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THE FINITE INTEGRAL METHOD IN DYNAMIC ANALYSIS: A REAPPRAISAL

by

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Synopsis

This paper makes further investigations into the Finite Integral Method (F.I.M.) as a means of solving the equilibrium equations which arise in the dynamic analysis of structures. Its purpose is to classify, comment upon and present an improvement of the method with special reference to its place in relation to some other, better known, Direct Integration schemes. The paper relies heavily on Reference 2 for its information regarding alternative schemes. The presentation is designed for people, like the writer, who do not have a background of experience in numerical methods of analysis but who have a curiosity about where their particular work "fits in".

CONTENTS

1. INTRODUCTION

When first presented by Brown and Trahair (3) the F.I.M. was described in the context of static structural analysis and, more particularly, in the solution of buckling problems. These problems are generally not of the "initial value" kind and require a numerical integration scheme which extends, so to speak, "throughout the length of the problem". The method appears to have been used successfully in several such applications.

A feature of the F.I.M. so presented is the use of the *same* integration operator when moving from second derivatives to first derivatives, and from first derivatives to basic unknowns. The essence of the technique is the assumption, firstly, that *curvatures* be considered to vary parabolically with respect to position over two integration steps. A pair of equations then follows giving the *gradients* at discrete positions x_1 and x_2, say, in terms of the curvatures at x_0, x_1 and x_2. In general $x_1 = x_0 + h$ and $x_2 = x_0 + 2h$ where h is the integration step size.

Thereafter it is again assumed (and inconsistently) that *displacements* may be obtained from gradients on the basis that the *gradients vary also parabolically* with respect to position over the two integration steps.

Such an assumption is convenient and provides elegant operator matrices. It is seen, Reference 3, that the integration may be extended over any number of steps by simple addition of the "one step" and "two step" results. Further, because of the assumption regarding similarity of operators, the double integration necessary to proceed from curvatures to displacements appears as a simple matrix product of the individual operators.

A second feature of the method is its use of the operators to write, in the case of a buckling or deflection analysis, the second order differential equation in displacements in terms of the *curvatures* only. The resulting set of linear algebraic equations is solved simultaneously to give curvature values at each node in the discretised system. Back-substitution into the operator relationships gives displacements and, if necessary, gradients. The solution procedure is adjusted to account for known boundary conditions which might in general be specified at any of the nodes of the system.

In Reference 7, O'Connor et al. report the use of the F.I.M. in the context of a small but non-linear *dynamic* analysis. The writer has summarised the application of the method to other dynamics problems in Reference 8.

In each of these references it has been recognised that an essential simplifying feature of the conventional dynamics problem is that it is an "initial value" problem. It is *not* necessary to solve a full set of simultaneous equations covering the complete (time) domain of the response. Instead, one can proceed in "two-step jumps" establishing accelerations velocities and displacements at the "present" time, t_0 say, and solving for unknown accelerations etc at "future" times $t_1 = t_0 + h$ and $t_2 = t_0 + 2h$. The solution procedure then "marches forward" two time-steps and repeats itself.

With this exception, the method has been used precisely as originally described i.e. using identical operators to move from accelerations to velocities and from velocities to displacements. Solution of the discretised governing equations has been in terms of *accelerations* with back-substitution to give velocities and displacements.

The F.I.M. need not be "classified" in order to use it successfully. However the writer has felt some concern that it *should* be possible to do this in relation to at least some of the many better known, and more frequently used, Direct Integration schemes. What follows is an achievement of that objective, which doubtless might have been obvious to those who possess backgrounds in numerical analysis!

Bonuses from the investigation have been the opportunity to observe that there is nothing fundamentally different about the Finite Integral Method and that, in a two-step initial value problem, the method can be improved without increasing its complexity. Investigations also show that both the "standard" and the "improved" F.I.M. bear favourable comparison with more commonly used procedures.

Some of what follows may be elementarily obvious but all is included for the sake of completeness.

2. THE STANDARD FINITE INTEGRAL METHOD

It is helpful to re-state the standard procedures in somewhat more detail than given in Reference 3. Figure 1 shows the general time continuum t with the discrete times t_{-2}, t_{-1}, t_0, t_1 etc each separated by a time interval h, here chosen as constant. Accelerations, velocities and displacements at each discrete time are indicated by appropriate suffices on $\ddot{y}$, $\dot{y}$ and y respectively. It may be helpful to regard t_0 as the "present time" with acceptable solutions for $\ddot{y}$, $\dot{y}$ and y already achieved at this and earlier times t_{-1}, t_{-2} etc. The objective of the analysis procedure is the achievement of acceptable solutions for $\ddot{y}$, $\dot{y}$ and y at "future times" t_1, t_2 etc.

Let it now be assumed that $\ddot{y}_t$, the acceleration at absolute time $t(t_0 \leqslant t \leqslant t_2)$, varies *parabolically* with respect to time.

Fitting the assumed parabolic variation to the specified values of $\ddot{y}_t$ at absolute times t_0, t_1 and t_2 leads directly to the following result for y_t in terms of interpolation polynomials γ_0, γ_1 and γ_2 with t now measured from t = 0 at absolute time t_0 to t = 2h at absolute time t_2 :-

$$\ddot{y}_t = \langle \gamma_0 \; \gamma_1 \; \gamma_2 \rangle \begin{Bmatrix} \ddot{y}_0 \\ \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} \qquad (1)$$

where $\gamma_0 = 1 - 3t/2h + t^2/2h^2$

$\gamma_1 = 2t/h - t^2/h^2$

$\gamma_2 = - t/2h + t^2/2h^2$

Hence, by integration of Equation 1, together with the boundary condition $\dot{y}_t = \dot{y}_0$ at $t = 0$,

$$\dot{y}_t = \dot{y}_0 + \langle \psi_0 \ \psi_1 \ \psi_2 \rangle \begin{Bmatrix} \ddot{y}_0 \\ \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} \tag{2}$$

where $\psi_0 = \int_0^t \gamma_0 \, dt = t - 3t^2/4h + t^3/6h^2$

$\psi_1 = \int_0^t \gamma_1 \, dt = t^2/h - t^3/3h^2$

$\psi_2 = \int_0^t \gamma_2 \, dt = - t^2/4h + t^3/6h^2$

The standard Finite Integral Method now proceeds, inconsistently, by writing displacements y_t ($t_0 \leqslant t \leqslant t_2$) in terms of velocities $\dot{y}_t$ on the basis that $\dot{y}_t$ varies *parabolically* with respect to time, i.e. by use of the same operator as used in Equation 2.

$$y_t = y_0 + \langle \psi_0 \ \psi_1 \ \psi_2 \rangle \begin{Bmatrix} \dot{y}_0 \\ \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} \tag{3}$$

Equations 2 and 3, together with the definitions of ψ_0, ψ_1 and ψ_2 define the basis of the standard F.I.M.

