

## Excitable Media in a Chaotic Flow

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The response to a localized perturbation of an excitable medium under stirring by chaotic advection is investigated. It is found that below a critical stirring rate a localized perturbation produces a coherent global excitation of the system. For very slow stirring, however, the coherence of the global excitation is gradually lost. We propose a simple model to describe the effect of the flow on the excitable dynamics, and explain the observed behavior as a consequence of a steady excited filament state found in the reduced problem.

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Excitable media [1–3] are extended systems exhibiting a variety of pattern formation phenomena relevant for different biological [4] and chemical systems. In some applications the excitable dynamics takes place in a fluid environment. One such example is the Belousov-Zhabotinsky reaction [5], that has been extensively investigated in laboratory experiments. Dynamics of oceanic plankton populations can also be represented by an excitable system [6] giving a possible explanation for the observed plankton bloom. Stirring by ocean currents may have an important influence on the excitable plankton population dynamics [7]. Therefore pattern formation in excitable media stirred by a fluid flow is a problem of theoretical interest relevant for natural processes and also accessible to laboratory experiments.

The effect of a steady shear flow on excitable media has been investigated recently by Biktashev *et al.* in [8], and it was found that the flow affects the excitable dynamics by blocking or disrupting excitation waves. In this Letter, we consider a different, more generic, type of fluid motion, *chaotic advection* [9], characterized by irregular trajectories of fluid elements, typical of a broad class of unsteady laminar flows. One important consequence of chaotic advection is the exponential separation of nearby fluid elements, that is in contrast to two-dimensional steady (e.g., shear) flows where advection is nonchaotic and the separation of fluid particles increases only linearly in time.

Here we consider the generic class of excitable systems with a stable homogeneous steady state—the “rest” state. A spatially homogeneous perturbation either dies out monotonically or, if it exceeds an excitation threshold, can drive the system temporarily to a slowly evolving “excited” state. In a motionless medium a localized perturbation exceeding the excitation threshold generates a solitary wave, which in two dimensions propagates in the form of an expanding circular ring. The waves that reach the boundary of the domain, or collide with each other, die out. Thus in a finite system the excitation persists only for a finite time. After that the system returns to the homogeneous rest state.

As a prototypical example for an excitable medium we consider the Fitz Hugh–Nagumo model [1,4,10], in

two dimensions, embedded in a time-dependent incompressible flow described by the “reaction”-advection-diffusion equations

$$\begin{aligned} \frac{\partial C_1}{\partial t} + \mathbf{v}(\mathbf{r}, t) \cdot \nabla C_1 &= \frac{1}{r} [C_1(a - C_1)(C_1 - 1) - C_2] \\ &+ D \Delta C_1, \\ \frac{\partial C_2}{\partial t} + \mathbf{v}(\mathbf{r}, t) \cdot \nabla C_2 &= \frac{\epsilon}{r} (C_1 - \gamma C_2) + D \Delta C_2, \end{aligned} \quad (1)$$

where  $D$  is the diffusivity,  $r$  is the characteristic time scale of the local dynamics, and  $a$  is the excitation threshold. It will be assumed that the constant  $\gamma$  is such that the homogeneous system has a single stable steady state ( $C_1 = C_2 = 0.0$ ). For  $\epsilon \ll 1$ , there is a separation between the fast evolution of the “active” component,  $C_1$ , and the slow evolution of  $C_2$ . The flow is assumed to be imposed externally, so that the chemical dynamics has no influence on the velocity field,  $\mathbf{v}(\mathbf{r}, t)$ . For the flow we use a simple kinematic model velocity field, formed by two alternating sinusoidal shear flows, oriented along the  $x$  and  $y$  direction for the first and second half of the period,  $T$ , respectively:

