On the Transduction of Various Noise Sources in Optical Microtoroids

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ABSTRACT

Optical microresonators constitute the basic building block for numerous precision measurements including single-particle detection, magnetometry, force and position sensing. The ability to resolve a signal of interest is limited however by various noise sources. In this tutorial style paper we provide a matrix formalism to analyze the effect of various modulations upon the optical cavity. The technique can in principle be used to estimate the sensitivity of microresonator based sensors and potentially to identify the optimal detection basis and cavity parameters to optimise the signal to noise ratio.

Keywords: Microresonator, whispering gallery mode, noise, sensing

1. INTRODUCTION

Optical microresonators, including spherical and toroidal whispering gallery mode (WGM) resonators, constitute an ultra-sensitive, on-chip platform for a large number of sensing applications. These encompass label-free detection of molecules[1,2] and nanoparticles in liquid and gaseous environments,[3,4] force and position sensing,[5] magnetic field sensing[6,7] and applications in the burgeoning field of quantum optomechanics.[8] All these techniques relay on the ultra-precise detection of resonance frequency shifts. To improve and optimize current sensor systems it is essential to study the impact of various noise sources on the detection of the frequency shift and to select the most favorable detection scheme. In this tutorial style paper we provide a coherent formalism to treat general noise sources in optical microresonatos, i.e. toroidal whispering gallery mode resonators.

2. HAMILTONIAN DESCRIPTION

The system of interest treated in this document is composed of an optical cavity, in practice a microtoroid, upon which small, sub-wavelength scattering particles are attached. Following a similar approach that used previously,[3,9] we approximate the cavity in isolation as supporting only two degenerate modes at frequency ω_0. Each mode corresponds to light circulating in either the clockwise or anti-clockwise direction in the toroid. Other resonances, although they exist in reality, are assumed to have sufficiently different eigenfrequencies as to be irrelevant in the description of the system dynamics in the regime of interest. Gain and loss, corresponding to non-unitary evolution of the system, will be introduced later.

The general form of Hamiltonian describing the system can then be written as follows:

$$\mathcal{H} = \mathcal{H}^0_+ + \mathcal{H}^0_- + \sum_{n=1}^{N} \sum_{i,j} V^n_{i,j} \quad | \quad i,j \in (+,-) \tag{1}$$

This Hamiltonian is fundamentally composed of terms that describe a cavity in the absence of scattering interactions ($\mathcal{H}^0$) and terms that describe the interaction with scatters (V). The summations are performed over each of the N scatters in the system, and for each, additional sums are performed considering the 4 possible coupling terms with the 2 cavity modes $(+,-)$. The effect of the scattering terms upon the system is both to effectively

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shift the optical length of the cavity and to couple light between modes resulting in a splitting of the cavity eigenmodes. Explicitly, the Hamiltonian of an electromagnetic field may be expressed as:

$$\mathcal{H}^0_{\pm} = \hbar\omega_0(\hat{a}^\dagger_{\pm}\hat{a}_{\pm} + 1/2) \tag{2}$$

where $\hat{a}$ is the field amplitude operator. Here we have found it convenient to normalise the field operators such that $[\hat{a}^\dagger, \hat{a}] = 1$. The interaction components are computed as follows:

$$\begin{aligned}\mathcal{V}^n_{i,j} &= -\epsilon_0\eta_n\mathcal{R}[E_i(\mathbf{r}_n)]\mathcal{R}[E_j(\mathbf{r}_n)] && (3)\\ &\approx -\frac{\epsilon_0\eta_n}{2}\mathcal{E}_i\mathcal{E}_j[\hat{a}^\dagger_i\hat{a}_j(\mathbf{r}_n) + \hat{a}_i\hat{a}^\dagger_j(\mathbf{r}_n)] && (4)\\ &\approx -\frac{g^n_{i,j}\hbar}{2}[\hat{a}^\dagger_i\hat{a}_j(\mathbf{r}_n) + \hat{a}_i\hat{a}^\dagger_j(\mathbf{r}_n)] && (5)\end{aligned}$$

Equation 3 follows from the definition of the energy of the dipole field interaction, namely $\mathbf{p}\cdot\mathbf{E} = \epsilon_0\eta\mathbf{E}\cdot\mathbf{E}$, where η is the polarisability of the particle. Practically the polarisability may be determined from a particle's volume and refractive index via the Clausius-Mossotti relation. Recasting equation 3 in terms of the field operators (equation 4) has been done with the removal of terms in the envelope that vary on frequency scales of the order of $2\omega_0$ (c.f. the rotating wave approximation). Note the normalisation parameter $\mathcal{E}$ is the vacuum fluctuation associated with each mode respectively, where:

$$\mathcal{E}_i = \sqrt{\frac{\hbar\omega_0}{2\epsilon_0 V_i}} \tag{6}$$

In equation 5, the parameters describing the system are conveniently condensed to a single variable g, which encompasses the fixed parameters of the scattering particle:

$$g_{i,j} = -\epsilon_0\eta\mathcal{E}_i\mathcal{E}_j/\hbar \tag{7}$$

From symmetry arguments, it can be shown that for any given particle n, the coupling rates between the various modes are equal, therefore reducing the number of variables: $g^n_{i,j} = g_n$. Hence we may write:

$$g_n = \frac{\omega_0\eta_n}{2V} \tag{8}$$

It follows from equation 8, that the interaction terms in the Hamiltonian are enhanced by *decreasing* the mode volume. If we allow all N scatters to be located at the same position, the scattering terms for all N particles may be trivially combined to the scattering terms for a single particle, with altered scattering rates. If instead we allow the scatters to be located anywhere within the spatial distribution of the optical mode, the position dependence can be well approximated by a complex phase factor:

$$\hat{a}(\mathbf{r}_n) = e^{i\varphi}\hat{a} \tag{9}$$

This approximation of *frequency independent phase shift* is only valid for detunings that are small compared to the free spectral range of the cavity. Given that the FSR of a typical microtoroid is about 1.4THz and cavity linewidths/coupling coefficients are of order 10MHz, this approximation is abundantly satisfied. Hence we can write equation 5 as:

$$\mathcal{V}^n_{i,j} = -\frac{g_n\hbar}{2}[\hat{a}^\dagger_i e^{-i\varphi^n_i}\hat{a}_j e^{i\varphi^n_j} - \hat{a}_i e^{i\varphi^n_i}\hat{a}^\dagger_j e^{-i\varphi^n_j}] \tag{10}$$

