pi(\mathcal{O}(j, C^{USMA}(Y)_I))$, for $j \leq q(1-\lambda)$. For the rest of the slots there are three cases possible:

Case 1: If all cadets in top $q(1-\lambda)$ have contracts with t_0 in Y, then both choice functions choose the same set of cadets, i.e. $C'(Y) = C^{USMA}(Y)$.

If there is at least one cadet in top $q(1-\lambda)$ who has no contract with t_0 in Y, then let Y' be the set of contracts of the remaining cadets and Y'_+ be the set of contracts in Y' with t_+. If $|Y'_+| \geq q\lambda$ then C^{USMA} terminates the procedure in the second phase and C' terminates the procedure in the first phase.

Case 2.1: If $|Y'_+| \geq q\lambda$ and the constraint for the alternative choice function never binds, then C^{USMA} chooses top $q\lambda$ cadets who has a contract with t_+ in Y', i.e. $\mathcal{O}(q(1-\lambda)+j, C^{USMA}(Y)_I) = \mathcal{O}(j, Y'_{+I})$ for $j = 1, \ldots, q\lambda$ and for the alternative choice function $\mathcal{O}(q(1-\lambda)+j, C'(Y)_I) = \mathcal{O}(j, Y'_I)$ for $j = 1, \ldots, q\lambda$. Since there is a set inclusion between Y' and Y'_+, i.e. $Y'_+ \subset Y'$, for the remaining slots alternative choice function chooses better cadets. Therefore, $\pi(\mathcal{O}(j, C^{USMA}(Y)_I)) \geq \pi(\mathcal{O}(j, C'(Y)_I))$ for $j = 1, \ldots, q$.

Case 2.2: If $|Y'_+| \geq q\lambda$ and the constraint for the alternative choice function binds in phase 1 when $q(1-\lambda)+j^*$ seats filled, then let Y''_+ be the set of remaining contracts with t_+. So we have $\mathcal{O}(q(1-\lambda)+j, C'(Y)_I) = \mathcal{O}(j, Y'_I)$ for $j = 1, \ldots, j^*$ and $\mathcal{O}(q(1-\lambda)+j, C'(Y)_I) = \mathcal{O}(j, Y''_{+I})$ for $j = j^*+1, \ldots, q\lambda$. Since we have $Y'_+ \subset Y'$ and any high cost contract of cadets in the set $\bigcup_{j=j^*+1}^{q\lambda} \mathcal{O}(j, Y'_{+I})$ is available in Y''_+, alternative choice function chooses at least as high ranked cadets as C^{USMA} for the remaining seats. Therefore, $\pi(\mathcal{O}(j, C^{USMA}(Y)_I)) \geq \pi(\mathcal{O}(j, C'(Y)_I))$ for $j = 1, \ldots, q$.

Case 3.1: If $|Y'_+| < q\lambda$ and the constraint for the alternative choice function never binds, then the alternative choice function chooses top $q\lambda$ cadets in the set Y'_I, i.e. $\mathcal{O}(q(1-\lambda)+j, C'(Y)_I) = \mathcal{O}(j, Y'_I)$ for $j = 1, \ldots, q\lambda$. On the other hand, C^{USMA} has a further constraint of choosing all the cadets in set Y'_{+I}. Therefore, $\pi(\mathcal{O}(j, C^{USMA}(Y)_I)) \geq \pi(\mathcal{O}(j, C'(Y)_I))$ for $j = 1, \ldots, q$.

Case 3.2: If $|Y'_+| < q\lambda$, the constraint for the alternative choice function binds in phase 1 when $q(1-\lambda)+j^*$ seats filled and the alternative choice function terminates the procedure in the first phase, then we have $\mathcal{O}(q(1-\lambda)+j, C'(Y)_I) = \mathcal{O}(j, Y'_I)$ for $j = 1, \ldots, j^*$ and $\mathcal{O}(q(1-\lambda)+j, C'(Y)_I) = \mathcal{O}(j, Y''_{+I})$ for $j = j^*+1, \ldots, q\lambda$. Since set of contracts $Y^* = C'(Y) \cap Y''_+$ is chosen under both choice functions and C' chooses top $q - |Y^*|$ cadets among the remaining cadets we have $\pi(\mathcal{O}(j, C^{USMA}(Y)_I)) \geq \pi(\mathcal{O}(j, C'(Y)_I))$ for $j = 1, \ldots, q$.

Case 3.3: If $|Y'_+| < q\lambda$, the constraint for the alternative choice function binds in phase 1 when $q(1-\lambda)+j^*$ seats filled and the alternative choice function terminates the procedure in the third phase, set of contracts Y''_+ is chosen under both choice functions and C' chooses top $q - |Y''_+|$ cadets among the remaining cadets. Therefore, $\pi(\mathcal{O}(j, C^{USMA}(Y)_I)) \geq \pi(\mathcal{O}(j, C'(Y)_I))$ for $j = 1, \ldots, q$. ■

Proof of Proposition 4. Let $x = (i, b, t) \in Y \subseteq X$ be the only contract in Y that involves cadet i and suppose $x \notin C_b(Y)$. Consider another contract $z \notin Y$ such that $z_I \notin Y_I$. Let $Y' = \{x \in X : x_I \notin \bigcup_{\eta \leq (1-\lambda)q_b} \{\mathcal{O}(\eta, Y_I)\}\}$. We have two cases to consider:

Case 1: $t = t_0$. Since $(i, b, t) \notin C_b(Y)$, either we have

$$i \notin \bigcup_{\eta \leq q_b} \{\mathcal{O}(\eta, Y_I)\}$$

therefore $(Y \cup \{z\}) \supset Y$ implies

$$i \notin \bigcup_{\eta \leq q_b} \{\mathcal{O}(\eta, Y_I \cup \{z_I\})\}$$

or we have

$$i \notin \bigcup_{\eta \leq (1-\lambda)q_b} \{\mathcal{O}(\eta, \{j \in I : (j, b, t_0) \in Y\})\} \text{ and}$$

$$|\{j \in I : (j, b, t_0) \in Y \text{ and } \pi(j) < \pi(i)\} \cup \{j \in I : (j, b, t_+) \in Y\}| \geq q_b$$

therefore $(Y \cup \{z\}) \supset Y$ implies

$$i \notin \bigcup_{\eta \leq (1-\lambda)q_b} \{\mathcal{O}(\eta, \{j \in I : (j,b,t_0) \in (Y \cup \{z\}\})\} \text{ and}$$

$$|\{j \in I : (j,b,t_0) \in (Y \cup \{z\}) \text{ and } \pi(j) < \pi(i)\} \cup \{j \in I : (j,b,t_+) \in (Y \cup \{z\})\}| \geq q_b$$

as well. Hence $x \notin C_b(Y \cup \{z\})$.

Case 2: $t = t_+$. Since $(i,b,t) \notin C_b(Y)$, either we have

$$i \notin \bigcup_{\eta \leq q_b} \{\mathcal{O}(\eta, Y_I)\} \text{ and}$$

$$\bigcup_{\eta \leq (1-\lambda)q_b} \{\mathcal{O}(\eta, \{j \in I : (j,b,t_0) \in Y\})\} \nsubseteq \bigcup_{\eta \leq q_b} \{\mathcal{O}(\eta, Y_I)\}$$

or we have

$$i \notin \bigcup_{\eta \leq q_b} \{\mathcal{O}(\eta, Y_I)\},$$

$$\bigcup_{\eta \leq (1-\lambda)q_b} \{\mathcal{O}(\eta, \{j \in I : (j,b,t_0) \in Y\})\} \subseteq \bigcup_{\eta \leq q_b} \{\mathcal{O}(\eta, Y_I)\} \text{ and}$$

$$|\{j \in I \backslash \bigcup_{\eta \leq (1-\lambda)q_b} \{\mathcal{O}(\eta, \{j \in I : (j,b,t_0) \in Y\})\} : (j,b,t_+) \in Y\}| \geq \lambda q_b$$

therefore $(Y \cup \{z\}) \supset Y$ implies either

$$i \notin \bigcup_{\eta \leq q_b} \{\mathcal{O}(\eta, Y_I \cup \{z_I\})\} \text{ and}$$

$$\bigcup_{\eta \leq (1-\lambda)q_b} \{\mathcal{O}(\eta, \{j \in I : (j,b,t_0) \in Y \cup \{z\}\})\} \nsubseteq \bigcup_{\eta \leq q_b} \{\mathcal{O}(\eta, Y_I \cup \{z_I\})\}$$

or

$$i \notin \bigcup_{\eta \leq q_b} \{\mathcal{O}(\eta, Y_I \cup \{z_I\})\},$$

$$\bigcup_{\eta \leq (1-\lambda)q_b} \{\mathcal{O}(\eta, \{j \in I : (j,b,t_0) \in Y \cup \{z\}\})\} \subseteq \bigcup_{\eta \leq q_b} \{\mathcal{O}(\eta, Y_I \cup \{z_I\})\} \text{ and}$$

$$|\{j \in I \backslash \bigcup_{\eta \leq (1-\lambda)q_b} \{\mathcal{O}(\eta, \{j \in I : (j,b,t_0) \in Y \cup \{z\}\})\} : (j,b,t_+) \in Y \cup \{z\}\}| \geq \lambda q_b$$

as well. Hence $x \notin C_b(Y \cup \{z\})$ for this case as well, completing the proof. ■

Proof of Proposition 5. For any sets of contracts $Y \subset Y' \subset X$, let $C'(Y') \subset Y \subset Y'$. Take a contract $x \in Y'$ such that $x \notin C'(Y')$. If owner of contract x has no chosen contract, then removing x is not going to change chosen set since contracts of higher ranked cadets are still available. If owner of contract x has a chosen contract, say y, then since removing x has no effect on choosing y and choice function never returns rejected contracts again removing x does not affect chosen set. Therefore, removing a rejected contract does not affect chosen set.

Since, removing x does not change chosen set, removing rejected contracts one at a time untill we reach set Y does not change chosen set which completes the proof.

■

Proof of Proposition 6. By the propositions 2 and 3 choice functions of branches satisfy bilateral substitutes and IRC conditions. By Theorem 1 of Aygun and Sonmez (2012b) cadet proposing cumulative offer algorithm produces stable allocation. ■

Proof of Lemma 4. First, think about C_{b_1}. For any $\langle x, s\rangle = \langle (i, b', t), s\rangle$, if $b' \neq b$ or $s = 2$, then $\langle x, s\rangle$ is not going to be chosen in $\tilde{Y}$ or $\tilde{Y}'$. Let $\langle x, 1\rangle = \langle (i, b, t), s\rangle \in \tilde{Y} \subseteq \tilde{Y}' \subseteq \tilde{X}$ be a contract in $\tilde{Y}$ and suppose $x \notin C_{b_1}(\tilde{Y})$. We have two cases to consider:

Case 1: $t = t_0$ or $t = t_+$ and $\nexists \langle (i, b, t_0), 1\rangle \in \tilde{Y}$. Since $\langle x, 1\rangle \notin C_{b_1}(\tilde{Y})$, we have

$$i \notin \bigcup_{\eta \leq q_{b_1}} \{\mathcal{O}(\eta, \tilde{Y}_I)\}$$

therefore $\tilde{Y}' \supset \tilde{Y}$ implies

$$i \notin \bigcup_{\eta \leq q_{b_1}} \{\mathcal{O}(\eta, \tilde{Y}'_I)\}$$

Hence $\langle x, 1\rangle \notin C_{b_1}(\tilde{Y}')$.

Case 2: $t = t_+$ and $\exists \langle (i, b, t_0), 1\rangle \in \tilde{Y}$. Since b_1 always gives priority to low cost contracts, $\langle (i, b, t_0), 1\rangle \in \tilde{Y}$ implies $\langle x, 1\rangle \notin C_{b_1}(\tilde{Y})$. Therefore, $\langle (i, b, t_0), 1\rangle \in \tilde{Y}' \supset \tilde{Y}$ implies $\langle x, 1\rangle \notin C_{b_1}(\tilde{Y}')$.

Hence $\langle x, 1\rangle \notin C_{b_1}(\tilde{Y}')$ for this case as well, completing the proof.

Now, consider C_{b_2}. For any $\langle x, s\rangle = \langle (i, b', t), s\rangle$, if $b' \neq b$ or $s = 1$, then $\langle x, s\rangle$ is not going to be chosen in $\tilde{Y}$ or $\tilde{Y}'$. Let $\langle x, 2\rangle = \langle (i, b, t), 2\rangle \in \tilde{Y} \subseteq \tilde{Y}' \subseteq \tilde{X}$ be a contract in $\tilde{Y}$ and suppose $x \notin C_{b_2}(\tilde{Y})$. We have two cases to consider:

Case 1: $t = t_+$. Since $\langle x, 2\rangle \notin C_{b_2}(\tilde{Y})$, we have

$$i \notin \bigcup_{\eta \leq q_{b_2}} \{\mathcal{O}(\eta, \{j \in I : \langle (j, b, t_+), 2\rangle \in \tilde{Y}_I\})\}$$

therefore $\tilde{Y}' \supset \tilde{Y}$ implies

$$i \notin \bigcup_{\eta \leq q_{b_2}} \{\mathcal{O}(\eta, \{j \in I : \langle (j, b, t_+), 2\rangle \in \tilde{Y}'_I\})\}$$

Hence $\langle x, 2\rangle \notin C_{b_2}(\tilde{Y}')$.

Case 2.1: $t = t_0$ and $\exists \langle (i, b, t_+), 2\rangle \in \tilde{Y}$. Since b_2 always gives priority to high cost contracts, $\langle (i, b, t_+), 2\rangle \in \tilde{Y}$ implies $\langle x, 2\rangle \notin C_{b_2}(\tilde{Y})$. Therefore, $\langle (i, b, t_+), 2\rangle \in \tilde{Y} \supset \tilde{Y}'$ implies $\langle x, 2\rangle \notin C_{b_2}(\tilde{Y}')$.

