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Free field realizations of 2D current algebras, screening currents and primary fields

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Abstract

In this paper we consider Wakimoto free field realizations of simple affine Lie algebras, a subject already much studied. We present three new sets of results. (i) Based on quantizing differential operator realizations of the corresponding Lie algebras we provide general universal very simple expressions for all currents, more compact than has been established so far. (ii) We supplement the treatment of screening currents of the first kind, known in the literature, by providing a direct proof of the properties for screening currents of the second kind. Finally (iii) we work out explicit free field realizations of primary fields with general non-integer weights. We use a formalism where the (generally infinite) multiplet is replaced by a generating function primary operator. These results taken together allow setting up integral representations for correlators of primary fields corresponding to non-integrable degenerate (in particular admissible) representations. © 1997 Elsevier Science B.V.

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1. Introduction

Since the work by Wakimoto [1] on free field realizations of affine $SL(2)$ current algebra much effort has been made in obtaining similar constructions in the general case, a problem in principle solved by Feigin and Frenkel [2] and further studied by

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many groups [3-13]. Free field realizations enable one in principle to build integral representations for correlators in conformal field theory [14-17]. In a recent series of papers we have carried out such a study for affine $SL(2)$ [18,19]. It turns out that screening operators of both the first and the second kinds are crucial for being able to treat the general case of degenerate representations [20] and admissible representations [21]. It is also necessary to be able to handle fractional powers of free fields. We have established well-defined rules for that [18,19]. We were particularly interested in this technique because of its close relationship with two-dimensional quantum gravity and string theory [22,23], although many other applications may be envisaged, see e.g. Ref. [24].

In this paper we provide the ingredients for generalizations to affine algebras based on any simple Lie algebra. That would enable one e.g. to treat the case of W-matter coupled to W-gravity.

Our new results consist first in presenting very explicit universal compact expressions for the affine currents. We use techniques based on "triangular" parameters on a representation spaces to treat in an efficient way any representation. Some of these expressions are new. Second, we have provided a proof of the properties of the screening currents of the second kind proposed without proof by Ito [7] in addition to the better known ones of the first kind. Our proof for the validity of this second kind so far works only for $SL(N)$ but it seems natural to expect the result to hold in general [25]. Finally, we have generalized the very compact form of the primary field used in Refs. [16,18,19] to the general case. A number of these results were given in preliminary form in Ref. [26]. Primary fields for integrable representations are described in Ref. [2]. Our treatment holds also for non-integrable, degenerate (including admissible) representations. The compactness of our result is due to the use of triangular parameters.

The paper is organized as follows. In Section 2 we fix our notation which we keep rather general. We define our "triangular" coordinates and we introduce a crucial matrix depending on them in the adjoint representation of the underlying algebra. All our explicit results are given in a very simple way in terms of that matrix.

In Section 3 we present differential operator realizations of simple Lie algebras. This technique is well known. The new aspect is that we work out in great detail certain Gauss decompositions of relevant group elements. These are the key to our explicit formulas. We then discuss differential operators later to become essential counterparts of the screening currents of both kinds. We provide several non-trivial polynomial identities later to be used.

In Section 4 we quantize the differential operator realization of a simple Lie algebra to a Wakimoto free field realization of the corresponding affine Lie algebra in the standard way. The non-trivial part is to take care of multiple contractions (or in other words the normal ordering) by adding anomalous terms to the lowering operators. These terms were recently discussed in the general case by de Boer and Fehér [13]. Our result again is somewhat more explicit. We end the section by listing further polynomial identities following from the quantum realization, to be used later on.

In Section 5 we discuss the screening currents. First, we review the known results for screening currents of the first kind and for completeness write them down in our notation and indicate an explicit straightforward proof. The new result concerns our proof of the properties of screening currents of the second kind, generalizing the idea of Ref. [27] from $SL(2)$ to any simple algebra [7]. In the case of $SL(N)$ we prove that our explicit expression fulfills the required properties.

In Section 6 we give a thorough discussion of primary fields using the formalism based on the "triangular" parameters. We derive simple and general free field realizations of primary fields with arbitrary, possibly non-integral weights, i.e. non-integer Dynkin labels and non-integer level.

Section 7 contains concluding remarks.

2. Notation

Let $\mathbf{g}$ be a simple Lie algebra of $\dim \mathbf{g} = d$ and $\operatorname{rank} \mathbf{g} = r$. $\mathbf{h}$ is a Cartan subalgebra of $\mathbf{g}$. The set of (positive) roots is denoted (Δ_+) Δ, and we write $\alpha > 0$ if $\alpha \in \Delta_+$. The simple roots are $\{\alpha_i\}_{i=1,\dots,r}$. θ is the highest root, while $\alpha^\vee = 2\alpha/\alpha^2$ is the root dual to α. Using the triangular decomposition

$$\mathbf{g} = \mathbf{g}_- \oplus \mathbf{h} \oplus \mathbf{g}_+, \tag{1}$$

the raising and lowering operators are denoted $e_\alpha \in \mathbf{g}_+$ and $f_\alpha \in \mathbf{g}_-$, respectively with $\alpha \in \Delta_+$, and $h_i \in \mathbf{h}$ are the Cartan operators. We let j_a denote an arbitrary Lie algebra element. For simple roots we sometimes write $e_i = e_{\alpha_i}$, $f_i = f_{\alpha_i}$. The $3r$ generators e_i, h_i, f_i are the Chevalley generators. Their commutator relations are

$$\begin{aligned} &[h_i, h_j] = 0, \qquad [e_i, f_j] = \delta_{ij} h_j, \\ &[h_i, e_j] = A_{ij} e_j, \qquad [h_i, f_j] = -A_{ij} f_j, \end{aligned} \tag{2}$$

where A_{ij} is the Cartan matrix. In the Cartan–Weyl basis we have

$$[h_i, e_\alpha] = (\alpha_i^\vee, \alpha) e_\alpha, \qquad [h_i, f_\alpha] = -(\alpha_i^\vee, \alpha) f_\alpha, \tag{3}$$

and

$$[e_\alpha, f_\alpha] = h_\alpha = G^{ij} (\alpha_i^\vee, \alpha^\vee) h_j, \tag{4}$$

where the metric G_{ij} is related to the Cartan matrix as $A_{ij} = \alpha_i^\vee \cdot \alpha_j = (\alpha_i^\vee, \alpha_j) = G_{ij}\alpha_j^2/2$, while the Cartan–Killing form (denoted by κ and tr) is

$$\operatorname{tr}(j_a j_b) = \kappa_{ab}, \qquad \kappa_{\alpha,-\beta} = \kappa(e_\alpha f_\beta) = \frac{2}{\alpha^2}\delta_{\alpha,\beta}, \qquad \kappa_{ij} = \kappa(h_i h_j) = G_{ij}. \tag{5}$$

The Weyl vector $\rho = \frac{1}{2}\sum_{\alpha>0} \alpha$ satisfies $\rho \cdot \alpha_i^\vee = 1$. We use the convention $f_{-\alpha,-\beta}{}^{-\gamma} = -f_{\alpha\beta}{}^{\gamma}$. The Dynkin labels Λ_k of the weight Λ are defined by

$$\Lambda = \Lambda_k \Lambda^{(k)}, \qquad \Lambda_k = (\alpha_k^\vee, \Lambda), \tag{6}$$

where $\{\Lambda^{(k)}\}_{k=1,\dots,r}$ is the set of fundamental weights satisfying

$$(\alpha_i^\vee, \Lambda^{(k)}) = \delta_i^k. \tag{7}$$

Elements in $\mathbf{g}_+$ (or $\mathbf{g}_-$) or vectors in representation spaces (see below) are parametrized using "triangular coordinates" denoted by x^α, one for each positive root. We introduce the Lie algebra elements

$$e(x) = x^\alpha e_\alpha \in \mathbf{g}_+, \qquad f(x) = x^\alpha f_\alpha \in \mathbf{g}_-, \tag{8}$$

and the corresponding group elements $g_+(x)$ and $g_-(x)$ by

$$g_+(x) = e^{e(x)}, \qquad g_-(x) = e^{f(x)}. \tag{9}$$

Also we introduce the matrix representation, $C(x)$, of $e(x)$ in the adjoint representation,

$$C_a^b(x) = C(x)_a{}^b = (x^\beta C_\beta)_a{}^b = -x^\beta f_{\beta a}{}^b \tag{10}$$

and use the following notation for the (block) matrix elements:

$$C = \begin{pmatrix} C_+{}^+ & 0 & 0 \\ C_0{}^+ & 0 & 0 \\ C_-{}^+ & C_-{}^0 & C_-{}^- \end{pmatrix}. \tag{11}$$

$C_+{}^+$ etc. are matrices themselves. In $C_+{}^+$ both row and column indices are positive roots, in $C_-{}^0$ the row index is a negative root and the column index is a Cartan algebra index, etc. One easily sees that (leaving out the argument x for simplicity)

$$\begin{aligned}
&(C^n)_+{}^+ = (C_+{}^+)^n, \\
&(C^n)_0{}^+ = C_0{}^+(C_+{}^+)^{n-1}, \\
&(C^n)_-{}^0 = (C_-{}^-)^{n-1}C_-{}^0, \\
&(C^n)_-{}^- = (C_-{}^-)^n, \\
&(C^n)_-{}^+ = \sum_{l=0}^{n-1}(C_-{}^-)^l C_-{}^+(C_+{}^+)^{n-l-1} + \sum_{l=0}^{n-2}(C_-{}^-)^l C_-{}^0 C_0{}^+(C_+{}^+)^{n-l-2}, \\
&\qquad 0 = (C^n)_+{}^0 = (C^n)_+{}^- = (C^n)_0{}^0 = (C^n)_0{}^-.
\end{aligned} \tag{12}$$

We shall use repeatedly that $C_\alpha^\beta(x)$ vanishes unless $\alpha < \beta$, corresponding to $C_+{}^+$ being upper triangular with zeros in the diagonal. Similarly, $C_-{}^-$ is lower triangular. It will turn out that we shall be able to provide remarkably simple universal analytic expressions for most of the objects we consider using the matrix $C(x)$. This will be one of the new results in this paper.