For the n^{th} particle, the four scattering terms in equation 10 evaluate as:

$$\begin{aligned}\mathcal{V}^n_{\pm,\pm} &= -\frac{g_n\hbar}{2}[\hat{a}^\dagger_{\pm}\hat{a}_{\pm} + \hat{a}_{\pm}\hat{a}^\dagger_{\pm}] && (11)\\ \mathcal{V}^n_{\pm,\mp} &= -\frac{g_n\hbar}{2}[\hat{a}^\dagger_{\pm}e^{-i\varphi^n_{\pm}}\hat{a}_{\mp}e^{i\varphi^n_{\mp}} + \hat{a}_{\pm}e^{i\varphi^n_{\pm}}\hat{a}^\dagger_{\mp}e^{-i\varphi^n_{\mp}}] &&\\ &= -\frac{g_n\hbar}{2}[\hat{a}^\dagger_{\pm}\hat{a}_{\mp}e^{i\phi_n} + \hat{a}_{\pm}\hat{a}^\dagger_{\mp}e^{-i\phi_n}] && (12)\end{aligned}$$

Where the difference in accumulated phases is given by the variable
$\phi_n = (\varphi_-^n - \varphi_+^n)$. This angle can be viewed as twice the phase shift accumulated by either the clockwise or anticlockwise mode when traveling from the origin to the scatter. Expanding out all the terms in equation 1, the total Hamiltonian for the system can now be expressed as:

$$\mathcal{H} = \hbar\omega'[(\hat{a}_+^\dagger \hat{a}_+ + 1/2) + (\hat{a}_-^\dagger \hat{a}_- + 1/2)] - \hbar[\beta \hat{a}_+^\dagger \hat{a}_- + \beta^* \hat{a}_+ \hat{a}_-^\dagger] \tag{13}$$

with the new 'shifted' cavity frequency $\omega' = \omega_0 - \sum_n g_n$, and the complex scattering term $\beta = \sum_n g_n e^{i\phi_n}$.

As a note of interest, equation 13 shows the net effect of adding scatters to the system is twofold:

1. In the absence of cross-coupling, the cavity eigenmodes are shifted to lower frequencies by an amount equal to the sum of the magnitudes of the scatter strengths g.
2. The cross-coupling terms scatter energy between the shifted modes, with a magnitude proportional to the complex sum of the scatter strengths, weighted by a position dependent complex phase factor.

It is also apparent the two effects may be *independently* adjusted by suitable choice of the position and strength of the scatters.

2.1 Time Evolution of the System

The unitary time evolution of the operators is given by the Heisenberg equations of motion. Since we are interested in the behavior of an open system, additional terms that couple the system to external fields are required. These equations are termed the quantum Langevin equations of the system:

$$\begin{aligned}\dot{\hat{a}}_\pm = {} & \frac{i}{\hbar}[\mathcal{H}, \hat{a}_\pm] - (\gamma_c \hat{a}_\pm + \sqrt{2\gamma_c}\hat{a}_{env}) - (\gamma \hat{a}_\pm + \sqrt{2\gamma}\hat{d}_\pm) \\ & - \sum_n (\frac{\gamma_n}{2}[\hat{a}_\pm(\mathbf{r}_n) + \hat{a}_\mp(\mathbf{r}_n)] + \sqrt{2\gamma_n}\hat{a}_{env})\end{aligned} \tag{14}$$

where γ is the coupling rate to the external field $\hat{d}$; γ_c is the intrinsic loss rate of the cavity and $\hat{a}_{env}$ is the vacuum field operator. The terms on the first line of equation 14 give the dynamics of a toroid where the only damping is intrinsic cavity damping. The terms on the second line include the effects of additional noise and damping produced by the scatters.

For the moment we will neglect the vacuum noise by removing all $\hat{a}_{env}$ terms, and the dissipative effects of the scatters by disregarding the second line of equation 14, so that the langevin equations are reduced to:

$$\dot{\hat{a}}_\pm = \frac{i}{\hbar}[\mathcal{H}, \hat{a}_\pm] - \gamma_t \hat{a}_\pm - \sqrt{2\gamma}\hat{d}_\pm \tag{15}$$

where γ_t is the total distributed loss rate of the system, equal to $\gamma_t = \gamma + \gamma_c$. Evaluating the commutators and transforming into the rotating reference frame i.e. $\hat{a} = \tilde{a}e^{i\omega t}$ and isolating the time derivative of the slowly oscillating envelope gives the following two equations of motion:

$$\dot{\tilde{a}}_+ = -(i\Delta + \gamma_t)\tilde{a}_+ - i\beta^* \tilde{a}_- - \sqrt{2\gamma}\tilde{l}_{in} \tag{16}$$

$$\dot{\tilde{a}}_- = -(i\Delta + \gamma_t)\tilde{a}_- - i\beta\, \tilde{a}_+ - \sqrt{2\gamma}\tilde{r}_{in} \tag{17}$$

where the detuning Δ has been defined as $\Delta = \omega - \omega'$. These equations constitute the equations of motion of the system in the rotating reference frame. By setting the time derivative of the mode amplitudes to zero, we arrive at the steady-state solutions to the system. The equations may be readily cast in matrix form as follows, where we have condensed $\eta = i\Delta + \gamma_t$ for convenience.:

$$\begin{pmatrix} \eta & i\beta^* \\ i\beta & \eta \end{pmatrix} \begin{bmatrix} \tilde{a}_+ \\ \tilde{a}_- \end{bmatrix} = -\sqrt{2\gamma} \begin{bmatrix} \tilde{l}_{in} \\ \tilde{r}_{in} \end{bmatrix} \tag{18}$$

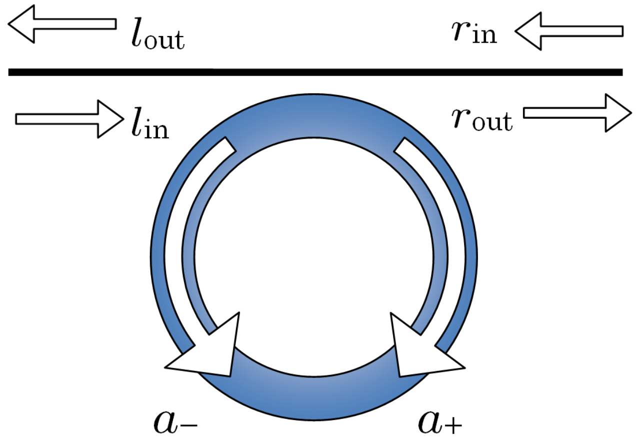

Figure 1. Illustration of the relevant fields in the taper-toroid system. Communication of the fields in the taper with the outside world is effected through two 'ports' here labeled left (l) and right (r) respectively. Each port supports both a forward and reverse propagating wave, which couple to either the clockwise (a_+) or anticlockwise (a_-) toroid mode respectively.