$$\begin{aligned} v_x(x, y, t) &= \frac{2A}{T} \Theta\left(\frac{T}{2} - t \bmod T\right) \sin\left(\frac{2\pi y}{L} + \phi_i\right), \\ v_y(x, y, t) &= \frac{2A}{T} \Theta\left(t \bmod T - \frac{T}{2}\right) \sin\left(\frac{2\pi x}{L} + \phi_{i+1}\right), \end{aligned} \quad (2)$$

where  $\Theta$  is the Heaviside step function. To avoid transport barriers, typically present in time-periodic flows [9], the velocity field is made aperiodic by introducing a random phase  $\phi_i$ , uniformly distributed in the range  $[0, 2\pi]$ , taking on different values in each half period. Note that the trajectories of fluid elements are set solely by the parameter  $A$ , while  $T$  defines the rate at which elements move along these trajectories.

We consider the case of weak diffusion, i.e., the diffusive time scale corresponding to the system size is assumed to be much larger than the time scale of the “local”

dynamics,  $L^2/D \gg r$ . The model (1) is nondimensionalized by assuming that the system is defined on the unit square  $S$  ( $L = 1$ ), and the unit of time is the reaction time scale ( $r = 1$ ).

The model (1) is integrated numerically on a doubly periodic domain, using a semi-Lagrangian scheme. The parameter values used in the numerical simulations are  $a = 0.25$ ,  $\gamma = 3.0$ ,  $\epsilon = 10^{-3}$ ,  $D = 10^{-5}$ ,  $A = 0.5$ . Initially, the system is in the homogeneous steady state, perturbed by a localized Gaussian pulse in the concentration of the active component

$$C_1(x, y, t = 0) = C_1^0 \exp[-(x^2 + y^2)/2l_0^2], \quad (3)$$

where  $C_1^0$  is chosen to be larger than the excitation threshold,  $a$ , and the size of the perturbation,  $l_0$ , is much smaller than the system size ( $C_1^0 = 0.5$ ,  $l_0 = 0.1$ ). We study the response of the system for different stirring rates, defined by  $\nu \equiv 1/T$ .

Snapshots of the spatial structure for  $\nu = 0.08$  are shown in Fig. 1. The patterns in Fig. 1 are quite different from the ones observed in motionless excitable media. The initially excited circular patch is strongly deformed into filaments by the advection. The time dependence of the mean concentration,  $\langle C_1 \rangle$ , and its maximum value in function of the stirring rate, are shown in Figs. 2 and 3, respectively.

The behavior of the system can be classified using a Lagrangian system of coordinates, comoving with the fluid elements, defined by labeling each fluid element by its initial coordinate  $\mathbf{r}_0$ , and let  $\hat{\mathbf{r}}(t; \mathbf{r}_0)$  represent its position at a later time  $t$ . We define *global excitation* as

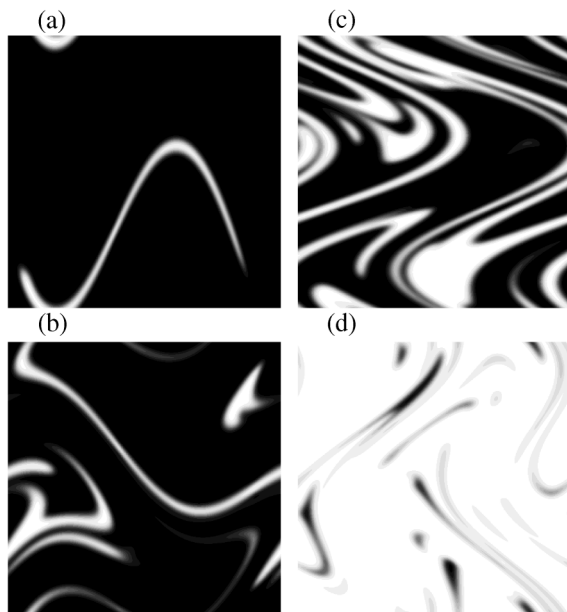


FIG. 1. Snapshots of the distribution  $C_1(x, y)$  for  $\nu = 0.08$  at  $t = 40, 80, 120, 160$ . The numerical resolution is  $600 \times 600$  points.

