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BUCKLING APPROXIMATIONS FOR INELASTIC BEAMS

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Synopsis

Hand methods of calculating buckling loads of inelastic moment gradient beams are developed. An inelastic parameter, the stiffness modification factor, j, is used to estimate equivalent uniform tangent modulus rigidities for partially yielded simply supported beams. From this is developed a buckling moment equation. For laterally continuous beams, a step-by-step procedure which allows for interaction between adjacent segments is proposed. The structure is reduced to a critical subassemblage of beam segments. The stiffness modification factor is used to quantify segment end interaction and an effective length factor, k, is found for the critical segment. The buckling moment equation is used to estimate the beam capacity. A worked example and comparisons with theoretical and experimental results show that the proposals are accurate and simple to apply.

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1. INTRODUCTION

Elastic and inelastic buckling of single span beams with a variety of support and loading conditions has been widely researched. From well documented solutions a number of approximate inelastic buckling formulae has been developed. Among these are several equations for simply-supported inelastic I-beams under end moments (8,14). In this paper a new inelastic buckling equation for such beams is developed from an approximate buckling theory. An approximate analysis for braced, determinate, inelastic beams with loads at braced points is also presented. The braces are assumed to prevent lateral movement and twisting.

The critical loads of beams continuous in the lateral plane are greatly influenced by interaction between adjacent segments. Nethercot and Trahair (13,14) first proposed approximate analyses for both elastic and inelastic beams, which attempted to account for this interaction. They introduced the concept of the critical segment and adjacent restraining segments. The end restraint offered to the critical segment was allowed for in an effective length factor, k. Their method for elastic laterally continuous beams has been refined by Dux and Kitipornchai (5,7) and has been extended to elastic beam-grids (9). The new method proposed in this paper extends the refined elastic analysis to inelastic laterally continuous beams. It incorporates the new equation for single span beams and includes multiple effective length charts for the critical segment. The method also introduces a more rigorous appraisal of the effects of yielding on segment interaction.

2. BUCKLING OF SIMPLY-SUPPORTED INELASTIC I-BEAMS UNDER END MOMENTS

2.1 Rigidities of Yielded Cross-Sections

Figure 1(a) shows a simply supported I-beam under end moments. While the beam remains elastic its buckling moment is given by (20),

$$M_E = m \frac{\pi}{L} \sqrt{EI_y GJ} \sqrt{1 + K^2} \tag{1}$$

where EI_y is the minor axis bending rigidity, GJ is the torsional rigidity, L is the beam length and m and K are the moment modification factor and beam parameter respectively. The beam parameter is

$$K = \sqrt{\frac{\pi^2}{L^2} \frac{EI_\omega}{GJ}} \tag{2}$$

where EI_ω is the warping rigidity. The moment modification factor is given by

$$m = 1.75 + 1.05\beta + 0.3\beta^2 \ngtr 2.56 \tag{3}$$

In Equation 3 the moment gradient, β, lies in the range $-1.0 \leqslant \beta \leqslant 1.0$ with $\beta = -1.0$ for uniform bending. At elastic buckling the larger major axis moment, M, occurring at end A of the beam in Figure 1 has the value M_E. Throughout the paper end A of a beam or beam segment always has the larger end moment.

As the in-plane loading increases, the combination of residual and applied stress causes partial yielding at more highly loaded cross-sections. Rigidities alter and the elastic buckling moment becomes unattainable.

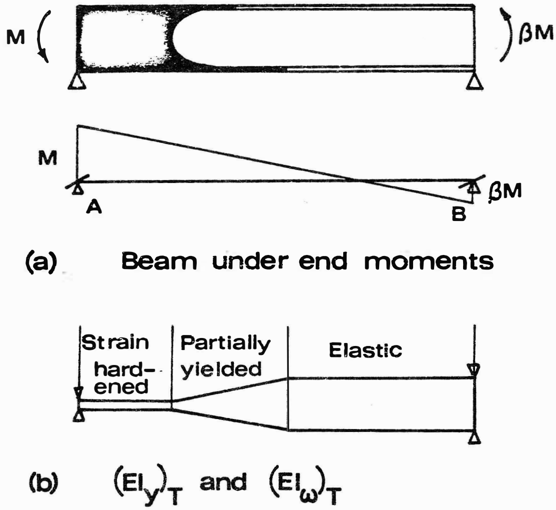


Figure 1: Simply supported inelastic beams

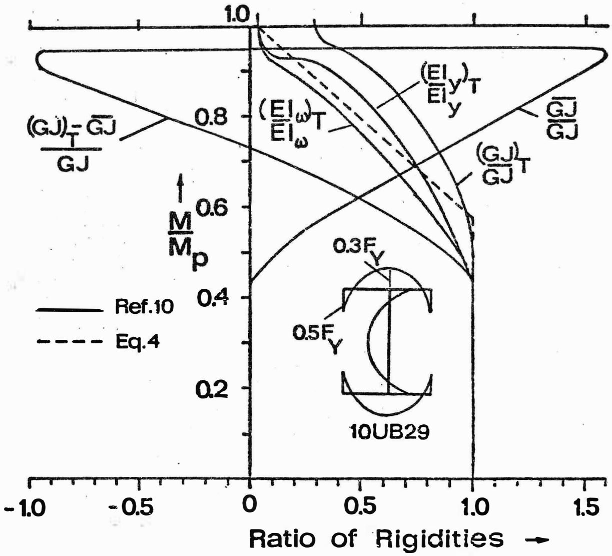


Figure 2: Rigidity variation for 10UB29

The reduced capacity is given by the inelastic buckling moment, M_I.

Many proposals have been made for assessing the rigidity variations of a cross-section under increasing major axis moment. Trahair and Kitipornchai (21) have examined the research in depth and have suggested a theoretical model comprising what they consider the best of the proposals. Their model features the tangent modulus theory and Lay's equation for the strain-hardening shear modulus (11).