For the associated affine algebra, the operator product expansion, OPE, of the associated currents is

$$J_a(z)J_b(w) = \frac{\kappa_{ab}k}{(z-w)^2} + \frac{f_{ab}{}^c J_c(w)}{z-w}, \tag{13}$$

where regular terms have been omitted. k is the central extension and $k^\vee = 2k/\theta^2$ is the level. In the mode expansion

$$J_a(z) = \sum_{n=-\infty}^{\infty} J_{a,n} z^{-n-1}, \tag{14}$$

we use the identification

$$J_{a,0} \equiv j_a. \tag{15}$$

The Sugawara energy momentum tensor is

$$\begin{aligned} T(z) &= \frac{1}{\theta^2(k^\vee + h^\vee)} \kappa^{ab} : J_a J_b : (z) \\ &= \frac{1}{t} : \sum_{\alpha>0} \frac{1}{\alpha^2}(E_\alpha F_\alpha + F_\alpha E_\alpha) + \frac{1}{2}(H,H) : (z), \end{aligned} \tag{16}$$

where we have introduced the parameter

$$t = \frac{\theta^2}{2}\left(k^\vee + h^\vee\right) \tag{17}$$

and where $h^\vee$ is the dual Coxeter number. This tensor has central charge

$$c = \frac{k^\vee d}{k^\vee + h^\vee}. \tag{18}$$

The standard free field construction [1–3,5–11] consists in introducing for every positive root $\alpha > 0$, a pair of free bosonic ghost fields $(\beta_\alpha, \gamma^\alpha)$ of conformal weights $(1,0)$ satisfying the OPE

$$\beta_\alpha(z)\gamma^\beta(w) = \frac{\delta_\alpha{}^\beta}{z-w}. \tag{19}$$

The corresponding energy-momentum tensor is

$$T_{\beta\gamma} =: \partial\gamma^\alpha \beta_\alpha : \tag{20}$$

with central charge

$$c_{\beta\gamma} = d - r. \tag{21}$$

We will understand "properly" repeated root indices as in (20) to be summed over the positive roots.

For every Cartan index $i = 1, \dots, r$ one introduces a free scalar boson φ_i with contraction

$$\varphi_i(z)\varphi_j(w) = G_{ij} \ln(z-w). \tag{22}$$

The energy-momentum tensor,

$$T_\varphi = \frac{1}{2} : \partial\varphi \cdot \partial\varphi : -\frac{1}{\sqrt{t}} \rho \cdot \partial^2 \varphi, \tag{23}$$

has central charge

$$c_\varphi = r - \frac{h^\vee d}{k^\vee + h^\vee}. \tag{24}$$

This follows from the Freudenthal–de Vries strange formula $\rho^2 = h^\vee \theta^2 d/24$. The total free field realization of the Sugawara energy-momentum tensor is $T = T_{\beta\gamma} + T_\varphi$ as is well known (see also Ref. [13]).

The vertex operator

$$V_\Lambda(z) = :\exp\left(\frac{1}{\sqrt{t}}\Lambda \cdot \varphi(z)\right):,$$
$$\Lambda \cdot \varphi(z) = \Lambda_i \varphi_j(z) G^{ij} \tag{25}$$

has conformal weight

$$\Delta(V_\Lambda) = \frac{1}{2t}(\Lambda, \Lambda + 2\rho). \tag{26}$$

It is also affine primary corresponding to highest weight Λ. A new result in this paper will be the explicit general construction of the full multiplet of primary fields, parametrized by the x^α coordinates in Section 6.

3. Differential operator realizations

Following an old idea (see e.g. Ref. [28]), elaborated on in Refs. [2,5,7,8,11], we here discuss a differential operator realization of a simple Lie algebra $\mathbf{g}$ on the polynomial ring $\mathbb{C}[x^\alpha]$. We introduce the lowest weight vector in the (dual) representation space,

$$\langle\Lambda| f_\alpha = 0, \qquad \langle\Lambda| h_i = \Lambda_i \langle\Lambda|. \tag{27}$$

An arbitrary vector in this representation space is parametrized as

$$\langle\Lambda, x| = \langle\Lambda| g_+(x). \tag{28}$$

The differential operator realizations $\widetilde{J}_a(x, \partial, \Lambda)$ with $\partial_\alpha = \partial_{x^\alpha}$ denoting partial derivative with respect to x^α, are then defined by

$$\langle\Lambda, x| j_a = \widetilde{J}_a(x, \partial, \Lambda)\langle\Lambda, x|. \tag{29}$$

Obviously these satisfy the Lie algebra commutation relations. It is convenient to have a similar notation for highest weight (ket-) vectors,

$$|\Lambda, x\rangle = g_-(x)|\Lambda\rangle,$$
$$j_a|\Lambda, x\rangle = -J_a(x, \partial, \Lambda)|\Lambda, x\rangle, \tag{30}$$

where the relation between the two sets of realizations of the Lie algebra, $\{\widetilde{J}_a(x, \partial, \Lambda)\}$ and $\{J_a(x, \partial, \Lambda)\}$, is as follows:

$$
\begin{aligned}
E_\alpha(x,\partial,\Lambda) &= -\widetilde{F}_\alpha(x,\partial,\Lambda),\\
F_\alpha(x,\partial,\Lambda) &= -\widetilde{E}_\alpha(x,\partial,\Lambda),\\
H_i(x,\partial,\Lambda) &= -\widetilde{H}_i(x,\partial,\Lambda).
\end{aligned}
\tag{31}
$$

We write the Gauss decomposition of $\langle\Lambda|g_+(x)e^{tJ_a}$ for t small as

$$
\begin{aligned}
\langle\Lambda|g_+(x)\exp(te_\alpha) &= \langle\Lambda|\exp\left(x^\gamma e_\gamma + tV_\alpha^\beta(x)e_\beta + \mathcal{O}(t^2)\right)\\
&= \langle\Lambda|\exp\left(tV_\alpha^\beta(x)\partial_\beta + \mathcal{O}(t^2)\right)g_+(x),\\
\langle\Lambda|g_+(x)\exp(th_i) &= \langle\Lambda|\exp(th_i)\exp\left(x^\gamma e_\gamma + tV_i^\beta(x)e_\beta + \mathcal{O}(t^2)\right)\\
&= \langle\Lambda|\exp\left(t\left(V_i^\beta(x)\partial_\beta + \Lambda_i\right) + \mathcal{O}(t^2)\right)g_+(x),\\
\langle\Lambda|g_+(x)\exp(tf_\alpha) &= \langle\Lambda|\exp\left(tQ_{-\alpha}^{-\beta}(x)f_\beta + \mathcal{O}(t^2)\right)\exp\left(tP_{-\alpha}^j(x)h_j + \mathcal{O}(t^2)\right)\\
&\quad\times\exp\left(x^\gamma e_\gamma + tV_{-\alpha}^\beta(x)e_\beta + \mathcal{O}(t^2)\right)\\
&= \langle\Lambda|\exp\left(t\left(P_{-\alpha}^j(x)\Lambda_j + V_{-\alpha}^\beta(x)\partial_\beta\right) + \mathcal{O}(t^2)\right)g_+(x).
\end{aligned}
\tag{32}
$$

It follows that the differential operator realization is of the form

$$
\begin{aligned}
\widetilde{E}_\alpha(x,\partial) &= V_\alpha^\beta(x)\partial_\beta,\\
\widetilde{H}_i(x,\partial,\Lambda) &= V_i^\beta(x)\partial_\beta + \Lambda_i,\\
\widetilde{F}_\alpha(x,\partial,\Lambda) &= V_{-\alpha}^\beta(x)\partial_\beta + P_{-\alpha}^j(x)\Lambda_j.
\end{aligned}
\tag{33}
$$

Since $\widetilde{E}_\alpha(x,\partial,\Lambda) = \widetilde{E}_\alpha(x,\partial)$ is independent of Λ it may be defined through a Gauss decomposition alone.

From the realization of $\widetilde{E}_\alpha$ we obtain

$$
V_\alpha^\beta(x)\,\mathrm{tr}\left(g_+^{-1}(x)\partial_\beta g_+(x)f_\gamma\right) = \frac{2}{\alpha^2}\delta_{\alpha,\gamma}.
\tag{34}
$$

In Ref. [13] essentially this trace is introduced as a key object in the explicit Wakimoto construction in those papers (and we see here that our V_α^β is related to the matrix inverse of that). In the present paper we explicitly evaluate this trace (or equivalently V_α^β) in terms of a simple universal analytic function of the matrix $C(x)$. A similar but somewhat more complicated expression was provided in Ref. [7]. Analogous and new results will be given for all the other objects occurring: the remaining V's as well as the P's and the Q's. These results are obtained by (laboriously) working out the Gauss decompositions (32) involved.