Referring to Figure 1 for relevant symbol definitions and by substituting t = h or t = 2h in the values of ψ_0, ψ_1 and ψ_2 we obtain the standard results,

$$\begin{Bmatrix} \dot{y}_0 \\ \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} = \begin{Bmatrix} \dot{y}_0 \\ \dot{y}_0 \\ \dot{y}_0 \end{Bmatrix} + \frac{h}{12} \begin{bmatrix} 0 & 0 & 0 \\ 5 & 8 & -1 \\ 4 & 16 & 4 \end{bmatrix} \begin{Bmatrix} \ddot{y}_0 \\ \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} \qquad (4)$$

and

$$\begin{Bmatrix} y_0 \\ y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} y_0 \\ y_0 \\ y_0 \end{Bmatrix} + \frac{h}{12} \begin{bmatrix} 0 & 0 & 0 \\ 5 & 8 & -1 \\ 4 & 16 & 4 \end{bmatrix} \begin{Bmatrix} \dot{y}_0 \\ \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} \qquad (5)$$

Thereafter, substitution for $\{\dot{y}_0 \; \dot{y}_1 \; \dot{y}_2\}$ in Equation 5 from Equation 4 gives displacements in terms of accelerations,

$$\begin{Bmatrix} y_0 \\ y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} y_0 \\ y_0 \\ y_0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ h\dot{y}_0 \\ 2h\dot{y}_0 \end{Bmatrix} + \frac{h^2}{144} \begin{bmatrix} 0 & 0 & 0 \\ 36 & 48 & -12 \\ 96 & 192 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{y}_0 \\ \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} \qquad (6)$$

At this stage it is helpful to note one further result. Using Equation 3 and substituting for $\{\dot{y}_0 \; \dot{y}_1 \; \dot{y}_2\}$ from Equation 4 gives the general, *assumed but inconsistent*,

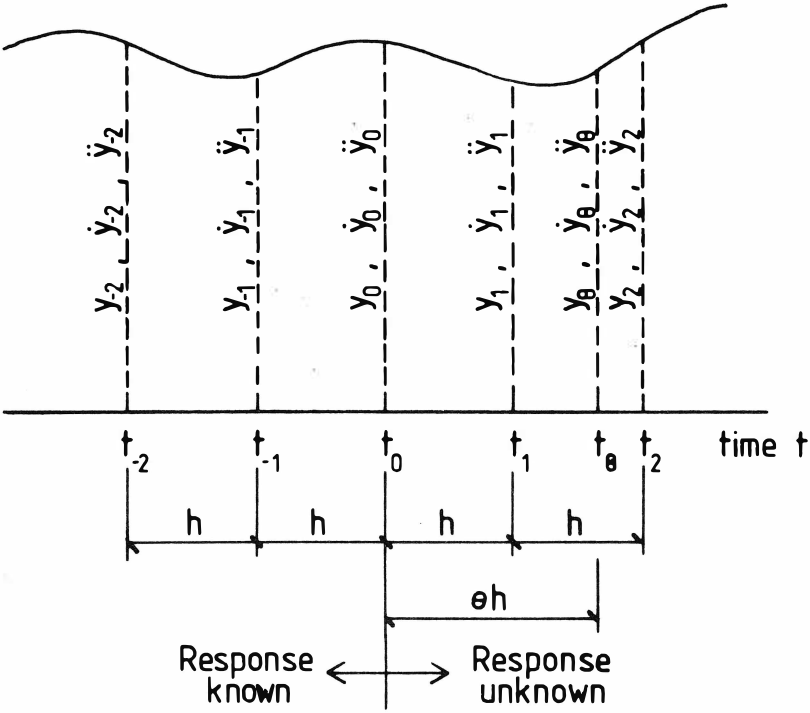

FIGURE 1 : Response at discrete times, general equation $M\ddot{Y} + C\dot{Y} + KY = R(t)$

variation of y_t over the time interval $(t_2 - t_0)$ in terms of the accelerations $\ddot{y}_0$, $\ddot{y}_1$ and $\ddot{y}_2$ and time t $(0 < t < 2h)$, viz:

$$y_t = y_0 + \langle \psi_0 \quad \psi_1 \quad \psi_2 \rangle \begin{Bmatrix} \dot{y}_0 \\ \dot{y}_0 \\ \dot{y}_0 \end{Bmatrix} + \frac{h}{12} \langle \psi_0 \quad \psi_1 \quad \psi_2 \rangle \begin{bmatrix} 0 & 0 & 0 \\ 5 & 8 & -1 \\ 4 & 16 & 4 \end{bmatrix} \begin{Bmatrix} \ddot{y}_0 \\ \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} \tag{7}$$

$$\text{i.e.} \quad y_t = y_0 + \dot{y}_0 \, t + \left(\frac{t^2}{3} - \frac{t^3}{12h}\right) \ddot{y}_0 + \frac{t^2}{3} \ddot{y}_1 + \left(\frac{t^3}{12h} - \frac{t^2}{6}\right) \ddot{y}_2 \tag{8}$$

Equation 8 inevitably confirms the *assumed cubic* distribution of y_t with respect to time.

In the standard method the solution proceeds in "two step jumps" using equations similar to Equations 4 and 6 to write the general equilibrium equations of a dynamic system in terms of *accelerations* only.

In general, with upper case symbols being used to describe *vectors* of displacements, velocities etc. we have,

$$[M]\{\ddot{Y}\} + [C]\{\dot{Y}\} + [K]\{Y\} = \{R\} \tag{9}$$

and, thence, at discrete times t_1 and t_2,

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{Bmatrix} \ddot{Y}_1 \\ \ddot{Y}_2 \end{Bmatrix} + \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{Bmatrix} \dot{Y}_1 \\ \dot{Y}_2 \end{Bmatrix} + \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} \tag{10}$$

Subscripts are here used to denote the discrete times t_1 and t_2. In an n degree-of-freedom system $\{Y_1\}$, $\{\dot{Y}_1\}$, $\{Y_2\}$, $\{\dot{Y}_2\}$ etc are all n x 1 vectors. Equations 4 and 6 are used in expanded form to write $\{Y_1\}$ $\{\dot{Y}_1\}$ $\{Y_2\}$ and $\{\dot{Y}_2\}$ in terms of $\{\ddot{Y}_1\}$, $\{\ddot{Y}_2\}$ and *known* responses at times t_0. The resulting equations, in terms of *unknown accelerations* $\{\ddot{Y}_1\}$ and $\{\ddot{Y}_2\}$ are then solved simultaneously.

The standard F.I.M. results are summarised in row 5 of the Tables 1 and 2. In Table 2, it is sometimes convenient to partition matrix [B] and re-write the results firstly in terms of $\{\ddot{Y}_1\}$ only. After solution for $\{\ddot{Y}_1\}$, the values of $\{\ddot{Y}_2\}$ are obtained by back-substitution in the partitioned equations.