Figure 2 shows typical rigidity variations for a 10UB29 cross-section (1) with residual stresses having a fourth order polynomial distribution and a flange tip compressive stress of 0.5 F_Y (10). It is seen that the section remains elastic until M/M_p = 0.45 and is fully strain-hardened when M/M_p = 1.0. The minor axis bending and warping rigidity curves are similar and are reasonably well approximated by the straight line found from Equation 4. The line departs from the elastic property line at an equivalent first yield moment M_{FY} and extends to the strain-hardened ratio of 0.03 at M/M_p = 1.0. The equivalent first yield moment is

$$M_{FY} = 0.9 \left(1 - 0.7 \frac{F_{rc}}{F_Y}\right) M_p \qquad (4)$$

where F_{rc} is the peak flange residual compressive stress. In the absence of other information, F_{rc}, can be estimated from the empirical formula of Young (22).

$$F_{rc} = 165 \left(1 - \frac{A_W}{1.2\ A_F}\right) \quad \text{MPa} \qquad (5)$$

where A_W is the beam web area and A_F is the total flange area. Equation 4 has been developed from a number of comparisons of straight line approximations with theoretical rigidity curves (3,4).

The torsional rigidity curves in Figure 2 differ markedly from the other curves. However, Trahair and Kitipornchai (21) have shown that the assumption $(GJ)_T = GJ$ and the consequent significant change to the effective torsional rigidity, $GJ_{eff} = (GJ)_T - \overline{GJ}$, do not lead to substantial changes in inelastic buckling moments. In the development of the approximate inelastic buckling formula it will be assumed that some freedom can be exercised when approximating the effective torsional rigidity.

2.2 Stiffness Modification Factors

For a given moment magnitude and residual stress distribution, the longitudinal distribution of tangent modulus rigidities can be determined (see Figure 1). The straight line approximation results in linear variations of $(EI_y)_T$ and $(EI_\omega)_T$ as indicated by the equivalent non-uniform beam in Figure 1(b).

Timoshenko and Gere (19) suggest using the minimum tangent modulus rigidities with the elastic buckling formula to obtain lower bound estimates of the inelastic buckling moment, M_I. This leads to gross overestimations except when the in-plane moment distribution is near uniform ($\beta = -1.0$) A number of studies has shown that the longitudinal distribution of yielding is a significant factor (e.g. References 12,14). In moment gradient beams, yielding is most pronounced at the maximum moment end. It is proposed here that approximate uniform rigidities can be found from the increased flexibility at the maximum moment or critical end of the partially yielded beam. Specifically, it is proposed that the equivalent uniform minor axis bending rigidity be found from the critical end response to a uniform minor axis moment. This loading has been chosen to take some account of

Figure 3: Beam under uniform minor axis moment

the yielding distribution. The out-of-plane components of major axis loading are omitted from the analysis.

Figure 3 shows a partially yielded beam under uniform minor axis moment. Through yielding, the minor axis bending stiffness, M_{yA}/θ_{yA}, at the critical end changes from its elastic value of $2EI_y/L$ to become

$$\frac{M_{yA}}{\theta_{yA}} = 2\,\frac{jEI_y}{L} \tag{6}$$

The factor, j, is called the stiffness modification factor (4) and jEI_y is taken as the equivalent uniform bending rigidity.

Similarly, it is assumed that the warping response of the beam at the critical end to a uniform bimoment indicates the effective uniform warping rigidity. If the bimoment is assumed to produce equal flange bending within the plane of each flange, the warping displacements at the critical end can be measured in terms of flange end rotations. The assumption that the distributions of $(EI_\omega)_T$ and $(EI_y)_T$ along the beam are identical leads to an equivalent uniform warping rigidity of jEI_ω.

As discussed by Trahair and Kitipornchai (21), the choice of GJ_{eff} is relatively unimportant for inelastic beams. For simplicity, the equivalent uniform torsional rigidity will be taken as jGJ. When the effective rigidities are substituted for elastic rigidities in Equation 1, the following approximate inelastic buckling equation emerges

$$M_I = m \frac{\pi}{L} \sqrt{jEI_y \; jGJ} \sqrt{1 + \frac{\pi^2}{L^2} \frac{jEI_\omega}{jGJ}} \tag{7}$$

which simplifies to

$$M_I = jM_E \tag{8}$$

where M_I is the inelastic buckling moment.

The lateral bending stiffness at the critical end of a beam with linearly varying rigidities can be calculated using a simple moment-area integration method. The values of j for a number of end moment ratios, β, are plotted against M/M_p in Figure 4 with $F_{rc} = 0.29\ F_Y$. This residual stress matches that used by Nethercot and Trahair (14) for a 254x146UB31 (2) section having a yield stress of 300 MPa. The residual stress is a little less than that predicted by Equation 5.

Figure 4 shows that the reduction in stiffness is most pronounced for uniform moment ($\beta = -1.0$). As β increases so does the value of j at any particular moment level. The inelastic buckling moment equation can be solved directly by plotting the line $M/M_p = jM_E/M_p$ as shown on Figure 4 where the corresponding modified slendernesses, $\sqrt{M_p/M_E}$ are given. The intersection point with the relevant j curve gives $M/M_p = M_I/M_p$, a solution for which the yield pattern is correct.

Good approximations of the stiffness modification factors for other values of F_{rc} can be obtained by superimposing on Figure 4 the new $\beta = -1.0$ line from Equation 4. At each j value, an amount equal to j times the abscissa separation of the $\beta = -1.0$ lines is then either added to or subtracted from the abscissas of the curves to form a new set. For $-1.0 \leq \beta < 0.5$, the full separation should be added or subtracted.

2.3 Inelastic Buckling Formula

Equation 9 approximates the stiffness modification factors from Figure 4 by a set of parallel lines.

$$j = 3.5\left(1 + \frac{\sqrt{1+\beta}}{8} - \frac{M}{M_p}\right) \qquad .03 \leqslant j \leqslant 1.0 \qquad (9)$$

From Figure 4 it is seen that Equation 9 will give inelastic capacities within a few percent of those from the true j curves.

