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Quasi-exact treatment of the relativistic generalized isotonic oscillator

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We investigate the pseudospin symmetry case of a spin-$\frac{1}{2}$ particle governed by the generalized isotonic oscillator, by presenting quasi-exact polynomial solutions of the Dirac equation with pseudospin symmetry vector and scalar potentials. The resulting equation is found to be quasi-exactly solvable owing to the existence of a hidden $sl(2)$ algebraic structure. A systematic and closed form solution to the basic equation is obtained using the Bethe ansatz method. Analytic expression for the energy is obtained and the wavefunctions are derived in terms of the roots to a set of Bethe ansatz equations. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4712298]

I. INTRODUCTION

In a recent study,[1] the non-relativistic one-dimensional quantum system described by the non-polynomial potential of the form

$$V(x) = \frac{\omega^2 x^2}{2} + g_a \frac{x^2 - a^2}{(x^2 + a^2)^2}, \tag{1}$$

where a is a positive parameter, was shown to be exactly solvable for the particular case of $g_a = 2$ and $\omega a^2 = 1/2$, with a general solution

$$\begin{cases} \Psi_n(x) = \frac{P_n(x)}{(2x^2+1)} e^{-x^2/2}, \\ E_n = -\frac{3}{2} + n, \quad n = 0, 3, 4, 5, \ldots \end{cases} \tag{2}$$

where $\Psi_n(x)$ is the wavefunction and E_n is the energy and the polynomial $P_n(x)$ relates to the Hermite polynomial as follows:

$$P_n(x) = \begin{cases} 1 & \text{for} \quad n = 0, \\ H_n(x) + 4n H_{n-2}(x) + 4n(n-3) H_{n-4}(x) & \text{for} \quad n = 3, 4, 5, \ldots \end{cases} \tag{3}$$

Also, using the supersymmetric approach[2] and for certain values of the parameters g_a, a, and ω, potential (1) has been shown to be a supersymmetric partner of the harmonic oscillator potential. Moreover, in a very recent work,[3] by employing the Möbius transformation, the Schrödinger equation for potential (1) was transformed into a confluent Heun equation and a simple and efficient algorithm to solve the problem numerically irrespective of the values of the parameters was presented. In addition, the 3D case of the potential was studied for the quasi-polynomial solutions in cases where the potential parameters satisfy certain conditions and using the asymptotic iterative method, the authors obtain numerical solutions to the problem for a more general case.[4]

On the other hand, due to the limited application of exactly solvable systems, recent attentions have been on the systems with partially solvable spectral. Such systems are said to be quasi-exactly solvable (QES). Thus, a quantum mechanical system is called quasi-exactly solvable, if only a finite number of eigenvalues and corresponding eigenvectors can be obtained exactly.[5] An essential

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feature of a QES system is that having separated the asymptotic behaviours of the system, one gets an equation for the part which can be expanded as a power series of the basic variable. This equation unlike an exactly solvable equation with two-step recursive relations, possesses at least three-step recursive relations for the coefficient of the power series. The complexity of the recursive relations does not allow one to guarantee the square integrability property of the wavefunction. However, by choosing a polynomial wavefunction, one can terminate the series at a certain order and then impose a sufficient condition for the normalization. By so doing, exact solutions to the original problem can be obtained but only for certain energies and for special values of the parameters of the problem.

Solutions to QES systems have mostly been discussed in terms of the recursion relations of the power series coefficients, which is mostly expressed in terms of the (generalized) Heun differential equations. Although the solutions obtained in connection with the Heun equations are exact but the procedures involved are quite ambiguous, thus expunging the closed form of the solutions. In a series of recent studies,[6–10] the Bethe ansatz method (BAM) has been used in obtaining the solutions to QES systems. This method did not only yield exact solutions, it also preserve the closed form representation of the solutions. For instance, the BAM has been used to obtain the solutions of QES difference equation[6] and the exact polynomial solutions of general quantum non-linear optical models[7,8] and recently, the method has also been used to obtain the exact solutions for a family of spin-boson systems.[10]

The purpose of this work is to extend the study of the generalized isotonic oscillator (GIO) to a relativistic case, within the framework of the pseudospin symmetry Dirac equation, using the Bethe ansatz method. In Sec. II, we reduce the Dirac equation with the GIO to a QES equation. A brief discussion of the Bethe ansatz method is given in Sec. III followed by the solution to the reduced equation. Section IV presents the Lie algebraic structure of the system and then some concluding remarks are given in Sec. V.