3. THE FINITE INTEGRAL METHOD IN TERMS OF DISPLACEMENTS

In expanded form, allowing for multiple degrees of freedom, Equations 5 and 6 provide the following results:-

$$\{Y_1\} = \{Y_0\} + \frac{h}{12}\{5\ \dot{Y}_0 + 8\ \dot{Y}_1 - \dot{Y}_2\} \qquad (11a)$$

$$\{Y_2\} = \{Y_0\} + \frac{h}{12}\{4\ \dot{Y}_0 + 16\ \dot{Y}_1 - 4\ \dot{Y}_2\} \qquad (11b)$$

and

$$\{Y_1\} = \{Y_0\} + h\ \{\dot{Y}_0\} + \frac{h^2}{144}\{36\ \ddot{Y}_0 + 48\ \ddot{Y}_1 - 12\ \ddot{Y}_2\} \qquad (12a)$$

$$\{Y_2\} = \{Y_0\} + 2h\ \{\dot{Y}_0\} + \frac{h^2}{144}\{96\ \ddot{Y}_0 + 192\ \ddot{Y}_1\} \qquad (12b)$$

Equations 11 may be used to write velocities in terms of displacements and Equations 12 similarly provide expressions for accelerations in terms of displacements. In this revised form the relationships can be used to write the discretised dynamics Equations 10 in terms of *displacements* only. It is, then, self-evident that the F.I.M. does *not* have, as a fundamental feature, the need to solve firstly in terms of accelerations. It becomes apparent that the F.I.M. is in every way "classifiable" alongside such common Direct Integration schemes as, for example, the Central Difference or Newmark Methods.

The revised forms of Equations 11 and 12 are,

$$\{\dot{Y}_1\} = -\frac{1}{2}\{\dot{Y}_0\} + \frac{1}{4h}\{-\ 5\ Y_0 + 4\ Y_1 + Y_2\} \qquad (13a)$$

$$\{\dot{Y}_2\} = \{\dot{Y}_0\} + \frac{2}{h}\{Y_0 - 2Y_1 + Y_2\} \quad (13b)$$

and $$\{\ddot{Y}_1\} = -\frac{1}{2}\{\ddot{Y}_0\} - \frac{3}{2h}\{\dot{Y}_0\} + \frac{3}{4h^2}\{-Y_0 + Y_2\} \quad (14a)$$

$$\{\ddot{Y}_2\} = \{\ddot{Y}_0\} + \frac{6}{h}\{\dot{Y}_0\} + \frac{3}{h^2}\{3Y_0 - 4Y_1 + Y_2\} \quad (14b)$$

The necessary steps for the solution of Equation 10 in terms of displacements are summarised in row six of Tables 1 and 2 (The Finite Integral Method (Revised)).

The computational effort required to implement the F.I.M. in this revised form is precisely similar to that of the standard form and solutions are identical for any given time-step. The sole purpose of the revision has been to highlight the fact that the F.I.M. is simply another Direct Integration scheme, taking its place alongside many existing schemes. It bases its procedure on the "forward projection" of accelerations and (unlike in other methods) a second *SIMILAR* forward projection of velocities each being described by the interpolation functions γ_0 γ_1 and γ_2 of Equation 1.

An obvious area of investigation is the improvement that might be gained by dispensing with the, formerly convenient, quite unnecessary constraint upon the velocity distribution having already prescribed an acceleration distribution.

4. THE FINITE INTEGRAL METHOD (IMPROVED TECHNIQUE)

For completeness, two aspects are worth further attention:

(i) A "first principles" derivation of y_t in terms of both $\{\ddot{y}_0\ \ddot{y}_1\ \ddot{y}_2\}$ and $\{\dot{y}_0\ \dot{y}_1\ \dot{y}_2\}$, noting that Equations 1 and 2 are retained, *without* assumption regarding the time variation of $\dot{y}_t$ when proceeding to displacement estimates.

(ii) A comparison of the resulting relationships with those of the standard method.

We commence with Equation 1 and note that Equations 4, remain valid. From Equations 4, making accelerations the subject of the equations,

$$\begin{Bmatrix} \ddot{y}_0 \\ \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} = \begin{Bmatrix} \ddot{y}_0 \\ -\frac{1}{2}\ddot{y}_0 \\ \ddot{y}_0 \end{Bmatrix} + \frac{1}{4h}\begin{bmatrix} 0 & 0 & 0 \\ -5 & 4 & 1 \\ 8 & -16 & 8 \end{bmatrix}\begin{Bmatrix} \dot{y}_0 \\ \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} \qquad (15)$$

Hence, in Equation 1,

$$\ddot{y}_t = (\gamma_0 - \gamma_1/2 + \gamma_2)\ \ddot{y}_0 + \frac{1}{4h} \langle \gamma_0\ \gamma_1\ \gamma_2 \rangle \begin{Bmatrix} 0 \\ -5\dot{y}_0 + 4\dot{y}_1 + \dot{y}_2 \\ 8\dot{y}_0 - 16\dot{y}_1 + 8\dot{y}_2 \end{Bmatrix}$$

After substitution for γ_0, γ_1 etc and rearrangement,

$$\ddot{y}_t = \ddot{y}_0\beta_0 + \frac{1}{4h}(\alpha_0\dot{y}_0 + \alpha_1\dot{y}_1 + \alpha_2\dot{y}_2) \qquad (16)$$

where
$$\beta_0 = 1 - 3t/h + 3t^2/2h^2$$
$$\alpha_0 = -\ 14t/h + 9t^2/h^2$$
$$\alpha_1 = 16t/h - 12t^2/h^2$$
$$\alpha_2 = -\ 2t/h + 3t^2/h^2$$

Thence, by integration of Equation 16 with the initial condition that $\dot{y}_t = \dot{y}_0$ at $t = 0$,

$$\dot{y}_t = \dot{y}_0 + \ddot{y}_0\beta_1 + \frac{1}{4h}(\phi_0\dot{y}_0 + \phi_1\dot{y}_1 + \phi_2\dot{y}_2) \qquad (17)$$

where $\beta_1 = \int_0^t \beta_0\,dt = t - 3t^2/2h + t^3/2h^2$

$$\phi_0 = \int_0^t \alpha_0 dt = -7t^2/h + 3t^3/h^2$$

$$\phi_1 = \int_0^t \alpha_1 dt = 8t^2/h - 4t^3/h^2$$

$$\phi_2 = \int_0^t \alpha_2 dt = -t^2/h + t^3/h^2$$

Equation 17 gives the velocity distribution resulting from an assumed parabolic variation of accelerations with respect to time. It is appropriate to both the Standard F.I.M. and the present investigation.