In Ref. [12] the V's are determined by an approach very similar to the one we have employed. However, again we have carried out explicitly the Gauss decomposition. In Ref. [12] functions similar to the P's are given by recursion relations while functions similar to the Q's are not discussed.

The Gauss decompositions rely on the Campbell-Baker-Hausdorff (CBH) formula (see e.g. Ref. [26] for a proof),

$$e^A e^{tB} = \exp\left\{ A + t \sum_{n\geqslant 0} \frac{B_n}{n!}(-\mathrm{ad}_A)^n B + \mathcal{O}(t^2) \right\}, \tag{35}$$

where the coefficients B_n are the Bernoulli numbers

$$\begin{aligned}
B(u) &= \frac{u}{e^u - 1} = \sum_{n\geqslant 0} \frac{B_n}{n!} u^n, \\
B(u) - B(-u) &= -u, \qquad B_{2m+1} = 0 \quad \text{for } m \geqslant 1, \\
B_0 &= 1, \qquad B_1 = -\frac{1}{2}, \qquad B_2 = \frac{1}{6}, \qquad B_4 = -\frac{1}{30}, \qquad \ldots, \\
B^{-1}(u) &= \frac{e^u - 1}{u} = \sum_{n\geqslant 1} \frac{u^{n-1}}{n!}.
\end{aligned} \tag{36}$$

We apply these repeatedly (infinitely many times) to the group element $g_+(x)e^{tj_a}$. The results are expressed in terms of the generating function of Bernoulli numbers (36) (and other even simpler analytic functions) evaluated on the matrix $C(x)$. Since for any given Lie algebra this matrix is nilpotent, the formal power series all become polynomials. The main new result of this section is then the following explicit expressions for the polynomials V and P (and Q) in the differential operator realization (33) of the Lie algebra $\mathbf{g}$:

$$\begin{aligned}
V_\alpha^\beta(x) &= [B(C(x))]_\alpha^\beta, \\
V_i^\beta(x) &= -[C(x)]_i^\beta, \\
V_{-\alpha}^\beta(x) &= \left[e^{-C(x)}\right]_{-\alpha}^\gamma [B(-C(x))]_\gamma^\beta, \\
P_{-\alpha}^j(x) &= \left[e^{-C(x)}\right]_{-\alpha}^j, \\
Q_{-\alpha}^{-\beta}(x) &= \left[e^{-C(x)}\right]_{-\alpha}^{-\beta}.
\end{aligned} \tag{37}$$

These matrix functions are defined in terms of universal power series expansions, valid for any Lie algebra, but ones that truncate and give rise to polynomials the orders of which do depend on the algebra, see Ref. [26] for details on how the truncations work and for an alternative explicit polynomial expression of $V_{-\alpha}^\beta(x)$.

Now we introduce a differential operator [2,5,8,11] which will turn out to be a building block in the free field construction of screening operators of both the first and the second kinds in Section 5. It is the differential operator S_α (which we may construct for any root, although only the ones for simple roots will be used). It is defined in terms of a left action

$$\begin{aligned}
\exp\{-te_\alpha\}g_+(x) &= \exp\{tS_\alpha(x,\partial) + \mathcal{O}(t^2)\}g_+(x), \\
S_\alpha(x,\partial) &= S_\alpha^\beta(x)\partial_\beta.
\end{aligned} \tag{38}$$

It is easily seen that

$$S_\alpha(x,\partial) = \widetilde{E}_\alpha(-x,-\partial), \tag{39}$$

so that

$$S_\alpha^\beta(x) = -\left[B(-C(x))\right]_\alpha^\beta. \tag{40}$$

From the associativity property of

$$e^{-se_\alpha} g_+(x) e^{tj_a} \tag{41}$$

and the Gauss decomposition (32) one deduces the following commutation relations:

$$\begin{aligned}
\left[\widetilde{E}_\alpha(x,\partial), S_\beta(x,\partial)\right] &= 0,\\
\left[\widetilde{H}_i(x,\partial,\Lambda), S_\beta(x,\partial)\right] &= (\alpha_i^\vee,\beta) S_\beta(x,\partial),\\
\left[\widetilde{F}_\alpha(x,\partial,\Lambda), S_\beta(x,\partial)\right] &= P^j_{-\alpha}(x)(\alpha_j^\vee,\beta)S_\beta(x,\partial)\\
&\quad + Q^{-\gamma}_{-\alpha}(x)(\delta_{\beta\gamma}(\beta^\vee,\Lambda) - f_{\beta,-\gamma}{}^\sigma S_\sigma(x,\partial)),\\
\left[S_\alpha(x,\partial), S_\beta(x,\partial)\right] &= f_{\alpha\beta}{}^\gamma S_\gamma(x,\partial).
\end{aligned} \tag{42}$$

The last commutator follows from the associativity of $e^{-se_\alpha}e^{-te_\beta}g_+(x)$.

We conclude this section by listing certain *classical polynomial identities* (as opposed to quantum polynomial identities established in Section 4) needed in the subsequent sections,

$$\begin{aligned}
(\alpha_i^\vee,\beta-\alpha)V_\alpha^\beta(x) &= (\alpha_i^\vee,\gamma)x^\gamma\partial_\gamma V_\alpha^\beta(x),\\
(\alpha_i^\vee,\gamma+\alpha)V^\gamma_{-\alpha}(x) &= (\alpha_i^\vee,\beta)x^\beta\partial_\beta V^\gamma_{-\alpha}(x),\\
V_\alpha^\gamma(x)\partial_\gamma V_\beta^\sigma(x) - V_\beta^\gamma(x)\partial_\gamma V_\alpha^\sigma(x) &= f_{\alpha\beta}{}^\gamma V_\gamma^\sigma(x),\\
V_\alpha^\gamma(x)\partial_\gamma V^\sigma_{-\beta}(x) - V^\gamma_{-\beta}(x)\partial_\gamma V_\alpha^\sigma(x) &= f_{\alpha,-\beta}{}^\gamma V_\gamma^\sigma(x)\\
&\quad + f_{\alpha,-\beta}{}^{-\gamma}V^\sigma_{-\gamma}(x) - \delta_{\alpha\beta}(\alpha^\vee,\sigma)x^\sigma,\\
V^\gamma_{-\alpha}(x)\partial_\gamma V^\sigma_{-\beta}(x) - V^\gamma_{-\beta}(x)\partial_\gamma V^\sigma_{-\alpha}(x) &= -f_{\alpha\beta}{}^\gamma V^\sigma_{-\gamma}(x),\\
V_\alpha^\beta(x)\partial_\beta P^j_{-\alpha}(x) &= G^{ij}(\alpha_i^\vee,\alpha^\vee),\\
V_\alpha^\gamma(x)\partial_\gamma P^j_{-\beta}(x) &= f_{\alpha,-\beta}{}^{-\gamma}P^j_{-\gamma}(x),\\
(\alpha_i^\vee,\beta)x^\beta\partial_\beta P^j_{-\alpha}(x) &= (\alpha_i^\vee,\alpha)P^j_{-\alpha}(x),\\
V^\gamma_{-\alpha}(x)\partial_\gamma P^j_{-\beta}(x) - V^\gamma_{-\beta}(x)\partial_\gamma P^j_{-\alpha}(x) &= -f_{\alpha\beta}{}^\gamma P^j_{-\gamma}(x).
\end{aligned} \tag{43}$$

They are obtained directly from the fact that E_α, H_i and F_α constitute a differential operator realization of **g**. Similarly, (42) gives the classical identities

$$\begin{aligned}
V_\alpha^\gamma(x)\partial_\gamma S_\beta^\sigma(x) - S_\beta^\gamma(x)\partial_\gamma V_\alpha^\sigma(x) &= 0,\\
(\alpha_i^\vee,\beta-\alpha)S_\alpha^\beta(x) &= (\alpha_i^\vee,\gamma)x^\gamma\partial_\gamma S_\alpha^\beta(x),
\end{aligned}$$

$$
\begin{aligned}
V^{\gamma}_{-\alpha}(x)\partial_\gamma S^{\sigma}_{\beta}(x) - S^{\gamma}_{\beta}(x)\partial_\gamma V^{\sigma}_{-\alpha}(x) &= P^{j}_{-\alpha}(x)(\alpha^\vee_j,\beta)S^{\sigma}_{\beta}(x) \\
&\quad -Q^{-\gamma}_{-\alpha}(x)f_{\beta,-\gamma}{}^{\mu}S^{\sigma}_{\mu}(x), \\
S^{\gamma}_{\alpha}(x)\partial_\gamma S^{\sigma}_{\beta}(x) - S^{\gamma}_{\beta}(x)\partial_\gamma S^{\sigma}_{\alpha}(x) &= f_{\alpha\beta}{}^{\gamma}S^{\sigma}_{\gamma}(x), \\
S^{\gamma}_{\beta}(x)\partial_\gamma P^{j}_{-\alpha}(x) &= -Q^{-\beta}_{-\alpha}(x)(\alpha^\vee_i,\beta^\vee)G^{ij}.
\end{aligned} \tag{44}
$$