Case 2.2: $t = t_0$ and $\nexists \langle (i, b, t_+), 2\rangle \in \tilde{Y}$. Since $\langle x, 2\rangle \notin C_{b_2}(\tilde{Y})$, either we have

$$i \notin \bigcup_{\eta \leq q_{b_2}} \{\mathcal{O}(\eta, \tilde{Y}_I)\}$$

therefore $\tilde{Y}' \supset \tilde{Y}$ implies

$$i \notin \bigcup_{\eta \leq q_{b_2}} \{\mathcal{O}(\eta, \tilde{Y}'_I)\}$$

or

$$|\{j \in I : \langle (j, b, t_+), 2\rangle \in \tilde{Y}\}| \geq q_{b_2}$$

therefore $\tilde{Y}' \supset \tilde{Y}$ implies

$$|\{j \in I : \langle (j, b, t_+), 2\rangle \in \tilde{Y}'\}| \geq q_{b_2}$$

as well. Hence $\langle x, 2\rangle \notin C_{b_2}(\tilde{Y}')$ for this case as well, completing the proof. ■

Proof of Lemma 5. Both C_{b_1} and C_{b_2} are in the class of lexicographic choice functions described in Kominers and Sonmez (2012). Therefore, Lemma 6 is corollary of Lemma D.1 of Kominers and Sonmez (2012). ■

Proof of Lemma 6. For any set of contracts $\tilde{Y}$, if the number of cadets who has at least one contract in $\tilde{Y}$ is greater than or equal to branch's quota, both choice functions fill all the seats. Therefore, adding more contract never decreases the number of chosen contracts, since number of cadets never decreases. If the number of cadets with at least one contract in $\tilde{Y}$ is less than branch's quota, both choice functions chooses one contract of each cadet. So, $|C_{b_j}(\tilde{Y})| = |\tilde{Y}_I|$ for $j = 1, 2$. Therefore, for any $\tilde{Y}' \supset \tilde{Y}$ and $j = 1, 2$, $|C_{b_j}(\tilde{Y})| < |C_{b_j}(\tilde{Y}')|$ if $\tilde{Y}_I \subset \tilde{Y}_I'$, $|C_{b_j}(\tilde{Y})| = |C_{b_j}(\tilde{Y}')|$ otherwise. Hence, the number of chosen contracts never decreases if the set of available contracts gets larger. ■

Proof of Proposition 7. First of all, for any cadet sub-branch matching problem with $(P_i^*)_{i\in I}$ induced by $(P_i)_{i\in I}$, let the outcome of cadet proposing cumulative offer algorithm be $\tilde{X}'$. By theorem 1 in Aygun Sonmez (2013) and by theorem 3 and 4 in Hatfield and Milgrom (2005) $\tilde{X}'$ is a stable allocation which is weakly preferred to any other stable allocation by all cadets.

We denote the outcome of the cumulative offer algorithm of original cadet-branch problem by X', and for any cadet $i \in X'_I$, let

$$\tilde{x}^i = \begin{cases} \langle (i,b,t_0),1\rangle & \text{if } (i,b,t_0) \in X' \text{ and } i \in \bigcup_{\eta \le (1-\lambda)q_b} \{\mathcal{O}(\eta, \{j \in I : (j,b,t_0) \in X'\})\} \\ \langle (i,b,t_0),2\rangle & \text{if } (i,b,t_0) \in X' \text{ and } i \notin \bigcup_{\eta \le (1-\lambda)q_b} \{\mathcal{O}(\eta, \{j \in I : (j,b,t_0) \in X'\})\} \\ \langle (i,b,t_+),1\rangle & \text{if } (i,b,t_+) \in X' \text{ and } i \notin \bigcup_{\eta \le \lambda q_b} \{\mathcal{O}(\eta, \{j \in I : (j,b,t_+) \in X'\})\} \\ \langle (i,b,t_+),2\rangle & \text{if } (i,b,t_+) \in X' \text{ and } i \in \bigcup_{\eta \le \lambda q_b} \{\mathcal{O}(\eta, \{j \in I : (j,b,t_+) \in X'\})\} \end{cases}$$

and let

$$\tilde{Y}' = \bigcup_{i \in X'_I} \{\tilde{x}^i\}.$$

By construction, we have $\varpi(\tilde{Y}') = X'$.

Now, first we are going to show that $\tilde{Y}'$ is a stable allocation in the cadet sub-branch matching problem. Since X' is a stable allocation, $\tilde{x}^i P_i^* \emptyset$ for all cadets and for any sub-branch, number of contracts does not exceed its quota. Therefore, $\tilde{Y}'$ is individually rational and satisfies first condition of stable allocations. Now assume that there is a sub-branch b_s and a set of contracts $\tilde{Y}'' \neq C_{b_s}(\tilde{Y}')$ such that

$$\tilde{Y}'' = C_{b_s}(\tilde{Y}' \cup \tilde{Y}'') \subseteq C_I(\tilde{Y}' \cup \tilde{Y}'').$$

Therefore, there exist a contract $\langle x,s\rangle = \langle (i,b,t),s\rangle \in \tilde{Y}'' \backslash C_{b_s}(\tilde{Y}')$ such that $\langle x,s\rangle\, P_i^* \tilde{Y}'_i$. If there exist a contract $\tilde{x}^i = \langle (i,b',t'),s'\rangle$ and $b' \neq b$ or there is no contract of cadet i in $\tilde{Y}'$, then by the fact that X' is the outcome of cumulative offer algorithm, we know that x is proposed in some step of cumulative offer algorithm and is rejected by branch b in the final step. So, we have four possible cases:

Case 1: If $s = 1$ and $t = t_0$, then either we have

$$i \notin \bigcup_{\eta \le q_b} \{\mathcal{O}(\eta, \{j \in I : X'(j) \in X'(b)\} \cup \{i\})\}$$

or we have

$$i \notin \bigcup_{\eta \leq (1-\lambda)q_b} \{\mathcal{O}(\eta, \{j \in I : (j, b, t_0) \in X'(b)\} \cup \{i\})\} \text{ and}$$
$$|\{j \in I : (j, b, t_0) \in X' \text{ and } \pi(j) < \pi(i)\} \cup \{j \in I : (j, b, t_+) \in X'\}| \geq q_b$$

by stability of X', which implies

$$|\{j \in I : \{\langle (j, b, t_0), 1\rangle \, \langle (j, b, t_+), 1\rangle\} \cap \tilde{Y}' \neq \emptyset \text{ and } \pi(j) < \pi(i)\}| \geq (1-\lambda)q_b = q_{b_1}$$

by construction of $\tilde{Y}'$. Therefore, $\langle x, s\rangle \notin C_{b_1}(\tilde{Y}' \cup \tilde{Y}'')$ by substitutes property of C_{b_1}. A contradiction.

Case 2: If $s = 2$ and $t = t_0$, then either we have

$$i \notin \bigcup_{\eta \leq q_b} \{\mathcal{O}(\eta, \{j \in I : X'(j) \in X'(b)\} \cup \{i\})\}$$

or we have

$$i \notin \bigcup_{\eta \leq (1-\lambda)q_b} \{\mathcal{O}(\eta, \{j \in I : (j, b, t_0) \in X'(b)\} \cup \{i\})\} \text{ and}$$
$$|\{j \in I : (j, b, t_0) \in X' \text{ and } \pi(j) < \pi(i)\} \cup \{j \in I : (j, b, t_+) \in X'\}| \geq q_b$$

by stability of X', which implies

$$|\{j \in I : \{\langle (j, b, t_0), 2\rangle \, \langle (j, b, t_+), 2\rangle\} \cap \tilde{Y}' \neq \emptyset \text{ and } \pi(j) < \pi(i)\}| \geq \lambda q_b = q_{b_2}$$

or

$$|\{j \in I : \langle (j, b, t_+), 2\rangle \in \tilde{Y}'\}| \geq \lambda q_b = q_{b_2}$$

by construction of $\tilde{Y}'$. Therefore, $\langle x, s\rangle \notin C_{b_2}(\tilde{Y}' \cup \tilde{Y}'')$ by substitutes property of C_{b_2}. A contradiction.

Case 3: If $s = 1$ and $t = t_+$, then we have

$$i \notin \bigcup_{\eta \leq q_b} \{\mathcal{O}(\eta, \{j \in I : X'(j) \in X'(b)\} \cup \{i\})\} \text{ and}$$

$$|X'(b) \cap \{j \in I : (j, b, t_0) \in X'\}| \leq (1-\lambda)q_b$$

by stability of X', which implies

$$|\{j \in I : \{\langle(j, b, t_0), 1\rangle \ \langle(j, b, t_+), 1\rangle\} \cap \tilde{Y}' \neq \emptyset \text{ and } \pi(j) < \pi(i)\}| \geq (1-\lambda)q_b = q_{b_1}$$

by construction of $\tilde{Y}'$. Therefore, $\langle x, s\rangle \notin C_{b_1}(\tilde{Y}' \cup \tilde{Y}'')$ by substitutes property of C_{b_1}. A contradiction.

Case 4: If $s = 2$ and $t = t_+$, then we have

$$i \notin \bigcup_{\eta \leq q_b} \{\mathcal{O}(\eta, \{j \in I : X'(j) \in X'(b)\} \cup \{i\})\} \text{ and}$$

$$|X'(b) \cap \{j \in I : (j, b, t_0) \in X'\}| \leq (1-\lambda)q_b$$

by stability of X', which implies

$$|\{j \in I : \{\langle(j, b, t_+), 2\rangle\} \in \tilde{Y}' \text{ and } \pi(j) < \pi(i)\}| \geq \lambda q_b = q_{b_2}$$

by construction of $\tilde{Y}'$. Therefore, $\langle x, s\rangle \notin C_{b_2}(\tilde{Y}' \cup \tilde{Y}'')$ by substitutes property of C_{b_2}. A contradiction. Hence if $X'_i = \emptyset$ or $b' \neq b$, then there is no contract $\langle x, s\rangle = \langle(i, b, t), s\rangle \in \tilde{Y}'' \backslash C_{b_s}(\tilde{Y}')$ such that $\langle x, s\rangle \in C_{b_s}(\tilde{Y}' \cup \tilde{Y}'')$.

Now consider the case where $b' = b$, $s' = s$ and $t' \neq t$. Since X' is the outcome of cumulative offer algorithm, contract x is proposed in some step and rejected in the final step. Therefore, we have four cases to consider:

Case 1: If $s = 1$ and $t = t_0$, then we have $x \notin C_b(X' \cup (x))$ by IRC condition and by stability of X'. Therefore, we have

$$i \notin \bigcup_{\eta \leq (1-\lambda)q_b} \{\mathcal{O}(\eta, \{j \in I : (j, b, t_0) \in X'\} \cup \{i\})\},$$

which implies

$$i \in \bigcup_{\eta \leq \lambda q_b} \{\mathcal{O}(\eta, \{j \in I : (j, b, t_+) \in X'\})\} \implies \tilde{x}^i = \langle (i, b, t_+), 2 \rangle$$

by construction of $\tilde{Y}'$. Therefore, $\langle (i, b, t_+), 1 \rangle \notin \tilde{Y}'$. A contradiction.

Case 2: If $s = 1$ and $t = t_+$, then $\langle x, s \rangle$ can not be chosen by b_1, since b_1 always gives priority to low cost contracts among the contracts of same cadet. Therefore, $\langle x, s \rangle \notin C_{b_1}(\tilde{Y}' \cup \tilde{Y}'')$ by substitutes property of C_{b_1}. A contradiction.

Case 3: If $s = 2$ and $t = t_0$, then $\langle x, s \rangle$ can not be chosen by b_2, since b_2 always gives priority to high cost contracts among the contracts of same cadet. Therefore, $\langle x, s \rangle \notin C_{b_2}(\tilde{Y}' \cup \tilde{Y}'')$ by substitutes property of C_{b_2}. A contradiction.

Case 4: If $s = 2$ and $t = t_+$, then we have $x \notin C_b(X' \cup (x))$ by IRC condition and by stability of X'. Therefore, we have

$$i \in \bigcup_{\eta \leq (1-\lambda) q_b} \{\mathcal{O}(\eta, \{j \in I : (j, b, t_0) \in X'\})\},$$

which implies

$$\tilde{x}^i = \langle (i, b, t_+), 1 \rangle$$

by construction of $\tilde{Y}'$. Therefore, $\langle (i, b, t_0), 2 \rangle \notin \tilde{Y}'$. A contradiction. Hence if $b' = b$ and $s' = s$, then there is no contract $\langle x, s \rangle = \langle (i, b, t), s \rangle \in \tilde{Y}'' \backslash C_{b_s}(\tilde{Y}')$ such that $\langle x, s \rangle \in C_{b_s}(\tilde{Y}' \cup \tilde{Y}'')$.

Now consider the case where $b' = b$, $t' = t$ and $s' \neq s$. We have four cases to consider:

Case 1: $s = 1$ and $t = t_0$. We have

$$i \notin \bigcup_{\eta \leq (1-\lambda) q_b} \{\mathcal{O}(\eta, \{j \in I : (j, b, t_0) \in X'\})\}$$

by construction of $\tilde{Y}'$, which implies

$$|\{j \in I : (j, b, t_0) \in X' \text{ and } \pi(j) < \pi(i)\}\}| \geq (1-\lambda)q_b = q_{b_1}$$

$$\implies |\{j \in I : \{\langle (j, b, t_0), 1\rangle \, \langle (j, b, t_+), 1\rangle\} \cap \tilde{Y}' \neq \emptyset \text{ and } \pi(j) < \pi(i)\}| \geq (1-\lambda)q_b = q_{b_1}$$

by construction of $\tilde{Y}'$. So, $\langle x, s\rangle \notin C_{b_1}(\tilde{Y}' \cup \{\langle x, s\rangle\})$. Therefore, $\langle x, s\rangle \notin C_{b_1}(\tilde{Y}' \cup \tilde{Y}'')$ by substitutes property of C_{b_1}. A contradiction.