II. DIRAC EQUATION WITH THE GIO

The Dirac equation for a single-nucleon with mass μ moving in a spherically symmetric attractive scalar $S(r)$ and repulsive vector $V(r)$ GIO, with $\hbar = c = 1$ is written as[11–13]

$$H\Psi(\mathbf{r}) = E_n\Psi(\mathbf{r}), \quad \text{where} \quad H = \sum_{j=1}^{3}\hat{\alpha}_j p_j + \hat{\beta}[\mu + S(r)] + V(r) \tag{4}$$

and E_n is the relativistic energy, $\{\hat{\alpha}_j\}$ and $\hat{\beta}$ are Dirac matrices defined as

$$\hat{\alpha}_j = \begin{pmatrix} 0 & \hat{\sigma}_j \\ \hat{\sigma}_j & 0 \end{pmatrix} \qquad \hat{\beta} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \tag{5}$$

where $\hat{\sigma}_j$ is the Pauli's 2×2 matrices and $\hat{\beta}$ is a 2×2 unit matrix, which satisfy anti-commutation relations

$$\begin{aligned} \hat{\alpha}_j\hat{\alpha}_k + \hat{\alpha}_k\hat{\alpha}_j &= 2\delta_{jk}\mathbf{1}, \\ \hat{\alpha}_j\hat{\beta} + \hat{\beta}\hat{\alpha}_j &= 0, \\ \hat{\alpha}_j^2 = \hat{\beta}^2 &= \mathbf{1}, \end{aligned} \tag{6}$$

and p_j is the three momentum which can be written as

$$p_j = -i\partial_j = -i\frac{\partial}{\partial x_j} \quad 1 \leqslant j \leqslant 3.$$

The orbital angular momentum operators L_{jk}, the spinor operators S_{jk}, and the total angular momentum operators J_{jk} can be defined as follows:

$$L_{jk} = -L_{jk} = ix_j\frac{\partial}{\partial x_k} - ix_k\frac{\partial}{\partial x_j}, \quad S_{jk} = -S_{kj} = i\hat{\alpha}_j\hat{\alpha}_k/2, \quad J_{jk} = L_{jk} + S_{jk},$$

$$L^2 = \sum_{j<k}^{3} L_{jk}^2, \quad S^2 = \sum_{j<k}^{3} S_{jk}^2, \quad J^2 = \sum_{j<k}^{3} J_{jk}^2, \quad 1 \leqslant j < k \leqslant 3. \tag{7}$$

For a spherically symmetric potential, total angular momentum operator J_{jk} and the spin-orbit operator $\hat{K} = -\hat{\beta}(J^2 - L^2 - S^2 + 1/2)$ commutate with the Dirac Hamiltonian. For a given total angular momentum j, the eigenvalues of $\hat{K}$ are $\kappa = \pm(j + 1/2)$; $\kappa = -(j + 1/2)$ for aligned spin $j = \ell + \frac{1}{2}$ and $\kappa = (j + 1/2)$ for unaligned spin $j = \ell - \frac{1}{2}$. Moreover, the spin-orbital quantum number κ is related to the orbital angular quantum number ℓ and the pseudo-orbital angular quantum number $\tilde{\ell} = \ell + 1$ by the expressions $\kappa(\kappa + 1) = \ell(\ell + 1)$ and $\kappa(\kappa - 1) = \tilde{\ell}(\tilde{\ell} + 1)$, respectively, for $\kappa = \pm 1, \pm 2, \ldots$. The spinor wavefunctions can be classified according to the radial quantum number n and the spin-orbital quantum number κ and can be written using the Dirac-Pauli representation

$$\Psi(\mathbf{r}) = \frac{1}{r}\begin{pmatrix} F(r)Y^{\ell}_{jm}(\theta, \phi) \\ iG(r)Y^{\tilde{\ell}}_{jm}(\theta, \phi) \end{pmatrix}, \tag{8}$$