The combination of Equations 8 and 9 gives the following inelastic buckling formula,

$$\frac{M_I}{M_p} = \frac{\left(1 + \frac{\sqrt{1+\beta}}{8}\right)}{\left(\frac{\chi^2}{3.5} + 1\right)} \qquad \frac{.03}{\chi^2} \leqslant \frac{M_I}{M_p} \leqslant \frac{M_E}{M_p} \qquad (10)$$

where χ is the modified slenderness, $\sqrt{M_p/M_E}$. Capacities from Equation 10 are plotted with theoretical capacities (14) and experimental results (3,6) in Figure 5. The formula gives reasonable accuracy over the moment range from first yield to $M_c/M_p = 1.1$. The approximations are conservative and are generally well within 10%. It is suggested that Equation 10 is suitable for use in the design office as a general inelastic buckling formula. If required, formulae corresponding to other F_{rc} values can be derived easily from the appropriate modification factors found by using the transformation described in the previous section.

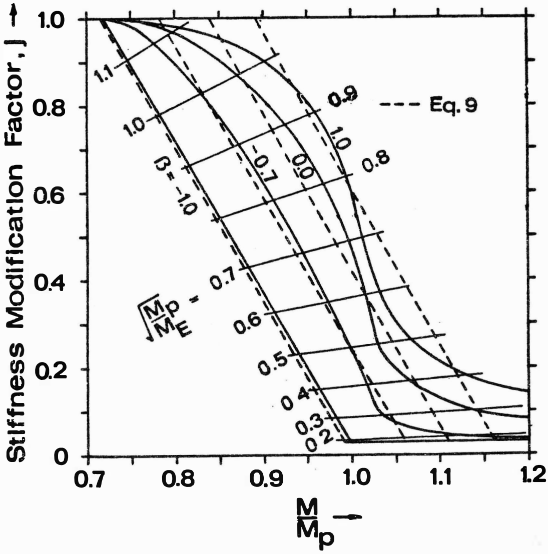


Figure 4: Stiffness modification factors

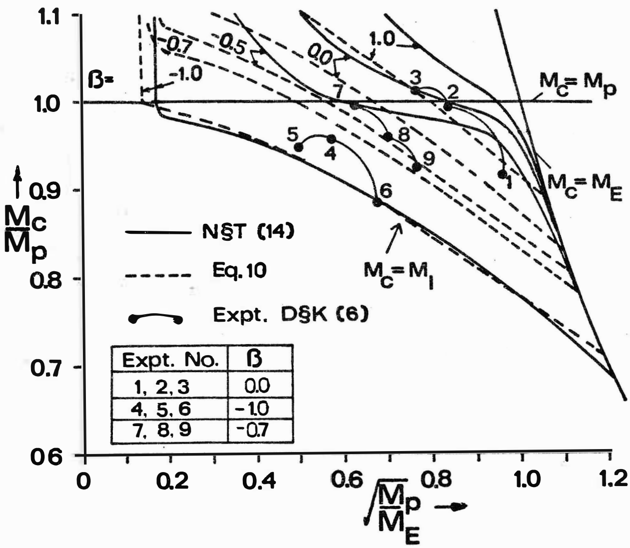


Figure 5: Equation 10 compared with theory and experiment

2.4 Comparison With Other Buckling Formulae

Several buckling formulae which approximate the theoretical capacity curves of Reference (14) have been proposed. The original formulae in References (14) and (15) are

$$\frac{M_I}{M_p} = 0.7 + 0.3 \frac{(1 - 0.7\ M_p/M_E)}{(0.61 - 0.3\ \beta + 0.07\ \beta^2)} \tag{11a}$$

for the range $0.7 \leq M_I/M_p \leq 1.0$ and

$$\frac{M_I}{M_p} = 1.2 - 0.2\ \chi \sqrt{\frac{0.7}{0.39 + 0.3\ \beta - 0.07\ \beta^2}} \tag{11b}$$

when $1.0 \leq M_I/M_p \leq 1.1$. Both Equations 10 and 11 closely predict the experimental results from Reference 6 with Equation 10 being perhaps the easier to apply. Equation 10 also offers some formula flexibility to the designer. Through the stiffness modification factor transformation, other formulae corresponding to a larger or a smaller characteristic imperfection, F_{rc}, can be developed by hand.

In References 8 and 17 the theoretical curves are approximated by series of straight lines. In Reference 8 it is observed that in beams under higher moment gradients, initial yielding is limited to portions near the supports and linear approximations are proposed which depart from the elastic buckling line at

$$\frac{M_I}{M_p} = 0.7 + 0.2\ \sqrt{1 + \beta} \tag{12a}$$

to intersect the line $M_c = M_p$ at a modified slenderness of

$$\chi = 0.17 + 0.55\ \sqrt{1 + \beta} \tag{12b}$$

The proposal in Reference 17 closely matches Equations 12a and b and extends the approximation beyond $M_c = M_p$. The equation to the approximation is

$$\frac{M_I}{M_p} = (1.06 + 1.16 \sqrt{1 + \beta}) - 0.3 \chi \qquad (13)$$

with $M_I/M_p \leqslant M_E/M_p$.

Although Equations 12 and 13 are simple to apply, Reference 3 shows that they tend to be unconservative as beam slenderness increases.

3. BUCKLING OF INELASTIC LATERALLY CONTINUOUS BEAMS

3.1 General

This section extends the method of estimating buckling loads of laterally continuous elastic beams (5,7) to determinate inelastic beams. Dux and Kitipornchai model the structure as a subassemblage comprising a critical segment and restraining segments. The restraining segments provide equal warping and minor axis bending restraints which are expressed in non-dimensional restraint parameters G_A and G_B. The dimensionless elastic buckling moment of a simply supported segment with end restraints can be expressed in the form,

$$\gamma_F = \frac{M_E L}{\sqrt{EI_y GJ}} = m \frac{\pi}{k} \sqrt{1 + \left(\frac{K}{k}\right)^2} \qquad (14)$$

$$= \text{fn}\ (K, \beta,\ G_A, G_B) \qquad (15)$$

The parameters G_A, G_B measure the minor axis bending and warping end stiffnesses of the adjacent segments relative to an initial end stiffness of the critical segment. For any given set of values of G_A, G_B, K and β, the effective length factor, k, may be obtained from charts presented in References (5) and (7). Some of the charts are reproduced in Appendix A. Because yielding alters segment stiffness it is proposed to modify the elastic analysis to include the effects of yielding for inelastic analysis.