Proceeding, now, *without* further assumption, the displacement distribution is obtained by integration of Equation 17 with the initial condition that $y_t = y_0$ at $t = 0$,

$$y_t = y_0 + \dot{y}_0 t + \ddot{y}_0 \beta_2 + \frac{1}{4h}(\lambda_0 \dot{y}_0 + \lambda_1 \dot{y}_1 + \lambda_2 \dot{y}_2) \qquad (18)$$

where $\beta_2 = \int_0^t \beta_1 dt = t^2/2 - t^3/2h + t^4/8h^2$

$$\lambda_0 = \int_0^t \phi_0 dt = -7t^3/3h + 3t^4/4h^2$$

$$\lambda_1 = \int_0^t \phi_1 dt = 8t^3/3h - t^4/h^2$$

$$\lambda_2 = \int_0^t \phi_2 dt = -t^3/3h + t^4/4h^2$$

Equation 18 gives the "correct" distribution of displacements based solely on the assumption regarding the acceleration distribtion. Substitution of $t = h$ and $t = 2h$ gives the replacement operator, comparable to that given in Equation 5 for the Standard F.I.M.,

$$\begin{Bmatrix} y_0 \\ y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} y_0 \\ y_0 \\ y_0 \end{Bmatrix} + \frac{h}{48} \begin{bmatrix} 0 & 0 & 0 \\ 29 & 20 & -1 \\ 16 & 64 & 16 \end{bmatrix} \begin{Bmatrix} \dot{y}_0 \\ \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} + \frac{h^2}{8} \begin{Bmatrix} 0 \\ \ddot{y}_0 \\ 0 \end{Bmatrix} \qquad (19)$$

Using Equation 4 and substituting for $\{\dot{y}_0 \ \dot{y}_1 \ \dot{y}_2\}$ gives the "correct" displacement distribution in terms of the discrete accelerations $\{\ddot{y}_0 \ \ddot{y}_1 \ \ddot{y}_2\}$.

$$y_t = y_0 + \dot{y}_0 t + \ddot{y}_0 \beta_2 + \frac{1}{4h} \langle \lambda_0 \ \lambda_1 \ \lambda_2 \rangle \begin{Bmatrix} \dot{y}_0 \\ \dot{y}_0 \\ \dot{y}_0 \end{Bmatrix}$$

$$+ \frac{1}{48} \langle \lambda_0 \ \lambda_1 \ \lambda_2 \rangle \begin{bmatrix} 0 & 0 & 0 \\ 5 & 8 & -1 \\ 4 & 16 & 4 \end{bmatrix} \begin{Bmatrix} \ddot{y}_0 \\ \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} \qquad (20)$$

Substitution for β_2, λ_0, λ_1 and λ_2 gives, after rearrangement,

$$y_t = y_0 + \dot{y}_0 t + \left(\frac{t^2}{2} - \frac{t^3}{4h} + \frac{t^4}{24h^2} \right) \ddot{y}_0 + \left(\frac{t^3}{3h} - \frac{t^4}{12h^2} \right) \ddot{y}_1$$

$$+ \left(\frac{-t^3}{12h} + \frac{t^4}{24h^2} \right) \ddot{y}_2 \qquad (21)$$

Equation 21 is directly comparable with Equation 8 of the standard method.

With t = h and t = 2h Equation 21 gives the result analogous to Equation 6 viz:

$$\begin{Bmatrix} y_0 \\ y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} y_0 \\ y_0 \\ y_0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ h\dot{y}_0 \\ 2h\dot{y}_0 \end{Bmatrix} + \frac{h^2}{144} \begin{bmatrix} 0 & 0 & 0 \\ 42 & 36 & -6 \\ 96 & 192 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{y}_0 \\ \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} \qquad (22)$$

In expanded form to include multiple degrees of freedom, the revised forms of Equations 19 and 22, written with velocities and accelerations as the subjects of the equations are as below. They are directly comparable with Equations 13 and 14 of the standard method:-

$$\{\dot{Y}_1\} = -\frac{5}{4}\{\dot{Y}_0\} + \frac{1}{8h}\{-17Y_0 + 16Y_1 + Y_2\} - \frac{h}{4}\ddot{Y}_0 \quad (23a)$$

$$\{\dot{Y}_2\} = 4\{\dot{Y}_0\} + \frac{1}{2h}\{11Y_0 - 16Y_1 + 5Y_2\} + h\ddot{Y}_0 \quad (23b)$$

and

$$\{\ddot{Y}_1\} = -\frac{1}{2}\{\ddot{Y}_0\} - \frac{3}{2h}\{\dot{Y}_0\} + \frac{3}{4h^2}\{-Y_0 + Y_2\} \quad (24a)$$

$$\{\ddot{Y}_2\} = 4\{\ddot{Y}_0\} + \frac{15}{h}\{\dot{Y}_0\} + \frac{3}{2h^2}\{13Y_0 - 16Y_1 + 3Y_2\} \quad (24b)$$

It will be observed that Equations 14a and 24a are identical as are the last of Equations 6 and 22. The necessity for such results is readily inferred from a comparison of Equations 8 and 21. In each equation the corresponding coefficients of $\dot{y}_0$, $\ddot{y}_0$, $\ddot{y}_1$ and $\ddot{y}_2$ become identical when t = 2h with, in fact, the last of these pairs of coefficients being zero at that time.

Notwithstanding this similarity it remains inevitable that the *simultaneous* solution of Equations 10 at discrete times t_1 *and* t_2 using Equations 23 and 24 rather than equations 13 and 14 will result in different predictions of displacements at times t_1 and t_2 from those obtained using the original operators. A similar comment obviously applies to the solution of Equations 10 in terms of accelerations using equations 4 and 22 rather than the original Equations 4 and 6. Table 2, rows 5 and 7, summarise the two alternative solutions in terms of accelerations. There is no obvious computational disadvantage in the "improved" formulation (row 7) and it might be expected that use of the more consistent set of operators would lead to somewhat better solution quality. This is inferred in the elementary but informative example investigated below.