where $F(r)$ and $G(r)$ are the radial wavefunction of the upper- and the lower-spinor components, respectively, $Y^{\ell}_{jm}(\theta, \phi)$ and $Y^{\tilde{\ell}}_{jm}(\theta, \phi)$ are the spherical harmonic functions coupled with the total angular momentum j. The orbital and the pseudo-orbital angular momentum quantum numbers for spin symmetry ℓ and pseudospin symmetry $\tilde{\ell}$ refer to the upper- and lower-component, respectively.

Substituting Eq. (8) into Eq. (4), and separating the variables we obtain the following coupled radial Dirac equation for the spinor components:

$$\left(\frac{d}{dr} + \frac{\kappa}{r}\right) F(r) = [\mu + E_n - \Delta(r)]G(r), \tag{9a}$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r}\right) G(r) = [\mu - E_n + \Sigma(r)]F(r), \tag{9b}$$

where $\Sigma(r) = V(r) + S(r)$ and $\Delta(r) = V(r) - S(r)$. Using Eq. (9a) as the upper component and substituting into Eq. (9b), we obtain the following second order differential equations:

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} - [\mu + E_n - \Delta(r)][\mu - E_n + \Sigma(r)] + \frac{\frac{d\Delta(r)}{dr}\left(\frac{d}{dr} + \frac{\kappa}{r}\right)}{[\mu + E_n - \Delta(r)]}\right] F(r) = 0, \tag{10a}$$

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} - [\mu + E_n - \Delta(r)][\mu - E_n + \Sigma(r)] - \frac{\frac{d\Sigma(r)}{dr}\left(\frac{d}{dr} - \frac{\kappa}{r}\right)}{[\mu - E_n + \Sigma(r)]}\right] G(r) = 0. \tag{10b}$$

To solve these equations, we employ the concept of pseudospin symmetry[12,14,15] in which $V(r) + S(r) = C_{ps}$, C_{ps} being the pseudospin constant. This implies $\frac{d\Sigma_q(r)}{dr} = 0$ and hence Eq. (10b) takes a simple form

$$\left\{\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} - [\mu + E_n - \Delta(r)]\left[\mu - E_n + C_{ps}\right]\right\} G(r) = 0. \tag{11}$$

If we take

$$\Delta(r) = \omega^2 r^2 + 2g\frac{r^2 - a^2}{(r^2 + a^2)^2}, \tag{12}$$

and introduce the dimensionless quantity $z = \beta_n \omega r^2$, Eq. (11) becomes

$$zG''(z) + \frac{1}{2}G'(z) - \left[\alpha + \frac{z}{4} + \frac{\kappa(\kappa-1)}{4z} + \frac{g\left(z - \omega a^2 \beta_n\right)}{2\left(z + \omega a^2 \beta_n\right)^2}\right] G(z) = 0, \tag{13}$$

where

$$-\beta_n^2 = \mu - E_n + C_{ps} \text{ and } \alpha = -\frac{\left(\beta_n^2 + 2\mu + C_{ps}\right)\beta_n}{4\omega}. \tag{14}$$

From the asymptotic behaviour of Eq. (13), one may seek a solution of the form

$$G(z) = (z + \beta_n \omega a^2)^{b+1} z^{\kappa/2} e^{-z/2} \psi(z), \tag{15}$$

to obtain

$$zG''(z)+\left[\left(\frac{5}{2}+2b+\kappa\right)-\frac{2(b+1)\beta_n\omega a^2}{z+\beta_n\omega a^2}-z\right]G'(z)$$
$$-\left[\frac{\beta_n^2 g/2-(b+1)(\kappa+b+\beta_n\omega a^2+1/2)}{z+\beta_n\omega a^2}\right]G(z)=\left[\alpha+b+\frac{\kappa}{2}+\frac{5}{4}\right]G(z), \qquad (16)$$

where we have assumed

$$b(b+1)-\beta_n^2 g=0\Rightarrow b=\frac{-1-\sqrt{1+4\beta_n^2 g}}{2}. \qquad (17)$$