Case 2: $s = 1$ and $t = t_+$. By Individual rationality of X', we have

$$(i, b, t_+)P_i\emptyset \implies \langle (i, b, t_+), 2\rangle \, P_i^* \, \langle (i, b, t_+), 1\rangle$$

by construction of P_i^*. Therefore, $\langle x, 2\rangle \, P_i^* \, \langle x, 1\rangle$. A contradiction.

Case 3: $s = 2$ and $t = t_0$. By Individual rationality of X', we have

$$(i, b, t_0)P_i\emptyset \implies \langle (i, b, t_0), 1\rangle \, P_i^* \, \langle (i, b, t_0), 2\rangle$$

by construction of P_i^*. Therefore, $\langle x, 1\rangle \, P_i^* \, \langle x, 2\rangle$. A contradiction.

Case 4: $s = 2$ and $t = t_+$. We have

$$i \notin \bigcup_{\eta \leq \lambda q_b} \{\mathcal{O}(\eta, \{j \in I : (j, b, t_+) \in X'\})\}$$

by construction of $\tilde{Y}'$, which implies

$$|\{j \in I : (j, b, t_+) \in X' \text{ and } \pi(j) < \pi(i)\}| \geq \lambda q_b = q_{b_2}$$

$$\implies |\{j \in I : \{\langle (j, b, t_+), 2\rangle\} \cap \tilde{Y}' \neq \emptyset \text{ and }$$

$$\pi(j) < \pi(i)\}| \geq \lambda q_b = q_{b_2}$$

So, $\langle x, s\rangle \notin C_{b_2}(\tilde{Y}' \cup \{\langle x, s\rangle\})$. Therefore, $\langle x, s\rangle \notin C_{b_2}(\tilde{Y}' \cup \tilde{Y}'')$ by substitutes property of C_{b_2}. A contradiction. Hence, $\tilde{Y}'$ is a stable allocation in the cadet sub-branch problem.

It suffices to show that no contract $z = (i, b, t) \in \varpi(\tilde{X}')$ is ever rejected during the cumulative offer process. To see this, we suppose the contrary, and consider the first step k at which some contract $z \in \varpi(\tilde{X}')$ is rejected. We let $s \in \{1,2\}$ be the sub-branch such that $\langle z, s\rangle \in \tilde{X}'$, and let $x \in C_b(A(k))$ be one of the contracts assigned to b in step k. Now, as z is the first contract in $\varpi(\tilde{X}')$ to be rejected, we know that for all $x \in C_b(A(k))$, $xR_{x_I}(\varpi(\tilde{X}'))(x_I)$ and $|C_b(A(k))| = q_b$.

First, assume that $\exists z' \in C_b(A(k))$ such that $z'_I = z_I$. By IRC property of C_b, we know that $\theta(k) \neq z_I$. So, we have $z \in C_b(A(k-1))$. Next, since $z, z' \in A(k)$ and $z \in C_b(A(k-1))$, we must have $z'_T = t_+$. Otherwise, we have $|\{y \in A(k-1) : y_T = t_0 \text{ and } \pi(y_I) < \pi(z_I)\}| \geq (1-\lambda)q_b$, which implies $|\{y \in A(k) : y_T = t_0$ and $\pi(y_I) < \pi(z_I)\}| \geq (1-\lambda)q_b$ and $z' \notin C_b(A(k))$. Let $k' < k-1$ be the step which z' is rejected for the first time. So, we have $z'_I \notin \bigcup\limits_{\eta \leq q_b} \{\mathcal{O}(\eta, (A(k'))_I)\}$, which implies $z_I \notin \bigcup\limits_{\eta \leq q_b} \{\mathcal{O}(\eta, (A(k-1))_I)\}$ and $z \notin C_b(A(k-1))$, a contradiction. Therefore, there is no $z' \in C_b(A(k))$ such that $z'_I = z_I$.

Now, assume that $\nexists z' \in C_b(A(k))$ such that $z'_I = z_I$. Therefore , as $z \notin C_b(A(k))$, for all $x \in C_b(A(k))$ we have on of the four cases

Case 1	$x_T = t_0, z_T = t_0$	then $\pi(x_I) < \pi(z_I)$
Case 2	$x_T = t_+, z_T = t_+$	then $\pi(x_I) < \pi(z_I)$
Case 3	$x_T = t_+, z_T = t_0$	then either $\pi(x_I) < \pi(z_I)$ or $\vert\{y \in C_b(A(k)) : y_T = t_+\}\vert \leq \lambda q_b$
Case 4	$x_T = t_0, z_T = t_+$	then $\pi(x_I) < \pi(z_I)$ and $\vert\{y \in C_b(A(k)) : y_T = t_+\}\vert \geq \lambda q_b$.

If for any $x \in C_b(A(k))$, we have $x \neq (\varpi(\tilde{X}'))(x_I)$, then we must have $xP_{x_I}(\varpi(\tilde{X}'))(x_I)$. In addition to this, if we have case 1 or case 2 then $(\{\langle x, s\rangle\} \cup \tilde{X}')\backslash\{\tilde{X}'(x_I)\}$ blocks $\tilde{X}'$. If we have case 3, then we must have $|\{y \in C_b(A(k)) : y_T = t_0$ and $\pi(y_I) < \pi(z_I)\}| \geq (1-\lambda)q_b$. In that case, if $x \neq (\varpi(\tilde{X}'))(x_I)$ and $\langle z, s\rangle \in \tilde{X}'$, then we have either $s = 1$ and there exists a contract $y \in C_b(A(k))$ such that

$y_T = t_0$, $\pi(y_I) < \pi(z_I)$ and $\langle y, 1\rangle \notin \tilde{X}'$, or $s = 2$. If the former happens then $(\{\langle y,1\rangle\} \cup \tilde{X}')\backslash\{\tilde{X}'(y_I)\}$ blocks $\tilde{X}'$ since b_1 gives priority to higher ranked cadets and $\langle y,1\rangle\, P^*_{y_I}\, \langle y,2\rangle\, P^*_{y_I} \tilde{X}'(y_I)$. If the latter happens then $(\{\langle x,2\rangle\} \cup \tilde{X}')\backslash\{\tilde{X}'(x_I)\}$ blocks $\tilde{X}'$ since b_2 gives priority to high cost contracts. If we have case 4, then we must have $|\{y \in C_b(A(k)) : y_T = t_+ \text{ and } \pi(y_I) < \pi(z_I)\}| \geq \lambda q_b$. In that case, if $x \neq (\varpi(\tilde{X}'))(x_I)$ and $\langle z, s\rangle \in \tilde{X}'$, then we have either $s = 2$ and there exists a contract $y \in C_b(A(k))$ such that $y_T = t_+$, $\pi(y_I) < \pi(z_I)$ and $\langle y, 2\rangle \notin \tilde{X}'$, or $s = 1$. If the former happens then $(\{\langle y,2\rangle\} \cup \tilde{X}')\backslash\{\tilde{X}'(y_I)\}$ blocks $\tilde{X}'$ since b_2 gives priority to higher ranked cadets among high cost contracts and $\langle y,2\rangle\, P^*_{y_I}\, \langle y,1\rangle\, P^*_{y_I} \tilde{X}'(y_I)$. If the latter happens then $(\{\langle x,1\rangle\} \cup \tilde{X}')\backslash\{\tilde{X}'(x_I)\}$ blocks $\tilde{Z}$ since b_1 gives priority to higher ranked cadets. Therefore, $x = (\varpi(\tilde{Z}))(x_I)$ for all cases. Finally, since $|C_b(A(k))| = q_b$ and $q_{b_1} + q_{b_2} = q_b$, we must have $z \notin \tilde{X}'$, a contradiction, which means no contract $z \in \varpi(\tilde{X}')$ is ever rejected during the cumulative offer process. Therefore, for all cadets $\tilde{Y}' R^*_i \tilde{X}'$. Hence, $\tilde{X}' = \tilde{Y}'$ and $\varpi(\tilde{X}') = X'$. ■

Proof of Theorem 5. The two choice functions defined for b_1 and b_2 are in the class of lexicographic choice functions described in Kominers and Sonmez (2012). Therefore, for any cadet sub-branch matching problem, cadet proposing cumulative offer algorithm is strategy-proof. Also, for any cadet and any given strategy profile chosen by the others in our original cadet branch matching problem, all the possible outcomes by choosing alternative strategies are achievable in the cadet sub-branch matching problem. Since for the cadet sub-branch matching problem, cadet proposing cumulative offer algorithm is strategy-proof, it would be weakly dominant strategy to submit actual preferences. Therefore no alternative strategy in the original problem makes cadet strictly better off. Hence, COSM induced by alternative choice function $C'()$ is strategy-proof. ■

CHAPTER 3.

COLLEGE ADMISSION WITH MULTIDIMENSIONAL PRIVILEGES: THE BRAZILIAN AFFIRMATIVE ACTION CASE

3.1 Introduction

Affirmative action policies in societies with heterogeneous populations are increasingly popular and are often considered necessary for equalizing opportunities for certain demographic groups. The United States and Brazil are examples of countries with greatly heterogeneous populations in terms of wealth and racial backgrounds. One way to mitigate the problem of inequality between individuals who belong to different racial or gender groups or come from families with different income levels is through affirmative action. Affirmative action is a method of positive discrimination in favor of a certain groups of people to close socioeconomic gaps that exist between different groups as a result of historic discriminatory practices. This paper studies affirmative action in college admission in Brazil where the goal is to give underrepresented groups increased chances of attending better universities.

The Brazilian federal higher education system comprises of 59 universities and 38 institutes of education, science and technology, with an annual inflow of about one million students to its undergraduate programs. Following an increasing role for affirmative action for students of African descent and of low-income families in terms of access to public universities[1], the Brazilian congress enacted in August 2012 a law

[1] For detailed information about history of affirmative action in Brazil, check Moehlecke (2003).

establishing the implementation of a series of affirmative action policies throughout said system.

The law established that 50% of the seats in each program offered in those institutions[2] should be used for the affirmative action policies. In order to claim the privilege of having higher priority in the access to those seats, a student must complete the three years of high-school in a public institution (being it local, state or federal). When assigning students to at least 50% of those seats, the university must also give higher priority to students who claim the privilege associated with being low-income (and give documentation proving such status as defined in law.) Additionally, when assigning a number of seats in the same proportion of the aggregate number of blacks, browns and indians (here referred to as "minorities") in the state in which the institution is, the university should give higher priority to students who claim the privilege associated with being a minority. We will throughout this chapter talk in terms of seats giving higher priority to students who claim some privileges, and denote those as "**public HS privilege**", "**low-income privilege**" and "**minority privilege**".

In a state where minorities constitute 25% of the population, for example, a program with capacity of 80 will have 40 seats giving higher priority for students claiming public HS privilege. At least 20 of those should give higher priority for those claiming low-income privilege, and 10 for those claiming minority privilege.

In October of the same year, Brazil's Ministry of Education published an

[2] In Brazil, like in the Turkish system studied in Balinski and Sonmez (1999), students apply directly to a specific program in the university, differently from other countries like the US where students simply apply to the university and once there chooses majors or programs to pursuit.

ordinance specifying some details on the implementation of the affirmative action law as well as a suggested mechanism for choosing students while satisfying those policies. Starting in the student selection processes of 2013, based on our observations, those recommendations were widely adopted as the new selection criteria.

The key distinctive issue presented by the privileges proposed in the law is the fact that they are multidimensional. That is, students may belong to one or more of the groups specified. For instance, a low-income white student from public high school qualifies for the low-income privilege but not for the minority privilege. Although the literature for affirmative action from a mechanism design perspective has seen many important contributions, as in Abdulkadiroglu and Sönmez (2003), Westkamp (2013) and Hafalir et al. (2013), to the best of our knowledge none of them are able to respond to the challenge introduced by these types of privileges.

Another unique aspect of this case is that students are not obligated to apply to the universities using any of the privileges for affirmative action to groups to which they belong. This is due to the fact that being selected through the affirmative action policies is an "opt-in" procedure, that is, those students who are object of those privileges may choose not to be selected through that special criterion. Therefore, some students may choose to "hide" whether they belong to some of the three groups mentioned above, depending on the mechanism used for the assignments.

Starting in 2010, a new centralized system[3] was put in place to match students to federal universities. Although the study of the characteristics of that system is outside of the scope of this paper, the problems identified here are still present in

[3] The Unified System of Selection, denoted SISU.

it, and moreover it shows that there is a tendency for centralization of that process. Methods that could improve upon the current system in a centralized way (as the one that we present in this chapter,) may therefore have a direct application and impact.

The problem of allocating indivisible goods in the absence of money is studied in many papers, starting from the seminal paper by Gale and Shapley (1962). They study a college admissions market where students have preferences over colleges and colleges have preferences over sets of students to be admitted. The market clearing condition that they defined, stability, is still in use (sometimes with variations) and considered as one of the most important goals that mechanism designers consider for matching problems. They also introduce the celebrated student-proposing deferred acceptance algorithm (DA) to find a stable allocation. The DA mechanism is also utilized in many applied and theoretical papers in the matching literature. The centralized algorithm we suggest in this paper, the cumulative offer algorithm, is also a variation of the DA algorithm.

The school choice with affirmative action problem consists of two parts. The first part is the schools' criteria for choosing students, which we denote a choice function. A choice function provides a set of students that are selected for any possible set of students that apply for a given school. The second part is the algorithm that the central authority uses to allocate school seats to students using the schools' choice functions.