a scenario, such that  $C_1$  exceeds the excitation threshold in all fluid elements before the excitation dies out completely, i.e., for any  $\mathbf{r}_0$  there is a time  $t_{\text{exc}}(\mathbf{r}_0)$ , such that  $C_1(\hat{\mathbf{r}}(t; \mathbf{r}_0), t_{\text{exc}}(\mathbf{r}_0)) > a$ . Note, that this does not necessarily mean that there is a time when the whole system is simultaneously in the excited state, since the excitation may die out in some fluid elements before it starts in others. We will call *coherent excitation* the situation, when there is a time,  $t_{\text{coh}}$ , such that  $C_1(\mathbf{r}, t_{\text{coh}}) > a$  for all points  $\mathbf{r} \in S$ . It follows that all coherent excitations are also global excitations. An example of a noncoherent global excitation is the case of a propagating excitation front, characteristic of excitable media in the absence of stirring.

For fast stirring,  $\nu > \nu_c \approx 0.1035$ , the excited filaments, marked by concentration  $C_1$  higher than the background, decay quickly and fade out well before they could reach all fluid elements. In this case, the maximum mean concentration of  $C_1$  is close to zero, and  $\langle C_1 \rangle$  typically decays monotonically.

When the stirring rate is below a critical value,  $\nu < \nu_c$ , the localized initial perturbation generates a *global excitation*. The transition is indicated by the sharp discontinuity in the maximum mean concentration of the active component (see Fig. 3). In a wide range of  $\nu$  below the critical stirring rate ( $0.04 \lesssim \nu < \nu_c$ ) the maximum of the mean concentration of  $C_1$  is about the same as the one that would result from an homogeneous superthreshold perturbation. This clearly indicates a *coherent global excitation*. The numerical simulations show that in this parameter range the excited filaments form a denser and denser structure until they fill the domain. The coherent excitation finally decays as in an homogeneous system. Note, that although the excitation starts at different times in different fluid parcels, it decays simultaneously everywhere.

For very slow stirring,  $\nu \lesssim 0.04$ , the coherence of the global excitation is gradually lost, as it is indicated by the decrease of  $\langle C_1 \rangle_{\text{max}}$ . The numerical simulations show, that although the excitation sooner or later reaches all fluid

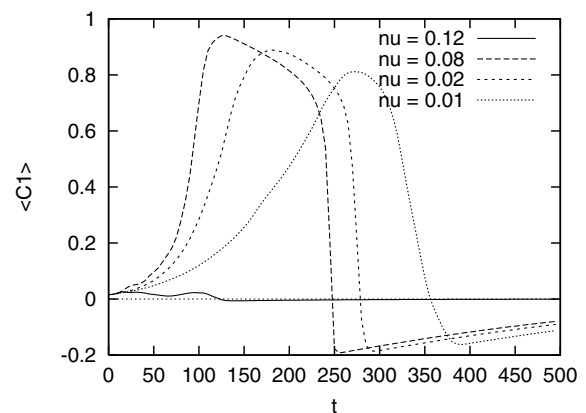


FIG. 2. Time dependence of the spatial mean concentration of the active component for four different values of the stirring rate,  $\nu = 0.12, 0.08, 0.02$ , and  $0.01$ .

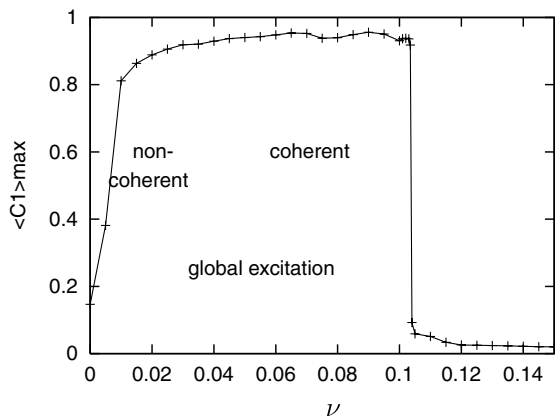


FIG. 3. The maximum mean concentration of the active component, represented as a function of the stirring rate.

parcels (i.e., global excitation), the area of the excited region at any one time is always smaller than the system size, as parts of the system are still in the preexcitation state while the excitation has already decayed elsewhere.