At this point we wish to introduce interference of out-coupled light within the taper. We first assume that the coupling rate into the toroid is small, and therefore that the field entering the taper is effectively un-depleted. This is known as the mean field approximation. In the rotating reference frame this may be written as follows:

$$\begin{bmatrix}\tilde{r}_{out}\\ \tilde{l}_{out}\end{bmatrix} = \begin{bmatrix}\tilde{l}_{in}\\ \tilde{r}_{in}\end{bmatrix} + \sqrt{2\gamma}\begin{bmatrix}\tilde{a}_{+}\\ \tilde{a}_{-}\end{bmatrix} \tag{19}$$

Equation 19 my now be inserted into equation 18 to obtain a relation between the incident and scattered fields in the taper:

$$\begin{bmatrix}\tilde{r}_{out}\\ \tilde{l}_{out}\end{bmatrix} = -2\gamma\begin{pmatrix}\eta & i\beta^{*}\\ i\beta & \eta\end{pmatrix}^{-1}\begin{bmatrix}\tilde{l}_{in}\\ \tilde{r}_{in}\end{bmatrix} + \begin{bmatrix}\tilde{l}_{in}\\ \tilde{r}_{in}\end{bmatrix} \tag{20}$$

This equation can be cast into the conventional scattering matrix form:

$$\begin{bmatrix}\tilde{r}_{out}\\ \tilde{l}_{out}\end{bmatrix} = \mathbf{S}\begin{bmatrix}\tilde{r}_{in}\\ \tilde{l}_{in}\end{bmatrix} \tag{21}$$

with:

$$\mathbf{S} = \frac{-2\gamma}{\eta^2 + |\beta|^2}\begin{pmatrix}-i\beta^{*} & \eta\\ \eta & -i\beta\end{pmatrix} + \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \tag{22}$$

Equation 22 defines the (experimentally accessible) steady-state scattering matrix for the taper-toroid system. Taking a moment to analyse the terms in the matrix, we see that considering a single input and output, the (diagonal) β terms are responsible for light reflected by the toroid, whilst transmitted light is an interference of the input (ones matrix) and the cavity modified light (the eta terms). When the toroid matrix (18) becomes nearly singular, the system response is at an extremum, which is manifested as a resonant dip or peak in transmission or reflection respectively. From the scattering matrix, the field transmission and reflection coefficients have the form:

$$T = 1 - \frac{2\gamma(i\Delta + \gamma_t)}{(i\Delta + \gamma_t)^2 + |\beta|^2} \qquad\qquad R = \frac{-2i\gamma\beta}{(i\Delta + \gamma_t)^2 + |\beta|^2} \tag{23}$$

where we have expanded out η to make the dependence upon detuning more explicit. A critical coupling condition can be defined where the transmitted amplitude decreases to zero at zero detuning. In this situation all the energy entering the system is scattered out of the toroid into free space or in the case of scattering particles on the surface of the toroid, additionally some light is reflected. Setting $T = 0$ and solving for the taper-toroid coupling rate we find:

$$\gamma = \sqrt{\gamma_c^2 + |\beta|^2} \tag{24}$$

In practice, this coupling rate depends upon the taper-toroid distance which may be varied until at some frequency, the transmitted light is seen to fall to zero. At at this point critical coupling has been achieved.

3. SIGNAL DETECTION AND NOISE

3.1 Perturbative Expansion

Having solved for the steady state, we can find the effect of modulation upon the system parameters by performing a perturbative expansion of the equations of motion. Let the field amplitude in the rotating wave basis be:

$$\tilde{a}(t) = \alpha + \delta\tilde{a}(t) \tag{25}$$

where α is the solution to the steady state equations of motion and $\delta\tilde{a}$ is a complex perturbation of the field. By substitution into the equations of motion, the perturbation must necessarily satisfy the following equations:

$$\delta\dot{\tilde{a}}_+ = -(i\Delta + \gamma_t)\delta\tilde{a}_+ - i\beta^*\delta\tilde{a}_- - \sqrt{2\gamma}\delta\tilde{l}_{in} \tag{26}$$

$$\delta\dot{\tilde{a}}_- = -(i\Delta + \gamma_t)\delta\tilde{a}_- - i\beta\,\delta\tilde{a}_+ - \sqrt{2\gamma}\delta\tilde{r}_{in} \tag{27}$$

and similarly, the consistency of the coupler equations is ensured only if:

$$\delta r_{out} = \delta l_{in} + \sqrt{2\gamma}\delta a_+ \qquad \delta l_{out} = \delta r_{in} + \sqrt{2\gamma}\delta a_- \tag{28}$$

Fourier transforming these equations with respect to the sideband modulation and writing as before the result in matrix form we obtain:

$$\begin{bmatrix}\delta\tilde{r}_{out}\\ \delta\tilde{l}_{out}\end{bmatrix} = \mathbf{S}(\Omega)\begin{bmatrix}\delta\tilde{r}_{in}\\ \delta\tilde{l}_{in}\end{bmatrix} \tag{29}$$

with:

$$\mathbf{S}(\Omega) = \frac{-2\gamma}{(\eta + i\Omega)^2 + |\beta|^2}\begin{pmatrix}-i\beta^* & \eta + i\Omega\\ \eta + i\Omega & -i\beta\end{pmatrix} + \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \tag{30}$$

where Ω is the sideband modulation frequency. It is apparent from (30) that the upper and lower sidebands produced by some modulation will have very different phase and amplitude response when there is a non-zero detuning of the cavity.

Completely analogous to the transmission and reflection coefficients described earlier in the steady state case (23), the appropriate scattering terms in the equations (30) give the generalised transmission and reflection coefficients:

$$T(\Omega) = 1 - \frac{2\gamma(\eta + i\Omega)}{(\eta + i\Omega)^2 + |\beta|^2} \qquad R(\Omega) = \frac{-2i\gamma\beta}{(\eta + i\Omega)^2 + |\beta|^2} \tag{31}$$

It is necessarily satisfied that in the limit as $\Omega \to 0$, equations 31 reduce to the steady-state coefficients (23).