4. Wakimoto free field realizations

The free field realization is well known to be obtained from the differential operator realization $\{\widetilde{J}_a\}$ by the substitution [2,3,5,8,10,11]

$$\partial_\alpha \to \beta_\alpha(z), \qquad x^\alpha \to \gamma^\alpha(z), \qquad \Lambda_i \to \sqrt{t}\,\partial\varphi_i(z) \tag{45}$$

and a subsequent normal ordering contribution or anomalous term, $F^{\text{anom}}_{\alpha}(\gamma(z),\partial\gamma(z))$, to be added to the lowering part. This term must have conformal dimension 1, and hence is bound to be of the form

$$F^{\text{anom}}_{\alpha}(\gamma(z),\partial\gamma(z)) = \partial\gamma^{\beta}(z)F_{\alpha\beta}(\gamma(z)), \tag{46}$$

giving rise to the following form of the free field realization:

$$
\begin{aligned}
&E_\alpha(z) = :V^{\beta}_{\alpha}(\gamma(z))\beta_\beta(z):, \\
&H_i(z) = :V^{\beta}_{i}(\gamma(z))\beta_\beta(z): + \sqrt{t}\,\partial\varphi_i(z), \\
&F_\alpha(z) = :V^{\beta}_{-\alpha}(\gamma(z))\beta_\beta(z): + \sqrt{t}\,\partial\varphi_j(z)P^{j}_{-\alpha}(\gamma(z)) + \partial\gamma^{\beta}(z)F_{\alpha\beta}(\gamma(z)), \\
&\Delta(J_a) = 1,
\end{aligned} \tag{47}
$$

where the normal ordering part for a *simple* root has been known for some time ([7]),

$$\partial\gamma^{\beta}(z)F_{\alpha_i\beta}(\gamma(z)) = \partial\gamma^{\alpha_i}(z)\left(\frac{k+t}{\alpha_i^2} - 1\right). \tag{48}$$

To find the result in the general case we first derive the *quantum polynomial identities* obtained by imposing the correct form of the OPE of the form JF and TF (we leave out the argument z),

$$
\begin{aligned}
\frac{2k}{\alpha^2}\delta_{\alpha,\beta} &= -\partial_\sigma V^{\gamma}_{\alpha}\partial_\gamma V^{\sigma}_{-\beta} + V^{\gamma}_{\alpha}F_{\beta\gamma}, \\
f_{\alpha,-\beta}{}^{-\gamma}\partial\gamma^{\delta}F_{\gamma\delta} &= -\partial\gamma^{\sigma}\partial_\sigma\partial_\mu V^{\gamma}_{\alpha}\partial_\gamma V^{\mu}_{-\beta} + V^{\gamma}_{\alpha}\partial\gamma^{\delta}\partial_\gamma F_{\beta\delta} + \partial\gamma^{\sigma}\partial_\sigma V^{\gamma}_{\alpha}F_{\beta\gamma}, \\
0 &= (\alpha^\vee_i,\sigma)\partial_\sigma V^{\sigma}_{-\beta} - (\alpha^\vee_i,\alpha)\gamma^{\alpha}F_{\beta\alpha} + tG_{ij}P^{j}_{-\beta}, \\
(\alpha^\vee_i,\beta)\partial\gamma^{\delta}F_{\beta\delta} &= (\alpha^\vee_i,\alpha)\gamma^{\alpha}\partial\gamma^{\delta}\partial_\alpha F_{\beta\delta} + (\alpha^\vee_i,\alpha)\partial\gamma^{\alpha}F_{\beta\alpha}, \\
0 &= 2(\rho,\alpha^\vee_j)P^{j}_{-\alpha} + \partial_\gamma V^{\gamma}_{-\alpha}, \\
\partial_\gamma V^{\sigma}_{-\alpha}\partial_\sigma V^{\gamma}_{-\beta} &= tG_{ij}P^{i}_{-\alpha}P^{j}_{-\beta} + V^{\gamma}_{-\alpha}F_{\beta\gamma} + V^{\gamma}_{-\beta}F_{\alpha\gamma},
\end{aligned}
$$

$$f_{\alpha\beta}{}^{\gamma}F_{\gamma\sigma} = \partial_\sigma\partial_\gamma V^{\mu}_{-\alpha}\partial_\mu V^{\gamma}_{-\beta} - V^{\gamma}_{-\alpha}\partial_\gamma F_{\beta\sigma} + V^{\gamma}_{-\beta}\partial_\gamma F_{\alpha\sigma}$$
$$-tG_{ij}\partial_\sigma P^{i}_{-\alpha}P^{j}_{-\beta} - \partial_\sigma V^{\gamma}_{-\alpha}F_{\beta\gamma} - V^{\gamma}_{-\beta}\partial_\sigma F_{\alpha\gamma}, \tag{49}$$

from the OPEs EF, HF, TF and FF. Not all the identities are independent, e.g. the second to last one follows from the last. The form of the normal ordering term is completely determined from the first identity, since we may introduce the inverse of $V_+^+ \sim V_\alpha^\beta$. Indeed we shall only need

$$\left((V_+^+)^{-1}\right)_\alpha^\beta \tag{50}$$

and it follows immediately that

$$(V_+^+(\gamma))^{-1} = B(C_+^+(\gamma))^{-1}$$
$$= \sum_{n\geqslant 0}\frac{1}{(n+1)!}(C_+^+(\gamma))^n. \tag{51}$$

Thus we have

$$F_{\alpha\beta}(\gamma) = \frac{2k}{\alpha^2}\left((V_+^+(\gamma))^{-1}\right)_\beta^\alpha + \left((V_+^+(\gamma))^{-1}\right)_\beta^\mu \partial_\sigma V_\mu^\gamma(\gamma)\partial_\gamma V^\sigma_{-\alpha}(\gamma). \tag{52}$$

A somewhat more involved form was given in Ref. [26]. The present result is similar to the one in Ref. [13], but as before, in this paper we have provided the explicit analytic results for all the polynomials of $\gamma(z)$ which enter.

5. Screening currents

5.1. Screening currents of the first kind

A screening current has conformal weight 1 and has the property that the singular part of the OPE with an affine current is a total derivative. These properties ensure that integrated screening currents (screening charges) may be inserted into correlators without altering the conformal or affine Ward identities. This in turn makes them very useful in construction of correlators, see e.g. Refs. [14,4,9,18,19]. The best known screening currents [2,5,7,10,11,13] are the following denoted screening currents of the first kind, one for each simple root:

$$s_j(z) =: S^\alpha_{\alpha_j}(\gamma(z))\beta_\alpha(z)\exp\left(-\frac{1}{\sqrt{t}}\alpha_j\cdot\varphi(z)\right):,$$
$$\alpha_j\cdot\varphi(z) = \frac{\alpha_j^2}{2}\varphi_j(z). \tag{53}$$

In this case we find

$$E_\alpha(z)s_j(w) = 0,$$
$$H_i(z)s_j(w) = 0,$$

$$F_\alpha(z)s_j(w) = -\frac{2t}{\alpha_j^2}\frac{\partial}{\partial w}\left(\frac{1}{z-w}Q_{-\alpha}^{-\alpha_j}(\gamma(w)) : \exp\left(-\frac{1}{\sqrt{t}}\alpha_j\cdot\varphi(w)\right) :\right),$$
$$T(z)s_j(w) = \frac{\partial}{\partial w}\left(\frac{1}{z-w}s_j(w)\right). \tag{54}$$

In our formalism the proof of these relations is a matter of direct verification and straightforward for E_α, H_i and T using the classical polynomial identities. In order for the OPE $F_\alpha(z)s_j(w)$ to be a total derivative we find that the following two relations are sufficient conditions:

$$\partial_\gamma V_{-\alpha}^\beta \partial_\beta S_{\alpha_j}^\gamma = 2\left(1-\frac{t}{\alpha_j^2}\right) S_{\alpha_j}^\beta \partial_\beta P_{-\alpha}^j + S_{\alpha_j}^\gamma F_{\alpha\gamma},$$
$$S_{\alpha_j}^\beta \partial_\beta F_{\alpha\sigma} = \partial_\gamma V_{-\alpha}^\beta \partial_\beta \partial_\sigma S_{\alpha_j}^\gamma - A_{ij}\partial_\sigma S_{\alpha_j}^\beta \partial_\beta P_{-\alpha}^i - \partial_\sigma S_{\alpha_j}^\gamma F_{\alpha\gamma}. \tag{55}$$

They are easily verified for α a simple root. In the case of a non-simple root α we have proven the conditions (55) by induction in addition of roots using various classical and quantum polynomial identities.

In Ref. [29] [4] screening currents of the first kind are considered and a proof of their properties is presented. In the recent work [13] a more direct proof similar to the one above is provided.

5.2. Screening currents of the second kind

The best known screening current of the second kind is the one by Bershadsky and Ooguri for $SL(2)$ [27]. For non-integral representations it involves non-integer powers of free ghost fields. Therefore, for some time discussions on its interpretation were only partly successful [9,16]. However, in the series of papers [18,19] we have provided techniques based on fractional calculus for handling such objects. Those techniques directly generalize to the present more general situation. Screening currents of both kinds are necessary for being able to treat correlators of primary fields belonging to degenerate (in particular admissible) representations [20,21].

