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Consolidation of Axi-Symmetric Bodies Subjected to Non-Axi-Symmetric Loading

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CONSOLIDATION OF AXI-SYMMETRIC BODIES SUBJECTED TO NON AXI-SYMMETRIC LOADING

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Synopsis

In many cases the analysis of the consolidation of axi-symmetric bodies subjected to non axi-symmetric loading may be carried out without recourse to a full three-dimensional treatment. Advantage may be taken of the axi-symmetric nature of the body and field quantities such as displacement and pore pressure can be given a Fourier representation. The problem is then reduced from one in three spatial dimensions (r, θ, z) and the time domain to one in two spatial dimensions (r, z) and the time domain. The method is illustrated with two examples; one is a tunnelling problem, the other involves consolidation due to lateral loading of a pile.

CONTENTS

1. INTRODUCTION

The consolidation of a saturated, elastic soil under one-dimensional conditions was investigated by Terzaghi (1943). Biot (1941a, 1941b) later extended the theory to three-dimensional situations and the governing equations he developed are quite complicated. They combine the complexities of an elastic problem coupled with those of a diffusion process. For this reason it has been possible to devise analytic solutions to only the simplest of problems (e.g. Gibson and McNamee, 1957; Mandel, 1953; McNamee and Gibson, 1960) and for more complicated situations it has been necessary to devise numerical techniques.

Several investigators have used a finite element approach (e.g. Booker, 1973; Christain and Boehmer, 1970; Hwang et al., 1971; Sandhu and Wilson, 1969) in which a marching technique is used to solve the finite element equations. In principle, it is possible to employ these numerical techniques for problems in one, two and even three spatial dimensions. In practice, however, solutions to three-dimensional problems can be costly in computer time and often require a large amount of data preparation.

There exists a certain class of problems where the loading may be three-dimensional in nature but the geometry of the consolidating body is axi-symmetric. Some practical situations where these conditions arise include the lateral loading of piles and caissons and some problems in tunnelling. In such cases it may be more economical to take into account the axi-symmetric nature of the body when investigating its three-dimensional deformation and loading. A method for doing so is presented in this paper. The method is illustrated with several example problems.

There have been earlier attempts to analyse the three-dimensional deformations of axi-symmetric elastic bodies (Wilson, 1965; Randolph, 1977) and there has been a recent application of the technique to non-linear problems (Winnicki and Zienkiewicz, 1979), but none of these has included a treatment of time dependent consolidation.

2. GOVERNING EQUATIONS

2.1 Fourier Representation

A cylindrical coordinate system (r, θ, z) is adopted and the symbol t is used to represent time.

The analysis is based on the assumption that field quantities such as displacement and pore pressure can be given a Fourier representation. In general, each of these quantities is a function of (r, θ, z, t) but because of the axi-symmetric nature of these problems it is possible to write

$$u_r\ (r,\theta,z,t) = \sum_{n=0}^{N} U_r^{(n)}\ (r,z,t)\ \cos\ (n\theta + \varepsilon_n) \tag{1a}$$

$$u_\theta\ (r,\theta,z,t) = \sum_{n=0}^{N} U_\theta^{(n)}\ (r,z,t)\ \sin\ (n\theta + \varepsilon_n) \tag{1b}$$

$$u_z\ (r,\theta,z,t) = \sum_{n=0}^{N} U_z^{(n)}\ (r,z,t)\ \cos\ (n\theta + \varepsilon_n) \tag{1c}$$

$$p\ (r,\theta,z,t) = \sum_{n=0}^{N} P^{(n)}\ (r,z,t)\ \cos\ (n\theta + \varepsilon_n) \tag{1d}$$

where (u_r, u_θ, u_z) are the (r, θ, z) components of displacement, respectively, p is the excess pore pressure, $n = 0, 1, 2, \ldots .N$, where N may be finite or infinite and ε_n is used to establish a reference point for the measurement of θ.

It is possible also to express the boundary loading in a form similar to that given in Equation 1. Hence the solution to any general problem thus reduces to one of finding the Fourier coefficients $U_r^{(n)}$ etc.

2.2 Constraints at $r = 0$

At the centreline ($r = 0$) of the axi-symmetric body all field quantities must be single valued. By considering the Cartesian components of these field quantities it may be shown that certain conditions at $r = 0$ need to be met. These conditions are:

$$\text{For } n = 0, \quad U_r^{(o)} = U_\theta^{(o)} = 0 \tag{2a}$$

$$\text{For } n = 1, \quad U_r^{(1)} + U_\theta^{(1)} = 0 \tag{2b}$$

$$P^{(1)} = U_z^{(1)} = 0 \tag{2c}$$

$$\text{For } n > 1, \quad U_r^{(n)} = 0 \tag{2d}$$

$$U_\theta^{(n)} = 0 \tag{2e}$$

$$U_z^{(n)} = 0 \tag{2f}$$

$$P^{(n)} = 0 \tag{2g}$$

2.3 Effective Stress Concept

It is assumed that the total normal stress transmitted across any surface can be thought of as the resultant of an effective stress and a pore pressure. This can be expressed as

$$\underset{\sim}{\sigma} = \underset{\sim}{\sigma}' - p\, \underset{\sim}{i} \tag{3}$$

where $\underset{\sim}{\sigma} = (\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{r\theta}, \sigma_{\theta z}, \sigma_{zr})^T$

$$\underset{\sim}{\sigma}' = (\sigma'_{rr}, \sigma'_{\theta\theta}, \sigma'_{zz}, \sigma_{r\theta}, \sigma_{\theta z}, \sigma_{zr})^T$$

$$\underset{\sim}{i} = (1, 1, 1, 0, 0, 0)^T$$

and tensile stress is taken as positive.