3.2 Effect of Yielding

It is proposed that a partially yielded critical segment behaves as a uniform beam with reduced rigidities jEI_y, jEI_ω and jGJ as suggested for single segment beams in the previous section. If end restraints are known, the effective length factor charts together with Equations 10 and 14 can be used to give the inelastic capacity. The end stiffnesses of an inelastic restraining segment can be expressed as

$$\text{Uncoupled stiffness} = n\left[\frac{jEI_{y,\omega}}{L}\right]_R\left(1-\left(\frac{M}{M_I}\right)^2\right)_R \qquad (16)$$

when j and M_I are from Equations 9 and 10 respectively. The subscript, R, refers to a restraining segment and n = 2, 3 or 4 depending on the far end conditions (14, 7). The restraint parameters G_A, G_B at the end of critical segment are

$$G_{A,B} = \frac{2\left[j\,\frac{EI_{y,\omega}}{L}\right]_c}{n\left[j\,\frac{EI_{y,\omega}}{L}\right]_R\left[1-\left(\frac{M}{M_I}\right)^2\right]_R} \qquad (17)$$

where the subscript c refers to the critical segment. The stiffness modification factor, j, in Equation 16 accounts for yielding excluding the enhanced out of plane effect of major axis moment distribution on the end stiffnesses as this is included in the destabilising factor, $1 - (M/M_I)^2$.

The stiffness modification factors for a restraining segment are shown in Figure 6 for the full range of far end support conditions and for various end moment ratios. Straight line approximations (Equation 9) obtained from the previous section are compared. It can be seen that when the critical segment connects to end A of a restraining segment (see left side of Figure 6) the error in using Equation 9 is not great. The largest errors occur when $\beta \to 1.0$ and end B of the restraining segment is fully fixed. However, under these circumstances Figure 6 shows that either the

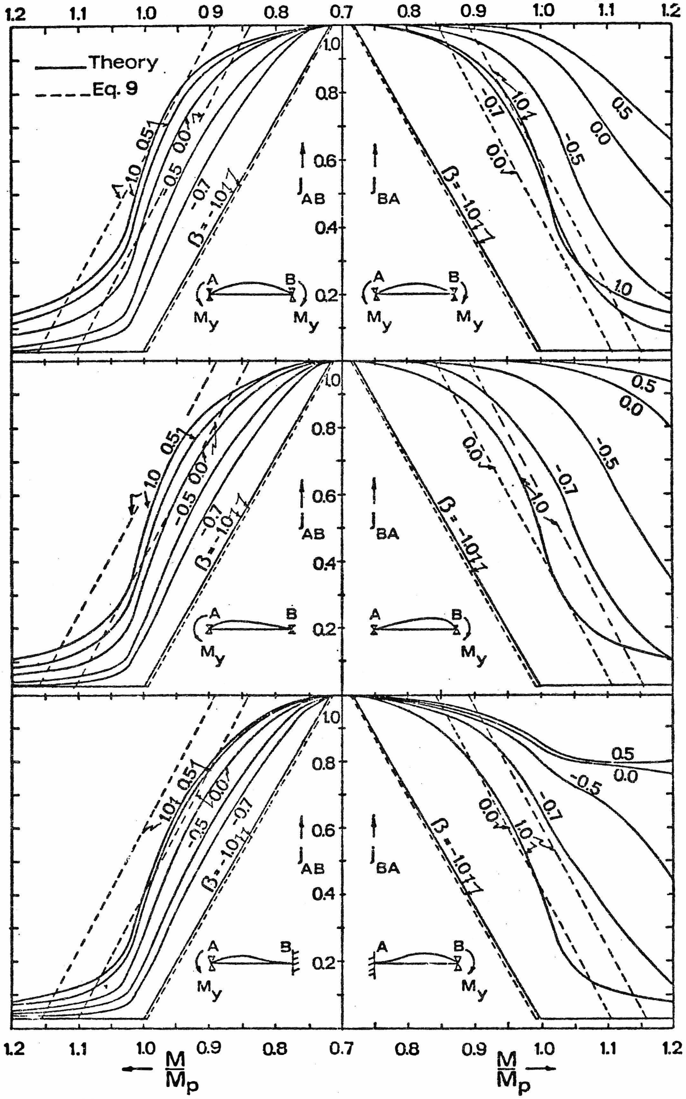


Figure 6: Stiffness modification factor for restraining segments

moments, M/M_p, corresponding to a given factor, j, do not differ significantly or the factors for a given moment are close. In the cyclical approach to solution which is outlined later, these two conditions should adequately contain unconservative errors resulting from the use of Equation 9. The equation generally underestimates modification factors at end B of a restraining segment. The effect is to increase G_B, a conservative change.

The inelastic buckling moment of the restraining segment, M_I, in the destabilising factor $1 - (M/M_I)^2$ is calculated assuming simply supported conditions. Usually the moment in a restraining segment at subassemblage buckling is less than M_I, and, as yielding is less severe, the term in the destabilising factor should be greater than M_I. However, it is not practical to introduce another buckling equation to predict capacities between M_E and M_I. It is recommended that M_I (Equation 10) be used. This reduces restraining segment stiffnesses which is conservative.

3.3 Analysis Procedure and Worked Example

The procedure for inelastic beam analysis is summarised below.

(i) Determine the major axis moment distribution.

(ii) Find β and K for each segment.