The first approach to this problem from a mechanism design perspective is the work of Abdulkadiroglu and Sönmez (2003). They analyze the system in Boston (denoted Boston Mechanism), which gave students higher priorities in schools in their neighborhoods or in schools in which students have a sibling already attending. By giving these priorities, the Boston Mechanism positively discriminates some students for certain schools. Abdulkadiroglu and Sönmez (2003) propose two algorithms, DA and top trading cycles (TTC), as alternatives for the Boston school choice algorithm, while keeping priorities of the schools as given. They show that the DA yield outcomes that are stable and efficient from the students' perspective. Also, DA is not manipulable, i.e. no student can manipulate their preferences and obtain a better school assignment. Subsequently, Abdulkadiroglu (2005) considers the college admission problem with affirmative action policy, and shows sufficient conditions on the schools' preferences to recover the properties of the DA algorithm.

In a recent paper, Westkamp (2013) studies the German university admission system in which reserved seats are transferred to different subpopulations in case of lack of applications. In this matching with complex constraints problem, the author specifies a method for schools to choose sets of students in any given case and designs a mechanism that gives a stable allocation under these circumstances. In another recent paper, Kamada and Kojima (2012) study the Japanese Residency Matching Program, where there are quotas for regions in order to help rural regions attract more residents. In the mechanism they study, the government sets a target capacity for each hospital to implement these quotas. They show that using target capacities may result in inefficiencies and that violating these targets may improve over the inefficiencies.

In 2012, Kojima showed that in affirmative action problems with two groups (majorities and minorities), using maximum quotas (that is, a maximum number of students for some types) for even one side may be inefficient and hurt all members of the minority group – the group which the policy intends to help. In a subsequent paper, Hafalir et al. (2013) study the school choice problems with affirmative action for minorities. They show the deficiencies of utilizing maximum quotas for school choice problems with affirmative action: welfare losses and wasted seats. Switching the system to DA with minority reserves instead solves the problem of wasted seats and signifficantly improves students' welfare.

Our model is built upon the matching with contracts model described by Hatfield and Milgrom (2005). Hatfield and Milgrom (2005) connect the matching problem of indivisible goods and the labor market model. They show that the foundations of a labor market model where workers can be hired by many alternative contracts (Kelso and Crawford, 1982) are also achievable in matching markets. This paper is very important because it not only subsumes and unifies these two problems but also relates the DA algorithm with fixed point techniques in lattice theory. In our problem, students do not have to declare their true demographic status through the privileges that they claim, i.e. a minority student can be admitted as a non-minority student. Hence, as in a matching with contracts problem, students can be admitted in different ways to schools.

The remaning of this chapter is structured as follows. In section 2 we present the mechanism suggested by the Ministry of Education and currently used by the universities surveyed. In section 3, we introduce the matching with contracts model that we apply to the school choice problem with affirmative action. In section 4, we introduce the Multidimensional Brazil Privileges Choice Function and we build upon the choice function defined to describe a mechanism – Student Optimal Stable Mechanism – that matches students to colleges in a centralized way, satisfies stability, is strategy-proof and fair. In section 5, we show that even for a single college, the currently used Brazil Reserves Choice Function induces a game with multiple Nash Equilibria in which strategically sophisticated students may obtain advantage by strategizing over the privileges that they claim. We also show that the current mechanism is not fair and cannot guarantee the satisfaction of the affirmative action objectives when they are feasible. In section 6, we conclude. All the proofs are given in the Appendix section.

3.2 Brazilian Reserves Choice Function

For the most part, until 2010, college admissions in Brazil worked essentially in a decentralized way. Students applied for a single program in each university that they desire to (Ex: History at University of Brasilia or Biology at Federal University of Minas Gerais). By using some combination of scores in a national exam and sometimes exams particular to those programs, the universities ranked them and accepted the top applicants to each program up to the programs' capacities, putting the remaining ones in waiting lists.

Among those accepted, typically some would not enroll because they were also accepted by other universities and courses of their preference. The universities would then proceed to a second round, accepting students from the waitlist following their ranking. Depending on the university this might be followed by third and fourth rounds.

The introduction of the reserves law has not changed the decentralized nature of the system yet. But the centralized online system used for some universities gives a strong signal that officials in charge of college admissions in Brazil are open to utilize a centralized method, which is shown in many papers to improve efficiency and reduce wasted seats in colleges. On the other hand, the affirmative action law changed the choice rules of universities in each step in an attempt to satisfy the affirmative action objectives. The rules used by the universities surveyed in this work are, essentially, strict implementations (or small variations) of the one suggested by Brazil's Ministry of Education. This rule tells the set of students to be chosen from any set of applicants and will be denoted as the class of Brazil Reserves Choice Function (BRCF). It suggests that the seats for each program should be split into five subsets. For any program with capacity Q, the five distinct subsets are:

- A set Q_{mi} with $\lceil \frac{Q}{4} r^m \rceil$ seats which give priority to students who claim public HS, minority and low-income privileges,
- A set Q_{Mi} with $\lceil \frac{Q}{4}(1 - r^m) \rceil$ seats which give priority to students who claim public HS and low-income privileges only,
- A set Q_{mI} with $\lceil \frac{Q}{4} r^m \rceil$ seats which give priority to students who claim public HS and minority privileges only,
- A set Q_{MI} with $\lceil \frac{Q}{4}(1 - r^m) \rceil$ seats which give priority to students who claim public HS privilege only,
- A set Q_- with the remaining seats.

where r^m is the ratio of minorities in the state where that program (college) is located.

Given the students who apply for each of those, the ones better ranked on the entrance exam are accepted up to the capacity of the set. If there are enough applicants for each of those sets, the affirmative action objectives, as described by the law, are satisfied. In case the number of students who apply for some of those sets is smaller than their capacity, those seats are filled following the priority structure below:

- If there are seats available in Q_{mi}, those are made available:
  - to students claiming low-income and public HS privileges only, then
  - to students claiming minority and public HS privileges only, then
  - to students claiming public HS privileges only, then
  - to any student
- If there are seats available in Q_{Mi}, those are made available:
  - to students claiming low-income, minority and public HS privileges, then
  - to students claiming minority and public HS privileges only, then
  - to students claiming HS privilege only, then
  - to any student
- If there are seats available in Q_{mI}, those are made available:

- to students claiming public HS privilege only, then
- to students claiming low-income, minority and public HS privileges, then
- to students claiming low-income and public HS privileges only, then
- to any student

- If there are seats available in Q_{MI}, those are made available:
  - to students claiming minority and public HS privileges only, then
  - to students claiming low-income, minority and public HS privileges, then
  - to students claiming low-income and public HS privileges only, then
  - to any student

It is not specified, however, in which order those seats are filled following those priorities[4].

3.3 The Model

We are dealing with a student-program matching problem where programs have complex privileges structures and students have more than one way to attend a program. Due to those characteristics of the problem we will use the matching with contracts model. There are finite sets $S = \{s_1, \ldots, s_n\}$ and $P = \{p_1, \ldots, p_m\}$ of students and programs. The set $S^P \subseteq S$ contains all students in S from public high-schools, $S^m \subseteq S^P$ contains the racial minority students from public schools and

[4] In section 5 we present two actual implementations being used by universities surveyed, clarifying the order in which those seats are filled.

$S^i \subseteq S^P$ contains the low-income students from public schools. Each program p has its own capacity level Q_p and minority reserve ratio r_p^m. Each student s has a vector of exam scores $z(s) = (z_{p_1}(s), \ldots, z_{p_m}(s))$ such that $z_p(s)$ indicates the score of student s for program p. For any two students s and s', $z_p(s)$ and $z_p(s')$ are assumed to be different, that is, $\forall s, s' \in S$ and $p \in P$, $z_p(s) = z_p(s') \iff s = s'$. Each student s has a vector of available privileges she can claim, $t_s = (t_s^P, t_s^m, t_s^i)$ where t_s^P, t_s^m, t_s^i represents public HS, minority and low-income privileges, respectively. Each element of t_s is binary and 1 means student is eligibile for the privilege and 0 means she is not eligible. For example, if a student is a low-income non-minority student from public high school, then $t_s = (1, 0, 1)$. In the Brazilian system, if a student claims public HS, minority or low-income privileges she is required to prove those classifications. Therefore, some students may opt not to claim a privilege associated to a group she belongs to, but students who don't belong to a group (and therefore can't prove belonging to it) are unable to claim that privilege.

Throughout this section we will make use of the *matching with contracts* notation. A contract x, in this context, is a tuple (s, p, t), where $s \in S$, $p \in P$ and $t = (t^P, t^m, t^i) \leq t_s$. Vector t represents the set of privileges student claims and t^P, t^m, t^i are binary and represents public HS, minority and low-income privileges she claims, respectively. For a contract x; x_S, x_P and x_T represent student, program and set of privileges s claims in contract x respectively. Let X be the set of all contracts. For ease of notation, for a set of contracts Y , Y_i is the subset of Y that contains only the contracts that include $i \in S \cup P$. Let $s(Y)$, moreover, be the set of students with contracts in Y, that is, $s(Y) = \{s \in S : \exists(s, p, t) \in Y\}$. An allocation is a set of contracts $X' \subset X$, such that for every $s \in S$ and every $p \in P$, $|X'_s| \leq 1$ and $|X'_p| \leq Q_p$. Let χ be the set of all possible allocations.

The null contract, meaning that the student has no contract, is denoted by $\emptyset$. Students have complete preferences, $\succeq$, over her contracts and the null contract, $X_s \cup \emptyset$. These preferences are derived from students' strict preferences, $\succ^*$, over programs and being unmatched, in addition to the fact that they consider irrelevant how they are accepted to a program:

$$\forall s \in S, \forall p, p' \in P \text{ and } t, t' \leq t_s : (s, p, t) \succ_s (s, p', t') \iff p \succ_s^* p'$$

Next, the choice function of program p, $C_p : 2^X \rightarrow 2^X$ is a function that chooses, that is, for $Y \subset X$, $C_p(Y) \subset Y_p$, $C_p(Y)$ has cardinality at most Q_p and has at most one contract for each student. The assumption about student preferences we mentioned above is one of the main differences of our paper with the current matching with contracts literature, since our model allows indifferences among contracts, in contrast with the usual assumption of strict preferences found in the literature so far. Due to indifferences students have between some contracts, we cannot derive choice functions of students as defined in the many to one matching with contracts models. As a result, instead of choice functions for students, we are going to use student preferences. Therefore, primitives of our model are student preferences over contracts and programs' choice functions.

A mechanism is a strategy space Δ_s for each student s along with an outcome function $\psi : \prod_{s \in S} \Delta_s \rightarrow \chi$ that selects an allocation for each strategy vector $\prod_{s \in S} \delta_s \in \prod_{s \in S} \Delta_s$. Given a student s and a strategy profile $\delta_s \in \Delta_s$, let δ_{-s} denote the strategy of all students except student s.

3.4 Student Optimal Stable Mechanism

The Multidimensional Brazil Privileges Choice Function One of our objectives is to find a choice function that satisfies the affirmative action objectives for each program, removes incentives for students to strategize over the privileges that they claim and guarantees the existence of a stable allocation. We also aim to design a mechanism that carries out our choice function's properties and finds a stable allocation.

We are proposing a new choice function, Multidimensional Brazil Privileges Choice Function (or MCF), in order to allocate students to seats in programs. Unlike the BRCF, our choice function C^{MCF} obtains the desired incentive characteristics by giving priority in a seat to any student who can claim the privileges associated with that seat. Also, by doing this, the choice function satisfies another important criterion: fairness.

Let q_p be the number of seats associated with students who claim low-income, minority and public HS in the BRCF, for program p. For any given set of contracts X, the algorithm which implements the choice function C^{MCF} is the following:

Phase 0: Program p rejects each contract that does not include itself ($x_P \neq p \implies x \notin C_p(X)$).

Phase 1: Program p considers only contracts with $x_T = (1, 1, 1)$. Program p accepts contracts including students with the highest scores z_p one at a time and continues until either all contracts are considered or q_p contracts are chosen. In any case, program proceeds with Phase 2. Let θ be $q_p - |\{\text{contracts accepted in Phase1}\}|$.

Phase 2: Program p considers remaining contracts with $x_T > (1,0,0)$. Program p accepts contracts including students with highest scores z_p one at a time. During the process, if constraint (1) or (2) below binds, program p tentatively rejects all the remaining contracts with the relevant vector of privileges. Then, the program continues accepting contracts one by one following the order of student scores. Phase 2 ends if all contracts are considered or $r_p^m \frac{Q_p}{2} + \frac{Q_p}{4} - q_p$ contracts are accepted. Then, the program proceeds with Phase 3.

Possible constraints to bind	Rel. vectors of priv.	
$\lvert\{\text{Contracts accepted with } x_T = (1,0,1)\}\rvert \leq \frac{Q_p}{4} + \theta - q_p$	$t = (1,0,1)$	(1)
$\lvert\{\text{Contracts accepted with } x_T = (1,1,0)\}\rvert \leq r_p^m \frac{Q_p}{2} + \theta - q_p$	$t = (1,1,0)$	(2)

Phase 3: In this phase, the program considers all tentatively rejected contracts and all the remaining contracts with $x_T \geq (1,0,0)$. Program p accepts contracts including students with highest scores z_p, one at a time. The program continues until either all contracts are considered or $\frac{Q_p}{2}$ students are chosen. In any case, it proceeds to Phase 4.

Phase 4: In this phase, the program considers all the remaining contracts. Program p accepts contracts including students with highest scores z_p, one at a time. It continues until either all contracts are considered or Q_P students are chosen. Then program terminates the procedure and rejects all the remaining contracts, if there are any.

Stability As in Gale and Shapley (1962) and most of the matching literature, we are interested in stable allocations. Intuitively, an allocation is stable if students or programs cannot improve upon the chosen allocation by either walking away from it or by bilaterally making arrangements outside of the allocation.

Definition 16 *An allocation X' is* ***stable*** *if*

i. for all $s \in S$ and for all $p \in P$, $X'_s \succ_s \emptyset$, $C_p(X') = X'_p$; and

ii. $\nexists (p, s) \in P \times S$, and contract $x \in X \setminus X'$, such that

$$x \in C_p((X' \setminus X'_s) \cup \{x\}), x \succ_s X'_s.$$

One can see that if students have strict preferences over contracts then our stability definition and the one used in the current literature would be equivalent. In order to show the existence of a stable allocation, we use the substitutes and law of aggregate demand properties defined in Hatfield and Milgrom (2005) and irrelevance of rejected contracts defined in Aygün and Sönmez (2013).