2.4 Stain Components and Hooke's Law

In a cylindrical coordinate system the strain - displacement relations are

$$\begin{aligned}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r} \quad , \quad & \gamma_{r\theta} &= \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \\
\varepsilon_{\theta\theta} &= \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, \quad & \gamma_{\theta z} &= \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta} \\
\varepsilon_{zz} &= \frac{\partial u_z}{\partial z} \quad , \quad & \gamma_{zr} &= \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}
\end{aligned} \tag{4}$$

The increment of effective stress is related to the increment of strain $\underset{\sim}{\varepsilon}$ by Hooke's law, viz.

$$\underset{\sim}{\sigma}' = D\,\underset{\sim}{\varepsilon} \tag{5}$$

For an isotropic elastic material

$$D = \begin{bmatrix} \lambda + 2G & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda + 2G & \lambda & 0 & 0 & 0 \\ & & \lambda + 2G & 0 & 0 & 0 \\ & & & G & 0 & 0 \\ & & & & G & 0 \\ \text{symmetric} & & & & & G \end{bmatrix}$$

λ and G are the Lamé parameters for the soil under fully drained contions,

$$\underset{\sim}{\varepsilon} = (\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{zz}, \gamma_{r\theta}, \gamma_{\theta z}, \gamma_{zr})^T$$

and $\underset{\sim}{\sigma}'$ is given above. However, the analysis is easily extended to cover the case where the soil is transversely isotropic.

2.5 Fluid Flow and Darcy's Law

It is assumed that the movement of pore fluid through the saturated elastic soil is governed by Darcy's law, viz.

$$\underset{\sim}{v} = - \frac{k}{\gamma_w} \underset{\sim}{\nabla} p \tag{6}$$

where $\underset{\sim}{v} = (v_r, v_\theta, v_z)^T$, are the components of the superficial velocity of the pore fluid.

k = isotropic permeability coefficient

γ_w = unit weight of pore fluid

$\underset{\sim}{\nabla} = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}\right)^T$, is the gradient operator.

Equation 6 is valid for an isotropic material but the analysis would also hold for a soil which had different vertical and horizontal permeabilities.

It is also assumed that the pore water is incompressible when compared to the soil mixture and thus the rate of volume decrease of a soil element equals the rate at which water is expelled. This condition is expressed as

$$\underset{\sim}{\nabla}^T \cdot \underset{\sim}{v} = - \frac{\partial \varepsilon_v}{\partial t} \tag{7}$$

where $\varepsilon_v = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}$

2.6 Equilibrium, Boundary Conditions and Initial Condition

At any time during the consolidation process the total stresses are in equilibrium with the loads applied at the boundaries and with any body forces.

The governing equations must be integrated subject to the stress, pore pressure and displacement boundary conditions and also subject to the initial condition.

$$\varepsilon_v = 0 \text{ when } t = 0^+ \tag{8}$$

This last equation follows from the assumption that the pore pressure is incompressible relative to the soil skeleton. Thus there can be no instantaneous volume change even though a load is applied suddenly.

2.7 Virtual Work

Let the soil body occupy a region V bounded by a surface S. Further, let $\underset{\sim}{u}$ and $\underset{\sim}{u} + d\underset{\sim}{u}$ be displacement fields which satisfy the boundary conditions of the problem. Then it is well known that the equilibrium equations and the stress boundary conditions are satisfied if and only if

$$\int_V \{d\underset{\sim}{\varepsilon}^T . \underset{\sim}{\sigma} + d\underset{\sim}{u}^T . \underset{\sim}{F}\} \, dV + \int_S d\underset{\sim}{u}^T . \underset{\sim}{T} \, dS = 0 \tag{8a}$$

where $\underset{\sim}{F}$ = vector body forces

$\underset{\sim}{T}$ = vector of surface tractions,

for all virtual displacement fields $d\underset{\sim}{u}$.

On inserting Hooke's law this equation becomes

$$\int_V \{d\underset{\sim}{\varepsilon}^T . D . d\underset{\sim}{\varepsilon} - p d\varepsilon_v\} dV + \int_V d\underset{\sim}{u}^T . \underset{\sim}{F} . dV + \int_S d\underset{\sim}{u}^T . \underset{\sim}{T} . dS = 0 \qquad (8b)$$

Similarly, let p and p + dp be two sets of pore pressures which satisfy the pore pressure boundary conditions, then Equation 7 is satisfied if and only if

$$\int_V \{\underset{\sim}{\nabla}^T dp . \underset{\sim}{v} - \frac{\partial \varepsilon_v}{\partial t} . dp\} \, dV = 0 \qquad (8c)$$

On inserting Darcy's law this becomes

$$- \int_V \{\underset{\sim}{\nabla}^T dp . \underset{\sim}{k} . \underset{\sim}{\nabla} p + \frac{\partial \varepsilon_v}{\partial t} \, dp\} \, dV = 0 \qquad (8d)$$

where $\underset{\sim}{k} = \left(\frac{k}{\gamma_w}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

It is also possible to incorporate different coefficients for vertical and horizontal permeability into the matrix $\underset{\sim}{k}$.

3. FINITE ELEMENT EQUATIONS

The governing equations for the consolidation of a soil with an elastic skeleton have been set out above - Equations 8b and 8d. An approximate solution of these equations may be obtained by an application of the finite element technique of spatial discretisation.

(a) Suppose that the continuous values of u_r, u_θ, u_z, p can be adequately represented by their values at selected nodes. Then it is also possible, for the axi-symmetric case, to represent the continuous values of the Fourier coefficient $U_r^{(n)}$, $U_\theta^{(n)}$, $U_z^{(n)}$, $P^{(n)}$ by their values at these same nodes, i.e. by $\underset{\sim}{\delta}_r^{(n)}$, $\underset{\sim}{\delta}_\theta^{(n)}$, $\underset{\sim}{\delta}_z^{(n)}$, $\underset{\sim}{q}^{(n)}$.