The existence of the critical stirring rate is a consequence of the competition between stirring, that tends to spread and dilute the excited patch, and the excitable dynamics, that increases the local concentration of the active component wherever the excitation threshold is exceeded. In order to understand the observed behavior, we introduce a reduced model of the problem based on the following observation. Chaotic advection acts as a deformation that stretches and folds fluid elements generating a filamental structure. The filaments are homogenized along the stretching direction, therefore they can be characterized by their transverse profile. We therefore replace the original problem with a one-dimensional reaction-diffusion system modified by a stretching term, that takes into account the effect of stirring. (In fact, the stretching term can be seen as advection by a pure strain flow,  $v_x = -\lambda x$ ,  $v_y = \lambda y$ , along the convergent direction,  $x$ .)

$$\frac{\partial}{\partial t} \mathbf{C}(x, t) - \lambda x \frac{\partial}{\partial x} \mathbf{C}(x, t) = \mathbf{F}(\mathbf{C}) + D \frac{\partial^2}{\partial x^2} \mathbf{C}(x, t), \quad (4)$$

where  $\lambda$  is the strain rate, and  $\mathbf{F}(\mathbf{C})$  represents the local excitable dynamics. A suitable choice for  $\lambda$  is the Lagrangian mean strain, given by the absolute value of the smallest (negative) Lyapunov exponent [9] of the advection dynamics. (For incompressible flows this is equal to the positive Lyapunov exponent, defined as the exponential rate of separation of nearby fluid elements.) For the flow used in our simulations the value of the numerically measured Lyapunov exponent is  $\lambda \approx 1.09/T$ . While the filaments are well separated from each other, so that they do not interact, the concentrations at the boundaries can be taken to be fixed to the equilibrium state. The independent filament approximation (4) breaks down when the excited filaments start to fill the domain.

We studied the evolution of perturbations of the form (3) (with  $y$  dependence suppressed), for different values of  $\lambda$ , integrating the one-dimensional model (4) on an interval of unit length ( $L = 1$ ). The numerical simulations indicate similar regimes as seen in the two-dimensional problem. For strong stretching ( $\lambda > \lambda_c \approx 0.081$ ), any perturbation decays to the homogeneous steady state,  $C_1 = C_2 = 0$ . Below the critical value,  $\lambda_c \approx 0.08$ , some perturbations converge to a *nonhomogeneous steady state*, representing a steady excited filament (SEF), while others decay. SEF solutions obtained for different stretching rates are shown in Fig. 4. Numerical continuation of the SEF solution for increasing  $\lambda$  confirms the existence of a critical point, where this solution disappears. In a certain range below the threshold,  $0.035 \leq \lambda < \lambda_c$  the SEF solution is unimodal. For weak stretching, however, the excited state may decay in the center of the filament, as it would in the absence of advection. This results in a two-humped steady filament profile. In the absence of stretching, the excited state propagates with a constant velocity,  $v_0$ . For the Fitz Hugh–Nagumo model, this can be obtained analytically in the  $\epsilon \rightarrow 0$  limit, as  $v_0 = \sqrt{D/2r} (1 - a)$ . Assuming that the width of the SEF solution,  $w_0$ , is determined by the balance between the strain and the speed of the propagating front, one obtains

$$\frac{w_0}{2} \sim \frac{v_0}{\lambda} = \sqrt{\frac{D}{2r}} \frac{1 - a}{\lambda}, \quad (5)$$

that compares reasonably well with the numerical results (see inset in Fig. 4).