Substitutes, IRC, Law of Aggregate Demand and the Student Optimal Stable Mechanism

In this section, we define the properties which are sufficient for existence of a stable allocation in our college admission problem and show that $C^{MCF}()$ has these properties.

Definition 17 *Elements of X are* ***substitutes*** *for program p if for all $Y' \subset Y'' \subset X$ we have $x \in Y' \setminus C_p(Y') \implies x \in Y'' \setminus C_p(Y'')$.*

The substitutes condition simply states that if a contract x is rejected, not chosen, in a set of contracts Y' then adding any other contract to Y' cannot make x desirable or x should remain rejected in bigger sets that contain Y'.

Lemma 7 *Elements of X are substitutes for each program p under the choice function C^{MCF}.*

Definition 18 *A choice function C satisfies the **Law of Aggregate Demand** if for all $Y' \subset Y'' \subset X$ we have $|C(Y')| \leq |C(Y'')|$.*

Under the law of aggregate demand, when more contracts are added to a set of contracts, the size of the chosen set never shrinks. Since, in any phase of the choice function unfilled seats are transferred to the next phases, and any student is acceptable to programs, we can state the following lemma.

Lemma 8 *The choice function C^{MCF} satisfies the Law of Aggregate Demand, as defined for each program p.*

For many to one matching problems that use choice functions of programs as a primitive, Aygün and Sönmez (2013) show that the substitutes condition is not sufficient to guarantee existence of stable allocations. Therefore, since our primitive of the model for programs is choice functions rather than preferences, we use the Irrelevance of Rejected Contracts[5] condition defined by Aygün and Sönmez (2013) along with the substitutes condition.

[5] The Irrelevance of Rejected Contracts condition was previously defined as "Consistency" in Alkan and Gale (2001).

Definition 19 *Given a set of contracts X, a choice function C satisfies the **Irrelevance of Rejected Contracts** (IRC) condition if*

$$\forall Y \subset X, \forall x \in X \setminus Y \quad x \notin C(Y \cup \{x\}) \implies C(Y) = C(Y \cup \{x\}).$$

The IRC condition simply states that an outcome of the choice function should not be affected by the removal of rejected contracts. With the help of this condition, Aygün and Sönmez (2013) show that we can guarantee the existence of stable allocation without the need for strict preferences of programs over sets of contracts.

Lemma 9 *The choice function C^{MCF} satisfies Irrelevance of Rejected Contracts for each program p.*

Finally, with the help of the conditions above, we can guarantee the existence of a stable allocation for our student-program matching problem.

Proposition 8 *If all programs use C^{MCF}, the set of stable allocations for student-program matching problem is not empty.*

The choice function defined above defines only how a single school should behave for a given set of students. Now, with the help of that choice function, we are ready to introduce the Student Optimal Stable Mechanism, ψ^{SOSM}. First, students submit a vector of privileges they want to claim and preferences $\succ$. We then use the student proposing cumulative offer algorithm with submitted vector of privileges $(t^s)_{s\in S}$, preferences $\succ$ and C^{MCF} for each program. The cumulative offer algorithm description we use here was previously introduced by Hatfield and Kojima (2010).

Step 1: One randomly selected student s_1 offers her first choice contract x^1 with the vector of privileges (t^{s_1}), according to her preferences $\succ_{s_1}$. The program that receives the offer, $p_1 = x^1_P$, holds the contract. Let $A_{p_1}(1) = x^1$, and $A_p(1) = \emptyset$ for all $p \neq p_1$.

In general,

Step $k \geq 2$: One of the students for whom no contract is currently held by a program, say s_k, offers the most preferred contract with the vector of privileges (t^{s_k}), according to her preferences $\succ_{s_k}$, that has not been rejected in previous steps. Call the new offered contract, x^k. Let $p_k = x^k_P$ hold $C_{p_k}(A_{p_k}(k-1) \cup \{x^k\})$ and reject all other contracts in $A_{p_k}(k-1) \cup \{x^k\}$. Let $A_{p_k}(k) = A_{p_k}(k-1) \cup \{x^k\}$, and $A_p(k) = A_p(k-1)$ for all $p \neq p_k$.

The algorithm terminates when either every student is matched to a program or every unmatched student has no contract left with the vector of privileges they submit to offer. The algorithm terminates in some finite number K of steps due to a finite number of contracts. At that point, the algorithm produces $X' = \bigcup_{p \in P} C_p(A_p(K))$, i.e., the set of contracts that are held by some program at the terminal step K.

We have already shown that the set of stable allocations is not empty if the choice functions satisfy the substitutes condition. Our result below shows that the student optimal stable mechanism gives us a stable allocation which is one of the main desired properties of a mechanism in the matching literature.

Proposition 9 *The Student Optimal Stable Mechanism, ψ^{SOSM}, produces a stable allocation for any given problem.*

Privilege Monotonicity, Fairness and Affirmative Action Objectives An ideal choice function should also satisfy Privilege Monotonicity and fairness. Privilege Monotonicity suggests that when a student applies to a program, claiming an additional privilege should not decrease her chance to be chosen. With this property, we can state that for any school, students do not have to gather information and strategize their application processes with respect to those privileges. Hence, we can level the playing field for students.

Definition 20 *Given a set of contracts X, a choice function $C : 2^X \to 2^X$ is **Privilege Monotonic** if for any given set of contracts $Y \subset X$, and any student s with no contract in Y,*

$$(s,p,t_s) \notin C_p(Y \cup \{(s,p,t_s)\}) \implies (s,p,t') \notin C_p(Y \cup \{(s,p,t')\}), \forall t' \leq t_s.$$

Proposition 10 *The choice function C^{MCF} is Privilege Monotonic.*

Unlike the BRCF, the choice function we design gives students no incentive to leave a privilege, associate to a group she belongs to, unclaimed. This property will have an important role in the strategic properties of the mechanism we suggest.

Definition 21 *Given a set of contracts X, a choice function $C : 2^X \to 2^X$ is **fair** if for any given subset $Y \subset X$, any program p and $x \in Y_p$,*

$$x \notin C_p(Y) \implies \forall y \in C(Y),\ \textit{either } z_p(y_S) > z_p(x_S) \textit{ or } x_T \ngeq y_T \geq (1,0,0).$$

Fairness of the choice function as we use here indicates that, if a contract is not chosen this means that chosen contracts either include students with higher test scores or they are chosen due to the affirmative action policy.

Proposition 11 *The choice function C^{MCF} is fair.*

The new law issued in Brazil requires some structure on the sets chosen by programs, with respect to the groups to which the students belong to. In other words, the ratios associated with public HS, low-income and minorities should be, when possible, satisfied by the students chosen for each program. We formalize this in the definition below.

Definition 22 *A choice function $C_p : 2^X \to 2^X$ **satisfies the affirmative action objectives** at program p if $\forall Y \subset X$:*

$$\begin{aligned} |\{x &\in C_p(Y) : x_T \geq (1,0,0)\}| \geq \min\{\frac{Q_p}{2}, |\{x \in Y : x_T \geq (1,0,0)\}|\}, \\ |\{x &\in C_p(Y) : x_T \geq (1,0,1)\}| \geq \min\{\frac{Q_p}{4}, |\{x \in Y : x_T \geq (1,0,1)\}|\}, \\ \text{and } |\{x &\in C_p(Y) : x_T \geq (1,1,0)\}| \geq \min\{\frac{r_p^m Q_p}{2}, |\{x \in Y : x_T \geq (1,1,0)\}|\}. \end{aligned}$$

The definition above states that a choice function must choose a sufficient number of students from all groups of students that are subject to affirmative action, whenever it is possible. One can check that when $q_p = 0$ our choice function satisfies the affirmative action objectives. However, when $q_p = 0$ and $r_p^m = \frac{1}{2}$, all the seats that give priority for those who claim public HS privilege will be reserved only for those who also claim low-income and/or minority privileges. In this case, those who claim only public HS privilege will in practice not have any privilege unless there are not enough applications from those claiming the other combinations of privileges. Also, students claiming all privileges may not enjoy this advantage unless their scores are high enough compared to those claiming only two. The current guidelines set by the

Brazilian government give priority to students claiming only public HS privilege for some seats. Due to this fact, one can argue that there is an implicit objective that programs should give priority to all combination of privileges which include public HS for some seats. Since giving priority to each group may cause incentive problems, our choice function, as a second best, prioritizes seats to students who claim each such combination of privileges along with all students who claim some subset of them. For a given program p, let q_p be the number of seats associated with students who claim low-income, minority and public HS in the BRCF. Therefore, if a program p, receives at least q_p contracts with the vector of privileges $(1,1,1)$, the program should accept at least q_p contracts with the vector of privileges $(1,1,1)$. Otherwise, the program should accept all contracts available with the vector of privileges $(1,1,1)$.

Definition 23 *A choice function $C_p : 2^X \to 2^X$* ***satisfies the affirmative action objectives conditional on*** *q_p at program p if $\forall Y \subset X$:*

$$\begin{aligned}
|\{x &\in Y : x_T = (1,1,1)\}| \geq q_p \textit{ implies} \\
|\{x &\in C_p(Y) : x_T \geq (1,0,0)\}| \geq \min\{\frac{Q_p}{2}, |\{x \in Y : x_T \geq (1,0,0)\}|\}, \\
|\{x &\in C_p(Y) : x_T \geq (1,0,1)\}| \geq \min\{\frac{Q_p}{4}, |\{x \in Y : x_T \geq (1,0,1)\}|\}, \\
\textit{and } |\{x &\in C_p(Y) : x_T \geq (1,1,0)\}| \geq \min\{\frac{r_p^m Q_p}{2}, |\{x \in Y : x_T \geq (1,1,0)\}|\}.
\end{aligned}$$

This second version includes a condition on the number of contracts claiming all privileges. This conditional satisfaction of the affirmative action objectives requires satisfying them only in situations where we have enough applications claiming all three privileges, as well as requiring that the satisfaction of all affirmative action objectives is possible.

Proposition 12 *The choice function C^{MCF} satisfies the affirmative action objectives conditional on q_p at any program p.*

Although depending on the set of contracts available C^{MCF} may not choose a set of contracts that satisfies the affirmative action objectives, q_p can be determined differently for different programs. While programs that set low q_p minimize the number of cases that fail to give enough seats to students claiming certain combinations of privileges, programs that set a higher value for q_p give more opportunity to students who claim only the public HS privilege. One possible way for setting q_p is to construct an expected number of applications claiming all three privileges based on past years' applications.

Incentives and Fairness of the Student Optimal Stable Mechanism Although we have shown that the choice function that we proposed satisfies the desired fairness and incentives properties, we are also interested in knowing whether corresponding properties are satisfied by the overall allocation when the SOSM mechanism is used to match students to programs. The first such property that we introduce is that of fairness.

Definition 24 *An allocation X' is **fair** if for any given pair of contracts $x, y \in X'$*

$$y_P \succ^*_{x_S} x_P \implies \text{either } z_{y_P}(y_S) > z_{y_P}(x_S) \text{ or } x_T \not\geq y_T \geq (1,0,0).$$

A mechanism is fair if for any given problem it chooses a fair allocation.

In the previous school choice and student placement literature, like for example in Balinski and Sönmez (1999), it is shown that stability is sufficient for the allocation to satisfy a fairness condition based on the priorities that students have at the schools. This idea comes from the fairness of the responsive preferences of schools. As opposed to the previous school choice and student placement literature, programs in our model do not have responsive preferences. The non existence of responsive preferences may result in allocations that are not fair as in Balinski and Sönmez (1999). Therefore, in our problem, the stability of the mechanism is not sufficient for fairness. That is the reason why the fairness satisfied by our mechanism comes from the fairness of the choice function.

Proposition 13 *The Student Optimal Stable Mechanism, ψ^{SOSM}, is fair.*

The next property that we discuss here is the incentive compatibility of the mechanism, which is a desired characteristic in mechanism design. Incentive compatibility in this context can be described as a property that guarantees that students cannot be better-off by strategizing over manipulations of the preferences being submitted or privileges being claimed. In our problem, students' strategy spaces do not consist only of preferences over schools but also the privileges claimed. Although it is tempting to conclude that the incentive compatibility of the SOSM immediately follows as a corollary of the well-known incentive properties of the SOSM mechanism, due to the wider strategy space for students the result must be obtained explicitly.

Definition 25 *A mechanism is* ***incentive compatible*** *if*

$$\nexists s \in S, \delta_{-s} \in \prod_{j \in S \setminus \{s\}} \Delta_j, (t_s, \succ_s), \delta' \in \Delta_s, \text{ such that } \psi(\delta', \delta_{-s}) \succ_s \psi((t_s, \succ_s), \delta_{-s}).$$

In other words, for any student that we consider, no matter what her true preferences are or which groups she belongs to, it will be in her best interest to reveal her true preferences and claim all privileges that she's eligible to. This is valid for any allocation problem and any strategies other students report.

Proposition 14 *The Student Optimal Stable Mechanism, ψ^{SOSM}, is incentive compatible.*

3.5 Current Mechanism Revisited

So far, we introduced some desired properties that a choice function and a mechanism should satisfy. In this section, first we formally describe two of the choice functions which are implementations of the guidelines published by the Ministry of Education and currently used by two of the largest federal universities in Brazil. Next, we show some deficiencies of those choice functions and any stable mechanism that uses these choice functions.

Two Examples of the BRCF Since the specification given by the guideline allows for different choice procedures, we can find variation on the universities' implementation of it. We will describe two instances: the choice function used by the Federal University of Minas Gerais (UFMG) and by the Federal University of Rio Grande do Sul (UFRGS) .

The implementations by UFMG and UFRGS are in the class of choice functions described in Westkamp (2013) and Kominers and Sonmez (2012). This relationship is helpful to analyze our properties.