Hence we may write

$$U_r = \underset{\sim}{N}_r^T . \underset{\sim}{\delta}_r \tag{9a}$$

$$U_\theta = \underset{\sim}{N}_\theta^T . \underset{\sim}{\delta}_\theta \tag{9b}$$

$$U_z = \underset{\sim}{N}_z^T . \underset{\sim}{\delta}_z \tag{9c}$$

$$P = \underset{\sim}{N}_p^T . \underset{\sim}{q} \tag{9d}$$

For convenience the superscript (n) has been dropped in Equations 9 and in the following.

(b) The vector of strain components $\underset{\sim}{\varepsilon}$ may be written in terms of the nodal values of Fourier coefficients $\underset{\sim}{\delta}^T = (\underset{\sim}{\delta}_r, \underset{\sim}{\delta}_\theta, \underset{\sim}{\delta}_z)$, i.e.

$$\underset{\sim}{\varepsilon} = \Lambda\ B\ \underset{\sim}{\delta} \tag{10}$$

where
$$B = \begin{bmatrix} \dfrac{\partial \underset{\sim}{N}_r^T}{\partial r} & 0 & 0 \\ \dfrac{1}{r}\underset{\sim}{N}_r^T & \dfrac{n}{r}\underset{\sim}{N}_\theta^T & 0 \\ 0 & 0 & \dfrac{\partial \underset{\sim}{N}_z^T}{\partial z} \\ -\dfrac{n}{r}\underset{\sim}{N}_r^T & \dfrac{\partial \underset{\sim}{N}_\theta^T}{\partial r} + \dfrac{1}{r}\underset{\sim}{N}_\theta^T & 0 \\ 0 & \dfrac{\partial \underset{\sim}{N}_\theta^T}{\partial z} & -\dfrac{n}{r}N_z^T \\ \dfrac{\partial \underset{\sim}{N}_r^T}{\partial z} & 0 & \dfrac{\partial \underset{\sim}{N}_z^T}{\partial r} \end{bmatrix}$$

$$\Lambda = \text{diag}\ (C,\ C,\ C,\ S,\ C,\ S)$$

$$C = \cos\ (n\theta + \varepsilon_n)$$

and $$S = \sin\ (n\theta + \varepsilon_n)$$

(c) The volume strain may be written as

$$\varepsilon_v = \underset{\sim}{d}^T .\ \underset{\sim}{\delta} \qquad (11)$$

where $\underset{\sim}{d}^T = \underset{\sim}{i}^T\ \Lambda B$

and $\underset{\sim}{i}^T = (1,\ 1,\ 1,\ 0,\ 0,\ 0)$

(d) The vector containing the gradients of excess pore pressure may be written as

$$\underset{\sim}{\nabla p} = \Gamma .\ E .\ \underset{\sim}{q} \qquad (12)$$

where $$E = \left(\frac{\partial \underset{\sim}{N}_p^T}{\partial r},\ -\frac{n}{r}\ \underset{\sim}{N}_p^T,\ \frac{\partial \underset{\sim}{N}_p^T}{\partial z} \right)^T$$

$$\Gamma = \text{diag}\ (C,\ S,\ C)$$

If Equations 9 to 12 are substituted into Equation 8b then it is found that

$$d\underset{\sim}{\delta}^T \{K\underset{\sim}{\delta} - L^T .\ \underset{\sim}{q} - \underset{\sim}{m}\} = 0 \qquad (13a)$$

where the form of K, L, and $\underset{\sim}{m}$ is discussed below. Equation 13a is true for arbitrary variations $d\underset{\sim}{\delta}$ and thus

$$K\underset{\sim}{\delta} - L^T\ \underset{\sim}{q} = \underset{\sim}{m} \qquad (13b)$$

In similar fashion if Equations 9 to 12 are substituted into Equation 8d then it is found that

$$d\underset{\sim}{q}^T \left\{ -\ L\ \frac{d\underset{\sim}{\delta}}{dt} - \Phi\ \underset{\sim}{q} \right\} = 0 \qquad (13c)$$

where Φ is also discussed below. Equation 13c is true for arbitrary variations $d\underset{\sim}{q}$ and thus

$$- L \frac{d\underset{\sim}{\delta}}{dt} - \Phi \underset{\sim}{q} = 0 \tag{13d}$$

For the special case where the displacements and the excess pore pressure can be given a Fourier representation, as in Equations 9, the matrices K, L, Φ and the vector $\underset{\sim}{m}$ have the following forms. For the nth Fourier component

$$K = \iiint (\Lambda.\ B)^T.\ D.(\Lambda\ B)\ r d\theta\ dr\ dz$$

$$= \iint B^T.\ \hat{D}.\ B\ r\ dr\ dz$$

$$\hat{D} = D.\ \underset{\sim}{I}$$

$$\underset{\sim}{I} = (I_c,\ I_c,\ I_c,\ I_s,\ I_c,\ I_s)^T$$

$$I_c = \int_0^{2\pi} \cos^2 (n\theta + \varepsilon_n)\ d\theta \quad = 2\pi\ \cos^2 \varepsilon_n,\ n = 0$$
$$= \pi,\ n = 1,\ 2.$$

$$I_s = \int_0^{2\pi} \sin^2 (n\theta + \varepsilon_n)\ d\theta \quad = 2\pi\ \sin^2 \varepsilon_n,\ n = 0$$
$$= \pi,\ n = 1,\ 2.$$

$$L^T = \iiint \underset{\sim}{d}.\ \underset{\sim}{N}_p^T\ r d\theta\ dr\ dz$$

$$= I_c \iint \underset{\sim}{d}.\ \underset{\sim}{N}_p^T\ r\ dr\ dz$$

$$\Phi = \iiint (\Gamma . \; E)^T . \; \underset{\sim}{k} . \; (\Gamma \; E)^T \; r d\theta \; dr \; dz$$

$$\Lambda = \iint E^T . \; \hat{\underset{\sim}{k}} . \; r dr \; dz$$

$$\hat{\underset{\sim}{k}} = \left(\frac{k}{\gamma_w}\right) . \; \text{diag} \; (I_c, \; I_s, \; I_c) \text{ for an isotropic material}$$

$\underset{\sim}{m}$ = vector of Fourier coefficients for applied nodal force.