(iii) For each segment calculate M_c where M_c is the lesser of M_E and M_I. Find the beam load factor to produce M_c. The segment with the lowest load factor λ_c is the critical segment. The two (at most) adjacent segments have higher load factors, λ_R.

(iv) Assume a trial value of λ_F the load factor at subassemblage buckling and calculate G_A and G_B from Equation 17, noting that

$$\left(\frac{M}{M_I}\right)_R^2 = \left(\frac{\lambda_F}{\lambda_R}\right)^2 \qquad (18)$$

Stiffness modification factors appearing in G_A and G_B correspond to the segment moments under load factor λ_F.

(v) Find the critical segment effective length factor using the appropriate chart (see Appendix A).

(vi) Revise the critical segment buckling moment from Equation 14 (for M_E) and from Equation 10 (for M_I). Obtain the beam load factor $\lambda_{F(new)}$ corresponding to the revised critical segment capacity.

(vii) Compare the new load factor with the value assumed at Step (iv) and repeat Steps (iv) to (vii) until good agreement is obtained.

Iteration (Step (vii)) ensures consistency between assumed and calculated yielding patterns, at least as far as the simple model permits. Often a close guess can be made at Step (iv) and only one or two cycles are required. The procedure is applied to the beam in Figure 7.

Beam properties and loading are those of Experiment 8 described by Dux and Kitipornchai (6). Much of the data necessary for analysis is provided in Table 1.

Step (i) Calculate bending moments (see Figure 7).

Step (ii) Find β and K for each segment (see Table 1).

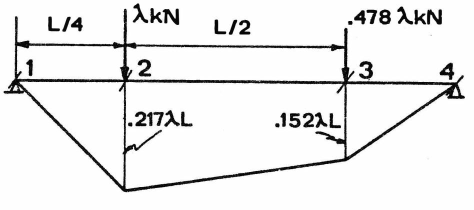


Figure 7: Test beam No. 8 (Reference 6)

TABLE 1. - Analysis Data

Segment	1-2	2-3	3-4
β	0.0	0.7	0.0
K	1.91	0.96	0.91
M_E (kN.m)	717.3	152.8	717.3
M_I (kN.m)	150.7	119.6	150.7
λ	86.8	68.9	123.9

Step (iii) Calculate M_c for each segment (in this instance $M_c = M_I$ for all segments). Segment 2-3 indicates the lowest load factor (λ = 68.9) and hence is the critical segment.

Step (iv) Assume a value of λ_F, say, $\lambda_F = 76$ and calculate G_A and G_B. End 2 of the critical segment is end A. When $\lambda_F = 76$, M/M_p = 0.932 at end A and M/M_p = 0.652 at end B. From Equation 10, $j_{1-2} = 0.68$, $j_{2-3} = 0.48$, and $j_{3-4} = 1.0$. Hence

$$G_A = \frac{2}{3} \times \frac{.25L}{.5L} \times \frac{.48}{.68} \times \frac{1}{1 - \left(\frac{76}{86.8}\right)^2} = 1.01$$

and since segment 3-4 has no yielding,

$$G_B = \frac{2}{3} \times \frac{.25L}{.5L} \times \frac{.48}{1} \quad \frac{1}{1 - \left(\frac{76}{590}\right)^2} = 0.163$$

Step (v) From Appendix A, effective lengths for $\beta = -1.0$ and $\beta = -0.5$, may be interpolated to give

$k = 0.675$.

Step (vi) From Equations 9 and 10, M_E = 284.4 kNm, and M_I = 132.3 kNm. The new load factor is

$$\lambda_{F(new)} = \frac{132.3}{.217 \times 8} = 76.2$$

Step (vii) Since calculated and assumed values of λ_R are close, the average value, λ_F = 76.1 will be taken as the solution. This underestimates the experimental value, λ_F = 78.2, by 2.7%.

The method has been applied to experimental results in Table 2 (6). It can be seen that predictions obtained are within 6% of the experimental values.

TABLE 2. - Comparison with Experiment

Experiment No.	β	Load Factor λ		Percentage Difference
		Experimental (6)	Predicted	
1	0.0	47.1	45.2	- 4.0%
2	0.0	62.6	58.9	- 5.9%
3	0.0	71.0	67.3	- 5.2%
4	- 1.0	89.8	85.2	- 5.1%
5	- 1.0	107.7	105.9	- 1.7%
6	- 1.0	71.6	71.4	- 0.3%
7	- 0.7	92.8	89.1	- 4.0%
8	- 0.7	78.2	76.2	- 2.7%
9	- 0.7	67.2	65.5	- 2.5%

4. COMPARISON WITH ALTERNATIVE APPROXIMATE METHODS

The simplest approximate analysis ignores lateral continuity. This is consistent with the general lower bound approach of Salvadori (18) which neglects interaction between segments. Inelastic buckling moments, M_I, in Table 1 indicate a lower bound buckling load factor for the beam in Figure 7 of λ_F = 68.9. This underestimates the experimental value by 12%. This approach is often overconservative particularly as slendernesses increase (see Figures 8 to 11). However, yielding reduces the differences in capacity between beams under different moment gradients and the tendency in laterally continuous beams is for segment interaction to reduce as yielding becomes more extensive. The accuracy of the lower bound solution is therefore likely to improve.

The approximate method developed in previous sections follows closely in concept that of Nethercot and Trahair (14,16). There are several differences some of which are similar to those between elastic analyses discussed in References 5 and 7. Nethercot and Trahair use a destabilising factor of $(1 - M/M_I)$ for restraining segments and take effective length factors, k, from the β = - 1.0 chart for all critical segments. Iteration is not suggested.