Let us now turn back to the original two-dimensional problem. In the fast stirring regime the absence of the SEF solution implies that the excited filaments decay. Therefore, a localized pulse, much smaller than the system size,

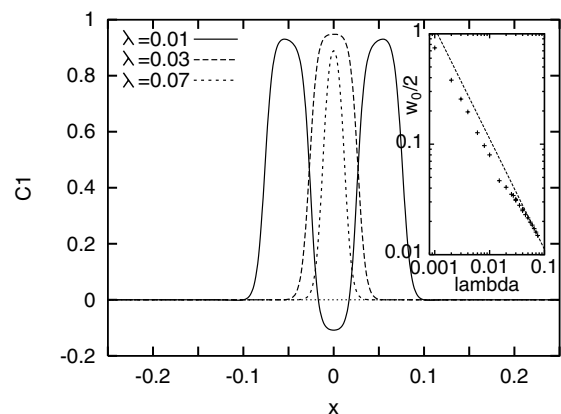


FIG. 4. Steady excited filament solutions of Eq. (4), obtained on an interval of length  $L = 1.0$  with boundary conditions  $c_1(\pm L/2) = c_2(\pm L/2) = 0.0$  for  $\lambda = 0.01, 0.03, 0.07$ . The inset shows the half width of the SEF solution,  $w_0$  [measured at the level of the excitation threshold,  $C_1(x) = 0.25$ ], as function of  $\lambda$  (crosses). The dashed line shows the analytical estimate (5).

cannot produce a global excitation. When the SEF solution exists, it allows the filaments to persist for an arbitrarily long time. Thus, they can cover the whole domain and produce a global excitation. (Note, that the advective propagation of the excited state along the unstable foliation of the chaotic advection is strongly anisotropic in contrast to the isotropic propagation mediated by diffusive transport in the absence of the flow.) If the SEF solution is one humped, the filaments covering the domain will produce a coherent excitation of the system. In the case of the two-humped SEF solution, the global excitation is non-coherent, since the central part of the filaments is in a post-excitation state.

The existence of a steady excited state is somewhat surprising in a system with local dynamics characterized by a transient excited state. In the absence of stirring, the accumulation of the slow component,  $C_2$ , leads to the aging and eventual decay of the excited state. Stirring, however, permanently dilutes the slow component by entraining fluid with low  $C_2$  concentration into the excited filament. This mechanism prevents the decay of the excited filaments until they become space filling.

The reduced problem (4) seems to capture, at least qualitatively, the main features of the interplay between stirring and excitable dynamics. Of course, the effect of stirring cannot be completely characterized by the single parameter,  $\lambda$ . As the stretching fluctuates in space and time the width of the excited filament also varies strongly. Moreover, certain parts of the excited filament may decay while others persist. Nonuniformity of the stretching and folding of the filaments, neglected in our “mean strain” approximation, should be taken into account in a more accurate analysis. We also note that the possibility of a large scale coherent response to a single localized pulslike perturbation may have some practical implications, particularly for chemical or biological processes taking place in environmental flows.

One example of such behavior is the plankton bloom produced by ocean fertilization experiments [7], in which the plankton ecosystem is perturbed by the supply of iron (a trace element that affects the growth rate of phytoplankton) in a patch of few km radius. The experiment produced a surprisingly long-lived bloom filament visible on satel-

lite images for more than 50 days. (After this time the area could not be observed due to cloud cover and later the bloom necessarily ended with the onset of the polar night.) The plankton ecosystem can be described as an excitable medium with phyto- and zoo-plankton representing the fast and the slow (recovery) components, respectively. This plankton ecosystem may have advectively propagating, steady bloom filament solutions [11]. As a consequence, a single fertilization experiment has the potential to produce an unexpected large scale change in the oceanic ecosystem. Similar phenomena may also appear in the atmosphere where the chemical equilibrium is perturbed by localized polluting sources.

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