3.2 Detection

Light falling on the detector produces a photocurrent in proportion to $i(t)$ as follows:

$$\begin{aligned} i(t) &= \hat{a}^\dagger(t)\hat{a}(t)\\ &= \tilde{a}^\dagger(t)\tilde{a}(t)\\ &= [\alpha^* + \delta\tilde{a}^\dagger(t)][\alpha + \delta\tilde{a}(t)]\\ &\approx |\alpha|^2 + \alpha\delta\tilde{a}^\dagger(t) + \alpha^*\delta\tilde{a}(t)\\ &\approx |\alpha|^2 + |\alpha|\delta\tilde{X}(t) \end{aligned} \tag{32}$$

where it is natural to define amplitude $\delta\tilde{X}$ and phase $\delta\tilde{Y}$ modulation operators respectively:

$$\delta\tilde{X}(t) = \frac{1}{|\alpha|}[\alpha\delta\tilde{a}^\dagger(t) + \alpha^*\delta\tilde{a}(t)] \tag{33}$$

$$\delta\tilde{Y}(t) = \frac{i}{|\alpha|}[\alpha\delta\tilde{a}^\dagger(t) - \alpha^*\delta\tilde{a}(t)] \tag{34}$$

Fourier transforming equation (32) gives:

$$i(\Omega) = |\alpha|^2\delta(\Omega) + |\alpha|\delta\tilde{X}(\Omega) \tag{35}$$

where Ω is the modulation frequency or detuning from the carrier and $\delta(\Omega)$ is the dirac delta. The spectral power density of the photocurrent (as measured by a spectrum analyser) can be well approximated as:

$$s(\Omega) = \langle\, |i(\Omega)|^2 \rangle \tag{36}$$

so for the photocurrent (35) we have a spectral density:

$$\begin{aligned} s(\Omega) = {} & |\alpha|^2 \langle \delta\tilde{X}^\dagger(\Omega)\delta\tilde{X}(\Omega)\rangle \\ & + \; \delta(\Omega)|\alpha|^3\langle \delta\tilde{X}^\dagger(\Omega) + \delta\tilde{X}(\Omega)\rangle + \delta^2(\Omega)|\alpha|^4 \end{aligned} \tag{37}$$

For the most part, the zero frequency terms may be disregarded, and we are simply left with:

$$s(\Omega) = |\alpha|^2\langle \delta\tilde{X}^\dagger(\Omega)\delta\tilde{X}(\Omega)\rangle \tag{38}$$

Thus the power spectrum measured by the spectrum analyser depends in a simple way only on amplitude modulation of the carrier.

3.3 The Effect of Laser Noise

In practice, the fields used to in the excitation of the toroid will always carry noise into the system. The effect of this noise on the measured output is modified by the action of the toroid. We can quantify the effect of the toroid upon the laser noise by expressing the output noise terms (in Fourier space) in terms of the input noise terms. We begin by combining 33 and 34, so that in Fourier space:

$$\delta\tilde{a}(\Omega) = \frac{\alpha}{2|\alpha|}[\delta\tilde{X}(\Omega) + i\delta\tilde{Y}(\Omega)] \tag{39}$$

$$\delta\tilde{a}^\dagger(-\Omega) = \frac{\alpha^*}{2|\alpha|}[\delta\tilde{X}(\Omega) - i\delta\tilde{Y}(\Omega)] \tag{40}$$

Additionally the Fourier transformed form of 33 gives:

$$\delta\tilde{X}(\Omega) = \frac{1}{|\alpha|}[\alpha\delta\tilde{a}^\dagger(-\Omega) + \alpha^*\delta\tilde{a}(\Omega)] \tag{41}$$

Then, as before the input and output fields are related by the appropriate terms in the scattering matrix:

$$\alpha_{ou,k} = S_k^m(\Delta)\alpha_{in,m} \tag{42}$$

$$\delta\tilde{a}_{ou,k}(\Omega) = S_k^m(\Delta+\Omega)\delta\tilde{a}_{in,m}(\Omega) \tag{43}$$

where we have used indexed notation to avoid explicitly writing the matrix products in the lines that follow. Rewriting 42 in indexed form we have:

$$|\alpha_{ou,k}|\delta\tilde{X}_{ou,k}(\Omega) = \alpha_{ou,k}\delta\tilde{a}^\dagger_{ou,k}(-\Omega) + \alpha^*_{ou,k}\delta\tilde{a}_{ou,k}(\Omega) \tag{44}$$

Then amplitude modulation of the output modes can then expressed in terms of the input modes using (42):

$$|\alpha_{ou,k}|\delta\tilde{X}_{ou,k}(\Omega) = S_k^m(\Delta)\alpha_{in,m}S_k^{n*}(\Delta-\Omega)\delta\tilde{a}_{in,n}^\dagger(-\Omega) + S_k^{m*}(\Delta)\alpha_{in,m}^* S_k^n(\Delta+\Omega)\delta\tilde{a}_{in,n}(\Omega) \tag{45}$$

Where summation over m,n is implied. Writing the input perturbation operators in terms of the input amplitude and phase modulations:

$$|\alpha_{ou,k}|\delta\tilde{X}_{ou,k}(\Omega) = S_k^m(\Delta)\alpha_{in,m}S_k^{n*}(\Delta-\Omega)\frac{\alpha_{in,n}^*}{2|\alpha_{in,n}|}[\delta\tilde{X}_{in,n}(\Omega) - i\delta\tilde{Y}_{in,n}(\Omega)] + S_k^{m*}(\Delta)\alpha_{in,m}^* S_k^n(\Delta+\Omega)\frac{\alpha_{in,n}}{2|\alpha_{in,n}|}[\delta\tilde{X}_{in,n}(\Omega) + i\delta\tilde{Y}_{in,n}(\Omega)] \tag{46}$$

Finally collecting terms:

$$|\alpha_{ou,k}|\delta\tilde{X}_{ou,k}(\Omega) = \left[S_k^m(\Delta)\alpha_{in,m}S_k^{n*}(\Delta-\Omega)\frac{\alpha_{in,n}^*}{2|\alpha_{in,n}|} + S_k^{m*}(\Delta)\alpha_{in,m}^* S_k^n(\Delta+\Omega)\frac{\alpha_{in,n}}{2|\alpha_{in,n}|}\right]\delta\tilde{X}_{in,n}(\Omega) + i\left[S_k^{m*}(\Delta)\alpha_{in,m}^* S_k^n(\Delta+\Omega)\frac{\alpha_{in,n}}{2|\alpha_{in,n}|} - S_k^m(\Delta)\alpha_{in,m}S_k^{n*}(\Delta-\Omega)\frac{\alpha_{in,n}^*}{2|\alpha_{in,n}|}\right]\delta\tilde{Y}_{in,n}(\Omega) \tag{47}$$

This is the most general relation connecting input phase and amplitude modulations to output amplitude modulations in a linear system.

Aside: pure phase or amplitude modulations on the carrier depend on the scattering parameters at both upper $(\omega+\Omega)$ and lower $(\omega-\Omega)$ sidebands. This naturally arises due to generation of two sideband frequencies produced when purely amplitude or phase modulating the carrier. Notably if the carrier is modulated such that $\delta\tilde{Y}_{in,j} = \pm i\delta\tilde{X}_{in,j}$, then only the lower or upper sideband is produced and the transfer function depends only upon $\omega-\Omega$ or $\omega+\Omega$ respectively.