The following expression for the screening currents of the second kind was written down without proof by Ito [7]:

$$\tilde{s}_j(w) =: \left(S_{\alpha_j}^\beta(\gamma(w))\beta_\beta(w)\exp\left(-\frac{1}{\sqrt{t}}\alpha_j\cdot\varphi(w)\right)\right)^{-2t/\alpha_j^2} :$$
$$=: \left(S_{\alpha_j}^\beta(\gamma(w))\beta_\beta(w)\right)^{-2t/\alpha_j^2} :: e^{\sqrt{t}\varphi_j(w)} : . \tag{56}$$

Here we will show that (at least in the case of $SL(N)$) they satisfy

$$E_\alpha(z)\tilde{s}_j(w) = 0,$$

[4] We thank E. Frenkel for pointing out this work to us.

$$
\begin{aligned}
H_i(z)\tilde{s}_j(w) &= 0, \\
F_\alpha(z)\tilde{s}_j(w) &= -\frac{2t}{\alpha_j^2}\frac{\partial}{\partial w}\left(\frac{1}{z-w} : Q_{-\alpha}^{-\alpha_j}(\gamma(w))\right. \\
&\quad \times \left.\left(S_{\alpha_j}^{\beta}(\gamma(w))\beta_\beta(w)\right)^{-(2t/\alpha_j^2)-1} e^{\sqrt{t}\varphi_j(w)} :\right), \\
T(z)\tilde{s}_j(w) &= \frac{\partial}{\partial w}\left(\frac{1}{z-w}\tilde{s}_j(w)\right).
\end{aligned} \tag{57}
$$

We employ the techniques discussed in Refs. [18,19] for how to perform contractions involving ghost fields raised to non-integer powers. Such techniques are necessary in the generic case where $-2t/\alpha_j^2$ is not integer.

In the case of $SL(N)$ where a simple root (here α_j) appears at most once in the decomposition of a positive root, it is straightforward to check that $H_i(z)\tilde{s}_j(w) = 0$ and $\Delta(\tilde{s}_j) = 1$.

Let us introduce the shorthand notation

$$
S_j^u(z) =: \left(S_{\alpha_j}^{\beta}(\gamma(z))\beta_\beta(z)\right)^u : \tag{58}
$$

and consider

$$
\begin{aligned}
E_\alpha\ (z)S_j^u(w) &= \sum_{l\geqslant 1}\frac{1}{(z-w)^l}(-1)^l\binom{u}{l}\Big(: \partial_{\gamma_1}\dots\partial_{\gamma_l}V_\alpha^\beta\beta_\beta(z)S_{\alpha_j}^{\gamma_1}\dots S_{\alpha_j}^{\gamma_l}S_j^{u-l} : \\
&\quad -l : \partial_{\gamma_1}\dots\partial_{\gamma_{l-1}}V_\alpha^\beta(z)S_{\alpha_j}^{\gamma_1}\dots S_{\alpha_j}^{\gamma_{l-1}}\partial_\beta S_{\alpha_j}^{\gamma_l}\beta_{\gamma_l}S_j^{u-l} :\Big) \\
&\quad +\sum_{l\geqslant 2}\frac{(-1)^{l-1}}{(z-w)^l}\binom{u}{l-1} : \partial_{\gamma_1}\dots\partial_{\gamma_{l-1}}V_\alpha^\beta(z)\partial_\beta(S_{\alpha_j}^{\gamma_1}\dots S_{\alpha_j}^{\gamma_{l-1}})S_j^{u-l+1} : .
\end{aligned} \tag{59}
$$

Here and in the following equations we have suppressed the argument $\gamma(z)$ of ${V_\alpha}^\beta$ and other fields. We use the explicit expressions for V_α^β and $S_{\alpha_j}^\gamma$ and find that all terms for $l > 1$ vanish. Hence, in that case we get

$$
\begin{aligned}
E_\alpha(z)S_j^u(w) &= \frac{u}{z-w} : \left(V_\alpha^\gamma\partial_\gamma S_{\alpha_j}^\beta - S_{\alpha_j}^\gamma\partial_\gamma V_\alpha^\beta\right)\beta_\beta S_j^{u-1} : (w) \\
&= 0.
\end{aligned} \tag{60}
$$

In a similar way, we have worked out the OPE $F_\alpha(z)\tilde{s}_j(w)$ and in the case of $SL(N)$ it reduces to ($u = -2t/\alpha_j^2$)

$$
\begin{aligned}
F_\alpha(z)\tilde{s}_j(w) &= \frac{1}{(z-w)^2}\Big(-u : \partial_\gamma V_{-\alpha}^\beta(z)\partial_\beta S_{\alpha_j}^\gamma S_j^{u-1} : \\
&\quad +\frac{u(u-1)}{2} : \partial_{\gamma_1}\partial_{\gamma_2}V_{-\alpha}^\beta\beta_\beta(z)S_{\alpha_j}^{\gamma_1}S_{\alpha_j}^{\gamma_2}S_j^{u-2} : \\
&\quad -u(u-1) : \partial_{\gamma_1}V_{-\alpha}^\beta(z)S_{\alpha_j}^{\gamma_1}\partial_\beta S_{\alpha_j}^{\gamma_2}\beta_{\gamma_2}S_j^{u-2} :
\end{aligned}
$$

$$
\begin{aligned}
& - utG_{ij} : \partial_\gamma P^i_{-\alpha}(z) S^\gamma_{\alpha_j} S_j^{u-1} : + u : F_{\alpha\sigma}(z) S^\sigma_{\alpha_j} S_j^{u-1} : \Big) : e^{\sqrt{t}\varphi_j(w)} : \\
& + \frac{1}{z-w} \Big(u : (V^\gamma_{-\alpha}\partial_\gamma S^\beta_{\alpha_j} - S^\gamma_{\alpha_j}\partial_\gamma V^\beta_{-\alpha})\beta_\beta S_j^{u-1} :: e^{\sqrt{t}\varphi_j(w)} : \\
& - u\sqrt{t} : \partial_\gamma P^i_{-\alpha} S^\gamma_{\alpha_j} S_j^{u-1} \partial\varphi_i e^{\sqrt{t}\varphi_j(w)} : \\
& + tG_{ij} : P^i_{-\alpha} S^u_j :: e^{\sqrt{t}\varphi_j(w)} : - u : \partial\gamma^\sigma \partial_\gamma F_{\alpha\sigma} S^\gamma_{\alpha_j} S_j^{u-1} :: e^{\sqrt{t}\varphi_j(w)} : \Big) .
\end{aligned}
\tag{61}
$$

A comparison with (57) yields the following consistency condition:

$$
\begin{aligned}
& (u-1)\left(\frac{1}{2}\partial_\sigma\partial_{\gamma_1}\partial_{\gamma_2} V^\beta_{-\alpha} S^{\gamma_1}_{\alpha_j} S^{\gamma_2}_{\alpha_j} - \partial_\sigma\partial_{\gamma_1} V^{\gamma_2}_{-\alpha} S^{\gamma_1}_{\alpha_j} \partial_{\gamma_2} S^\beta_{\alpha_j}\right) \\
& + \left(-\partial_\sigma\partial_\gamma V^\mu_{-\alpha}\partial_\mu S^\gamma_{\alpha_j} + S^\gamma_{\alpha_j}\partial_\sigma F_{\alpha\gamma} - tG_{ij} S^\gamma_{\alpha_j}\partial_\gamma\partial_\sigma P^i_{-\alpha} - S^\gamma_{\alpha_j}\partial_\gamma F_{\alpha\sigma}\right) S^\beta_{\alpha_j} \\
& = (u-1) Q^{-\alpha_j}_{-\alpha} \partial_\sigma S^\beta_{\alpha_j} + \partial_\sigma Q^{-\alpha_j}_{-\alpha} S^\beta_{\alpha_j}
\end{aligned}
\tag{62}
$$

besides more trivial relations such as

$$
\begin{aligned}
S^\gamma_{\alpha_j}\partial_\gamma P^i_{-\alpha} &= -\delta^i_j Q^{-\alpha_j}_{-\alpha}, \\
\frac{1}{2} S^{\gamma_1}_{\alpha_j} S^{\gamma_2}_{\alpha_j} \partial_{\gamma_1}\partial_{\gamma_2} V^\beta_{-\alpha} &= Q^{-\alpha_j}_{-\alpha} S^\beta_{\alpha_j}, \\
S^{\gamma_1}_{\alpha_j}\partial_{\gamma_1} V^{\gamma_2}_{-\alpha}\partial_{\gamma_2} S^\beta_{\alpha_j} &= 0,
\end{aligned}
\tag{63}
$$

which are easily seen to be satisfied. One can verify the less trivial part (62) using the polynomial identities together with the consistency conditions (55). Hence, we conclude that in the case of $SL(N)$ the screening currents of the second kind (57) exist. As already mentioned it seems natural that the expression (57) should hold for all simple groups; we refer to Ref. [25] (and Ref. [26]) for further details. In Ref. [25] a quantum group structure based on both kinds of screening currents will also be discussed (see also the presentation in Ref. [26]), along the lines of Gomez and Sierra [30].