Additional differences are found in the inelastic moment equations and in the estimation of yielding effects. The moment equations have been discussed in Section 2 where those of Nethercot and Trahair appear as Equations 12a and 12b. Nethercot and Trahair (14) propose that yielding be allowed for by multiplying the segment end stiffnesses by a yielding factor, F, where

$$F = \frac{M_c}{M_E} \qquad (19)$$

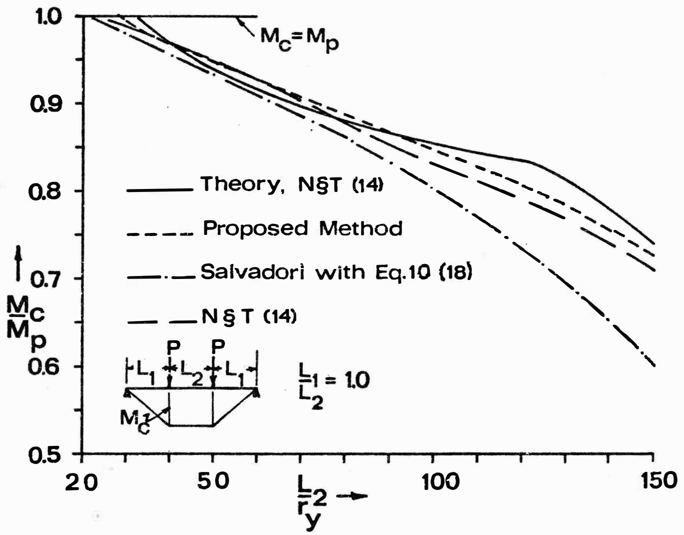


Figure 8: Comparison, with critical segment β = - 1.0

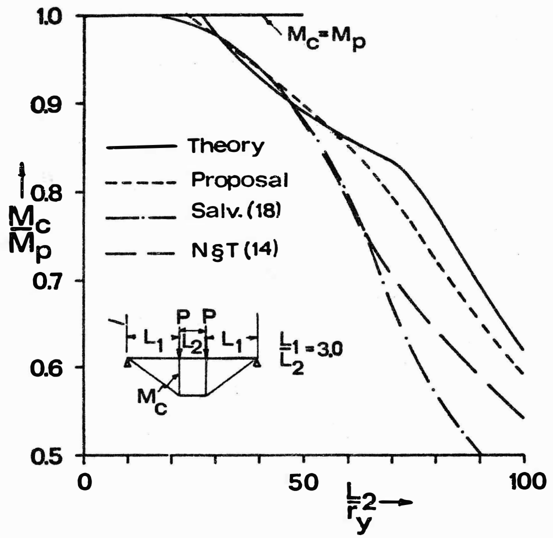


Figure 9: Comparison, with critical segment β = 0

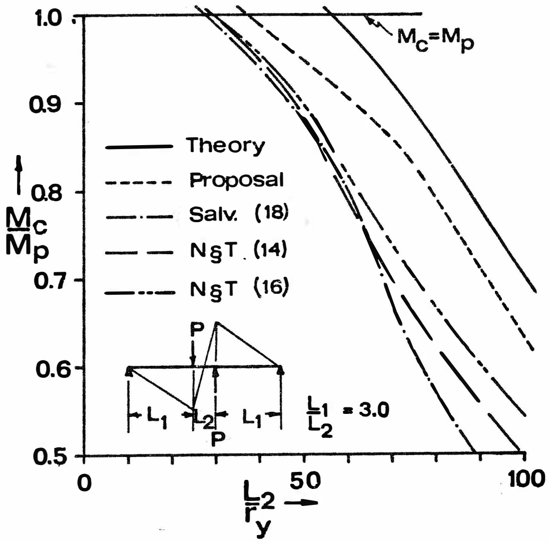


Figure 10: Comparison, with critical segment β = 0

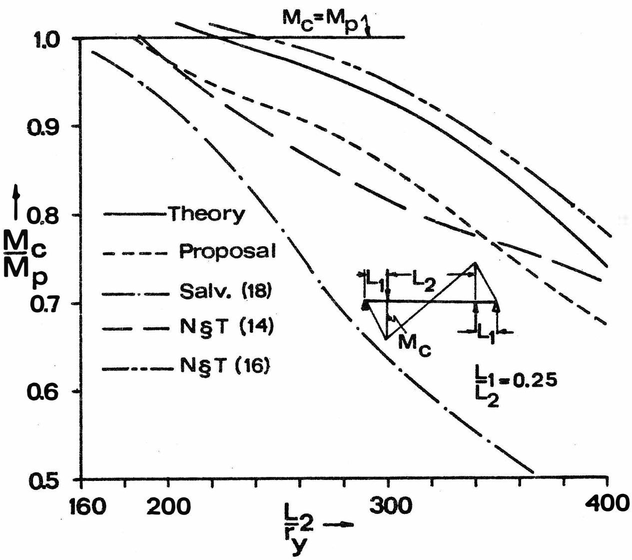


Figure 11: Comparison, with critical segment β = + 1.0

In Equation 19, M_c is the smaller of the inelastic and the elastic buckling moments of the segment when simply supported. The simple form of the yielding factor can be compared with Equation 9 which gives the equivalent, the stiffness modification factor, j. However, the simplicity of Equation 19 is at the expense of accuracy. The equation predicts some reduction in stiffness except for slender segments for which $M_c = M_E$. If $M_c = M_I$, the yielding factor, F, is less than unity whether or not the segment carries moments sufficient to cause yielding. Furthermore, the factor is often smaller for a restraining segment than for the critical segment. The elastic buckling moment, M_E, of a restraining segment tends to be relatively large. As discussed earlier in this section, inelastic moment differences tend to be less than differences in elastic buckling moments. The combination of these two features through Equation 19 can lead to excessive reductions in restraining segment end stiffnesses. In Figures 8 to 11 Nethercot and Trahair's predictions (14) vary from being within a few percent of theoretical capacities to underestimating these capacities by up to 30%. Because of this inconsistency they have revised the yielding factor, F, for restraining segments to (16)

$$F = \frac{M_c}{M_{ER}} \cdot \frac{\lambda_c}{\lambda_R} \tag{20}$$

where

$$M_{ER} = \frac{0.21\,M_p}{0.3 - \left(\frac{M_c\lambda_c}{M_p\lambda_R} - 0.7\right)\left(0.61 - .3\beta + .97\beta^2\right)} \tag{21}$$

The quantity, M_{ER}, is the elastic buckling moment of a segment with $M_I = M_c\lambda_c/\lambda_R$. Equation 21 is derived by making this substitution in Equation 12a. If $M_c\lambda_c/\lambda_R\,M_p > 1.0$ substitution into Equation 12b would be required for the appropriate expression. This empirical revision eliminates the possibility of F being less than unity in elastic segments as permitted by Equation 19. Equations 20 and 21 give F = 1.0 where $M_c\lambda_c/\lambda_R \leqslant 0.7\ M_p$. Figures 10 and 11 show that the new factor results in higher restraining segment stiffnesses and improved accuracy.