This transfer function contains many interference terms, as modulation terms originating from each of the inputs are mixed with each other. However, if we have only have a single input and are concerned with just a single output, then we can remove the indices in equation (47) and write (without the Δ dependence made explicit):

$$|\alpha_{ou}|\delta\tilde{X}_{ou}(\Omega) = \frac{|\alpha_{in}|}{2}[\sigma(0)\sigma^*(-\Omega) + \sigma^*(0)\sigma(+\Omega)]\delta\tilde{X}_{in} + \frac{i|\alpha_{in}|}{2}[\sigma^*(0)\sigma(+\Omega) - \sigma(0)\sigma^*(-\Omega)]\delta\tilde{Y}_{in} \tag{48}$$

where σ is the relevant scattering parameter between the input and output modes. Equation (48) may be written in a more compact fashion:

$$|\alpha_{ou}|\delta\tilde{X}_{ou}(\Omega) = |\alpha_{in}|\left[A\delta\tilde{X}_{in} + iB\delta\tilde{Y}_{in}\right] \tag{49}$$

with the following parameters defined:

$$A = [\sigma(0)\sigma^*(-\Omega) + \sigma^*(0)\sigma(+\Omega)]/2 \tag{50}$$

$$B = [\sigma^*(0)\sigma(+\Omega) - \sigma(0)\sigma^*(-\Omega)]/2 \tag{51}$$

For the toroid-taper system under consideration, the two particular cases of interest are $\sigma(\Omega) = T(\Omega)$; $\sigma(\Omega) = R(\Omega)$ (from (31)) for the forward and reverse scattered modes.

The detected signal spectral density can now be found straight-forwardly, in terms of the input variances:

$$s(\Omega) = |\alpha_{ou}|^2\langle\delta\tilde{X}_{ou}^\dagger(\Omega)\delta\tilde{X}_{ou}(\Omega)\rangle \tag{52}$$

$$= |\alpha_{in}|^2\left[A^*A\langle\delta\tilde{X}_{in}^\dagger\delta\tilde{X}_{in}\rangle + B^*B\langle\delta\tilde{Y}_{in}^\dagger\delta\tilde{Y}_{in}\rangle + A^*B\langle\delta\tilde{X}_{in}^\dagger\delta\tilde{Y}_{in}\rangle + B^*A\langle\delta\tilde{Y}_{in}^\dagger\delta\tilde{X}_{in}\rangle\right] \tag{53}$$

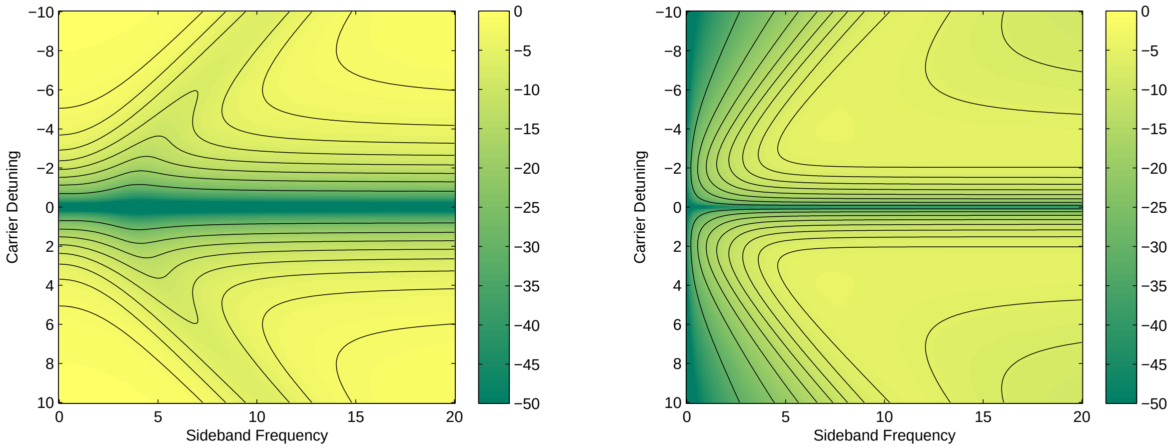

Figure 2. Normalised spectral density of detected signal for forward-scattered (transmitted) light. Shown on the left and right are sensitivity to amplitude modulation and phase modulation respectively. A scattering rate equal to the intrinsic decay rate is present. All frequencies normalised to the cavity decay rate and the cavity is critically coupled to the source.

For many noise generating processes, the phase and amplitude quadratures of the perturbation are not correlated, hence we may write:

$$s(\Omega) = |\alpha_{in}|^2 \left[A^* A \langle \delta\tilde{X}_{in}^\dagger \delta\tilde{X}_{in} \rangle + B^* B \langle \delta\tilde{Y}_{in}^\dagger \delta\tilde{Y}_{in} \rangle \right] \tag{54}$$

Plots of the coefficients preceding the phase modulation and amplitude modulation terms for transmission are shown in figure (2)

3.4 Cavity Noise

In addition to noise on the laser, noise will be introduced into the output by fluctuations of the cavity length. These fluctuations may pricipally be induced by thermal fluctuations, c.f. thermo-refractive noise.[10] By modifying the equations of motion to allow for a modulation in the cavity length as a function of time we have the following:

$$\dot{\tilde{a}}_+ = -(i\Delta(t) + \gamma_t)\tilde{a}_+ - i\beta^*\tilde{a}_- - \sqrt{2\gamma}\tilde{l}_{in} \tag{55}$$

$$\dot{\tilde{a}}_- = -(i\Delta(t) + \gamma_t)\tilde{a}_- - i\beta\ \tilde{a}_+ - \sqrt{2\gamma}\tilde{r}_{in} \tag{56}$$

now making the making the expansion about some stationary state:

$$\tilde{a}(t) = \alpha + \delta\tilde{a}(t) \tag{57}$$

$$\Delta(t) = \Delta + \delta\omega(t) \tag{58}$$

After linearisation and disregarding all other possible modulations, we may write:

$$\dot{\delta\tilde{a}}_+ = -(i\Delta + \gamma_t)\delta\tilde{a}_+ - i\beta^*\delta\tilde{a}_- - i\alpha_+\delta\omega \tag{59}$$

$$\dot{\delta\tilde{a}}_- = -(i\Delta + \gamma_t)\delta\tilde{a}_- - i\beta\ \delta\tilde{a}_+ - i\alpha_-\delta\omega \tag{60}$$

Linearity ensures we may additively introduce additional modulation terms later if required. Fourier transforming these equations into matrix form gives:

$$\begin{pmatrix} \eta + i\Omega & i\beta^* \\ i\beta & \eta + i\Omega \end{pmatrix} \begin{bmatrix} \delta\tilde{a}_+(\Omega) \\ \delta\tilde{a}_-(\Omega) \end{bmatrix} = -i\delta\omega(\Omega) \begin{bmatrix} \alpha_+ \\ \alpha_- \end{bmatrix} \tag{61}$$