TABLE 1 Integration Operators for Various Direct Integration Schemes

Method	Basis (1)	Integration Operators (2)	Remarks
Central Difference (1)	Uses standard Central Difference Formulae to write $\ddot{y}_o$ and $\dot{y}_o$ in terms of y_{-1}, y_o, y_1. Solves equilibrium equations at time t_o for unknowns y_1 and hence back substitutes in Difference Formulae to obtain $\ddot{y}_o$ and $\dot{y}_o$. A special "starting procedure" is required.	$\ddot{y}_o = \frac{1}{h^2}(y_{-1} - 2y_o + y_1)$ $\dot{y}_o = \frac{1}{2h}(-y_{-1} + y_1)$	Note that equations are solved at time t_o to give displacements at time t_1. Accels. and velocities at time t_o are found only after displacements at time t_1 have been found.
The Houbolt Method (2)	Uses Standard Backward Difference Formulae to write $\ddot{y}_1$ and $\dot{y}_1$ in terms of y_1, y_o, y_{-1} and y_{-2}. Solves equilibrium equations at time t_1 for unknowns y_1 and hence back substitutes in Difference Formulae to obtain $\ddot{y}_1$ and $\dot{y}_1$. A special "starting procedure" is required.	$\ddot{y}_1 = \frac{1}{h^2}(2y_1 - 5y_o + 4y_{-1} - y_{-2})$ $\dot{y}_1 = \frac{1}{6h}(11y_1 - 18y_o + 9y_{-1} - 2y_{-2})$	Note that equations are solved at time t_1 to give displacements at time t_1, and hence accelerations and velocities.
The Wilson θ Method (3)	Assumes linear accel. profile between t_o and t_θ. Solves equilibrium equations at time t_θ for unknowns y_θ, using a linearly projected load vector $R_\theta = R_o + \theta(R_1 - R_o)$. Thence back-substitutes into intermediate results to give $\ddot{y}_\theta$ and $\dot{y}_\theta$ and hence $\ddot{y}_1$, $\dot{y}_1$ and y_1. $\theta = 1$ is the standard "Linear Acceleration Method". Use of $\theta = 1.4$ is favoured in literature.	$\ddot{y}_\theta = \frac{6}{\theta^2 h^2}(y_\theta - y_o) - \frac{6}{\theta h}\dot{y}_o - 2\ddot{y}_o$ $\dot{y}_\theta = \frac{3}{\theta h}(y_\theta - y_o) - 2\dot{y}_o - \frac{\theta h}{2}\ddot{y}_o$	Note that equations are solved at time t_θ to give displacements at time t_θ. Thereafter, $y_1 = y_o + \dot{y}_o h + \frac{1}{2}\ddot{y}_o h^2 + \frac{h^2}{6\theta}(\ddot{y}_\theta - \ddot{y}_o)$ by double integration from the linear acceleration profile.
The Newmark Method (4)	An extension of the "linear acceleration" method which uses the following assumed velocity and displacement approximations. $\dot{y}_1 = \dot{y}_o + h(1-\delta)\ddot{y}_o + \delta\ddot{y}_1)$ $y_1 = y_o + \dot{y}_o h + h^2((\frac{1}{2} - \alpha)\ddot{y}_o + \alpha\ddot{y}_1)$ Manipulates these to give expressions (see column (2)) for $\ddot{y}_1$ and $\dot{y}_1$ in terms of y_1 and solves equilibrium equations at time t_1 for unknowns y_1. $\delta = \frac{1}{2}$, $\alpha = 1/6$ corresponds to the "linear acceleration" method (Wilson θ Method, $\theta = 1$). $\delta = \frac{1}{2}$, $\alpha = \frac{1}{4}$ corresponds to "constant average acceleration method".	$y = \frac{1}{\alpha h^2}(y - y_o - \dot{y}_o h) - \frac{y_o}{2\alpha}(1 - 2\alpha)$ $\dot{y} = \ddot{y}_o h(1 - \frac{\delta}{2\alpha}) + \dot{y}_o(1 - \frac{\delta}{\alpha}) + \frac{\delta}{\alpha h}(y_1 - y_o)$	Note that equations are solved at time t_1 to give displacements and time t_1. Note that, just as in the Standard Finite Integral Method the governing equation could have been written in terms of $\ddot{y}_1$ and solved, therefore, firstly for acceleration. To do this, the relationships in Column (1) would have been used directly.

The Standard Finite Integral Method (5)	Assumes a parabolic acceleration - time profile over two forward time intervals. Further assumes the same operator to give displacements ex. velocities i.e. a parabolic velocity - time profile. Solves equilibrium equations simultaneously at time t_1 and time t_2, in terms of unknown accelerations $\ddot{y}_1$, $\ddot{y}_2$. Thence back substitutes to obtain $\dot{y}_1$, y_1 and $\dot{y}_2$, y_2.	$\dot{y}_1 = \frac{h}{12}(5\ddot{y}_o + 8\ddot{y}_1 - \ddot{y}_2) + \dot{y}_o$ $y_2 = \frac{h}{12}(4\ddot{y}_o + 16\ddot{y}_1 + 4\ddot{y}_2) + \dot{y}_o$ $y_1 = \frac{h}{12}(5\dot{y}_o + 8\dot{y}_1 - \dot{y}_o) + y_o$ $y_2 = \frac{h}{12}(4\dot{y}_o + 16\dot{y}_1 + 4\dot{y}_2) + y_o$ alternatively, in terms of accelerations $y_1 = y_o + h\dot{y}_o + \frac{h^2}{144}(36\ddot{y}_o + 48\ddot{y}_1 - 12\ddot{y}_2)$ $y_2 = y_o + 2h\dot{y}_o + \frac{h^2}{144}(96\ddot{y}_o + 192\ddot{y}_1)$	Note that equations are solved at times t_1 and t_2 to give, simultaneously, $\ddot{y}_1$ and $\ddot{y}_2$ at times t_1 and t_2. Thence back-substitution gives $\dot{y}_1$, $\dot{y}_2$ etc.
The Finite Integral Method (Revised) (6)	Assumes acceleration and velocity profiles as above but rewrites the operators (column 2) in terms of displacements. Solves equilibrium equations simultaneously at time t_1 and time t_2 in terms of unknown displacements y_1 and y_2. Thence back-substitutes to obtain $\dot{y}_1$, y_1, $\dot{y}_2$, y_2.	$\ddot{y}_1 = \frac{1}{4h^2}(3y_2 - 3y_o - 6h\dot{y}_o - 2h^2\ddot{y}_o)$ $\ddot{y}_2 = \frac{1}{h^2}(3y_2 - 12y_1 + 9y_o + 6h\dot{y}_o + h^2\ddot{y}_o)$ $\dot{y}_1 = \frac{1}{4h}(y_2 + 4y_1 - 5y_o - 2h\dot{y}_o)$ $\dot{y}_2 = \frac{1}{h}(2y_2 - 4y_1 + 2y_o + h\dot{y}_o)$	Note that equations are solved at times t_1 and t_2 to give, simultaneously, y_1 and y_2 at times t_1 and t_2. Results obtained are, of course, identicsl to those obtained by the Standard Finite Integral Method.
The Finite Integral Method (Improved Technique) (7)	Assumes a parabolic acceleration - time profile over two forward time intervals. Does not assume that the second operator is similar but obtains "correct" operator by double integration. Solves equilibrium equations simultaneously at times t_1 and t_2. THIS MAY BE DONE EITHER IN TERMS OF ACCELERATIONS OR DISPLACEMENTS i.e. as in (5) above, or (6) above. Operators shown are for solution in terms of accelerations.	$\dot{y}_1 = \frac{h}{12}(5\ddot{y}_o + 8\ddot{y}_1 - \ddot{y}_2) + \dot{y}_o$ $\dot{y}_2 = \frac{h}{12}(4\ddot{y}_o + 16\ddot{y}_1 + 4\ddot{y}_2) + \dot{y}_o$ $y_1 = y_o + \frac{h^2}{8}\ddot{y}_o + \frac{h}{48}(29\dot{y}_o + 20\dot{y}_1 - \dot{y}_2)$ $y_2 = y_o + \frac{h}{12}(4\dot{y}_o + 16\dot{y}_1 + 4\dot{y}_2)$ alternatively, in terms of acceleration, $y_1 = y_o + h\dot{y}_o + \frac{h^2}{144}(42\ddot{y}_o + 36\ddot{y}_1 - 6\ddot{y}_2)$ $y_2 = y_o + 2h\dot{y}_o + \frac{h^2}{144}(96\ddot{y}_o + 192\ddot{y}_1)$	Note that equations are solved at times t_1 and t_2 simultaneously. Note that the operator giving y_2 is unchanged from that of the Standard Method. The simultaneous use of the operator giving y_1 and y_2 does, of course, produce unique and different estimates of y_1, y_2 etc from those given by the standard method. See text for further information.