4. INTEGRATION OF THE FINITE ELEMENT EQUATIONS

A solution is required to the finite element Equations 13b and 13d.

Suppose that the solution $(\underset{\sim}{\delta}_o, q_o)$ at time t_o is known and it is required to evaluate the solution (δ_1, q_1) at time t_1. Equation 13d can be integrated approximately in the form

$$- L\Delta\underset{\sim}{\delta} - \Phi \; \{\beta\Delta\underset{\sim}{q} + \underset{\sim}{q}_o\} \; \Delta t = 0 \tag{14}$$

where $\Delta\delta = \underset{\sim}{\delta}_1 - \underset{\sim}{\delta}_o$

$\Delta\underset{\sim}{q} = \underset{\sim}{q}_1 - \underset{\sim}{q}_o$

and β defines the particular integration rule used.

Equations 13b and 13d may now be written in the form

$$\begin{bmatrix} K & - L^T \\ - L & - \beta\Delta t\Phi \end{bmatrix} \begin{bmatrix} \Delta\underset{\sim}{\delta} \\ \Delta\underset{\sim}{q} \end{bmatrix} = \begin{bmatrix} \Delta\underset{\sim}{m} \\ \Delta t\Phi\underset{\sim}{q}_o \end{bmatrix} \tag{15}$$

where $\Delta\underset{\sim}{m} = \underset{\sim}{m}_1 - m_o$

A solution for the nodal displacement and pore pressure coefficients $\underset{\sim}{\delta}_r$ etc, may be obtained at a discrete number of times by solving Equations 15 and using a marching process. The whole process is repeated for the n Fourier terms which are of interest and the solution for the nodal displacements and excess pore pressure can then be assembled using Equations 1 and 9. For stability of the marching process $\beta \geqslant \frac{1}{2}$ (Booker and Small, 1975).

There are some important problems where the complete solution can be found using only one Fourier component, i.e. only one value of n. In other problems only a few Fourier terms are necessary for an adequate approximation to the solution. In both cases the present analysis is much more economical than a full three-dimensional finite element analysis of consolidation.

5. EXAMPLES

In order to illustrate the theory presented above, two example problems are considered. The first deals with the excavation of a long tunnel in saturated elastic soil, while the second investigates the behaviour of a pile subjected to lateral loading.

5.1 Tunnel Problem

The problem of determining the displacement, stress and pore pressure changes in an elastic medium, due to the removal of material to form a long tunnel, has been studied by the present authors (Carter and Booker, 1981). For this problem it has been assumed that tunnel cutting occurs instantaneously under conditions of plane strain ($\varepsilon_z = 0$), and that the in situ stress state, before cutting, can be completely described by vertical and horizontal components of total and effective stress. It is also assumed that the tunnel is at large depth so that, to sufficient accuracy, the in situ stress in the region of the tunnel is homogeneous and given by

$\sigma'_v = \sigma'_{vo}$ = vertical effective stress

$\sigma'_H = K_o \sigma'_v$ = horizontal effective stress

$p = p_o$ = pore pressure.

The following set of boundary conditions at the tunnel wall ($r = r_o$) are used to model the removal of material

$$\left.\begin{aligned} \Delta\sigma_{rr} &= \sigma_m + \sigma_d \cos 2\theta \\ \Delta\sigma_{r\theta} &= -\sigma_d \sin 2\theta \\ \Delta p &= -p_o \end{aligned}\right\} \text{ when } r = r_o \tag{16}$$

where $\sigma_m = \frac{1}{2}(\sigma'_v + \sigma'_H) + p_o$

$\sigma_d = \frac{1}{2}(\sigma'_v - \sigma'_H)$

and θ = the anticlockwise angle measured from the horizontal (see Fig. 1a).

For an elastic soil the principle of superposition may be used to separate out the following components of the problem.

<u>Case 1</u>:

$$\left.\begin{aligned} \Delta\sigma_{rr} &= -\sigma_m \\ \Delta\sigma_{r\theta} &= 0 \\ \Delta p &= 0 \end{aligned}\right\} \text{ when } r = r_o \tag{17a}$$

<u>Case II</u>:

$$\left.\begin{aligned} \Delta\sigma_{rr} &= 0 \\ \Delta\sigma_{r\theta} &= 0 \\ \Delta p &= -p_o \end{aligned}\right\} \text{ when } r = r_o \tag{17b}$$