This result is then combined with the steady-state cavity equations and the coupling equations:

$$\begin{pmatrix} \eta & i\beta^* \\ i\beta & \eta \end{pmatrix} \begin{bmatrix} \alpha_+ \\ \alpha_- \end{bmatrix} = -\sqrt{2\gamma} \begin{bmatrix} \tilde{l}_{in} \\ \tilde{r}_{in} \end{bmatrix} \tag{62}$$

$$\begin{bmatrix} \delta\tilde{r}_{out} \\ \delta\tilde{l}_{out} \end{bmatrix} = \sqrt{2\gamma} \begin{bmatrix} \delta\tilde{a}_{+} \\ \delta\tilde{a}_{-} \end{bmatrix} \tag{63}$$

ultimately resulting in the following equation:

$$\begin{bmatrix} \delta\tilde{r}_{out} \\ \delta\tilde{l}_{out} \end{bmatrix} = \delta\mathbf{W}(\Omega) \begin{bmatrix} \tilde{r}_{in} \\ \tilde{l}_{in} \end{bmatrix} \tag{64}$$

with:

$$\delta\mathbf{W}(\Omega) = \frac{2i\gamma\delta\omega(\Omega)}{((\eta+i\Omega)^2+|\beta|^2)(\eta^2+|\beta|^2)} \begin{pmatrix} \eta+i\Omega & -i\beta^* \\ -i\beta & \eta+i\Omega \end{pmatrix} \begin{pmatrix} -i\beta^* & \eta \\ \eta & -i\beta \end{pmatrix} \tag{65}$$

Of particular interest are the transmitted and reflected light transfer functions. For the transmitted light, that is the term taking l_{in} to δr_{out}, the function may be explicitly written as follows:

$$\delta a_T(\Omega) = \frac{2i\gamma[\eta(\eta+i\Omega)-|\beta|^2]}{(\eta^2+|\beta|^2)((\eta+i\Omega)^2+|\beta|^2)}\alpha_{in}\delta\omega(\Omega) \tag{66}$$

and in the limit of small modulation frequency this becomes:

$$\delta a_T(\Omega) \approx \frac{-2i\gamma}{(\eta^2+|\beta|^2)}\left[\frac{|\beta|^2-\eta^2}{|\beta|^2+\eta^2}\right]\alpha_{in}\delta\omega(\Omega) \tag{67}$$

The same equation may be obtained by the substitution $\eta \rightarrow \eta + i\delta\omega$ and linearising the steady state equation (23) in $\delta\omega$.

Similarly for the reflected light, corresponding to the scattering element transfering l_{in} to δl_{out} we have:

$$\delta a_R(\Omega) = \frac{2\gamma\beta[2\eta+i\Omega]}{(\eta^2+|\beta|^2)((\eta+i\Omega)^2+|\beta|^2)}\alpha_{in}\delta\omega(\Omega) \tag{68}$$

and in the limit of small modulation frequency:

$$\delta a_R(\Omega) \approx \frac{-2i\gamma}{\eta^2+|\beta|^2}\left[\frac{2i\beta\eta}{\eta^2+|\beta|^2}\right]\alpha_{in}\delta\omega(\Omega) \tag{69}$$

which again may be shown to be equal to the perturbative expansion of the relevant stationary state transfer function. It is convenient to label the coefficient in front of the input field-modulation product (e.g. $\alpha_{in}\delta\omega(\Omega)$) as a new variable $\delta T(\Omega)$. We can then easily find the detected signal caused by the modulation by writing the output modulation in terms of the amplitude quadrature. That is:

$$\begin{aligned} \delta\tilde{X}_{ou}(\Omega) &= \frac{1}{|T||\alpha_{in}|}[T\delta T^*(-\Omega)\alpha_{in}\alpha_{in}^*\delta\omega^\dagger(-\Omega) + T^*\delta T(\Omega)\alpha_{in}^*\alpha_{in}\delta\omega(\Omega)] \\ &= \frac{|\alpha_{in}|}{|T|}[T\delta T^*(-\Omega) + T^*\delta T(\Omega)]\delta\omega(\Omega) \qquad (70) \\ &= \frac{|\alpha_{in}|}{|T|}C_T\delta\omega(\Omega) \qquad (71) \end{aligned}$$

Hence for the transmitted light we have the signal:

$$s(\Omega) = |\alpha_{in}|^4|T|^2 C_T C_T^* \langle\delta\omega^\dagger(\Omega)\delta\omega(\Omega)\rangle \tag{72}$$

with:

$$C_T = T\delta T^*(-\Omega) + T^*\delta T(\Omega) \tag{73}$$

Similarly in the case of reflection the detected signal becomes:

$$s(\Omega) = |\alpha_{in}|^4|R|^2 C_R C_R^* \langle\delta\omega^\dagger(\Omega)\delta\omega(\Omega)\rangle \tag{74}$$

with:

$$C_R = R\delta R^*(-\Omega) + R^*\delta R(\Omega) \tag{75}$$

Figure (3) is a plot of $|R|^2 C_R C_R^*$ and $|T|^2 C_T C_T^*$ respectively.

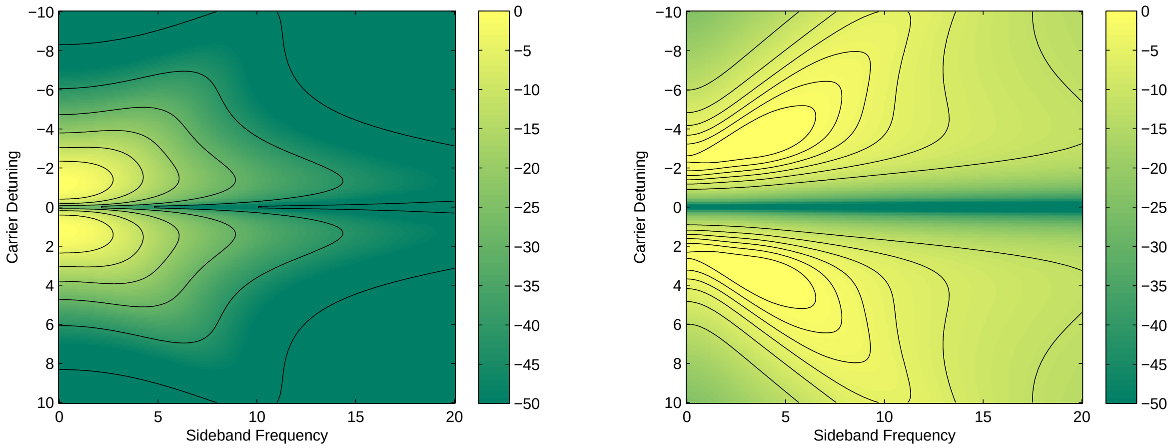

Figure 3. Cavity-jitter induced output modulation. On the left and right are the normalised modulation amplitudes of the reflected and transmitted light respectively. A scattering rate equal to the intrinsic decay rate is present. All frequencies normalised to the cavity decay rate and the cavity is critically coupled to the source. Since modulations of the output amplitude on resonance become second order, the transduction there approaches zero.