TABLE 2 Discrete Form of $M\ddot{Y} + C\dot{Y} + KY = R$ for Various Direct Integration Schemes

Method	Solution Form	Definitions
Central Difference (1)	$B_o Y_1 = \bar{R}_o$	$B_o = \frac{1}{h^2} M + \frac{1}{2h} C$ i.e. independent of K $\bar{R}_o = R_o - (K - \frac{2}{h^2} M) Y_o - (\frac{1}{h^2} M - \frac{1}{2h} C) Y_{-1}$
The Houbolt Method (2)	$B_1 Y_1 = \bar{R}_1$	$B_1 = \frac{2}{h^2} M + \frac{11}{6h} C + K$ $\bar{R}_1 = R_1 + (\frac{5}{h^2} M + \frac{3}{h} C) Y_o - (\frac{4}{h^2} M + \frac{3}{2h} C) Y_{-1} + (\frac{1}{h^2} M + \frac{1}{3h} C) Y_{-2}$
The Wilson Method (3)	$B_\theta Y_\theta = \bar{R}_\theta$	$B_\theta = \frac{6}{h^2\theta^2} M + \frac{3}{h\theta} C + K$ $\bar{R}_\theta = (1-\theta) R_o + \theta R_1 + (\frac{6}{h^2\theta^2} M + \frac{3}{h\theta} C) Y_o + (\frac{6}{h\theta} M + 2C) \dot{Y}_o + (2M + \frac{h\theta}{2} C)\ddot{Y}_o$
The Newmark Method (4)	$B_1 Y_1 = \bar{R}_1$	$B_1 = \frac{1}{\alpha h^2} M + \frac{\delta}{\alpha h} C + K$ $\bar{R}_1 = R_1 + \left(\frac{1}{\alpha h^2} M + \frac{\delta}{\alpha h} C\right) Y_o + \left(\frac{1}{\alpha h} M - (1 - \frac{\delta}{\alpha}) C\right) \dot{Y}_o$ $+ \left((\frac{1}{2\alpha} - 1) M - h (1 - \frac{\delta}{2\alpha}) C\right) \ddot{Y}_o$
The Standard Finite Integral Method (5)	$B \ddot{Y} = \bar{R}$ $\ddot{Y} = \{\ddot{Y}_1 \; \ddot{Y}_2\}$	$B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where $A_{11} = M + \frac{2h}{3} C + \frac{h^2}{3} K$, $A_{12} = \frac{-h}{12} C - \frac{h^2}{12} K$ $A_{21} = \frac{4h}{3} C + \frac{4h^2}{3} K$, $A_{22} = M + \frac{h}{3} C$ $\bar{R} = \{\bar{R}_1 \; \bar{R}_2\}$ where $\bar{R}_1 = R_1 - K Y_o - (h K + C) \dot{Y}_o - (\frac{h^2}{4} K + \frac{5h}{12} C) \ddot{Y}_o$ $\bar{R}_2 = R_2 - K Y_o - (2h K + C) \dot{Y}_o - (\frac{2h^2}{3} K + \frac{h}{3} C) \ddot{Y}_o$

The Finite Integral Method (Revised) (6)	$BY = \bar{R}$ $Y = \{Y_1 Y_2\}$	$B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where $A_{11} = \frac{1}{h} C + K$, $A_{12} = \frac{3}{4h^2} M + \frac{1}{4h} C$ $A_{21} = \frac{-12}{h^2} M - \frac{4}{h} C$, $A_{22} = \frac{3}{h^2} M + \frac{2}{h} C + K$ $\bar{R} = \{\bar{R}_1 \bar{R}_2\}$ where $\bar{R}_1 = R_1 + (\frac{3}{4h^2} M + \frac{5}{4h} C) Y_o + (\frac{3}{2h} M + \frac{1}{2} C) \dot{Y}_o + \frac{1}{2} M \ddot{Y}_o$ $\bar{R}_2 = R_2 - (\frac{9}{h^2} M + \frac{2}{h} C) Y_o - (\frac{6}{h} M + C) \dot{Y}_o - M \ddot{Y}_o$
The Finite Integral Method (Improved Technique) (7)	$B \ddot{Y} = \bar{R}$ $\ddot{Y} = \{\ddot{Y}_1 \ddot{Y}_2\}$	$B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where $A_{11} = M + \frac{2h}{3} C + \frac{h^2}{4} K$, $A_{12} = \frac{-h}{12} C - \frac{h^2}{24} K$ $A_{21} = \frac{4h}{3} C + \frac{4}{3} h^2 K$, $A_{22} = M + \frac{h}{3} C$ $\bar{R} = \{\bar{R}_1 \bar{R}_2\}$ where $\bar{R}_1 = R_1 - K Y_o - (h K + C) \dot{Y}_o - (\frac{7h^2}{24} K + \frac{5h}{12} C) \ddot{Y}_o$ $\bar{R}_2 = R_2 - K Y_o - (2h K + C) \dot{Y}_o - (\frac{2h^2}{3} K + \frac{h}{3} C) \ddot{Y}_o$

NOTES: (a) Suffices -2, -1, o, 1, 2, θ refer to times $t_o - 2\Delta t$, $t_o - \Delta t$, t_o, $t_o + \Delta t$, $t_o + 2\Delta t$ and $t_o + \theta\Delta t$ respectively.

(b) When M, C and/or K are time-dependent they should be constructed at the time indicated by the suffix of B or $\bar{R}$.

(c) In the case of matrices A_{11}, A_{12} etc the first suffix indicates the time at which M, C and K should be constructed.

5. ALTERNATIVE DIRECT INTEGRATION SCHEMES

An overwhelming volume of published research exists concerning Direct Integration methods. Felippa and Park [4] for example, provide an excellent over-view of available methods in non-linear structural dynamics and provide forty-two further references. Bathe and Wilson [2] discuss various schemes used in linear problems and quote from a range of investigations regarding the convergence and stability of some of the more popular methods. Weeks [9] and Nickell [6] are sources of some forty further references. Argyris et al. [1] describe a family of unconditionally stable algorithms for use with large linear systems. Newmark's classic paper [5] remains as one of the earliest and best sources of information. Tables 1 and 2 have been developed with the aid of Reference 2 in an attempt to provide a brief summary of the appropriate equations of some commonly used schemes. In particular the Tables facilitate comparison of the Finite Integral Method with other methods when used to solve Initial Value problems.

6. A SIMPLE EXAMPLE

The Finite Integral Method, in its standard, revised and improved forms, has been used to solve a simple two-degree-of-freedom lumped mass problem discussed by Bathe and Wilson [2]. Solutions to the problem using common direct integration schemes are presented by Bathe and Wilson.