It can be checked that a further transformation

$$G(z)=z^{\nu}\varphi(t), \quad t=z+\beta_n\omega a^2 \qquad (18)$$

do not change the structure of the differential equation (16) provided $\nu=0$ and $\nu=-\kappa+1/2$. This indicates that the solutions are of double algebraic sectors, with the even solution corresponding to $\nu=0$ ($\kappa>0$) and the odd solution corresponding to $\nu=-\kappa+1/2$ ($\kappa<0$). Thus, if we use the change in variable $t=z+\beta_n\omega a^2$, Eq. (17) takes the form

$$t(t-\beta_n\omega a^2)\varphi''(t)+\left[\left(\frac{5}{2}+2b+\kappa+2\nu+\beta_n\omega a^2\right)t-t^2-2\beta_n\omega a^2(b+1)\right]\varphi'(t)$$
$$-\left[t\left(\alpha+b+\nu+\frac{\kappa}{2}+\frac{5}{4}\right)\right]\varphi(t)=\left[\beta_n^2 g/2-(b+1)(\kappa+b+2\nu+\beta_n\omega a^2+1/2)\right]\varphi(t). \qquad (19)$$

III. THE BETHE ANSATZ SOLUTIONS TO RELATIVISTIC GIO

In this section, we give a brief description of the BAM of solving QES equation and then use the results to solve Eq. (19). For interested reader, detailed account of the method can be found in Ref. 10. We consider the differential equation of the form

$$\left[P(t)\frac{d^2}{dt^2}+Q(t)\frac{d}{dt}+R(t)\right]S(t)=0, \qquad (20)$$

where $P(t)$, $Q(t)$ are polynomials of degree 2 and $R(t)$ is a polynomial of degree 1, which we write as

$$P(t)=\sum_{k=0}^{2}p_k t^k, \quad Q(t)=\sum_{k=0}^{2}q_k t^k, \quad R(t)=\sum_{k=0}^{1}r_k t^k, \qquad (21)$$

where p_k, q_k, and r_k are constants. This equation is quasi-exactly solvable for certain values of its parameters, and exact solutions are given by degree n polynomials in t with n being non-negative integers. In fact, this equation is a special case of the general second order differential equations solved in Ref. 10 by means of the BAM. Applying the results in Ref. 10, we have

Proposition 1. Given a pair of polynomials $P(t)$ and $Q(t)$, then the values of the coefficients r_0 and r_1 of polynomial $R(t)$ such that the differential equation (20) has a degree n polynomial solution

$$S(t)=\prod_{i=1}^{n}(t-t_i), \quad S(t)\equiv 1 \text{ for } n=0 \qquad (22)$$

with distinct roots $t_1, t_2, \ldots, t_n$ given by

$$-r_1=nq_2, \qquad (23)$$

$$-r_0=q_2\sum_{i=1}^{n}t_i+n(n-1)p_2+nq_1, \qquad (24)$$

where the roots $t_1, t_2, \ldots, t_n$ satisfy the Bethe ansatz equations

$$\sum_{i\neq j}^{n} \frac{2}{t_i - t_j} + \frac{q_2 t_i^2 + q_1 t_i + q_0}{p_2 t_i^2 + p_1 t_i + p_0} = 0, \quad i = 1, 2, \ldots, n. \tag{25}$$

The above equations (23)–(25) give all polynomial $R(t)$ such that the Ordinary Differential Equation (ODE) (20) has a polynomial solution (22).

It is interesting to note that in line with the recent work,[16] Eqs. (23)–(25) satisfy the necessary and sufficient conditions for the differential equation (20) to have polynomial solutions. It is easy to show that the necessary condition (2.10) of Ref. 16 reduces to Eq. (23) for $a_{3,0} = 0$ and Eqs. (24) and (25) are the sufficient conditions for differential equation (20) to have a exact polynomial solution (22). For instance, if we consider the case $n = 1$, then tridiagonal determinant (Eq. (2.11) of Ref. 16) for Eq. (20) takes the form