In Figures 8 to 11 the beam analysed is taken to be a 254x146UB31 (2) with a yield stress of 280 MPa. Unfortunately Reference (14), from which the Finite Element results in Figures 8 to 11 have been taken, does not identify the beam. However a 254 x 146UB31 with a yield stress of 300 MPa was used in that reference to obtain capacity curves for simply supported moment gradient beams. It has been assumed that a similar section was used for the other theoretical analyses. A yield stress of 280 MPa was needed in order to reproduce the approximate solutions given in Refererence 14.

It can be seen that results using the proposed method (see Figures 8 to 11) are consistently accurate. The largest discrepancy occurs in Figure 11 (critical segment $\beta \doteqdot +1.0$) where theory is underestimated by around 8%. This accuracy is consistent with that of Equation 10 which is in error by a similar amount as β approaches + 1.0. As the critical segment moment gradient reduces, the approximate solutions improve. This trend can be seen also in Table 2. The accuracy of the new proposal supports the use of multiple effective length factor charts and the attempt to account more rigorously for the effects of yielding through the stiffness modification factor, j.

The alternative methods from References 14, 16 and 18 are less predictable as can be seen from the figures. Of these three, the method of Reference 16 seems preferable. However, the equations involved in its application are more complicated than Equations 9 and 10 of the new proposal. Furthermore, the yield factor, F, is of less obvious origin than is the stiffness modification factor.

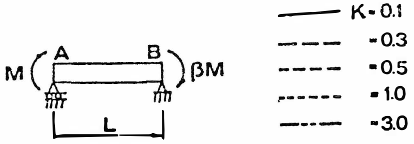
5. CONCLUSIONS

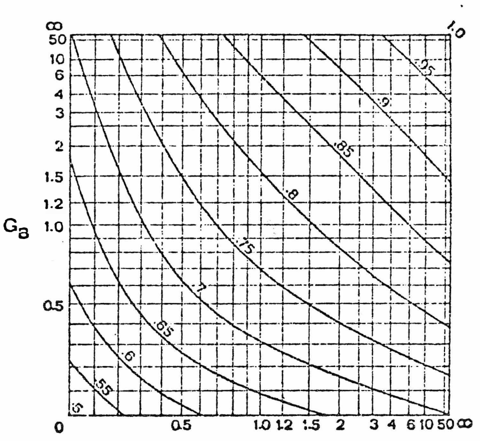
An inelastic buckling equation for simply supported beams under end moments has been developed. The stiffness modification factor, j, is introduced and used to estimate the effects of yielding, thereby to provide equivalent reduced uniform rigidities for use with the elastic buckling equation. The resulting inelastic buckling equation is flexible, simple to apply and is of good accuracy. A hand method is suggested for altering factors, j, for varying levels of residual stresses.

The analysis method for determinate inelastic laterally continuous beams is an extension of the elastic beam analysis previously proposed by the authors (4,6). The critical segment is assumed to have reduced uniform rigidities, the change being determined by a stiffness modification factor. Stiffness modification factors and a revised destabilisation factor are used to estimate restraining segment end stiffness reductions due to yielding. Inelastic restraint parameters G_A, G_B are used with a range of effective length factor charts to obtain an effective length factor for the critical segment.

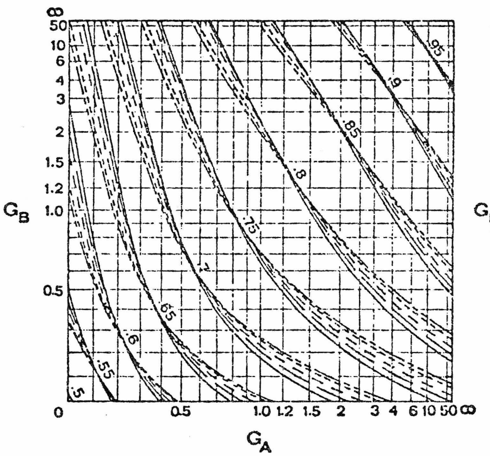
The proposed method predicts the experimental results from Reference 5 to an average accuracy of - 3.5%, the largest underestimation being - 5.9% (see Table 2). Predictions are compared against theory with those of other approximate methods. Only the new method is consistently accurate to the same order of accuracy as its simple inelastic beam equation. The method is easy to apply and is more rigorous in its modelling of the effects of yielding than are the alternatives.