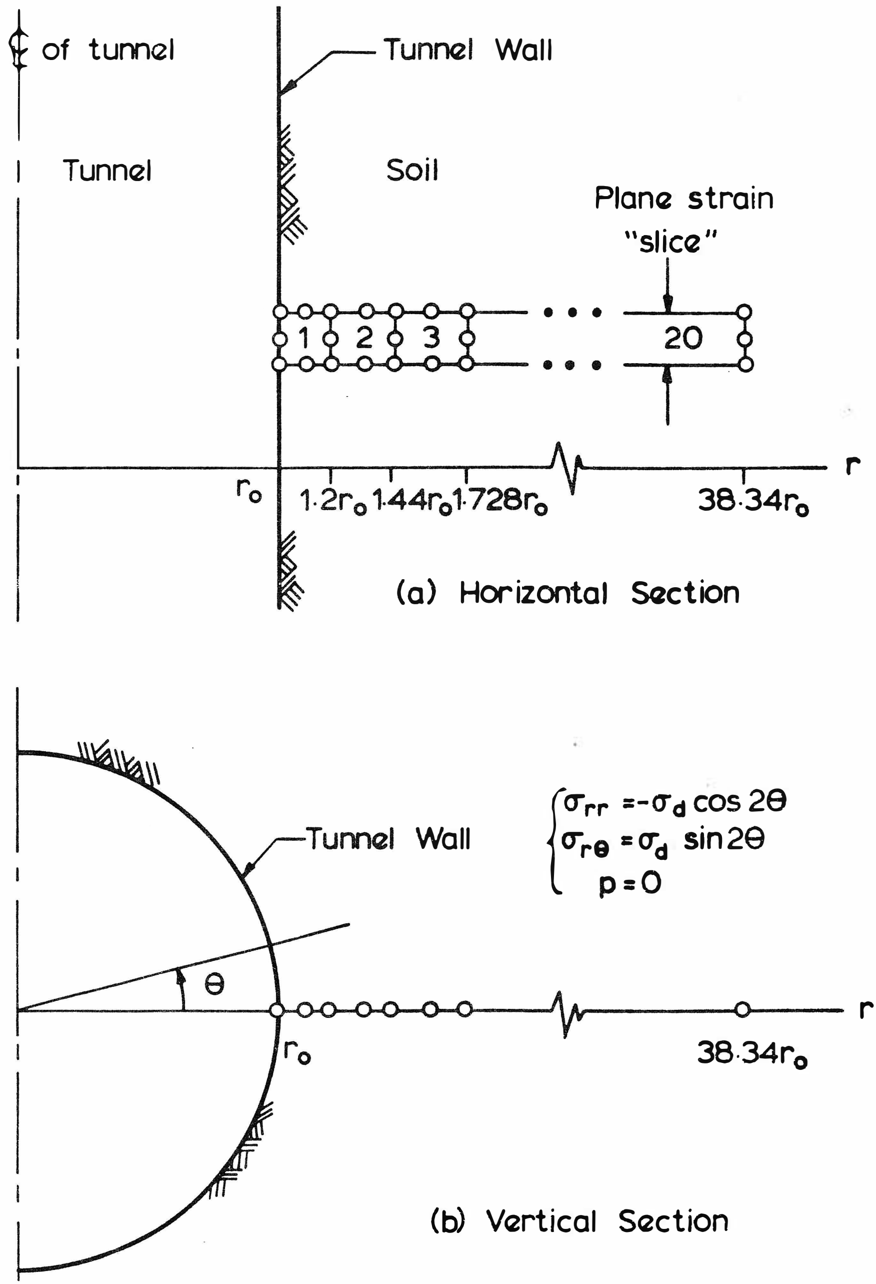

FIGURE 1 : Schematic representation of tunnel problem and finite element mesh

Case III:

$$\left.\begin{aligned} \Delta\sigma_{rr} &= -\ \sigma_d \cos 2\theta \\ \Delta\sigma_{r\theta} &= \sigma_d \sin 2\theta \\ \Delta p &= 0 \end{aligned}\right\} \text{ when } r = r_o \qquad (17c)$$

In what follows only Case III is considered. Physically this component of the overall problem arises only when the in situ stress state is anisotropic, i.e. $\sigma'_V \neq \sigma'_H$. It is only in such circumstances that there will be a variation with θ of the displacements and stress changes.

Case III has been analysed using the finite element technique described above; the mesh used is shown in Fig. 1. Quadrilateral elements with eight nodes were employed. Each node was assigned four degrees of freedom, corresponding to the Fourier coefficients U_r, U_θ, U_z and P. In the general case, the Fourier coefficients are assumed to vary within each element as quadratic functions of the coordinates r and z. In this particular problem, however, there is no variation of the field quantities in the z coordinate direction and, indeed, at all points and at all times $U_z = 0$. It is also obvious that for Case III only one Fourier term, corresponding to n = 2, is required.

Finite element solutions for this problem have been plotted in non-dimensional form in Figs 2-4. Time t is measured from the instant that the material within the tunnel is removed, and it has been non-dimensionalised according to the following expression

$$T = \frac{k}{\gamma_w} .\ 2G .\ \frac{(1 - \nu')}{(1 - 2\nu')} .\ \frac{t}{r_o^2} \qquad (18)$$

Also shown in Figs. 2 to 4 are the solutions for U_r, U_θ and P obtained using the analytical method. In all cases the agreement between the analytic and finite element solutions is good. At any time t, and at any location (r, θ, z) the displacement components and the excess pore pressure may be recovered from