As we mentioned in section 2, for any program, seats are partitioned into five: Q_{mi}, Q_{Mi}, Q_{mI}, Q_{MI} and Q_{-}. For any given program, numbers of seats and priority structure of Q_{mi}, Q_{Mi}, Q_{mI} and Q_{MI} are determined by the current guideline and are as we discussed in section 2. Since it is not possible to know actual demographic backgrounds of students for the priority structure, both implementations we discussed here takes claims of privileges as demographic backgrounds of students. For any given set of contracts, the choice function used by UFMG, $C^{UFMG}()$, works as the following:

Choice function fills seats in the following order: Q_{mi}, Q_{Mi}, Q_{mI}, Q_{MI} and Q_{-}. For the priorities of the first four group of seats choice function uses priorities described by the current guideline and for the last group, Q_{-}, it gives priority to contracts with privilege vector $(0, 0, 0)$. If there are seats available in Q_{-} choice function gives priority

- to contracts with privilege vector $(1, 1, 1)$, then
- to contracts with privilege vector $(1, 0, 1)$, then
- to contracts with privilege vector $(1, 1, 0)$, then
- to contracts with privilege vector $(1, 0, 0)$

During this procedure, choice function either accepts all the contracts or fills all the seats. In any case, choice function stops the procedure and rejects all the remaining contracts, if there is any.

On the other hand, the choice function used by UFRGS, $C^{UFRGS}()$, works as the following:

Choice function fills seats in the following order: Q_{-}, Q_{MI}, Q_{mI}, Q_{Mi} and Q_{mi}. For the priorities of the last four group of seats choice function uses priorities described by the current guideline and for the first group, Q_{-}, it accepts contracts one at a time based on student scores starting with the contract of student with highest score. During this procedure, choice function either accepts all the contracts or fills all the seats. In any case, choice function stops the procedure and rejects all the remaining contracts, if there is any.

Once we define these two implementations of the BRCF guidelines, the bilateral substitutes property of contracts directly comes from the second proposition of Kominers and Sönmez (2012). Also, since there is only one possible contract for each student to offer to a given program, the choice over contracts satisfies the substitutes condition. Moreover, since each contract is acceptable to all slots, with a bigger contract sets the set of contract chosen never shrinks. Therefore, C^{UFMG} and C^{UFGRS} satisfy the Law of Aggregate Demand. Hence, if all programs use one of the implementations above, the existence of a stable allocation is guaranteed by Proposition 1 of Aygün and Sönmez (2013).

Two Examples of the BRCF The two implementations of the guidelines designed by the Brazilian government are instances of choice functions described in Westkamp (2013) and Kominers and Sönmez (2012). Since these choice functions are designed for a single contract for each student, like C^{MCF}, contracts are not only bilateral substitutes, a weak version of substitutes condition, as shown in Kominers and Sönmez (2012) but also substitutes for each program. But these choice functions, unlike C^{MCF}, fail to satisfy the fairness and privilege monotonicity properties. They also don't satisfy the affirmative action objectives conditional on q_p. We show, using examples, how these choice functions violate these three conditions. We start with privilege monotonicity.

Example 5 *[Privilege Monotonicity] For a given program p let $Q_p = 8$, $r_p^m = \frac{1}{2}$ and let the set of contracts be $Y = \{x^1, \ldots, x^8\}$ such that $x_T^1 = x_T^2 = x_T^3 = x_T^4 = (0,0,0)$, $x_T^5 = (1,0,0)$, $x_T^6 = (1,1,1)$, $x_T^7 = (1,1,0)$ and $x_T^8 = (1,0,1)$. Also let $z_p(x_S^i) > z_p(x_S^j) \iff i < j$. Consider a low-income minority student from public high school $s \notin s(Y)$ with score $z_p(s) > z(x_S^8)$. If she applies with a contract that includes all of her privileges, i.e. $(s,p,(1,1,1))$, no matter which example of the BCRF program p uses, she will be rejected:*

$$(s,p,(1,1,1)) \notin C_p(Y \cup \{(s,p,(1,1,1))\}) = \{x^1,x^2,x^3,x^4,x^5,x^6,x^7,x^8\}$$

However, if she claims only low-income and public HS privileges, i.e. $(s,p,(1,0,1))$, no matter which implementation of BRCF program p uses, her contract will be accepted:

$$(s,p,(1,0,1)) \in C_p(Y \cup \{(s,p,(1,0,1))\}) = \{x^1,x^2,x^3,x^4,x^5,x^6,x^7,(s,p,(1,0,1))\}$$

Therefore, the two examples of the BRCF are not privilege monotonic.

The example above shows that since the choice function gives priority to students who claim low-income and public HS only, the choice function gives student s incentive not to claim her minority privilege. This problem can be solved by using C^{MCF} instead. C^{MCF} gives students equal or higher chances to be chosen when their contracts compete with others that has a subset of the privileges that she claims. Hence students have no incentive not to claim privileges. The second example we give regards the fairness property of choice functions.

Example 6 *[Fairness] For a given program p let $Q_p = 8$, $r_p^m = \frac{1}{2}$ and let the set of contracts be $Y = \{x^1, \ldots, x^9\}$ such that $x_T^1 = x_T^2 = x_T^3 = x_T^4 = (0,0,0)$, $x_T^5 = x_T^6 = (1,1,1)$, $x_T^7 = (1,0,1)$, $x_T^8 = (1,1,0)$ and $x_T^9 = (1,0,0)$. Also let $z_p(x_S^i) > z_p(x_S^j) \iff i < j$. In this case, no matter which example of the BCRF program p uses, the chosen set will be:*

$$C_p(Y) = \{x^1, x^2, x^3, x^4, x^5, x^7, x^8, x^9\}$$

Let $x_S^6 = j$. Since student j can offer x^6, we can say that $t_j = (1,1,1)$ and $(1,0,0) < t_j$. Also, by assumption, she has higher score than owner of contract x^9. Therefore, rejecting x^6 while accepting x^9, violates fairness of the choice function.

In this second example, the program p chooses x^9, although student j has higher score and claims more privileges than privileges claimed in x^9. This example tells us that the guideline provided by the government implicitly tries to provide diversity in the chosen students even when the law does not require it. On the other hand, C^{MCF} only gives priority to students to which the affirmative action is addressed to. Therefore, C^{MCF} prevents any fairness problems. The next example is about the relationship between choice functions and the affirmative action objectives.

Example 7 *[Affirmative Action conditional on q_p] For a given program p let $Q_p = 8$, $r_p^m = \frac{1}{2}$ and let the set of contracts be $Y = \{x^1, \ldots, x^9\}$ such that $x_T^1 = x_T^2 = x_T^3 = x_T^4 = (0,0,0)$, $x_T^5 = x_T^6 = (1,0,0)$, $x_T^7 = x_T^8 = (1,1,1)$ and $x_T^9 = (1,0,1)$. Also let $z_p(x_S^i) > z_p(x_S^j) \iff i < j$. In both implementations of the BRCF guidelines, the number of seats with priority for students who claim all the 3 privileges is 1 and one seat accepts a contract with privilege vector $(1,0,0)$ since there is no contract claiming minority and public HS privileges only. If the set of contracts is Y, no matter which example of the BRCF program p uses, the chosen set will be:*

$$C_p(Y) = \{x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^9\}$$

Therefore, the choice function chooses only one student claiming minority and public HS privileges, although it is possible to choose two, which is the number of seats with priority for students claiming those privileges.

Another problem with the BRCF is that it considers students claiming public HS privilege only as the first order substitutes for students claiming minority and public HS privileges only. Therefore, when there is an absence of applications from contracts with privilege vector $(1,1,0)$, the choice function turns to contracts with privilege vector $(1,0,0)$ and ignores the priority for minorities. In the example above, one of the students claiming only public HS privilege receives the seat with priority for those claiming minority and public HS privileges. Hence, implementations of the BRCF fail to satisfy the affirmative action objectives conditional on q_p.

Now, we will show that if programs adopt one of the implementations of BRCF above, no matter what algorithm one chooses in order to create a stable mechanism, the mechanism violates the properties we defined above. Previous papers have shown us that some of the deficiencies of choice functions can be corrected by choosing the right algorithm. One example of this is the choice function used by the U.S. Military Academy (USMA). Sönmez and Switzer (2013) have shown us that the USMA priorities may fail to satisfy fairness, but than when they use the cumulative offer algorithm the outcome of the mechanism is always fair. However, the following two examples show that violations of incentive compatibility and fairness are carried by any stable mechanism.

Example 8 *[Incentive Compatibility] There is one program p with capacity of eight seats and nine students $S = \{s_1, \ldots, s_9\}$. Let $r_p^m = \frac{1}{2}$ and p be preferred to the null contract by every student. The score order of students is given by $z_p(s_i) > z_p(s_j) \iff i < j$. Also, vectors of privileges available to students are given by*

$$t_{s_1} = t_{s_2} = t_{s_3} = t_{s_4} = (0,0,0)$$

$$t_{s_5} = t_{s_6} = (1,1,1)$$

$$t_{s_7} = (1,0,0)$$

$$t_{s_8} = (1,1,0)$$

$$t_{s_9} = (1,0,1)$$

For this problem, if every student claims all of the privileges that she is eligible to, there is only one stable allocation, X', that we can achieve if program p uses one of the implementations of the current BRCF. The set of students assigned is the following:

$$s(X') = \{s_1, s_2, s_3, s_4, s_5, s_7, s_8, s_9\}$$

Now, assuming that the other students use the same strategy as before, if s_6 claims only public HS privilege and submits $(s_6, p, (1,0,0))$, there is again only one stable allocation, say X'', that we can achieve if the program p uses one of the implementations of the current BRCF and the set of students assigned is the following:

$$s(X'') = \{s_1, s_2, s_3, s_4, s_5, s_6, s_8, s_9\}$$

Therefore, any stable mechanism with these two examples of the BRCF are not incentive compatible.

The example above shows that since these choice functions give priority to students who claim a subset of the privileges that s_6 is eligible to for some of the seats available, they may give student s_6 an incentive not to claim all of her privileges. This not only puts a burden on students to gather more information about their peers and strategize their behavior in order to get better assignments, but also gives some students an unfair advantage in their college applications. Also, violation of incentive compatibility causes an allocation to be chosen which is actually (with respect to the groups to which the students belong to) unstable. It also makes it harder to observe the effect of this affirmative action policy for future decisions over it. The last example we give relates to the fairness property of mechanisms.

Example 9 *[Fairness] There are one program p with capacity of eight seats and nine students $S = \{s_1, \ldots, s_9\}$. Let $r_p^m = \frac{1}{2}$ and p be preferred to the null contract for each student. The score order of students is given as $z_p(s_i) > z_p(s_j) \iff i < j$. Also, the vectors of privileges available to students are given by*

$$t_{s_1} = t_{s_2} = t_{s_3} = t_{s_4} = (0,0,0)$$

$$t_{s_5} = t_{s_6} = (1,1,1)$$

$$t_{s_7} = (1,0,0)$$

$$t_{s_8} = (1,1,0)$$

$$t_{s_9} = (1,0,1)$$

For this problem, if every student claims all the privileges that they are eligible to, there is only one stable allocation, say X', that we can achieve if the program p uses one of the implementations of the current BRCF and the set of students assigned is the following:

$$s(X') = \{s_1, s_2, s_3, s_4, s_5, s_7, s_8, s_9\}$$

Since student s_6 is eligible to claim all privileges and she has higher score than s_7,s_8 and s_9, rejecting $(s_6, p, (1,1,1))$ while accepting $(s_7, p, (1,0,0))$, violates fairness. This result holds no matter what kind of algorithm we use that gives stable allocation with these two implementations of the BRCF.

3.6 Concluding Remarks

In this paper, we presented a new market design application of university program-student matching that emerged as result of the affirmative action policy that was designed by the Brazilian government to aid minority and low-income students from public high schools. This problem is particularly interesting in the sense that the freedom of not claiming all of the privileges that a student is eligible to during the application process combines the matching and the adverse selection problems. Due to this fact, we defined the property of privilege monotonicity for choice functions for the first time in this literature.

This paper shows that the current guidelines for designing choice functions for programs have avoidable deficiencies, such as generating unfair allocations and giving sophisticated students an advantage over others by manipulating the system.

We proposed a new choice function, denoted the multidimensional Brazil privileges choice function, that can also be used together with the student optimal stable mechanism to generate student assignments. The choice function is privilege monotonic and fair unlike the current choice functions which are implementations of the guidelines designed by the Brazilian government. Moreover, the mechanism we suggest is incentive compatible, fair and yields a stable allocation for any problem.

With a complex privileges structure like we have in this problem, it is hard to satisfy the affirmative action objectives in all cases. We showed that the current choice functions used by programs in Brazil not only fails to satisfy the affirmative action objectives when they are possible but also fails to satisfy a weaker condition that imposes some restrictions over the population of students applying to a program. On the other hand, the choice function we suggest always satisfies that weaker condition and if the parameters for the choice function is selected correctly, the diversity targets in the programs are reached by our procedure.

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3.8 Appendix

Proof. [Proof of Lemma 7] For any set of contracts Y and any phase i, let Y_k be set of contracts that is considered in phase k. Think about the procedure:

Phase 1. First observe that $Y_1' \subseteq Y_1''$. If a contract x is not accepted in the first phase then either $x \notin Y_1'$ or we have

$$|\{y \in Y_1' : z_p(y_S) > z_p(x_S)\}| \geq q_p.$$

Therefore, either $x \notin Y_1''$, or $Y' \subseteq Y''$ implies

$$|\{y \in Y_1'' : z_p(y_S) > z_p(x_S)\}| \geq q_p.$$

Hence contract x can not be accepted from Y'' in the first phase as well. So, we have $Y_2' \subseteq Y_2''$.