$$\begin{vmatrix} -r_0 & -q_0 \\ -r_1 & -r_0 - q_1 \end{vmatrix} = 0 \qquad \Rightarrow \qquad r_0 = \frac{-q_1 \pm \sqrt{q_1^2 - 4q_0 q_2}}{2}, \tag{26}$$

where we have used the necessary condition (23). This result can be easily obtained by solving for parameter t in Bethe ansatz equation (25) and substituting the value into Eq. (24). However, it is important to note that one of the main tasks in the application of BAM is obtaining the roots of the n algebraic Bethe ansatz equation (25). For an arbitrary n, the equation is very difficult, if not impossible, to solve algebraically. However, numerical solutions to the Bethe ansatz equations have also been discussed in many applications.[17–21]

By comparing Eqs. (19) and (20), we have $p_2 = 1$, $p_1 = -\beta_n \omega a^2$, $q_2 = -1$, $q_1 = \left(\frac{5}{2} + 2b + \kappa + 2\nu + \beta_n \omega a^2\right)$, $q_0 = -2\beta_n \omega a^2 (b + 1)$, $r_1 = -\left(\alpha + b + \nu + \frac{\kappa}{2} + \frac{5}{4}\right)$, and $r_0 = -\left[\beta_n^2 g/2 - (b+1)(\kappa + b + 2\nu + \beta_n \omega a^2 + 1/2)\right]$. Thus, by Eqs. (14), (17), and (23), we immediately have the energy equation

$$4\omega\left(n + \nu + \frac{\kappa}{2} + \frac{3}{4}\right) = \left(\beta_n^2 + 2\mu + C_{ps}\right)\beta_n + 2\omega\sqrt{1 + 4g\beta_n^2} \tag{27}$$

and Eq. (24) yields

$$n\left(n + 2b + 2\nu + \kappa + \beta_n \omega a^2 + 3/2\right) - \sum_{i=1}^{n} t_\alpha = \frac{\beta_n^2 g}{2} - (b+1)(\kappa + b + 2\nu + \beta_n \omega a^2 + 1/2), \tag{28}$$

with the roots $t_1, t_2, \ldots t_n$ satisfying the Bethe ansatz equation

$$\sum_{i\neq j}^{n} \frac{2}{t_i - t_j} + \frac{\left(2b + \kappa + 2\nu + \beta_n \omega a^2 + 5/2\right) t_i - t_i^2 - 2\beta_n \omega a^2 (b+1)}{t_i (t_i - \beta_n \omega a^2)} = 0 \tag{29}$$

for $i = 1, 2, \ldots, n$.

It is obvious from the energy equation (27) that one deals with solutions with positive energy states. Moreover, the rhs of the energy equation (27) remain unchanged for quantum states (n, κ) and $(n - 1, \kappa + 2)$ thereby signifying degeneracy of the energy levels between these states. This energy degeneracy does not depend on the potential parameters as it can be seen from the numerical energy values of the ground state and some excited states for the exact pseudospin case (Table I). And we also note that for a given state, the energy values are inversely proportional to parameter g. Moreover, for the ground state, $n = 0$, we have from Eq. (28)

$$\frac{\beta_0^2 g}{2} - (b+1)(\kappa + b + 2\nu + \beta_0 \omega a^2 + 1/2) = 0, \tag{30}$$

TABLE I. Relativistic energy values of the GIO for various n and κ and a special case of $C_{ps}=0$, $g=\left\{\frac{1}{2},2,4\right\}$, and $\mu=\omega=1$.

n	$\kappa>0$	$g=\frac{1}{2}$	$g=2$	$g=4$
	1	0.7779142	0.5545547	0.4385638
	2	1.0709485	0.7934184	0.6363539
0	3	1.3018103	1.0000000	0.8157561
	4	1.4942015	1.1824728	0.9806115
	5	1.6601782	1.3457688	1.1328206
	1	1.3018103	1.0000000	0.8157561
	2	1.4942015	1.1824728	0.9806115
1	3	1.6601782	1.3457688	1.1328206
	4	1.8068160	1.4934769	1.2738521
	5	1.9386413	1.6283325	1.4049965
	1	1.6601782	1.3457688	1.1328206
	2	1.8068160	1.4934769	1.2738521
2	3	1.9386413	1.6283325	1.4049965
	4	2.0587311	1.7524556	1.5274033
	5	2.1692751	1.8675089	1.6420843
	1	1.9386413	1.6283325	1.4049965
	2	2.0587311	1.7524556	1.5274033
3	3	2.1692751	1.8675089	1.6420843
	4	2.2718886	1.9748119	1.7499209
	5	2.3677989	2.0754239	1.8516754