6. Primary fields

The final new result reported in this paper is the explicit construction in this section of primary fields for arbitrary representations, integral or non-integral (for integral representations, see also Ref. [2]). We find it particularly convenient to replace the traditional multiplet of primary fields (which generically would be infinite for non-integrable representations) by a generating function for that, namely the primary field $\phi_\Lambda(w,x)$ which must satisfy

$$
\begin{aligned}
J_a(z)\phi_\Lambda(w,x) &= \frac{-J_a(x,\partial,\Lambda)}{z-w}\phi_\Lambda(w,x), \\
T(z)\phi_\Lambda(w,x) &= \frac{\Delta(\phi_\Lambda)}{(z-w)^2}\phi_\Lambda(w,x) + \frac{1}{z-w}\partial\phi_\Lambda(w,x).
\end{aligned}
\tag{64}
$$

Here $J_a(z)$ are the affine currents, whereas $J_a(x, \partial, \Lambda)$ are the differential operator realizations Eqs. (30), (31), (33) and (37). For the simplest case of affine $SL(2)$ the result is known [16,18]. In that case we have only one positive root, one x, one ghost pair $(\beta(z), \gamma(z))$ and one scalar field $\varphi(z)$ while Λ is given by the spin j ($2j = \Lambda_1$ is non-integral in general). The result is

$$\phi_j(w, x) = (1 + x\gamma(w))^{2j} : \exp\left(\frac{2j}{\sqrt{t}}\varphi(w)\right) : . \tag{65}$$

In this section we show how to generalize this sort of expression to an arbitrary Lie algebra, with particularly explicit prescriptions in the case of affine $SL(N)$. We shall find the result in the form

$$\begin{aligned} \phi_\Lambda(w, x) &= \phi'_\Lambda(\gamma(w), x) V_\Lambda(w), \\ V_\Lambda(w) &=: \exp\left(\frac{1}{\sqrt{t}}\Lambda \cdot \varphi(z)\right) :, \\ \phi'_\Lambda(\gamma(w), 0) &= 1. \end{aligned} \tag{66}$$

Indeed such a field is conformally primary and has conformal dimension $\Delta(\phi_\Lambda) = \frac{1}{2t}(\Lambda, \Lambda + 2\rho)$. In order to comply with (64) for $J_a = H_i$, it seems very plausible that ϕ'_Λ must be symmetric in $\gamma(w)$ and x. Below we shall show this by explicit construction. Due to the fact that the anomalous or normal ordering part of $F_\alpha(z)$ does not give singular contributions when contracting with ϕ'_Λ, it is then sufficient to consider OPEs with E_α. The point is that the two OPEs $E_\alpha(z)\phi_\Lambda(w, x)$ and $F_\alpha(z)\phi_\Lambda(w, x)$ are obtained from one another by interchanging x and $\gamma(w)$. Because of the above symmetry it is enough to verify one of them. We therefore obtain the following sufficient conditions on $\phi'_\Lambda(\gamma(w), x)$, one for each $\alpha > 0$:

$$V_\alpha^\beta(\gamma)\partial_{\gamma^\beta}\phi'_\Lambda = V_{-\alpha}^\beta(x)\partial_{x^\beta}\phi'_\Lambda + \Lambda_j P_{-\alpha}^j(x)\phi'_\Lambda. \tag{67}$$

Further, one can use the classical polynomial identities (43) to demonstrate that if ϕ_Λ is a primary field with respect to E_α and E_β, then it is a primary field with respect to $f_{\alpha\beta}{}^\gamma E_\gamma$. Effectively, this amounts to prove (67) for a sum $\alpha = \beta + \gamma$ of two roots under the assumption that it is satisfied for both $\alpha = \beta$ and $\alpha = \gamma$. This means that there are only r sufficient conditions a primary field (66) must satisfy,

$$V_{\alpha_i}^\beta(\gamma)\partial_{\gamma^\beta}\phi'_\Lambda = V_{-\alpha_i}^\beta(x)\partial_{x^\beta}\phi'_\Lambda + \Lambda_i x^{\alpha_i}\phi'_\Lambda. \tag{68}$$

It is very hard to solve this set of partial differential equations directly. However, we have found an alternative way to obtain the primary fields. The construction goes as follows.

First we directly construct primary fields for each fundamental representation $M_{\Lambda^{(k)}}$. Such representation spaces are finite-dimensional modules and $\phi'_{\Lambda^{(k)}}(\gamma(w), x)$ will be polynomial in $\gamma(w)$ and x. Then finally, for a general representation with highest weight $\Lambda = \Lambda_k\Lambda^{(k)}$ we use (68) to immediately obtain that[5]

[5] In Ref. [26] a discussion of this point is partly incorrect.

$$\phi'_{\Lambda}(\gamma(w),x) = \prod_{k=1}^{r} [\phi'_{\Lambda^{(k)}}(\gamma(w),x)]^{\Lambda_k}. \tag{69}$$

We emphasize here that the Dynkin labels, Λ_k, may be non-integers as is required for degenerate (including admissible) representations. We proceed to explain how to construct the building blocks,

$$\phi'_{\Lambda^{(k)}}(\gamma(w),x). \tag{70}$$

The strategy goes as follows. First we concentrate on the case $w = 0$ where the object reduces to

$$\phi'_{\Lambda^{(k)}}(\gamma_0,x) \tag{71}$$

when acting on the highest weight state $|\Lambda^{(k)}\rangle$. γ_0 is the zero mode in the mode expansion

$$\gamma(w) = \sum_n \gamma_n w^{-n}, \qquad \beta(w) = \sum_n \beta_n w^{-n-1}. \tag{72}$$

Conformal covariance requires $\phi'_{\Lambda^{(k)}}(\gamma(w),x)$ to be obtained just by replacing γ_0 by $\gamma(w)$. The function (71) in turn is obtained from

$$|\Lambda^{(k)},x\rangle = g_-(x)|\Lambda^{(k)}\rangle = \phi'_{\Lambda^{(k)}}(\gamma_0,x)|\Lambda^{(k)}\rangle. \tag{73}$$

Indeed, it is an immediate consequence of the formalism, that the primary field constructed this way will satisfy the OPE (64). The construction is now simply achieved by expanding the state $|\Lambda^{(k)},x\rangle$ on an appropriate basis which is convenient to obtain using the free field realization.

Let the orthonormal basis elements in the kth fundamental highest weight module $M_{\Lambda^{(k)}}$ be denoted by $\{|b,\Lambda^{(k)}\rangle\}$ such that the identity operator may be written as

$$I = \sum_b |b,\Lambda^{(k)}\rangle\langle b,\Lambda^{(k)}|. \tag{74}$$

The state $|\Lambda^{(k)},x\rangle$ may then be written as

$$|\Lambda^{(k)},x\rangle = \sum_b |b,\Lambda^{(k)}\rangle\langle b,\Lambda^{(k)}|\Lambda^{(k)},x\rangle. \tag{75}$$

One of the basis vectors will always be taken to be the highest weight vector $|\Lambda^{(k)}\rangle$ itself.

Now concentrate on one particular basis vector. It will be of the form (see also Section 8: Note added in proof)

$$|b,\Lambda^{(k)}\rangle = f_{\beta_1^{(b)}} \cdots f_{\beta_{n(b)}^{(b)}}|\Lambda^{(k)}\rangle \tag{76}$$

and the expansion of $|\Lambda^{(k)},x\rangle$ will be

$$|\Lambda^{(k)},x\rangle = \sum_b f_{\beta_1^{(b)}} \cdots f_{\beta_{n(b)}^{(b)}}|\Lambda^{(k)}\rangle\langle\Lambda^{(k)}|e_{\beta_{n(b)}^{(b)}} \cdots e_{\beta_1^{(b)}}|\Lambda^{(k)},x\rangle. \tag{77}$$

For each term in the sum we treat the two factors differently. First consider the second factor. We may use the differential operator realizations to write

$$\begin{aligned}
&\langle \Lambda^{(k)} | e_{\beta^{(b)}_{n(b)}} \dots e_{\beta^{(b)}_1} | \Lambda^{(k)}, x\rangle \\
&= (-1)^{n(b)} E_{\beta^{(b)}_1}(x, \partial, \Lambda^{(k)}) \dots E_{\beta^{(b)}_{n(b)}}(x, \partial, \Lambda^{(k)}) \langle \Lambda^{(k)} | \Lambda^{(k)}, x\rangle \\
&= b(x, \Lambda^{(k)}),
\end{aligned} \tag{78}$$

where

$$\begin{aligned}
b(x, \Lambda^{(k)}) = &\left[V^{\gamma_1}_{-\beta^{(b)}_1}(x) \partial_{\gamma_1} + P^k_{-\beta^{(b)}_1}(x) \right] \dots \\
&\times \left[V^{\gamma_{n(b)-1}}_{-\beta^{(b)}_{n(b)-1}}(x) \partial_{\gamma_{n(b)-1}} + P^k_{-\beta^{(b)}_{n(b)-1}}(x) \right] P^k_{-\beta^{(b)}_{n(b)}}(x).
\end{aligned} \tag{79}$$

In the last step we used that clearly

$$\langle \Lambda^{(k)} | \Lambda^{(k)}, x\rangle \equiv 1. \tag{80}$$

In the first factor in (77),

$$f_{\beta^{(b)}_1} \dots f_{\beta^{(b)}_{n(b)}} | \Lambda^{(k)}\rangle, \tag{81}$$

we use the free field realizations. The state $|\Lambda^{(k)}\rangle$ is a vacuum for the β, γ system, so it is annihilated by γ_n, $n \geqslant 1$ and by β_n, $n \geqslant 0$. The f_β's are the zero modes of the affine currents. It follows that only γ_0's and β_0's need be considered. Also the normal ordering term will not give a contribution, and we obtain

$$\begin{aligned}
f_{\beta^{(b)}_1} \dots f_{\beta^{(b)}_{n(b)}} | \Lambda^{(k)}\rangle = &\left[V^{\gamma_1}_{-\beta^{(b)}_1}(\gamma_0) \beta_{\gamma_1, 0} + P^k_{-\beta^{(b)}_1}(\gamma_0) \right] \dots \\
&\times \left[V^{\gamma_{n(b)-1}}_{-\beta^{(b)}_{n(b)-1}}(\gamma_0) \beta_{\gamma_{n(b)-1}, 0} + P^k_{-\beta^{(b)}_{n(b)-1}}(\gamma_0) \right] P^k_{-\beta^{(b)}_{n(b)}}(\gamma_0) | \Lambda^{(k)}\rangle \\
= &\, b(\gamma_0, \Lambda^{(k)}) | \Lambda^{(k)}\rangle.
\end{aligned} \tag{82}$$

By the remarks above this completes the construction in general. Explicit expressions for the V's and the P's have already been provided.