APPENDIX A - EFFECTIVE LENGTH FACTOR CHARTS

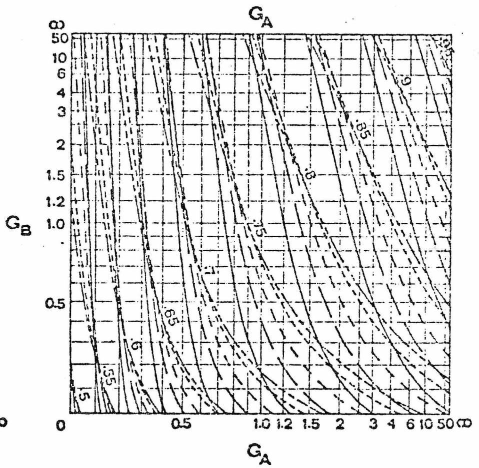




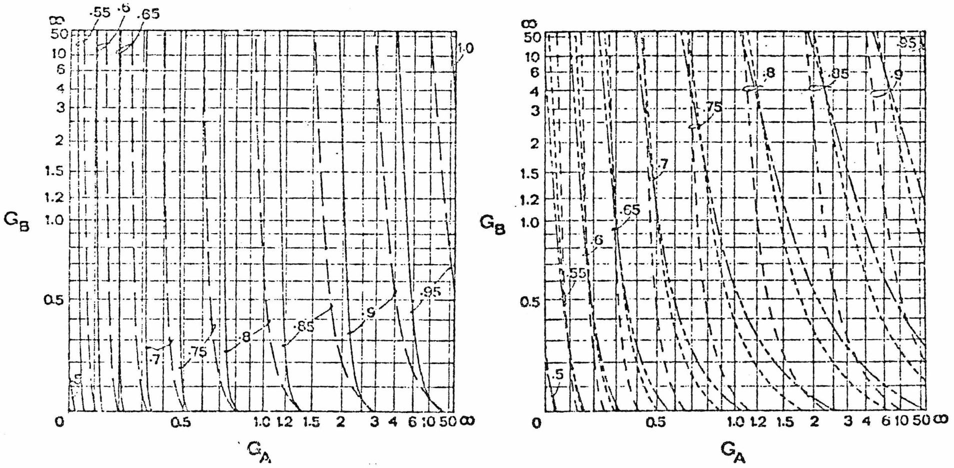
$\beta = -1.0$



$\beta = -.5$

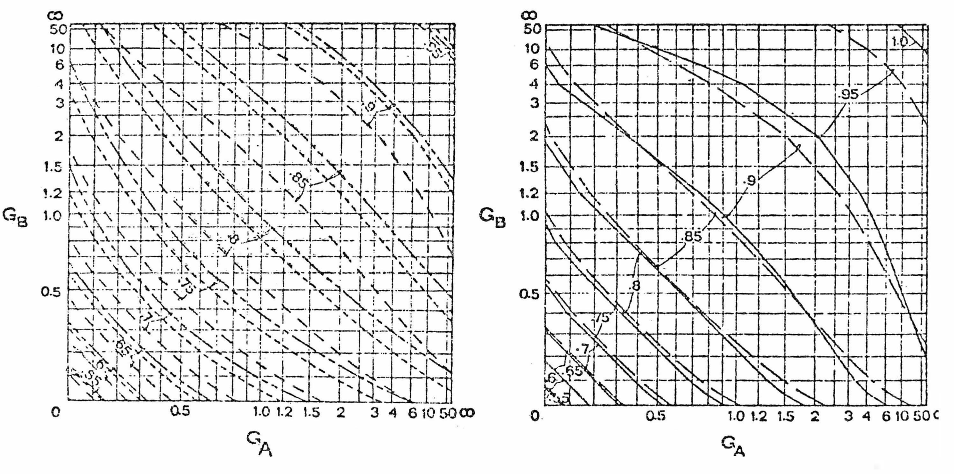


$\beta = 0.0$



.55 .6 .65
1.0
.7 .75 .8 .85 .9 .95
∞ 50 10 6 4 3 2 1.5 1.2 1.0 0.5
G_B
0 0.5 1.0 1.2 1.5 2 3 4 6 10 50 ∞
G_A
.5 .55 .6 .65 .7 .75 .8 .85 .9 .95

β = +.5



.9 .85 .8 .75 .7 .65 .6 .55
∞ 50 10 6 4 3 2 1.5 1.2 1.0 0.5
G_B
0 0.5 1.0 1.2 1.5 2 3 4 6 10 50 ∞
G_A
1.0 .95 .9 .85 .8 .75 .7 .65 .6 .55

β = +1.0

APPENDIX B - NOMENCLATURE

The following symbols are used in this paper:

A = beam end with larger axis end moment; or subscript referring to end A of beam

A_F = total flange area of beam

A_W = web area of beam

B = beam end with smaller major axis end moment or subscript referring to end B of beam

c = subscript referring to critical segment

E = Young's modulus of elasticity

EI_y = minor axis bending rigidity

$(EI_y)_T$ = tangent modulus minor axis bending rigidity

EI_ω = warping rigidity

$(EI_\omega)_T$ = tangent modulus warping rigidity

F = yielding factor

F_{rc} = compressive residual stress at flange tip

F_Y = yield stress

G = shear modulus of elasticity

G_A, G_B = minor axis bending and warping end restraint parameters at ends A and B

GJ = St Venant torsional rigidity

GJ_{eff} = effective torsional rigidity = $(GJ)_T - \overline{GJ}$

$\overline{GJ}$ = reduction in torsional rigidity due to Wagner effect

$(GJ)_T$ = tangent modulus torsional rigidity

I_y = minor axis second moment of area

I_ω = warping section constant

J = St Venant torsion constant

j = stiffness modification factors

j_{AB} = stiffness modification factor at end A

j_{BA} = stiffness modification factor at end B

K = beam parameter = $\sqrt{\pi^2 EI_\omega / GJL^2}$

k = effective length factors

L = length of beam or of beam segment

M = major axis moment

M_c = elastic or inelastic buckling moment

M_E = elastic buckling moment

M_{FY} = equivalent first yield moment

M_I = inelastic buckling moment

M_p = plastic moment

M_y = minor axis moment

M_{yA}, M_{yB} = minor axis moments at ends A and B

m = moment modification factors

n = end stiffness factor commonly 2, 3 or 4

R = subscript referring to restraining segment

β = ratio of major axis end moments

γ_F = dimensionless elastic buckling moment of restrained critical segment or substructure

θ_{yA} = minor axis end rotation at A

λ = load factor

λ_R = beam load factor indicated by adjacent segment

λ_c = beam load factor indicated by unrestrained critical segment

$\lambda_F, \lambda_{F(new)}$ = beam load factor indicated by restrained critical segment or subassemblage

χ = modified slenderness = $\sqrt{M_p/M_E}$

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