which yields the following condition:

$$\left(4\omega^2a^4-g\right)\beta_0^2+2\omega a^2(1+4\kappa)\beta_0+2\kappa(2\kappa+1)=0 \text{ for } \nu=0, \tag{31a}$$

$$\left(4\omega^2a^4-g\right)\beta_0^2+2\omega a^2(5-4\kappa)\beta_0+(4\kappa-6)(\kappa-1)=0 \text{ for } \nu=-\kappa+1/2. \tag{31b}$$

Hence, the ground state wavefunctions for even sector ($\nu=0$) ($\kappa>0$) can be written as

$$\begin{pmatrix} F_0(r) \\ G_0(r) \end{pmatrix}^{even} \sim r^{\kappa}\left(r^2+a^2\right)^b e^{-\frac{\beta_0\omega r^2}{2}}\begin{pmatrix} \frac{\beta_0\omega r^3-\left(2b-\beta_0\omega a^2+2\right)r}{\beta_0^2} \\ r^2+a^2 \end{pmatrix}, \tag{32}$$

with the parameters satisfying Eq. (31a). Similarly, the odd sector ($\nu=-\kappa+1/2$) ($\kappa<0$) solutions are

$$\begin{pmatrix} F_0(r) \\ G_0(r) \end{pmatrix}^{odd} \sim r^{-\kappa}\left(r^2+a^2\right)^b e^{-\frac{\beta_0\omega r^2}{2}}\begin{pmatrix} \frac{\beta_0\omega r^4-\left(2b-2\kappa-\beta_0\omega a^2+3\right)r^2+(2\kappa-1)a^2}{\beta_0^2} \\ r\left(r^2+a^2\right) \end{pmatrix}, \tag{33}$$

with the parameters satisfying Eq. (31b) and β_0 is related to ground state energy and obtained from Eq. (27) as

$$4\omega\left(\nu+\frac{\kappa}{2}+\frac{3}{4}\right)=\left(\beta_0^2+2\mu+C_{ps}\right)\beta_0+2\omega\sqrt{1+4g\beta_0^2}. \tag{34}$$

The wavefunctions $F_0(r)$ and $G_0(r)$ do not have nodes and so the states described by them are ground states of the system.

Similarly, for $n=1$, the Bethe ansatz equation (29) becomes

$$(2b+\kappa+2\nu+\beta_1\omega a^2+5/2)t_1-t_1^2-2\beta_1\omega a^2(b+1)=0, \tag{35}$$

which yields

$$t_1 = b + \nu + \frac{\kappa}{2} + \frac{\beta_1 \omega a^2}{2} + \frac{5}{4} \pm$$

$$\frac{1}{2}\sqrt{2(b+\nu)(2b+2\nu+2\kappa+5) - \beta_1\omega a^2(4b+8\kappa+16\nu-\beta_1\omega a^2+3)+(\kappa+5/2)^2}.$$

We substitute the roots into Eq. (28) and solve the resulting algebraic equation to obtain the following condition on the parameter:

$$\begin{aligned}
&g\beta_1^6\left(16\omega^4a^8+g^2-8g\omega^2a^4\right)+16g\omega a^2\beta_1^5\left(4\omega^2a^4-g\right)(\kappa+1)\\
&+\beta_1^4\left[4\omega^2a^4g\left(24\kappa^2+48\kappa+11-8\omega^2a^4\right)-2g^2(4\kappa^2+8\kappa-9)\right]\\
&+\beta_1^3\left[4g\omega a^2(16\kappa^3+48\kappa^2-38\kappa-9)-16\omega^3a^6(8\kappa+11)\right]\\
&+\beta_1^2\left[2g\left(8\kappa^4+32\kappa^3+54\kappa^2+26\kappa+3\right)-24\omega^2a^4\left(8\kappa^2+18\kappa+11\right)\right]\\
&-4\omega a^2\beta_1\left(32\kappa^3+84\kappa^2+64\kappa+15\right)-4\kappa\left(8\kappa^3+20\kappa^2+16\kappa+3\right)=0,
\end{aligned} \tag{36a}$$