It remains to account in detail for how to obtain a convenient basis for the fundamental representations. This part will depend on the algebra. Here, for completeness and illustration, we indicate the construction for $SL(N)$ where the fundamental representations are conveniently realized in terms of N fermionic creation and annihilation operators,

$$q^\dagger_i, q_i, \qquad i = 1, \dots, N. \tag{83}$$

An orthonormal basis in the kth fundamental representation is provided by the sets of states where k fermionic creation operators act on the Fermi vacuum, giving dimension

$$\binom{N}{k}. \tag{84}$$

The $r = N - 1$ dimensional root and weight space may be represented as the $N - 1$ dimensional hyperplane in N-dimensional Euclidean space,

$$\{\sum_{j=1}^{N} y^j \boldsymbol{e}_j \mid \sum_j y^j = 0\}, \tag{85}$$

where $\{\boldsymbol{e}_j\}$ is an orthonormal basis for the N-dimensional space. The roots of $SL(N)$ are of the form

$$\alpha_{ij} = \boldsymbol{e}_i - \boldsymbol{e}_j. \tag{86}$$

We may take

$$\begin{aligned} e_{ij} &\equiv e_{\alpha_{ij}} = q_i^\dagger q_j, \qquad i < j, \\ f_{ij} &\equiv f_{\alpha_{ij}} = q_j^\dagger q_i, \qquad i < j. \end{aligned} \tag{87}$$

The highest weight vector is

$$|\Lambda^{(k)}\rangle = q_1^\dagger q_2^\dagger \dots q_k^\dagger |0\rangle. \tag{88}$$

A basis with a minimal set of lowering operators is then easily seen to be the set

$$\begin{aligned} &|\Lambda^{(k)}\rangle, \\ &f_{ij}|\Lambda^{(k)}\rangle, \qquad i \leqslant k < j \leqslant N, \\ &\vdots \\ &f_{i_1 j_1} \dots f_{i_p j_p} |\Lambda^{(k)}\rangle, \qquad i_1 < \dots < i_p \leqslant k < j_1 < \dots < j_p \leqslant N, \\ &\vdots \end{aligned} \tag{89}$$

where altogether $p = 0, 1, \dots, N - k$.

The primary field for the kth fundamental representation is then of the form

$$\begin{aligned} \phi_{\Lambda^{(k)}}(z,x) &= \phi'_{\Lambda^{(k)}}(\gamma(z),x) V_{\Lambda^{(k)}}(z), \\ \phi'_{\Lambda^{(k)}}(\gamma(z),x) &= \sum_{p=0}^{N-k} \sum_{i_1<\dots<i_p\leqslant k<j_1<\dots<j_p\leqslant N} \\ &\quad \times b_p(\{i_l\},\{j_l\},\gamma(z),\Lambda^{(k)}) b_p(\{i_l\},\{j_l\},x,\Lambda^{(k)}), \\ b_0(x,\Lambda^{(k)}) &= 1, \\ b_p(\{i_l\},\{j_l\},x,\Lambda^{(k)}) &= (-1)^p E_{\alpha_{i_1 j_1}}(x,\partial,\Lambda^{(k)}) \dots E_{\alpha_{i_p j_p}}(x,\partial,\Lambda^{(k)}) 1. \end{aligned} \tag{90}$$

Actually in this particular case of $SL(N)$ (and possibly with a suitable generalization, for more general groups), an even more explicit realization is possible, one not involving derivatives. Indeed the basis for the kth fundamental representation (89) may be equivalently obtained as the set

$$|b; I(k)\rangle = q_{i_1}^\dagger \dots q_{i_k}^\dagger |0\rangle. \tag{91}$$

Here we have denoted by $I(k)$ the subset $\{i_1,\ldots,i_k\}$ of the set $\{1,2,\ldots,N\}$ and we shall denote by $M(N,k)$ the set of all these subsets, so that

$$|M(N,k)| = \binom{N}{k}. \tag{92}$$

Now we may evaluate

$$\begin{aligned} |\Lambda^{(k)},x\rangle &= e^{f(x)} q_1^\dagger \ldots q_k^\dagger |0\rangle \\ &= \left(e^{F^{(N)}(x)}\right)_1^{\ j_1} \ldots \left(e^{F^{(N)}(x)}\right)_k^{\ j_k} q_{j_1}^\dagger \ldots q_{j_k}^\dagger |0\rangle, \end{aligned} \tag{93}$$

where $F^{(N)}(x)$ is the matrix representation of $f(x)$ in the N-dimensional defining representation of $SL(N)$,

$$\begin{aligned} F^{(N)}(x)_i^{\ j} &= x^{ij}, \qquad i<j, \\ F^{(N)}(x)_i^{\ j} &= 0, \qquad i \geqslant j. \end{aligned} \tag{94}$$

We have to evaluate the overlap between the state (93) and the basis vector in (91). The result is the well-known determinant. Namely, denote by

$$e^{F^{(N)}(x)}(I(k)) \tag{95}$$

the $k \times k$ matrix obtained from the $N \times N$ matrix $e^{F^{(N)}(x)}$ by using the first k rows and the k columns given by the set $I(k)$. Then the sought overlap (up to a sign which will be irrelevant) is the determinant of that reduced $k \times k$ matrix. From the discussion above we know that in the primary field, the polynomial in x thus obtained will be multiplied by exactly the same polynomial in $\gamma(z)$ (hence the irrelevance of the above sign). Thus we have the simplified version of Eq. (90),

$$\phi'_{\Lambda^{(k)}}(\gamma(z),x) = \sum_{I(k)\in M(N,k)} \det\left(e^{F^{(N)}(x)}(I(k))\right) \det\left(e^{F^{(N)}(\gamma(z))}(I(k))\right). \tag{96}$$

A similar expression is obtained when states are represented in terms of fermion *annihilation* operators acting on the "filled Dirac sea" state,

$$|\bar{0}\rangle = q_1^\dagger \ldots q_N^\dagger |0\rangle. \tag{97}$$

The new form is similar to Eq. (96) but k is replaced by $N-k$ and $F^{(N)}$ by $-F^{(N)}$, and now we have to use the *last* $N-k$ rows. This second form is the most convenient one for $k \geqslant N/2$.

It is not difficult to check that the two expressions Eqs. (90) and (96) (of course) agree with each other in the cases $k=1$ and $k=r=N-1$. Thus Eq. (90) gives

$$\begin{aligned} \phi'_{\Lambda^{(1)}}(\gamma(z),x) &= 1 + \sum_{j=2}^{N} P^1_{-\alpha_{1j}}(\gamma(z)) P^1_{-\alpha_{1j}}(x), \\ \phi'_{\Lambda^{(N-1)}}(\gamma(z),x) &= 1 + \sum_{i=1}^{N-1} P^{N-1}_{-\alpha_{iN}}(\gamma(z)) P^{N-1}_{-\alpha_{iN}}(x). \end{aligned} \tag{98}$$

As an illustration, we get in the case of $SL(3)$ ($M_{\Lambda^{(1)}} = \{3\}$, $M_{\Lambda^{(2)}} = \{\bar{3}\}$)

$$\phi'_{\Lambda^{(1)}}(\gamma(z),x) = 1 + \gamma^{12}(z)x^{12} + \left(\gamma^{13}(z) + \frac{1}{2}\gamma^{12}(z)\gamma^{23}(z)\right)\left(x^{13} + \frac{1}{2}x^{12}x^{23}\right),$$
$$\phi'_{\Lambda^{(2)}}(\gamma(z),x) = 1 + \gamma^{23}(z)x^{23} + \left(\gamma^{13}(z) - \frac{1}{2}\gamma^{12}(z)\gamma^{23}(z)\right)\left(x^{13} - \frac{1}{2}x^{12}x^{23}\right). \tag{99}$$

The last two results were obtained already in Ref. [26].