3.5 Defect Modulation

Similarly to the case of cavity jitter, the equations of motion are changed appropriately to allow for a modulation in the scatters as a function of time:

$$\dot{\tilde{a}}_+ = -(i\Delta + \gamma_t)\tilde{a}_+ - i\beta(t)^*\tilde{a}_- - \sqrt{2\gamma}\tilde{l}_{in} \tag{76}$$

$$\dot{\tilde{a}}_- = -(i\Delta + \gamma_t)\tilde{a}_- - i\beta(t)\ \tilde{a}_+ - \sqrt{2\gamma}\tilde{r}_{in} \tag{77}$$

now making the making the expansion about some stationary state:

$$\tilde{a}(t) = \alpha + \delta\tilde{a}(t) \tag{78}$$

$$\beta(t) = \beta_0 + \delta\beta(t) \tag{79}$$

and disregarding modulations on the input, the linearised equations of motion may be written as follows:

$$\dot{\delta\tilde{a}}_+ = -(i\Delta + \gamma_t)\delta\tilde{a}_+ - i\beta^*\delta\tilde{a}_- - i\alpha_-\delta\beta^* \tag{80}$$

$$\dot{\delta\tilde{a}}_- = -(i\Delta + \gamma_t)\delta\tilde{a}_- - i\beta\ \delta\tilde{a}_+ - i\alpha_+\delta\beta \tag{81}$$

Then Fourier transforming and recasting in matrix form gives:

$$\begin{pmatrix} \eta + i\Omega & i\beta^* \\ i\beta & \eta + i\Omega \end{pmatrix} \begin{bmatrix} \delta\tilde{a}_+(\Omega) \\ \delta\tilde{a}_-(\Omega) \end{bmatrix} = -i \begin{pmatrix} 0 & \delta\beta^*(-\Omega) \\ \delta\beta(\Omega) & 0 \end{pmatrix} \begin{bmatrix} \alpha_+ \\ \alpha_- \end{bmatrix} \tag{82}$$

combining this result with the steady-state cavity equations and coupler equations respectively:

$$\begin{pmatrix} \eta & i\beta^* \\ i\beta & \eta \end{pmatrix} \begin{bmatrix} \alpha_+ \\ \alpha_- \end{bmatrix} = -\sqrt{2\gamma} \begin{bmatrix} \tilde{l}_{in} \\ \tilde{r}_{in} \end{bmatrix} \tag{83}$$

$$\begin{bmatrix} \delta\tilde{r}_{out} \\ \delta\tilde{l}_{out} \end{bmatrix} = \sqrt{2\gamma} \begin{bmatrix} \delta\tilde{a}_+ \\ \delta\tilde{a}_- \end{bmatrix} \tag{84}$$

the transfer function may ultimately be written in the following matrix form:

$$\begin{bmatrix} \delta\tilde{r}_{out} \\ \delta\tilde{l}_{out} \end{bmatrix} = \delta\mathbf{G}(\Omega) \begin{bmatrix} \tilde{r}_{in} \\ \tilde{l}_{in} \end{bmatrix} \tag{85}$$

where the transfer matrix (which contains the modulation terms) is given by:

$$\delta\mathbf{G}(\Omega) = \frac{-2i\gamma}{((\eta+i\Omega)^2+|\beta|^2)(\eta^2+|\beta|^2)}\begin{pmatrix}\eta+i\Omega & -i\beta^*\\ -i\beta & \eta+i\Omega\end{pmatrix}\begin{pmatrix}0 & \delta\beta^*(-\Omega)\\ \delta\beta(\Omega) & 0\end{pmatrix}\begin{pmatrix}-i\beta^* & \eta\\ \eta & -i\beta\end{pmatrix} \tag{86}$$

Writing out the modulation term specifically responsible for transmission from l_{in} to δr_{out} we have:

$$\delta a_T(\Omega) = \frac{-2\gamma[(\eta+i\Omega)\beta\delta\beta^*(-\Omega)+\eta\beta^*\delta\beta(\Omega)]}{(\eta^2+|\beta|^2)((\eta+i\Omega)^2+|\beta|^2)}\alpha_{in} \tag{87}$$

and, in the limit of $\Omega << \Delta$, the transfer function becomes:

$$\delta a_T(\Omega) = \frac{-2\gamma\eta[\beta\delta\beta^*(-\Omega)+\beta^*\delta\beta(\Omega)]}{(\eta^2+|\beta|^2)^2}\alpha_{in} \tag{88}$$

Equation 88 implies that slow modulation of the azimuthal position of the defect is to first order *undetectable*, while slow modulations of the defect strength produces first order modulation of the output. Similarly for the reflected light, corresponding to the scattering element transferring l_{in} to δl_{out} we have:

$$\delta a_R(\Omega) = \frac{-2i\gamma[\eta(\eta+i\Omega)\delta\beta(\Omega)-\beta^2\delta\beta^*(-\Omega)]}{(\eta^2+|\beta|^2)((\eta+i\Omega)^2+|\beta|^2)}\alpha_{in} \tag{89}$$

which in the limit of small Ω becomes:

$$\delta a_R(\Omega) = \frac{-2i\gamma[\eta^2\delta\beta(\Omega)-\beta^2\delta\beta^*(-\Omega)]}{(\eta^2+|\beta|^2)^2}\alpha_{in} \tag{90}$$