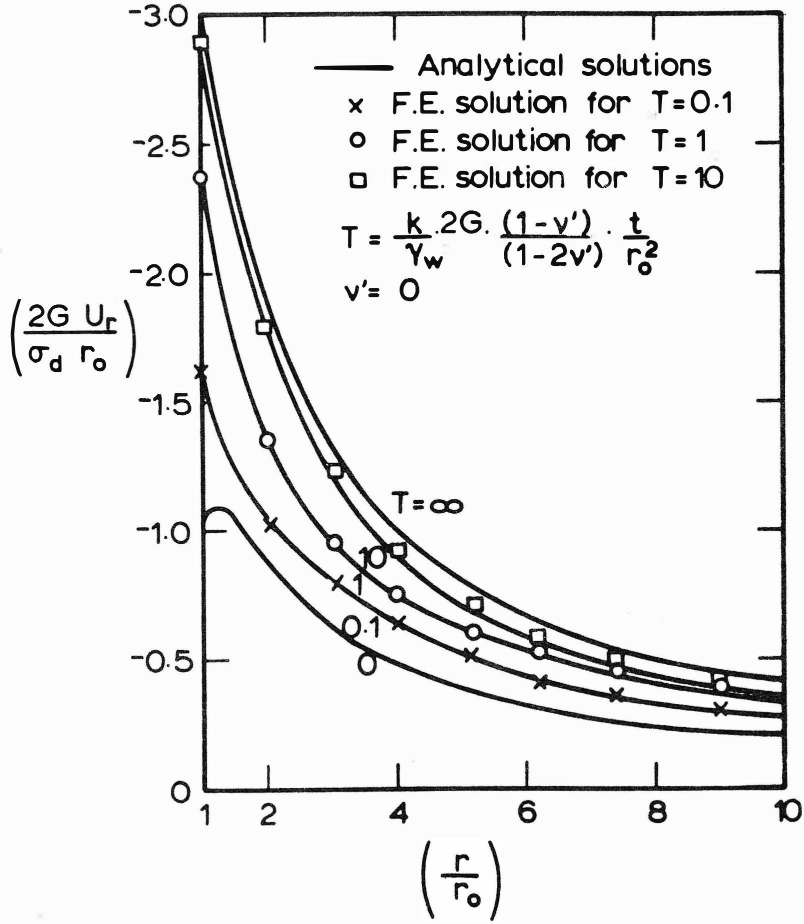

FIGURE 2 : Comparison of analytical and finite element solutions for isochrones of radial desplacement co-efficient

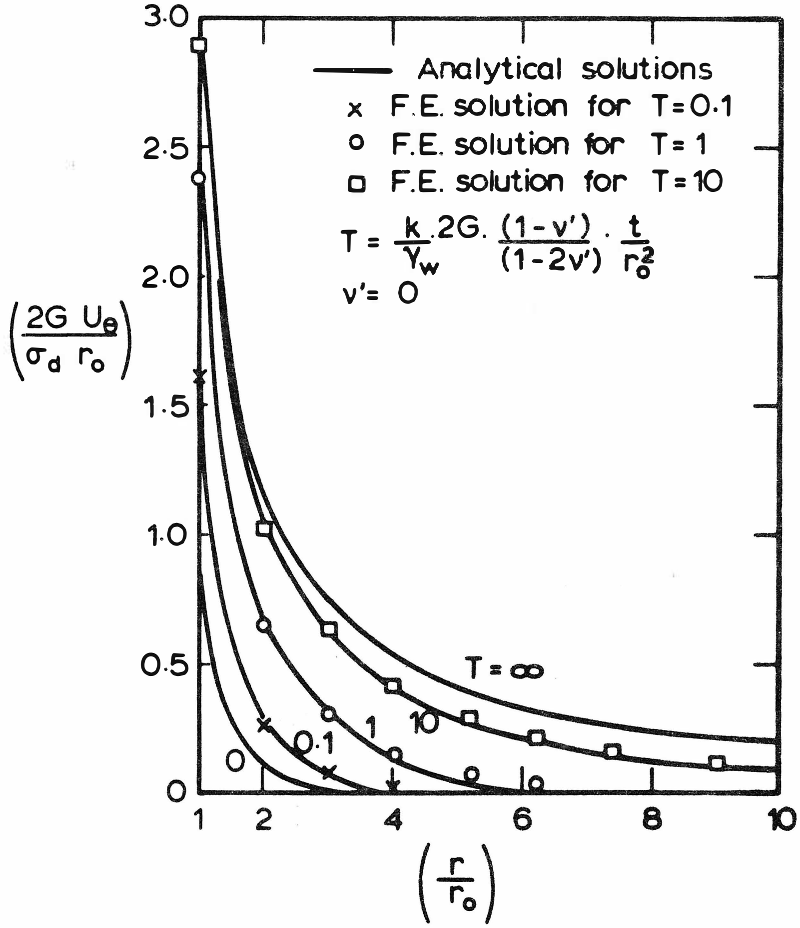

FIGURE 3 : Comparison of analytical and finite element solutions for isochrones of circumferential displacement co-efficient

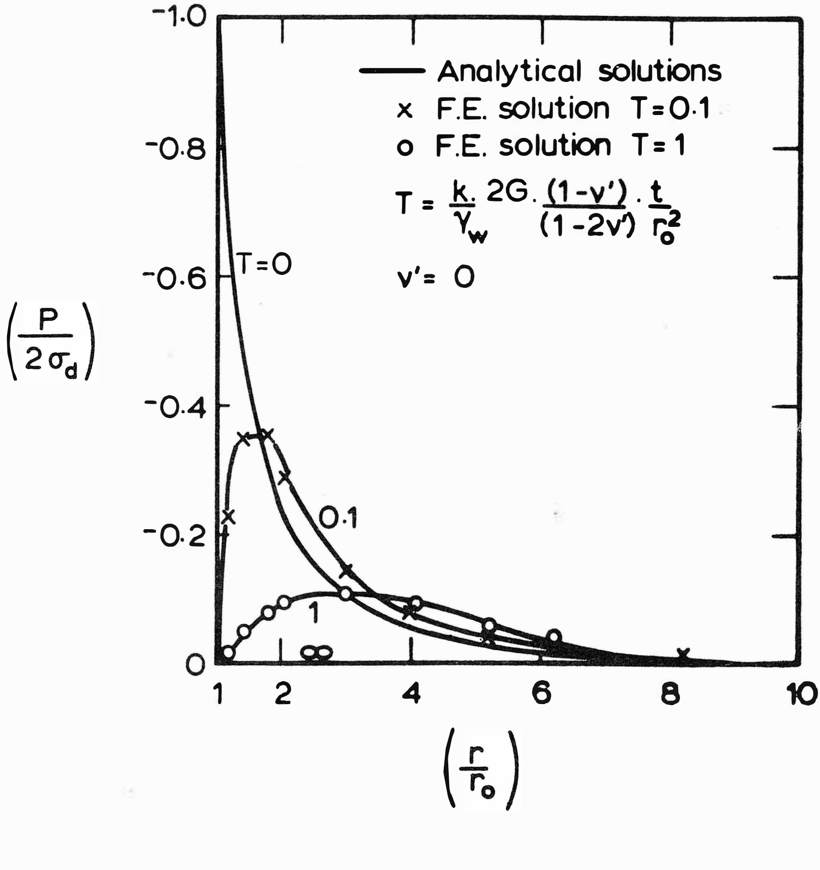

FIGURE 4 : Comparison of analytical and finite element solutions for isochrones of pore pressure co-efficient

the results of Figs. 2 to 4, together with the Fourier expressions of Equations 1 where n = 2.