Phase 2. Let θ' and θ'' be number of unused seats in Phase 1 when we use Y' and Y'', respectively. As $Y_1' \subseteq Y_1''$, we have $\theta' \geq \theta''$. If a contract x is not accepted in the second phase then either $x \notin Y_2'$ which means $x \notin Y_2''$, or we have three cases

Case 1: If $x_T = (1,1,1)$, we have

$$\begin{aligned}
&\min\{|\{y \in Y_2' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,0)\}|, r_p^m \frac{Q_p}{2} + \theta' - q_p\} + \\
&\min\{|\{y \in Y_2' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,0,1)\}|, \frac{Q_p}{4} + \theta' - q_p\} + \\
&\quad |\{y \in Y_2' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,1)\}| \geq r_p^m \frac{Q_p}{2} + \frac{Q_p}{4} + \theta' - 2q_p
\end{aligned}$$

Therefore, $Y_2' \subseteq Y_2''$ implies

$$\begin{aligned}
&\min\{|\{y \in Y_2'' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,0)\}|, r_p^m \frac{Q_p}{2} + \theta'' - q_p\} + \\
&\min\{|\{y \in Y_2'' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,0,1)\}|, \frac{Q_p}{4} + \theta'' - q_p\} + \\
&\quad |\{y \in Y_2'' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,1)\}| \geq r_p^m \frac{Q_p}{2} + \frac{Q_p}{4} + \theta'' - 2q_p
\end{aligned}$$

as well. Hence, contract x can not be accepted from Y'' in the second phase as well.

Case 2: If $x_T = (1,1,0)$, we have either

$$|\{y \in Y_2' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,0)\}| \geq r_p^m \frac{Q_p}{2} + \theta' - q_p, \text{ or}$$

$$\begin{aligned}
&\quad |\{y \in Y_2' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,0)\}| + \\
&\min\{|\{y \in Y_2' : z_p(y_S) > z_p(x_S) \text{ s.t. } y_T = (1,0,1)\}|, \frac{q_c}{4} + \theta' - q\} + \\
&\quad |\{y \in Y_2' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,1)\}| \geq r_p^m \frac{Q_p}{2} + \frac{Q_p}{4} + \theta' - 2q_p.
\end{aligned}$$

Therefore, $Y_2' \subseteq Y_2''$ implies

$$\begin{aligned}
&|\{y \in Y_2'' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,0)\}| \geq r_p^m \frac{Q_p}{2} + \theta'' - q_p, \text{ or} \\
&|\{y \in Y_2'' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,0)\}| + \\
\min\{&|\{y \in Y_2'' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,0,1)\}|, \frac{q_c}{4} + \theta'' - q\} + \\
&|\{y \in Y_2'' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,1)\}| \geq r_p^m \frac{Q_p}{2} + \frac{Q_p}{4} + \theta'' - 2q_p
\end{aligned}$$

as well. Hence, contract x can not be accepted from Y'' in the second phase as well.

Case 3: If $x_T = (1,0,1)$, we have either

$$\begin{aligned}
&|\{y \in Y_2' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,0,1)\}| \geq \frac{q_c}{4} + \theta' - q, \text{ or} \\
&|\{y \in Y_2' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,0,1)\}| + \\
\min\{&|\{y \in Y_2' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,0)\}|, r_p^m \frac{Q_p}{2} + \theta' - q_p\} + \\
&|\{y \in Y_2' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,1)\}| \geq r_p^m \frac{Q_p}{2} + \frac{Q_p}{4} + \theta' - 2q_p.
\end{aligned}$$

Therefore, $Y_2' \subseteq Y_2''$ implies

$$\begin{aligned}
&|\{y \in Y_2'' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,0,1)\}| \geq \frac{q_c}{4} + \theta'' - q, \text{ or} \\
&|\{y \in Y_2'' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,0,1)\}| + \\
\min\{&|\{y \in Y_2'' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,0)\}|, r_p^m \frac{Q_p}{2} + \theta'' - q_p\} + \\
&|\{y \in Y_2'' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,1)\}| \geq r_p^m \frac{Q_p}{2} + \frac{Q_p}{4} + \theta'' - 2q_p
\end{aligned}$$

as well. Hence, contract x can not be accepted from Y'' in the second phase as well. So any contract x that is not accepted from Y' in Phase 2, is not accepted from Y'' in Phase 2. Moreover, that guarantees $Y_3' \subseteq Y_3''$.

Phase 3. Let θ_1' and θ_1'' be the number of unused seats in Phase 2 when we use Y' and Y'', respectively. As $Y_2' \subseteq Y_2''$, we have $\theta_1' \geq \theta_1''$. If a contract x is not accepted in the third phase then either $x \notin Y_3'$ which means $x \notin Y_3''$, or we have

$$|\{y \in Y_3' : z_p(y_S) > z_p(x_S)\}| \geq (1 - r_p^m)\frac{Q_p}{2} - \frac{Q_p}{4} + q_p + \theta_1'.$$

Therefore, $Y_3' \subseteq Y_3''$ implies

$$|\{y \in Y_3'' : z_p(y_S) > z_p(x_S)\}| \geq (1 - r_p^m)\frac{Q_p}{2} - \frac{Q_p}{4} + q_p + \theta_1''$$

as well. Hence, contract x can not be accepted from Y'' in the third phase as well. So any contract x that is not accepted from Y' in Phase 3, is not accepted from Y'' in Phase 3. Moreover, that guarantees $Y_4' \subseteq Y_4''$.

Phase 4. Let θ_2' and θ_2'' be number of unused seats in Phase 3 when we use Y' and Y'', respectively. As $Y_3' \subseteq Y_3''$, we have $\theta_2' \geq \theta_2''$. If a contract x is not accepted in the fourht phase then we have

$$|\{y \in Y_4' : z_p(y_S) > z_p(x_S)\}| \geq \frac{Q_p}{2} + \theta_2'.$$

Therefore, $Y_4' \subseteq Y_4''$ implies

$$|\{y \in Y_4'' : z_p(y_S) > z_p(x_S)\}| \geq \frac{Q_p}{2} + \theta_2''$$

as well. Hence, contract x can not be accepted from Y'' in the last phase as well. So, any contract x that is not accepted from Y' in Phase 4 is not accepted from Y'' in Phase 4.

A contract x is rejected in set Y' means that x must not be accepted in any phase of the procedure. Above, we showed that for any phase if a contract is not accepted from Y', it can not be accepted from Y''. Therefore, if a contract is rejected from set Y' it must be rejected from set Y''. Hence, contracts are substitutes for any program. ■

Proof. [Proof of Lemma 8] By construction of the choice function $C^{MCF}()$, all contracts of a given student can be rejected from a set only when school reaches full capacity. Hence, the size of the chosen set can never shrink as the set of available contracts grows. ■

Proof. [Proof of Lemma 9] The choice function for any program p satisfies the substitutes condition by Lemma 1 and satisfies the Law of Aggregate Demand by Lemma 2. Hence, Lemma 3 is a corollary of Proposition 1 in Aygun and Sonmez (2013) ■

Proof. [Proof of Proposition 8] To proof this proposition we use a parallel problem where each student s has preference, $\succ_s^{t_s}$, over contracts with t_s and all other contracts are unacceptable for s. The choice function for any program p satisfies the substitutes condition by Lemma 1 and satisfies Irrelevance of Rejected Contracts by Lemma 3. Therefore, as a corollary of Theorem 1 in Aygun and Sonmez (2013), there is a stable allocation for a problem consists of $(\succ_s^{t_s})_{s\in S}$ and $(C_p^{MCF}())_{p\in P}$. Let one of possible stable allocations for the parallel problem be X'. We next show that X' is a stable allocation for our original problem consists of $(\succ_s)_{s\in S}$ and $(C_p^{MCF}())_{p\in P}$.

Assume this is not true. Then there exists a student-program pair (s, p) and a contract x such that

$$x \in X \setminus X', x_S = s \text{ and } x_P = p$$
$$x \in C_p((X' \setminus X'_s) \cup \{x\}) \text{ and } x \succ_s X'_s.$$

Due to privilege monotonicity property of C_p^{MCF}, we can find a contract y such that

$$y \in X \setminus X', y_S = s, y_P = p \text{ and } y_T = t_s$$
$$y \in C_p((X' \setminus X'_s) \cup \{y\}) \text{ and } y \succ_s X'_s$$

which contradicts with the stability of X' for the parallel problem. Hence, X' is a stable allocation for original matching problem consists of $(\succ_s)_{s \in S}$ and $(C_p^{MCF}())_{p \in P}$.

■

Proof. [Proof of Proposition 9] Think about five cases:

Case 1: Let $t_s = (1,1,1)$. Assume that her contract, x', such that $x'_T = t_s$, is rejected. Now, we are going to show that another contract of her, x, such that $x_T < t_s$, must be rejected. For a given program p, let $x' = (s, p, t_s)$ and $x = (s, p, t')$ where $t' < t_s$ and let $Y' = Y \cup \{x'\}$ and $Y'' = Y \cup \{x\}$. First, observe that if her contract x' is rejected from set Y', then her contract is not chosen in any phase. Therefore, θ', θ'_1 and θ'_2 are all zero since she is considered in all phases. Assume that she offers contract x instead of x'.

Phase 1: If $t' < (1,1,1)$ then x is not considered in the first phase. Moreover, since her contract x' is rejected from set Y', there are at least q_s contracts in Y with the privilege vector (1,1,1). Therefore, $\theta' = \theta'' = 0$ and $= (Y'_2 \setminus \{x'\}) \subseteq Y''_2$.

Phase 2: Observe that if x is rejected from set Y', then we have

$$\min\{|\{y \in Y_2' : z_p(y_S) > z_p(s) \text{ and } y_T = (1,1,0)\}|, r_p^m\frac{Q_p}{2} + \theta' - q_p\} +$$
$$\min\{|\{y \in Y_2' : z_p(y_S) > z_p(s) \text{ and } y_T = (1,0,1)\}|, \frac{Q_p}{4} + \theta' - q_p\} +$$
$$|\{y \in Y_2' : z_p(y_S) > z_p(s) \text{ and } y_T = (1,1,1)\}| \geq r_p^m\frac{Q_p}{2} + \frac{Q_p}{4} - 2q_p$$

If $t' = (1,1,0)$, in the second phase we have either

$$|\{y \in Y_2'' : z_p(y_S) > z_p(s) \text{ and } y_T = (1,1,0)\}| \geq r_p^m\frac{Q_p}{2} + \theta'' - q_p \text{ or}$$
$$|\{y \in Y_2'' : z_p(y_S) > z_p(s) \text{ and } y_T = (1,1,0)\}| +$$
$$\min\{|\{y \in Y_2'' : z_p(y_S) > z_p(j) \text{ and } y_T = (1,0,1)\}|, \frac{Q_p}{4} + \theta'' - q_p\} +$$
$$|\{y \in Y_2'' : z_p(y_S) > z_p(s) \text{ and } y_T = (1,1,1)\}| \geq r_p^m\frac{Q_p}{2} + \frac{Q_p}{4} - 2q_p$$

Therefore, x can not be accepted in the second phase. If $t' = (1,0,1)$, in the second phase we have either

$$|\{y \in Y_2'' : z_p(y_S) > z_p(s) \text{ and } y_T = (1,0,1)\}| \geq \frac{Q_p}{4} + \theta'' - q_p \text{ or}$$
$$|\{y \in Y_2'' : z_p(y_S) > z_p(s) \text{ and } y_T = (1,0,1)\}| +$$
$$\min\{|\{y \in Y_2'' : z_p(y_S) > z_p(s) \text{ and } y_T = (1,1,0)\}|, r_p^m\frac{Q_p}{2} + \theta'' - q_p\} +$$
$$|\{y \in Y_2'' : z_p(y_S) > z_p(s) \text{ and } y_T = (1,1,1)\}| \geq r_p^m\frac{Q_p}{2} + \frac{Q_p}{4} + \theta'' - 2q_p$$

Therefore, x can not be accepted in the second phase. If $t' \ngeq (1,1,0)$ or $t' \ngeq (1,0,1)$, x will not be considered in the second phase, therefore it cannot be accepted in this phase. Hence, no other available contract of student s can be chosen in this phase. Also, $\theta_1' = \theta_1'' = 0$ and $(Y_3' \setminus \{x'\}) \subseteq Y_3''$.

Phase 3: Observe that if x is rejected from set Y', then we have

$$|\{y \in Y_3' : z_p(y_S) > z_p(s)\}| \geq (1 - r_p^m)\frac{Q_p}{2} - \frac{Q_p}{4} + q_p$$

If $(1,0,0) \leq t' < (1,1,1)$, in the third phase we have

$$|\{y \in Y_3'' : z_p(y_S) > z_p(s)\}| \geq (1 - r_p^m)\frac{Q_p}{2} - \frac{Q_p}{4} + q_p$$

Therefore, x can not be accepted in the third phase. If $t' \ngeq (1,0,0)$, x will not be considered in the third phase, therefore it cannot be accepted in this phase. Hence, no other available contract of student s can be chosen in this phase. Also $\theta_2' = \theta_2'' = 0$ and $(Y_4' \setminus \{x'\}) \subseteq Y_4''$.

Phase 4: First, observe that if x is rejected from set Y', then we have

$$|\{y \in Y_4' : z_p(y_S) > z_p(s)\}| \geq \frac{Q_p}{2}$$

If $t' < (1,1,1)$, in the fourth phase we have

$$|\{y \in Y_4'' : z_p(y_S) > z_p(s)\}| \geq \frac{Q_p}{2}$$

Therefore, x can not be accepted in the fourth phase. Hence, no other available contract of student s can be chosen.

Case 2: If $t_s = (1,1,0)$ and her contract x' is rejected we can show that x is not chosen in any phase.

Phase 1 and 2: If $t' < (1,1,0)$, then x is not considered in the first two phases. So, it can not be accepted in the these phases. Also $\theta_1' = \theta_1''$ and $(Y_3' \setminus \{x'\}) \subseteq Y_3''$.