This function implies the modulation on the reflected light is first order sensitive to both the position and strength of the scatting particles. In practice we wish to find the effect of the modulation of a single scatterer on a system possessing a number of scattering particles. For an amplitude and phase modulation of a scatter respectively we therefore have:

$$\delta\beta(\Omega) = e^{i\phi_n}\delta g_n(\Omega) \tag{91}$$

$$\delta\beta(\Omega) = ie^{i\phi_n}g_n\delta\phi(\Omega) \tag{92}$$

where we have applied a modulation on the n^{th} scatterer, with scattering magnitude g_n and phase ϕ_n. Hence for the amplitude modulation from equations (87; 89) we have:

$$\delta a_T(\Omega) = \frac{-2\gamma[\eta\beta^*e^{i\phi_n}+(\eta+i\Omega)\beta e^{-i\phi_n}]}{(\eta^2+|\beta|^2)((\eta+i\Omega)^2+|\beta|^2)}\delta g_n(\Omega)\alpha_{in} \tag{93}$$

and

$$\delta a_R(\Omega) = \frac{2i\gamma[\beta^2e^{-i\phi_n}-\eta(\eta+i\Omega)e^{i\phi_n}]}{(\eta^2+|\beta|^2)((\eta+i\Omega)^2+|\beta|^2)}\delta g_n(\Omega)\alpha_{in} \tag{94}$$

where we have identified $\delta g_n(\Omega) = \delta g_n^*(-\Omega)$. Similarly for the phase modulation:

$$\delta a_T(\Omega) = \frac{-2i\gamma g_n[\eta\beta^*e^{i\phi_n}-(\eta+i\Omega)\beta e^{-i\phi_n}]}{(\eta^2+|\beta|^2)((\eta+i\Omega)^2+|\beta|^2)}\delta\phi_n(\Omega)\alpha_{in} \tag{95}$$

and

$$\delta a_R(\Omega) = \frac{2\gamma g_n[\beta^2e^{-i\phi_n}+\eta(\eta+i\Omega)e^{i\phi_n}]}{(\eta^2+|\beta|^2)((\eta+i\Omega)^2+|\beta|^2)}\delta\phi_n(\Omega)\alpha_{in} \tag{96}$$

where again we have noted $\delta\phi_n(\Omega) = \delta\phi_n^*(-\Omega)$. The detected signals for the possible phase and amplitude modulations can be found as before using equation (71). Specifically, for the transmitted light we have:

$$s(\Omega) = |\alpha_{in}|^4|T|^2C_T^{\phi}C_T^{\phi*}\langle\delta\phi^\dagger(\Omega)\delta\phi(\Omega)\rangle \tag{97}$$

$$s(\Omega) = |\alpha_{in}|^4|T|^2C_T^{g}C_T^{g*}\langle\delta g^\dagger(\Omega)\delta g(\Omega)\rangle \tag{98}$$

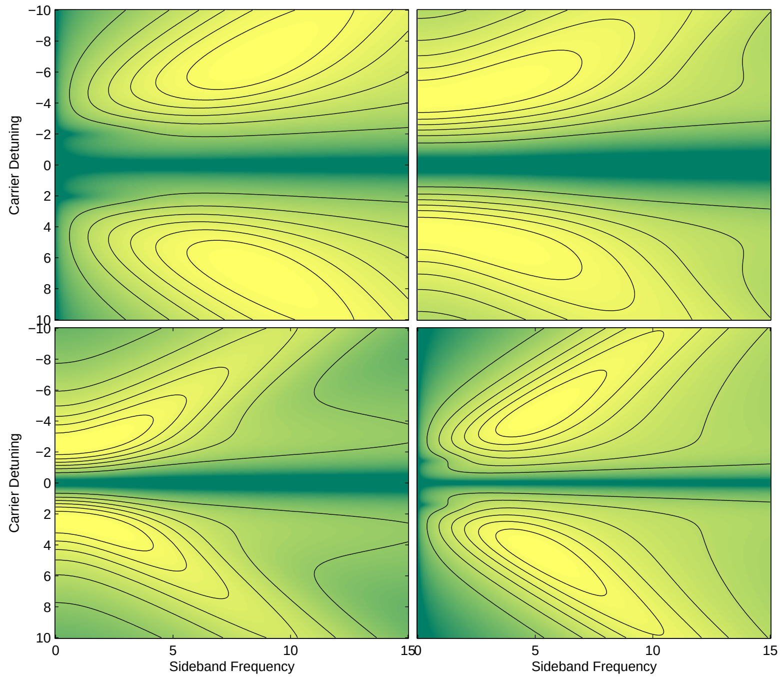

Figure 4. Output modulation induced by defect motion. On the left and right are the normalised modulation amplitudes of the transmitted light when the scatter is phase or amplitude modulated respectively. A scattering rate equal to the intrinsic decay rate is present in addition to a time dependent scatter, of equal mean scattering rate. The phase angle between the two scatters is changed by $3\pi/4$ between the top and bottom plots.

with:

$$C_T^{\phi} = T(0)\delta T^{\phi *}(-\Omega) + T^*(0)\delta T^{\phi}(\Omega) \tag{99}$$

$$C_T^{g} = T(0)\delta T^{g *}(-\Omega) + T^*(0)\delta T^{g}(\Omega) \tag{100}$$

where the δT's are the coefficients preceding the respective modulation terms in (93-96).

Some examples of the detected signals for the transmitted light under both phase and amplitude modulation of the scatter are shown in figure (4). It is evident from the figure that under certain circumstances, phase and amplitude modulation of the scatterer can produce near identical spectra on the detector. This is due to the fact that, as stated earlier, the transmitted signal depends largely on the amplitude quadrature of the modulation of the scattering parameter β. Dependent upon the relative orientation of the scatters, either a modulation in the scattering strength *or* a modulation in the phase of the scatter will produce a modulation in the amplitude of β.

3.6 Combined Noise at the Detector

Due to the linearity of the effects previously outlined, they may be combined additively to produce the total output modulation. That is we may write:

$$\begin{bmatrix} \delta\tilde{r}_{out} \\ \delta\tilde{l}_{out} \end{bmatrix} = \mathbf{S}(\Omega) \begin{bmatrix} \delta\tilde{r}_{in} \\ \delta\tilde{l}_{in} \end{bmatrix} + \delta\mathbf{G}(\Omega) \begin{bmatrix} \tilde{r}_{in} \\ \tilde{l}_{in} \end{bmatrix} + \delta\mathbf{W}(\Omega) \begin{bmatrix} \tilde{r}_{in} \\ \tilde{l}_{in} \end{bmatrix} \tag{101}$$

In the context of detection of the change of state of a scattering particle, the signal to noise may be computed by comparing the magnitude of the appropriate quadratures of the $\delta\mathbf{G}$ (scatter) terms to the other terms in the equation. Alternatively, in the context of detection of cavity resonance shifts in the presence of laser noise, the magnitude of $\delta\mathbf{W}$ (cavity) terms may be compared to the laser noise terms.

3.7 Summary

Using a matrix formalism we have shown the linearised influence of both laser and cavity noise on the output signal of a fibre-coupled optical microresonator. Addtionally, using the same formalism, we have derived general expressions for the signal strength produced by a modulated interaction with a scattering paticle. This treatment includes the effect of both polarisability *and* binding location of the scattering particle and the effect of interference with existing (fixed) scatters. The formalism provides the tools to compare the sensitivity of different detection methods, i.e. by comparing the signal-to-noise ratio.

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