$$\begin{aligned}
&g\beta_1^6\left(16\omega^4a^8+g^2-8g\omega^2a^4\right)+16g\omega a^2\beta_1^5\left(-4\kappa\omega^2a^4+8\omega^2a^4+\kappa g-2g\right)\\
&+2\beta^4\left[g^2\left(4\kappa^2+16\kappa-21\right)+2g\omega^2a^4\left(24\kappa^2-96\kappa+83\right)-16\omega^4a^8\right]\\
&+\beta_1^3\left[4g\omega a^2(-16\kappa^3+96\kappa^2-182\kappa+93)-16\omega^3a^6(8\kappa+19)\right]\\
&+\beta_1^2\left[4g\left(4\kappa^4-32\kappa^3+99\kappa^2-131\kappa+66\right)+8\omega^2a^4\left(-24\kappa^2+102\kappa-111\right)\right]\\
&+4\omega a^2\beta_1\left(32\kappa^3-180\kappa^2+328\kappa-195\right)+4\left(-8\kappa^4+52\kappa^3-122\kappa^2+123\kappa-45\right)=0,
\end{aligned} \tag{36b}$$

where β_1 is related to the first excited energy and is obtained from Eq. (27) as

$$4\omega\left(\nu+\frac{\kappa}{2}+\frac{7}{4}\right)=\left(\beta_1^2+2\mu+C_{ps}\right)\beta_1+2\omega\sqrt{1+4g\beta_1^2}. \tag{37}$$

Thus, the wavefunctions for the first excited state for the even sector ($\nu = 0$) ($\kappa > 0$) can be written as

$$\begin{pmatrix} F_1(r) \\ G_1(r) \end{pmatrix}^{even} \sim r^{\kappa}\left(r^2+a^2\right)^b e^{-\frac{\beta_1\omega r^2}{2}} \times$$

$$\begin{pmatrix} \frac{\beta_1\omega r^5-\beta_1\omega\left(2b-2\beta_1\omega a^2+t_1^e+4\right)r^3-\left[\beta_1\omega a^2(2b-\beta_1\omega a^2+t_1^e+4)-2t_1^e(b+1)\right]r}{\beta_1^2} \\ (r^2+a^2)\left(r^2+a^2-t_1^e\right) \end{pmatrix}, \tag{38}$$

with the parameters satisfying Eq. (36a) and the root given as

$$t_1^e = b+\frac{\kappa}{2}+\frac{\beta_1\omega a^2}{2}+\frac{5}{4}\pm\frac{1}{2}\sqrt{2b(2b+2\kappa+5)-\beta_1\omega a^2(4b+8\kappa-\beta_1\omega a^2+3)+(\kappa+5/2)^2}.$$

Similarly, the odd sector ($\nu = -\kappa + 1/2$) ($\kappa < 0$) solutions are

$$\begin{pmatrix} F_1(r) \\ G_1(r) \end{pmatrix}^{odd} \sim r^{-\kappa}\left(r^2+a^2\right)^b e^{-\frac{\beta_1\omega r^2}{2}} \times$$

$$\begin{pmatrix} \frac{(r^2+a^2)\left[\beta_1^2\omega^2r^4+\beta_1\omega\left(\beta_1\omega a^2+\kappa-t_1^o-2\right)r^2+\kappa\left(\beta_1\omega a^2-t_1^o\right)\right]-2r(b+1)}{\beta_1^2} \\ r(r^2+a^2)\left(r^2+a^2-t_1^o\right) \end{pmatrix}, \tag{39}$$

with the parameters satisfying Eq. (36b) and the root given as

$$t_1^o = b - \frac{\kappa}{2} + \frac{\beta_1\omega a^2}{2} + \frac{7}{4} \pm \frac{1}{2}\sqrt{3(b+2)(2b-2\kappa+1) - \beta_1\omega a^2(4b-8\kappa-\beta_1\omega a^2+11) + (\kappa+5/2)^2}.$$