The problem is described in Figure 2 and has been synthesised from the data given in Chapter 8 of Reference [2]. The governing dynamic equilibrium equations are given below and are identical to those quoted by Bathe and Wilson,

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{y}_a \\ \ddot{y}_b \end{Bmatrix} + \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{Bmatrix} y_a \\ y_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix} \qquad (15)$$

The exact solution of this trivial problem is readily obtained,

$$y_a = 1 - \frac{1}{3}(5 \cos \sqrt{2}t - 2 \cos \sqrt{5}t)$$

$$y_b = 3 - \frac{1}{3}(5 \cos \sqrt{2}t + 4 \cos \sqrt{5}t) \qquad (26)$$

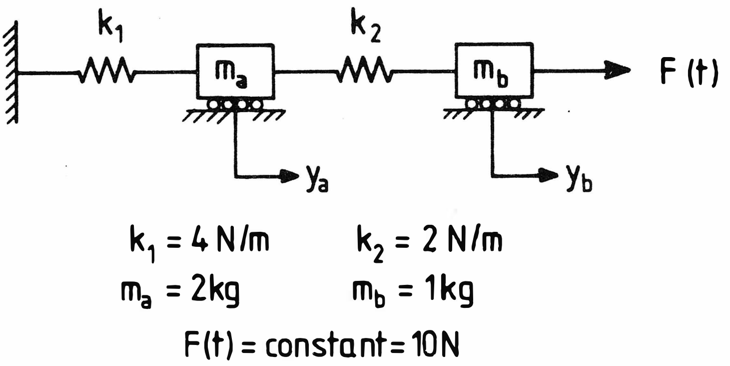

FIGURE 2 : Bathe and Wilson's[2] example problem

The displacements obtained by Bathe and Wilson, using a time-step equal to 0.28 secs and various integration schemes, are reproduced in Table 3. They are compared with results obtained by use of the improved Finite Integral Method using time-steps equal to 0.28 secs, 0.14 secs and 0.56 secs. Table 4 compares the solutions obtained by the standard and improved Finite Integral Methods for displacements, velocities and accelerations using a time-step equal to 0.28 secs.

Unfortunately the *exact* results quoted by Bathe and Wilson in Reference 2 are incorrect. Whilst they correctly quoted equations identical to Equations 26 their subsequent evaluation of displacements from these equations is in error. It follows that certain conclusions drawn by them regarding the accuracy of *any* of the quoted schemes are not strictly valid. The relative accuracy of results obtained by them using various schemes remains of considerable interest.

The correct exact results (obtained by substitution in Equations 26 and their derivatives) are quoted in Tables 3 and 4. An inspection of all the results unambiguously shows the excellence of the predictions obtained by the Standard and, particularly, the Improved Finite Integral Methods using a time-step equal to 0.28 secs. Clearly all the schemes give good predictions over at least part of the response time but a characteristic of the Improved F.I.M. results is their accuracy throughout the whole range.

The natural periods of the system are 4.44 secs and 2.81 secs and the time step chosen for comparison purposes is therefore 10% of the shorter natural period. It is self-evident that the chosen problem could hardly be better chosen in respect of its simplicity or suitability for numerical solution with a relatively large time-step! It will also be noted that the Finite Integral Method proceeds in "two-step" jumps with therefore twice the number of equations to be manipulated in every "solution cycle" (and half the number of "solution cycles" in any given time range).

7. CONCLUSION

It would not be valid to draw general conclusions about the relative suitability of the various schemes. References quoted earlier, and many others seek to do that from a much more generalised stance. Rather, the purpose of the paper has been to clarify the Finite Integral Method and its relationship with other schemes.

In the context of Initial Value problems typified by common dynamic analysis models the nature of the F.I.M. is clear. In its standard form it is merely a direct integration scheme in which accelerations and velocities are assumed to *vary periodically* with respect to time over two time-steps projected forward from the time at which satisfactory responses

TABLE 3 Comparison of Displacement Solutions Using Various Integration Schemes

Method	Central Difference h=0.28secs	Houbolt Integration h=0.28secs	Wilson θ, θ = 1.4 h=0.28secs	Newmark $\delta = \frac{1}{2}$ $\alpha = \frac{1}{4}$ h=0.28secs	Improved F.I.M. h=0.28secs	Improved F.I.M. h=0.14secs	Improved F.I.M. h=0.56secs	Exact Solution
Time seconds	Displacement y_a							
0	0	0	0	0	0	0	0	0
0.28	0	0	0.0061	0.0067	0.0031	0.0026		0.0025
0.56	0.0307	0.0307	0.0525	0.0504	0.0388	0.0380	0.061	0.0381
0.84	0.168	0.167	0.196	0.189	0.177	0.176		0.176
1.12	0.487	0.461	0.490	0.485	0.487	0.486	0.504	0.486
1.40	1.02	0.923	0.952	0.961	0.995	0.996		0.996
1.68	1.70	1.50	1.54	1.58	1.655	1.657	1.608	1.657
1.96	2.40	2.11	2.16	2.23	2.335	2.337		2.338
2.24	2.91	2.60	2.67	2.76	2.857	2.860	2.743	2.861
2.52	3.07	2.86	2.92	3.00	3.050	3.051		3.052
2.80	2.77	2.80	2.82	2.85	2.807	2.806	2.767	2.806
3.08	2.04	2.40	2.33	2.28	2.135	2.131		2.131
3.36	1.02	1.72	1.54	1.40	1.164	1.158	1.378	1.157
Time Seconds	Displacement y_b							
0	0	0	0	0	0	0	0	0
0.28	0.392	0.392	0.366	0.364	0.380	0.382		0.382
0.56	1.45	1.45	1.34	1.35	1.410	1.411	1.391	1.412
0.84	2.83	2.80	2.64	2.68	2.778	2.780		2.781
1.12	4.14	4.08	3.92	4.00	4.091	4.093	3.997	4.094
1.40	5.02	5.02	4.88	4.95	4.997	4.996		4.996
1.68	5.26	5.43	5.31	5.34	5.293	5.291	5.279	5.291
1.96	4.90	5.31	5.18	5.13	4.992	4.987		4.985
2.24	4.17	4.77	4.61	4.48	4.284	4.278	4.515	4.277
2.52	3.37	4.01	3.82	3.64	3.463	3.459		3.457
2.80	2.78	3.24	3.06	2.90	2.807	2.806	2.943	2.806
3.08	2.54	2.63	2.52	2.44	2.479	2.483		2.484
3.36	2.60	2.28	2.29	2.31	2.479	2.487	2.225	2.489