5.2 Lateral Loading of a Pile

In the second example the time dependent behaviour of a vertical pile, embedded in a saturated elastic soil and subjected to a lateral load applied at the mudline, has been studied. The problem is defined in Fig. 5, which also indicates the Finite Element mesh used in the computations.

For piles subjected to lateral load it is necessary to find only one set of Fourier terms, i.e. $U_r^{(1)}$ etc, corresponding to a value of n = 1. It may also be noted here that for a pile subjected to pure axial load, is also only necessary to find the one set of Fourier terms, i.e. $U_r^{(o)}$ etc, corresponding to n = 0.

In order to be specific, the following values were adopted for the example problem

$$\frac{\ell}{r_o} = 20$$

$$\frac{E_p}{G_s} = 10^3$$

$$\frac{H}{G_s\ r_o^2} = 1$$

$$\nu'_s = 0.4$$

where ℓ = embedded pile length

r_o = radius of pile

E_p = Young's modulus of pile material

G_s = shear modulus of elastic soil

ν_s' = drained Poisson's ratio of elastic soil

H = magnitude of applied lateral load.

Predictions of the lateral displacement of the pile head ρ in the direction of the applied force H are plotted against time in Fig. 6. As before, a non-dimensional time is plotted as the abscissa, using Equation 18 with $\nu' = \nu_s'$. Fig. 6 shows that ρ increases with time from some immediate (undrained) response ρ_u to a final (fully drained) response ρ_d. The values of ρ_u and ρ_d obtained here are in good agreement with finite element solutions obtained, independently, by Randolph (1977).

Fig. 7 shows a plot of contours of the pore pressure coefficient P at a time $T = 10^{-2}$, i.e. 'early', when very little consolidation has occurred. It is clear that the significant values of excess pore pressure generated in the soil by the application of the horizontal load are mostly confined to a region close to the pile and close to the soil surface. Fig. 8 gives an indication of the distribution of excess pore pressure at an 'intermediate' time, T = 3, when some of the original excess pore pressures have dissipated.

6. CONCLUSIONS

A numerical technique has been presented which provides an efficient analysis of the consolidation of axi-symmetric bodies subjected to non-axi-symmetric loading. The technique is suitable for problems in which the field quantities such as displacement, pore pressure and stress and the applied boundary loading can be expressed in the form of a Fourier series in the spatial coordinate θ. The method may prove useful for the analysis of problems involved in tunnelling and piling.

7. ACKNOWLEDGEMENT

The authors would like to acknowledge the value of helpful discussions with Dr M.F. Randolph.