Phase 3: As contract x' is rejected from set Y', we have

$$|\{y \in Y_3' : z_p(y_S) > z_p(s)\}| \geq (1 - r_p^m)\frac{Q_p}{2} - \frac{Q_p}{4} + q_p + \theta_1'$$

If $t' \ngeq (1,0,0)$, then x is not considered in this phase, so it can not be accepted in phase 3. If $t' = (1,0,0)$, then in the third phase we have

$$|\{y \in Y_3'' : z_p(y_S) > z_p(s)\}| \geq (1 - r_p^m)\frac{Q_p}{2} - \frac{Q_p}{4} + q_p + \theta_1''$$

Therefore, x can not be accepted in the third phase. Hence, no other available contract of student s is chosen. Also $\theta_2' = \theta_2''$ and $(Y_4' \setminus \{x'\}) \subseteq Y_4''$.

Phase 4: As contract x is rejected from set Y', then we have

$$|\{y \in Y_4' : z_p(y_S) > z_p(s)\}| \geq \frac{Q_p}{2} + \theta_2'$$

If $t' < (1,1,0)$, in the fourth phase we have

$$|\{y \in Y_4'' : z_p(y_S) > z_p(s)\}| \geq \frac{Q_p}{2} + \theta_2''$$

Therefore, x can not be accepted in the fourth phase. Hence, no other available contract of student s is chosen.

Case 3: If $t_s = (1,0,1)$ and her contract x' is rejected we can show that x is not chosen in any phase.

Phase 1 and 2: If $t' < (1,0,1)$, then x is not considered in the first two phases. So, it can not be accepted in the these phases. Also $\theta_1' = \theta_1''$ and $(Y_3' \setminus \{x'\}) \subseteq Y_3''$.

Phase 3: As contract x' is rejected from set Y', we have

$$|\{y \in Y_3' : z_p(y_S) > z_p(s)\}| \geq (1 - r_p^m)\frac{Q_p}{2} - \frac{Q_p}{4} + q_p + \theta_1'$$

If $t' \ngeq (1,0,0)$, then x is not considered in this phase, so it can not be accepted in phase 3. If $t' = (1,0,0)$, then in the third phase we have

$$|\{y \in Y_3'' : z_p(y_S) > z_p(s)\}| \geq (1 - r_p^m)\frac{Q_p}{2} - \frac{Q_p}{4} + q_p + \theta_1''$$

Therefore, x can not be accepted in the third phase. Hence, no other available contract of student s is chosen. Also $\theta'_2 = \theta''_2$ and $(Y'_4 \setminus \{x'\}) \subseteq Y''_4$.

Phase 4: As contract x is rejected from set Y', then we have

$$|\{y \in Y'_4 : z_p(y_S) > z_p(s)\}| \geq \frac{Q_p}{2} + \theta'_2$$

If $t' < (1, 0, 1)$, in the fourth phase we have

$$|\{y \in Y''_4 : z_p(y_S) > z_p(s)\}| \geq \frac{Q_p}{2} + \theta''_2$$

Therefore, x can not be accepted in the fourth phase. Hence, no other available contract of student s is chosen.

Case 4: If $t_s = (1, 0, 0)$ and her contract x' is rejected we can show that x is not chosen in any phase.

Phase 1,2 and 3: If $t' < (1, 0, 0)$, then x is not considered in the first three phases. So, it can not be accepted in the these phases. Also $\theta'_2 = \theta''_2$ and $(Y'_4 \setminus \{x'\}) \subseteq Y''_4$.

Phase 4: As contract x is rejected from set Y', then we have

$$|\{y \in Y'_4 : z_p(y_S) > z_p(s)\}| \geq \frac{Q_p}{2} + \theta'_2$$

If $t' < (1, 0, 1)$, in the fourth phase we have

$$|\{y \in Y''_4 : z_p(y_S) > z_p(s)\}| \geq \frac{Q_p}{2} + \theta''_2$$

Therefore, x can not be accepted in the fourth phase. Hence, no other available contract of student s is chosen.

Case 5: If $t_s \ngeq (1,0,0)$, then x, like x', is only considered in the last phase and can not be chosen since the set of other contracts considered in this phase are identical for Y' and Y''.

Therefore, $(s,p,t_s) \notin Y'$ guarantees $(s,p,t') \notin Y''$, for any $t' < t_s$. Hence, Choice function is privilege monotonic. ■

Proof. [Proof of Proposition 10] For any arbitrary set of contracts Y, owner of any rejected contract x such that $x_T = (1,1,1)$, has lower score than owners of chosen contracts. So, $x \notin C_p^{MCF}(Y)$ and $x_T = (1,1,1) \implies \forall y \in C_p^{MCF}(Y), z_p(y_S) > z_p(x_S)$.

For any rejected contract x such that $x_T = (1,0,1)$, the only possible two types of contracts that is chosen and with lower score than x are contracts with privilege vector $(1,1,1)$ or $(1,1,0)$. But, since $x_T \ngeq (1,1,1)$, $x_T \ngeq (1,1,0)$ and owners of other chosen contracts have higher scores than owner of x, we have $x \notin C_p^{MCF}(Y)$ and $x_T = (1,0,1) \implies \forall y \in C_p^{MCF}(Y), z_p(y_S) > z_p(x_S)$ or $x_T \ngeq y_T \geq (1,0,0)$.

For any rejected contract x such that $x_T = (1,1,0)$, the only possible two types of contracts that is chosen and with lower score than x are contracts with privilege vector $(1,1,1)$ or $(1,0,1)$. But, since $x_T \ngeq (1,1,1)$, $x_T \ngeq (1,0,1)$ and owners of other chosen contracts have higher score than owner of x, we have $x \notin C_p^{MCF}(Y)$ and $x_T = (1,1,0) \implies \forall y \in C_p^{MCF}(Y), z_p(y_S) > z_p(x_S)$ or $x_T \ngeq y_T \geq (1,0,0)$.

For any rejected contract x such that $x_T = (1,0,0)$, the only possible types of contracts that is chosen and with lower score than x are contracts with privilege vector $(1,1,1), (1,1,0)$ or $(1,0,1)$. But, since $x_T \ngeq (1,1,1)$, $x_T \ngeq (1,1,0)$, $x_T \ngeq (1,0,1)$ and owners of other chosen contracts have higher score than owner of x, we have $x \notin C_p^{MCF}(Y)$ and $x_T = (1,0,0) \implies \forall y \in C_p^{MCF}(Y), z_p(y_S) > z_p(x_S)$ or $x_T \ngeq y_T \geq (1,0,0)$.

For any rejected contract such that $x_T \ngeq (1,0,0)$, owners of chosen contracts with privilege vector greater than or equal to $(1,0,0)$ may have lower score than owner of x. Also, owners of other chosen contracts have higher score than owner of x. Therefore, we have $x \notin C_p^{MCF}(Y)$ and $x_T = \ngeq (1,0,0) \implies \forall y \in C_p^{MCF}(Y), z_p(y_S) > z_p(x_S)$ or $x_T \ngeq y_T \geq (1,0,0)$. Hence for any type of contract, $x \notin C_p^{MCF}(Y) \implies \forall y \in C_p^{MCF}(Y), z_p(y_S) > z_p(x_S)$ or $x_T \ngeq y_T \geq (1,0,0)$. ■

Proof. [Proof of Proposition 11] For a given program p and given set of contracts Y, let

$$|\{x \in Y : x_T = (1,1,1)\}| \geq q_p.$$

In the first phase q_p contracts with privilege vector $x_T = (1,1,1)$ will be accepted. In the second phase, a contract will be accepted whenever it is in top $\frac{r_p^m Q_p}{2} - q_p$ among contracts claiming minority and public HS privilege, i.e. $x_T \geq (1,1,0)$, in Y_2. Therefore, in the second phase at least $\frac{r_p^m Q_p}{2} - q_p$ and in total at least $\frac{r_p^m Q_p}{2}$ contracts with $x_T \geq (1,1,0)$ will be accepted, otherwise all contracts with $x_T \geq (1,1,0)$ will be accepted. Hence,

$$|\{x \in C_p(Y) : x_T \geq (1,1,0)\}| \geq \min\{\frac{r_p^m Q_p}{2}, |\{x \in Y : x_T \geq (1,1,0)\}|\}.$$

will be satisfied.

Next, consider contracts with $x_T \geq (1,0,1)$. In the first phase q_p contracts with privilege vector $x_T = (1,1,1)$ will be accepted. In the second phase, a contract will be accepted whenever it is in top $\frac{Q_p}{4} - q_p$ among contracts claiming low-income and public HS privilege, i.e. $x_T \geq (1,0,1)$, in Y_2. Therefore, in the second phase at least $\frac{Q_p}{4} - q_p$ and in total at least $\frac{Q_p}{4}$ contracts with $x_T \geq (1,0,1)$ will be accepted, otherwise all contracts with $x_T \geq (1,0,1)$ will be accepted. Hence,

$$|\{x \in C_p(Y) : x_T \geq (1,0,1)\}| \geq \min\{\frac{Q_p}{4}, |\{x \in Y : x_T \geq (1,0,1)\}|\}.$$

will be satisfied.

Finally, consider contracts with $x_T \geq (1,0,0)$. In the first two phases $\frac{r_p^m Q_p}{2} + \frac{Q_p}{4} - q_p - \theta_1'$ contracts with with privilege vector $x_T > (1,0,0)$, will be accepted. In the third phase, all the contracts with $x_T = (1,0,0)$ and all the tentatively rejected contracts in phase 2 are considered. In this phase, a contract will be accepted whenever it is in top $\frac{Q_p}{4} - \frac{r_p^m Q_p}{2} + q_p$ among contracts with $x_T \geq (1,0,0)$ in Y_3. Therefore, in the third phase at least $\frac{Q_p}{4} - \frac{r_p^m Q_p}{2} + q_p$ and in total at least $\frac{Q_p}{2}$ contracts with $x_T \geq (1,0,0)$ will be accepted, otherwise all contracts with $x_T \geq (1,0,0)$ will be accepted. Hence,

$$|\{x \in C_p(Y) : x_T \geq (1,0,0)\}| \geq \min\{\frac{Q_p}{2}, |\{x \in Y : x_T \geq (1,0,0)\}|\}.$$

will be satisfied. ■

Proof. [Proof of Proposition 12] The contracts are substitutes for any program p by Lemma 1 and choice functions satisfy IRC condition by Lemma 3. Therefore, as a corollary of Theorem 3 in Hatfield and Milgrom (2005) and Theorem 1 in Aygun and Sonmez (2013), SOSM produces a stable allocation for student preferences for a problem consists of $(\succ_s^{t_s})_{s \in S}$ and $(C_p^{MCF}())_{p \in P}$. Moreover, as we showed in the proof of Proposition 1, the stable allocation SOSM produces is also stable for the original problem consists of $(\succ_s^{t_s})_{s\in S}$ and $(C_p^{MCF}())_{p\in P}$. Hence, for any problem, the outcome of SOSM is stable. ■

Proof. [Proof of Proposition 13] Assume that is not true. So, we can find $x, y \in X'$ such that $y_P \succ_{x_S}^* x_P$, $z_{y_P}(y_S) < z_{y_P}(x_S)$ and $x_T > y_T$. Since we have $y_P \succ_{x_S}^* x_P$, there exist a contract x' such that $x' = (x_S, y_P, t_{x_S})$ and $x' \succ_{x_S}^* x$. By the design of cumulative offer algorithm, x' must be offered by x_S and be rejected before the final step K. Therefore, at step K, we have $y, x' \in A_{y_P}(K)$ and $X'_{y_P} = C_{y_P}^{MCF}(A_{y_P}(K))$. Since contracts are substitutes for each program and x' is rejected before the final step K, $x' \notin C_{y_P}^{MCF}(A_{y_P}(K))$ must be true. By fairness condition of choice function

$$x' \notin C_{y_P}^{MCF}(A_{y_P}(K)) \implies z_{y_P}(y_S) > z_{y_P}(x'_S) \text{ or } x_T \nleq y_T$$

a contradiction. Hence ψ^{SOSM}, is fair. ■

Proof. [Proof of Proposition 14] For an arbitrary student s, assume that $\delta' = (t', \succ'_s) \neq (t_s, \succ_s)$. Let her assigned program from $\psi^{SOSM}(\delta', \delta_{-s})$ be p^*. Also, let δ'' be a strategy with privilege vector t' and preference with only contract (s, p^*, t') is acceptable. Since choice functions satisfies substitutes condition by Lemma 1 and Law of Aggregate Demand by Lemma 2, student s gets same assignment from $\psi^{SOSM}(\delta'', \delta_{-s})$. This part is a corollary of Theorem 10 in Hatfield and Milgrom (2005).

Now, let δ''' be a strategy with privilege vector t_s and preference with only (s, p^*, t_s) is acceptable. Due to privilege monotonicity of choice functions, her assignment from $\psi^{SOSM}(\delta''', \delta_{-s})$ must be (s, p^*, t_s).

Finally, since for any given type profile choice function satisfies substitutes condition by Lemma 1 and Law of Aggregate Demand by Lemma 2, we know that students can not manipulate student optimal stable mechanism by submitting different preferences, i.e. $\psi^{SOSM}((t_s, \succ_s), \delta_{-s}) \succeq_s \psi^{SOSM}(\delta''', \delta_{-s})$, by Theorem 11 in Hatfield and Milgrom (2005). So we have;

$$\psi^{SOSM}((t_s, \succ_s), \delta_{-s}) \succeq_s \psi^{SOSM}(\delta''', \delta_{-s}) \succeq_s \psi^{SOSM}(\delta'', \delta_{-s}) \succeq_s \psi^{SOSM}(\delta', \delta_{-s})$$

Therefore for any δ',

$$\psi^{SOSM}(\delta', \delta_{-s}) \not\succ_s \psi^{SOSM}((t_s, \succ_s), \delta_{-s})$$

Hence ψ^{SOSM} is incentive compatible. ∎