TABLE 4 Comparative Values Using Standard and Improved Finite Integral Methods

		Displacements			Velocities			Accelerations		
Time (secs)	Node	Exact Solution	"Standard" F.I. Δt=0.28sec	"Improved" F.I. Δt=0.28sec	Exact Solution	"Standard" F.I. Δt=0.28sec	"Improved" F.I. Δt=0.28sec	Exact Solution	"Standard" F.I. Δt=0.28sec	"Improved" F.I. Δt=0.28sec
0.28	Node A	2.51459E-3	1.60545E-3	3.05323E-3	3.55931E-2	4.14765E-2	3.91360E-2	0.374332	0.382958	0.371297
0.28	Node B	0.381875	0.387774	0.389457	2.65621	2.64044	2.64681	8.47753	8.45211	8.48428
0.56	Node A	3.80705E-2	4.00318E-2	3.88129E-2	0.261874	0.263007	0.259309	1.29739	1.2861	1.29312
0.56	Node B	1.4116	1.40619	1.40956	4.50895	4.50462	4.51514	4.42974	4.45529	4.4394
0.84	Node A	0.175595	0.180045	0.176468	0.76559	0.768276	0.768534	2.25417	2.23083	2.24863
0.84	Node B	2.78095	2.77097	2.77804	5.02795	5.01456	5.01711	-0.772611	-0.723788	-0.759208
1.12	Node A	0.486026	0.487732	0.48667	1.46993	1.46068	1.46502	2.63548	2.62279	2.6307
1.12	Node B	4.09356	4.08598	4.09071	4.1306	4.14917	4.14303	-5.40219	-5.36847	-5.38949
1.4	Node A	0.996351	0.997948	0.99536	2.14597	2.13056	2.13953	2.00717	1.99249	2.01042
1.4	Node B	4.99623	4.98633	4.9965	2.1956	2.22305	2.2053	-7.99221	-7.94942	-7.99527
1.68	Node A	1.65696	1.65108	1.65473	2.49362	2.48151	2.49183	0.319616	0.34471	0.329137
1.68	Node B	5.29051	5.29795	5.29334	-8.66230E-2	-5.60347E-2	-7.88522E-2	-7.84811	-7.88963	-7.86389
1.96	Node A	2.3382	2.32852	2.33455	2.26217	2.25001	2.25479	-2.02889	-1.98766	-2.01145
1.96	Node B	4.98571	4.99791	4.99221	-1.9691	-1.93765	-1.9524	-5.26645	-5.33459	-5.29974
2.24	Node A	2.86081	2.85096	2.8571	1.36385	1.37437	1.37148	-4.30579	-4.25633	-4.28713
2.24	Node B	4.27665	4.29656	4.28416	-2.91327	-2.92252	-2.92415	-1.38497	-1.48432	-1.42246
2.52	Node A	3.05171	3.04146	3.04994	-6.57443E-2	-4.53269E-2	-5.93460E-2	-5.69765	-5.64703	-5.68719
2.52	Node B	3.45748	3.47736	3.46264	-2.76615	-2.79211	-2.77251	2.2735	2.17347	2.24932
2.8	Node A	2.80572	2.80784	2.80655	-1.68736	-1.65509	-1.67568	-5.61095	-5.61395	-5.61223
2.8	Node B	2.80622	2.80957	2.80742	-1.78661	-1.84107	-1.80805	4.38658	4.3774	4.38341
3.08	Node A	2.13058	2.13843	2.13536	-3.05525	-3.02024	-3.03987	-3.90742	-3.93764	-3.92721
3.08	Node B	2.48433	2.47766	2.47889	-0.515699	-0.573601	-0.539951	4.32383	4.36622	4.35519
3.36	Node A	1.15723	1.17596	1.16431	-3.76018	-3.74842	-3.76025	-0.982921	-1.06409	-1.01365
3.36	Node B	2.48876	2.46378	2.47928	0.454766	0.43579	0.452078	2.35943	2.49679	2.4115

NOTE! Standard and Improved F.I. results have been obtained by solution of the equilibrium equations in terms of accelerations, and checked by re-solution in terms of displacements.

are known. It is immaterial whether the dynamic equations of equilibrium, in discrete form, are formulated in terms of accelerations or displacements. There is no computational advantage or improvement in solution accuracy achievable by use of one formulation rather than the other. Solving the basic equations in terms of accelerations or displacements merely alters the order in which the calculations proceed and, in each case, accuracy is determined by the size of the time-step chosen and the *actual* (a priori unknown) acceleration and velocity profiles.

Any advantage apparent in the use of identical operators for synthesising velocities from accelerations and displacements from velocities disappears in the context of the simple "two-step" initial value problem. The evidence of the simple example is that an improved technique in which a *consistent* synthesis of displacements and velocities, on the basis of the single assumption of a parabolic acceleration-time relationship, gives an improvement in solution accuracy for a particular time step. There is, further, no computational disadvantage, whatsoever, in using the more consistent formulation.

There is every reason to be influenced by computational efficiency when formulating solution procedures for systems involving a very large number of degrees of freedom. There can possibly be an over-emphasis in this regard, however, when studying problems with relatively few degrees of freedom where the transient initial response of the system is of interest. A method which is manageable and capable of modelling a rapidly changing acceleration response with a relatively large time-step is a useful tool even if it does not appeal to "Big System" designers. The F.I.M. appears to be one such method in an increasing range of applications.

APPENDIX A - REFERENCES

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APPENDIX B - NOTATION

h	Integration step size	
t	Time variable	
t_{-2}, t_{-1}, t_0 etc.	Discrete values of t	
x_0, x_1, x_2	Position variables in F.I. static analysis	
y_{-2}, y_{-1}, y_0 etc.	Displts. at times t_{-2}, t_{-1}, t_0 etc	
$\dot{y}_{-2}$, $\dot{y}_{-1}$, $\dot{y}_0$ etc.	Velocities at times t_{-2}, t_{-1}, t_0 etc.	
$\ddot{y}_{-2}$, $\ddot{y}_{-1}$, $\ddot{y}_0$ etc.	Accelerations at times t_{-1}, t_{-1}, t_0 etc.	
$\ddot{y}_a$, $\ddot{y}_b$	Nodal accels. in example problem	
[C]	Structure damping matrix	Suffices indicate value at times t_0, t_1, t_2
[K]	Structure stiffness matrix	Suffices indicate value at times t_0, t_1, t_2
[M]	Structure mass matrix	Suffices indicate value at times t_0, t_1, t_2
{R}	Time-dependent load vector	Suffices indicate value at times t_0, t_1, t_2
$\{Y\}$, $\{\dot{Y}\}$, $\{\ddot{Y}\}$	Displt., vel. and accel. vectors. Suffices indicate values at times t_0, t_1 and t_2	
θ	Integration step size multiplier, (Wilson θ)	
β_0, β_1, β_2	Interpolation functions (Equations 16, 17 and 18)	
γ_0, γ_1, γ_2	Interpolation functions (Equation 1)	
ψ_0, ψ_1, ψ_2	Interpolation functions (Equation 2)	
α_0, α_1, α_2	Interpolation functions (Equation 16)	
ϕ_0, ϕ_1, ϕ_2	Interpolation functions (Equation 17)	
λ_0, λ_1, λ_2	Interpolation functions (Equation 18)	

All other symbols, Tables 1 and 2, are defined in these Tables.

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