IV. HIDDEN LIE ALGEBRAIC STRUCTURE

One way to understand the QES theory is to demonstrate that the Hamiltonian can be expressed in terms of generator of a Lie algebra

$$J^- = \frac{d}{dt} \quad J^+ = t^2\frac{d}{dt} - nt, \quad J^0 = t\frac{d}{dt} - \frac{n}{2}, \tag{40}$$

which are differential operator realization of the $n + 1$ dimensional representation of the $sl(2)$ algebra. Moreover, if we write the basic equation (20) in the Schrödinger form

$$HS(t) = -r_0 S(t), \tag{41}$$

where $-r_0$ is the eigenvalue of the Hamiltonian H, then it can easily be shown that if $r_1 = -nq_2$, with n being any non-negative integer, the differential operator H is an element of the enveloping algebra of Lie algebra $sl(2)$

$$H = J^0J^0 + p_1J^0J^- + q_2J^+ + (q_1+n-1)J^0 + \left(q_0 + \frac{np_1}{2}\right)J^- + \frac{n}{2}\left(\frac{n}{2} + q_1 - 1\right). \tag{42}$$

Thus, for Eq. (19), we have

$$\begin{aligned} H = t(t-\beta_n\omega a^2)\frac{d^2}{dt^2} + \left[\left(\frac{5}{2} + 2b + \kappa + 2\nu + \beta_n\omega a^2\right)t - t^2 - 2\beta_n\omega a^2(b+1)\right]\frac{d}{dt} \\ - \left[t\left(\alpha + b + \nu + \frac{\kappa}{2} + \frac{5}{4}\right)\right] \end{aligned} \tag{43}$$

and

$$r_0 = (b+1)(\kappa + b + 2\nu + \beta_n\omega a^2 + 1/2) - \beta_n^2 g/2 \tag{44}$$

with $sl(2)$ algebraization

$$\begin{aligned} H = J^0J^0 - \beta_n\omega a^2J^0J^- - J^+ + \left(n + 2b + \kappa + 2\nu + \beta_n\omega a^2 + \frac{3}{2}\right)J^0 \\ -\beta_n\omega a^2\left(2b + 2 - \frac{n}{2}\right)J^- + \frac{n}{2}\left(2b + \kappa + 2\nu + \beta_n\omega a^2 + \frac{n}{2} + \frac{3}{2}\right). \end{aligned} \tag{45}$$

V. CONCLUDING REMARKS

In this paper, we have extended the works on the GIO to a relativistic case by constructing the Bethe ansatz solutions to GIO, within the framework of the relativistic Dirac equation. We showed that the governing equation is reducible to a QES differential equation which has an exact solution, provided the parameters satisfy certain constraints. Unlike previous non-relativistic cases the quasi-exact solvability of the equations has enabled us to use Proposition 1 to obtain closed form expressions for the energies and eigenfunctions. It is interesting to note that with the limits $\beta_n \to 1$, $\kappa \to \ell + 1$, the spinor component of the wavefunction, $G(z)$ gives the non-relativistic wavefunction of the GIO, which is in agreement with previous works.[1–4,16,22]

Moreover, we reported the existence of degeneracies between the energy levels and the energy is inversely related with the potential parameter g. We also showed that the relativistic GIO possesses an underlying $sl(2)$ algebraic structure, which is responsible for the quasi-exact solvability of this model. Let us remark, however, that the existence of a underlying Lie algebraic structure in a differential equation is only a sufficient condition for the differential equation to be quasi-exactly solvable. In fact, there are more general (than the Lie-algebraically based) differential equations which do not possess a underlying Lie algebraic structure but are nevertheless quasi-exactly solvable (i.e., have exact polynomial solutions).[10] Finally, it is pertinent to note that our method gives a more general closed form expression for the solutions, however the determination of the roots of the Bethe ansatz equations for higher excited states may be a major difficulty in the application of the method.

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