We note that with the two sets of screening operators constructed in Section 5, in principle we shall be able to use the standard free field techniques to provide integral representations of correlators of primary fields with weights given by non-integer Dynkin labels of the form

$$\Lambda_k = A_{ki}r^i - G_{ki}s^i t = \hat{r}_k - \hat{s}_k \frac{\theta^2}{\alpha_k^2}\hat{t},$$
$$\hat{t} = \frac{2}{\theta^2}t = k^\vee + h^\vee, \tag{100}$$

where $i, k = 1, \dots, r$ and $r^i, s^i, \hat{r}_k, \hat{s}_k$ are integers corresponding to degenerate representations [20]. For $\hat{t}$ rational of the form $\hat{t} = p/q$ with p and q co-prime, this corresponds to admissible representations [21].

In our previous work on $SL(2)$ [18,19] we used a notation different from the one employed in this paper. The correspondence is the following, where "hats" refer to our old notation:

$$\hat{J}^+ = E, \qquad \hat{J}^3 = \frac{1}{2}H, \qquad \hat{J}^- = F,$$
$$\hat{k} = k^\vee, \qquad \hat{t} = \frac{2}{\theta^2}t, \qquad 2\hat{j} = \Lambda_1,$$
$$\hat{\varphi} = -\sqrt{\frac{\theta^2}{4}}\varphi_1 \tag{101}$$

and where $G^{11} = \theta_2/4$, such that $\Delta(\phi) = \hat{j}(\hat{j}+1)/\hat{t}$. Furthermore, there are additional phases on the screening currents.

7. Conclusions

In this paper we have provided missing ingredients needed in order to use free field realizations of affine algebras for setting up integral representations of conformal (chiral) blocks for arbitrary degenerate representations [20] generalizing the famous treatments for minimal models [14] and the more recent ones for $SL(2)$ [18,19]. Our new results are (i) very explicit and universal formulas for the free field realizations of currents, Eqs. (47), (52), (37), (36) and (10); (ii) a proof of the properties of the screening currents of the second kind Eq. (56) [7], at least for affine $SL(N)$ based on the screening currents of the first kind Eq. (53); (iii) free field realizations for full multiplets of primary fields using the triangular parameters, and valid for arbitrary weights (integral and non-integral), Eqs. (66), (69) and (79). In particular, we now have ingredients for building correlators for degenerate (and admissible) representations with weights obtained from Eq. (100). The realization is particularly explicit for $SL(N)$, Eqs. (90) and (96).

8. Note added in proof

A normalization constant is missing in the general discussion of primary fields in Section 6. It concerns the construction of the unit operator (77) built from *normalized* basis vectors. Thus, (76) and (77) should be replaced by

$$|b,\Lambda^{(k)}\rangle = N_b^{-\frac{1}{2}} f_{\beta_1^{(b)}} \dots f_{\beta_{n(b)}^{(b)}} |\Lambda^{(k)}\rangle\,,$$

$$N_b = \langle\Lambda^{(k)}| e_{\beta_{n(b)}^{(b)}} \dots e_{\beta_1^{(b)}} f_{\beta_1^{(b)}} \dots f_{\beta_{n(b)}^{(b)}} |\Lambda^{(k)}\rangle \tag{102}$$

and

$$|\Lambda^{(k)},x\rangle = \sum_b \frac{1}{N_b} f_{\beta_1^{(b)}} \dots f_{\beta_{n(b)}^{(b)}} |\Lambda^{(k)}\rangle\langle\Lambda^{(k)}| e_{\beta_{n(b)}^{(b)}} \dots e_{\beta_1^{(b)}} |\Lambda^{(k)},x\rangle\,. \tag{103}$$

The subsequent construction of primary fields for $SL(N)$ is correct, since $N_b = 1$ in that case.

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References

[1] M. Wakimoto, Commun. Math. Phys. 104 (1986) 60.

[2] B.L. Feigin and E.V. Frenkel, Usp. Mat. Nauk. 43 (1988) 227, in Russian; Russ. Math. Surv. 43 (1989) 221;
B.L. Feigin and E.V. Frenkel, Commun. Math. Phys. 128 (1990) 161; Lett. Math. Phys. 19 (1990) 307; *in* Physics and Mathematics of Strings, ed. L. Brink et al. (World Scientific, Singapore, 1990);
E. Frenkel, Free field realizations in representation theory and conformal field theory, preprint hep-th/9408109.

[3] A. Morozov, JETP Lett. 49 (1989) 345;
A. Gerasimov, A. Marshakov, A. Morozov, M. Olshanetskii and S. Shatashvili, Int. J. Mod. Phys. A 5 (1990) 2495.

[4] D. Bernard and G. Felder, Commun. Math. Phys. 127 (1990) 145.
[5] P. Bouwknegt, J. McCarthy and K. Pilch, Phys. Lett. B 234 (1990) 297; Commun. Math. Phys. 131 (1990) 125; Prog. Theor. Phys. Suppl. 102 (1990) 67; *in* Strings and Symmetries 1991, ed. N. Berkovits et al. (World Scientific, Singapore, 1992).
[6] M. Kuwahara and H. Suzuki, Phys. Lett. B 235 (1990) 52;
M. Kuwahara, N. Ohta and H. Suzuki, Phys. Lett. B 235 (1990) 57; Nucl. Phys. B 340 (1990) 448; Nucl. Phys. B 332 (1990) 146.
[7] K. Ito, Phys. Lett. B 252 (1990) 69.
[8] K. Ito and Y. Kazama, Mod. Phys. Lett. A 5 (1990) 215; A 6 (1991) 581.
[9] Vl.S. Dotsenko, Nucl. Phys. B 338 (1990) 747; B 358 (1991) 547.
[10] G. Kuroki, Commun. Math. Phys. 142 (1991) 511.
[11] H. Awata, A. Tsuchiya and Y. Yamada, Nucl. Phys. B 365 (1991) 680;
H. Awata, Prog. Theor. Phys. Suppl. 110 (1992) 303.
[12] W. Taylor IV, LBL-34507, hep-th/9310040, Ph.D. thesis.
[13] J. de Boer and L. Fehér, Mod. Phys. Lett. A 11 (1996) 1999; Wakimoto realizations of current algebras: an explicit construction, LBNL-39562, UCB-PTH-96/49, BONN-TH-96/16, hep-th/9611083, preprint.
[14] Vl.S. Dotsenko and V.A. Fateev, Nucl. Phys. B 240 [FS12] (1984) 312; B 251 [FS13] (1985) 691.
[15] V.A. Fateev and A.B. Zamolodchikov, Sov. J. Nucl. Phys. 43 (1986) 657.
[16] P. Furlan, A.Ch. Ganchev, R. Paunov and V.B. Petkova, Phys. Lett. B 267 (1991) 63; Nucl. Phys. B 394 (1993) 665;
A.Ch. Ganchev and V.B. Petkova, Phys. Lett. B 293 (1992) 56.
[17] O. Andreev, Int. J. Mod. Phys. A 10 (1995) 3221; Phys. Lett. B 363 (1995) 166.
[18] J.L. Petersen, J. Rasmussen and M. Yu, Nucl. Phys. B 457 (1995) 309; B 457 (1995) 343; *in* Proc. Workshop Gauge Theories, Applied Supersymmetry and Quantum Gravity, Leuven July 1995, Leuven Notes in Mathematical and Theoretical Physics, Vol. 6, ed. B. de Wit et al. (Leuven, 1996) hep-th/9510059; Nucl. Phys. B (Proc. Suppl.) 49 (1996) 27.
[19] J.L. Petersen, J. Rasmussen and M. Yu, Nucl. Phys. B 481 (1996) 577; *in* Proc. Inauguration Conf. of the Asia Pacific Center for Theoretical Physics (APCTP), Seoul, Korea, 1996.
[20] V.G. Kac and D.A. Kazhdan, Adv. Math. 34 (1979) 97.
[21] V.G. Kac and M. Wakimoto, Proc. Natl. Acad. Sci. USA 85 (1988) 4956; Adv. Ser. Math. Phys., Vol. 7 (World Scientific, 1989) p. 138.
[22] H.-L. Hu and M. Yu, Phys. Lett. B 289 (1992) 302; Nucl. Phys. B 391 (1993) 389.
[23] O. Aharony, O. Ganor, J. Sonnenschein and S. Yankielowicz, Nucl. Phys. B 399 (1993) 527.
[24] J. Schnittger, Nucl. Phys. B 471 (1996) 521.
[25] J. Rasmussen, Screening currents of both kinds in 2D current algebra, and a quantum group structure, preprint in preparation.
[26] J. Rasmussen, hep-th/9610167, Ph.D. thesis (The Niels Bohr Institute).
[27] M. Bershadsky and H. Ooguri, Commun. Math. Phys. 126 (1989) 49.
[28] B. Kostant, Lect. Notes Math. 466 (1974) 101.
[29] B.L. Feigin, E.V. Frenkel and N. Reshetikhin, Commun. Math. Phys. 166 (1994) 27.
[30] C. Gomez and G. Sierra, Phys. Lett. B 240 (1990) 149; Nucl. Phys. B 352 (1991) 791;
C. Ramirez, H. Ruegg and M. Ruiz-Altaba, Phys. Lett. B 247 (1990) 499; Nucl. Phys. B 364 (1991) 195;
C. Gomez and G. Sierra, *in* Proc. the Trieste 1990 Summer School (World Scientific, Singapore, 1991);
C. Gomez, M. Ruiz-Altaba and G. Sierra, Quantum Groups in Two-dimensional Physics (Cambridge University Press, Cambridge, 1996).