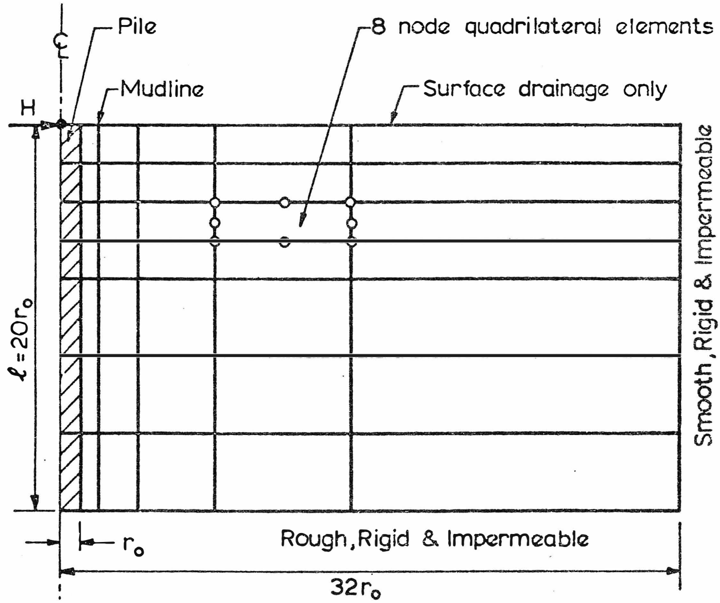

FIGURE 5 : Axi-symmetric finite element mesh for pile problem

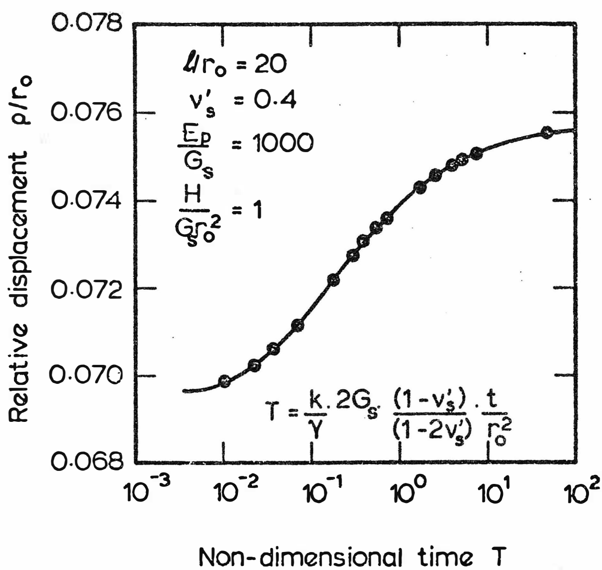

FIGURE 6 : Lateral displacement of pile at mudline versus time

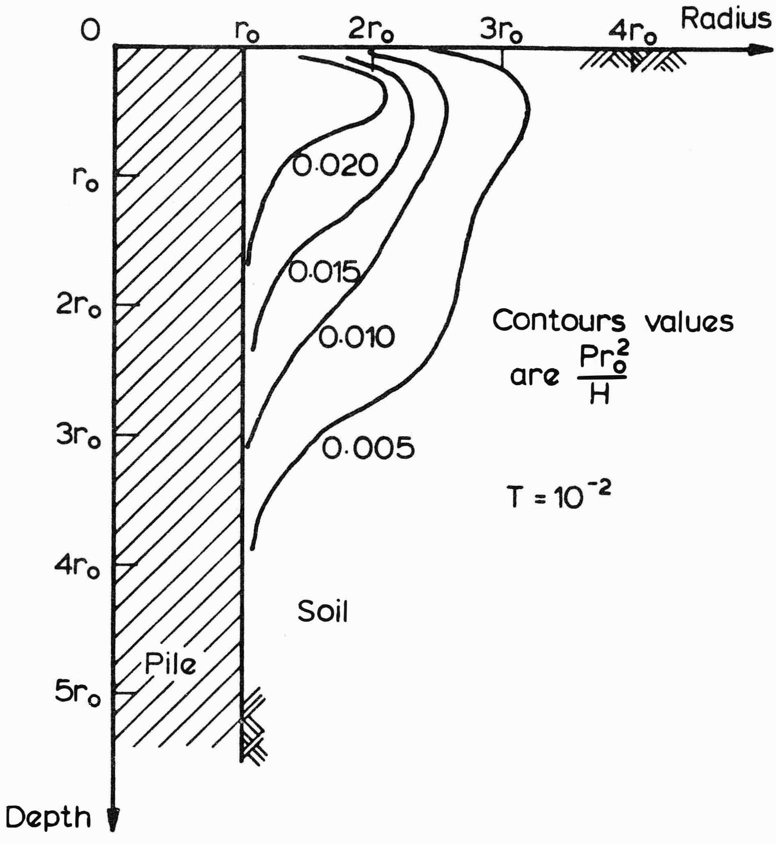

FIGURE 7 : Contours of pore pressure co-efficient P at 'early' time, T = 0.01

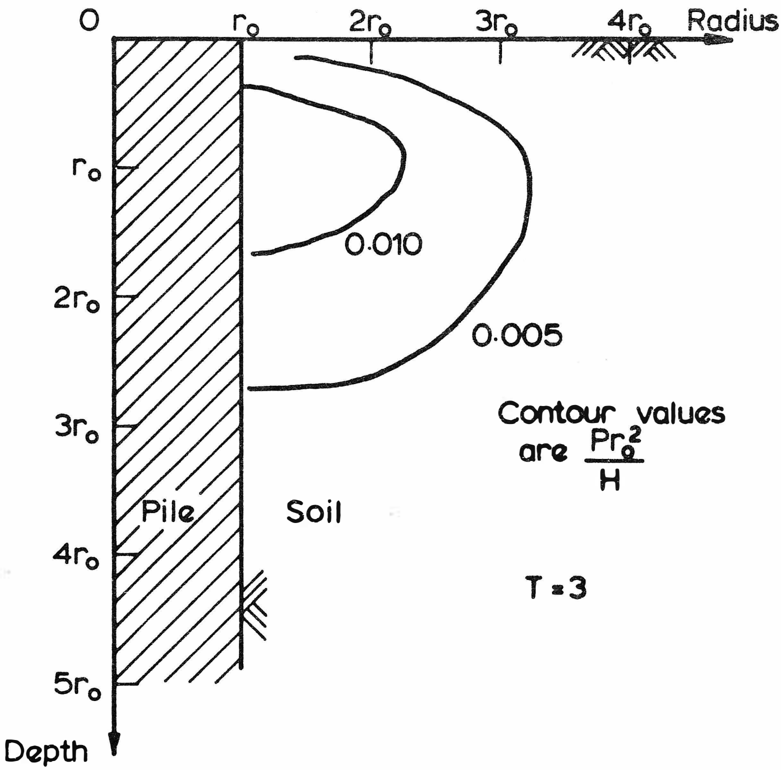

FIGURE 8 : Contours of pore pressure co-efficient P at 'intermediate' time, T = 3

APPENDIX A - NOMENCLATURE

Symbol	Meaning
r, θ, z	cylindrical coordinates
u_r, u_θ, u_z	cylindrical displacement components
p	excess pore pressure
U_r, U_θ, U_z	Fourier coefficients for displacement
P	Fourier coefficient for excess pore pressure
$\underset{\sim}{\delta}_r, \underset{\sim}{\delta}_\theta, \underset{\sim}{\delta}_z$	vector of nodal displacements
$\underset{\sim}{q}$	vector of nodal pore pressures
$\underset{\sim}{N}_r, \underset{\sim}{N}_\theta, \underset{\sim}{N}_z$	vector of shape functions for Fourier displacement coefficients
$\underset{\sim}{N}_p$	vector of shape functions for Fourier pore pressure coefficients
$\underset{\sim}{\varepsilon}$	vector of strain components
$\underset{\sim}{\sigma}, \underset{\sim}{\sigma}'$	vectors of total and effective stress components
σ_v, σ_v'	total and effective vertical stress
σ_m	mean total stress
σ_d	deviator stress
r_o	tunnel radius or pile radius
ℓ	pile length
H	lateral load applied to pile
E_p	Young's modulus of pile
G, G_s	shear modulus of soil
ν', ν_s'	drained Poisson's ratio of soil
t	time
T	non-dimensional time

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