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On the energy-momentum of light: An all optical view of the Abraham-Minkowski Controversy

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Momentum of light in medium

The momentum of light in the linear vacuum is an unambiguous concept. Due to the simple constitutive relations $\mathbf{D} = \varepsilon_0 \mathbf{E}$, and $\mathbf{H} = \mathbf{B}/\mu_0$, the momentum density can be expressed as either $\mathbf{g}_{vac} = \mathbf{D} \times \mathbf{B}$ or $\mathbf{g}_{vac} = \mathbf{E} \times \mathbf{H}/c^2$.

The presence of a medium with polarization $\mathbf{P}$ and magnetization $\mathbf{M}$ changes the constitutive relations,

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \qquad \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}.$$

There are now two possibilities for the momentum density, due to Minkowski and Abraham (see review [1]).

Minkowski

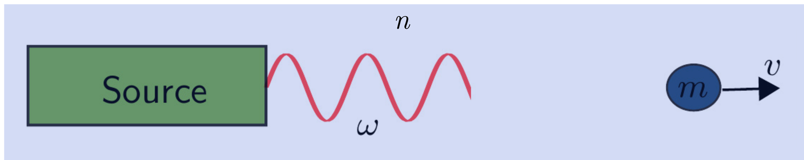

Fig. 1: Atom of mass m and transition frequency ω_0 moving away from light source with velocity v. Beam frequency ω. System is in a medium of refractive index n .

Atom will absorb photons from field if matching occurs between the Doppler-shifted and transition frequencies,

$$\omega_0 \approx \omega\left(1 - \frac{nv}{c}\right).$$

After absorption the atom will have a velocity v' and conservation of energy and momentum gives,

$$\frac{1}{2}mv'^2 + \hbar\omega_0 = \frac{1}{2}mv^2 + \hbar\omega, \qquad mv' = mv + p_{photon}.$$

These equations can then be solved to give [2],

$$p_{photon} = \frac{\hbar\omega n}{c}\frac{2v}{v+v'} \approx \frac{\hbar\omega n}{c} \equiv p_M.$$

This corresponds to momentum density,

$$\mathbf{g}_M = \mathbf{D} \times \mathbf{B}.$$

Abraham

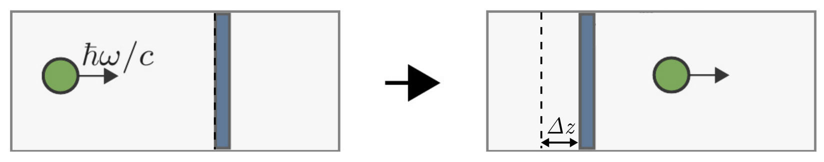

Fig. 2: A photon with energy $\hbar\omega$ travels in the z-direction with speed c towards a transparent block at rest (mass M, thickness L, refractive index n).

Photon speed becomes c/n in the medium, and uniform motion of centre of mass gives block displacement,

$$\Delta z = (n-1)L\frac{\hbar\omega}{Mc^2}.$$

Block must have acquired momentum from photon,

$$p_{block} = M\frac{\Delta z}{L(n/c)} = \left(1 - \frac{1}{n}\right)\frac{\hbar\omega}{c}.$$

Global conservation of momentum gives the total momentum of the system as $\hbar\omega/c$ and so [2],

$$p_{photon} = \frac{\hbar\omega}{cn} \equiv p_A.$$

This corresponds to momentum density,

$$\mathbf{g}_A = \mathbf{E} \times \mathbf{H}/c^2.$$

A possible resolution?

Both forms of the momentum in a medium have experimental and theoretical support. It has been postulated that the total energy-momentum in either case must be the same, and that obtaining Abraham or Minkowski is dependent on how we partition the total energy-momentum between the light and the medium. If the energy-momentum of the wave is $\Theta^\mu{}_\nu$ for Minkowski, and $\Omega^\mu{}_\nu$ for Abraham, then this can be expressed as,

$$\Theta^{\mu\nu} + T^{\mu\nu}_{M,\text{material}} \stackrel{?}{=} \Omega^{\mu\nu} + T^{\mu\nu}_{A,\text{material}}.$$

Calculating the energy-momentum for real materials is extremely difficult to achieve. Can we approach this from a different perspective?

Non-linear vacuum electrodynamics

Spontaneously created and annihilated particle-antiparticle pairs in the quantum vacuum can become polarized by the presence of a strong electromagnetic field. This can cause the vacuum to exhibit **non-linear properties analogous to that of a dielectric material** such as acquiring an effective refractive index (see e.g. review [3]).

A probe field propagating on a strong background electromagnetic field gives us an analogous all optical set up.

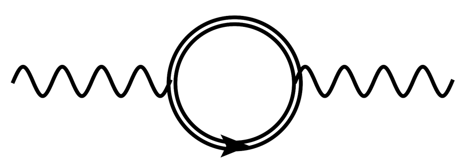

Fig. 3: Vacuum fluctuation.

Energy-momentum tensor for non-linear electrodynamics

Non-linear vacuum electrodynamics is defined by the class of Lagrangian densities of the form $\mathcal{L}(X,Y)$ where $X = -F^{\mu\nu}F_{\mu\nu}/4$ and $Y = -\widetilde{F}^{\mu\nu}F_{\mu\nu}/4$ are the only Lorentz invariant contractions of the electromagnetic field $F^{\mu\nu}$. This ensures that such theories maintain Lorentz invariance.
Using variational principles the full energy-momentum tensor is found to be,

$$T^\mu{}_\nu = H^{\mu\alpha}F_{\nu\alpha} + \delta^\mu_\nu\mathcal{L}.$$

The function $H^{\mu\nu}$ is the excitation tensor which describes the *constitutive relations*. In non-linear vacuum electrodynamics this is defined by,

$$H^{\mu\nu} \equiv \frac{\partial\mathcal{L}}{\partial X}F^{\mu\nu} + \frac{\partial\mathcal{L}}{\partial Y}\widetilde{F}^{\mu\nu}.$$

To compare more directly with the light-matter case, we separate the electromagnetic field into a strong background $\mathcal{F}_{\mu\nu}$ and weak probe $f_{\mu\nu}$,

$$F_{\mu\nu} = \mathcal{F}_{\mu\nu} + f_{\mu\nu},$$

such that the excitation tensor can be expanded as,

$$H^\mu{}_\nu = \mathcal{H}^\mu{}_\nu + h^\mu{}_\nu + \mathscr{H}^\mu_\nu + \mathcal{O}(f^3).$$

Here $\mathcal{H}^\mu{}_\nu = O(f^0)$, $h^\mu{}_\nu = O(f^1)$ and $\mathscr{H}^\mu_\nu = O(f^2)$. This can then be used to expand the full energy-momentum tensor and compare this with Abraham and Minkowski.

Equivalence of the full energy-momentum in Abraham and Minkowski

The energy-momentum tensor of the probe in Minkowski theory is given by,

$$\Theta^\mu{}_\nu = h^{\mu\alpha}f_{\nu\alpha} - \frac{1}{4}\delta^\mu_\nu h^{\alpha\beta}f_{\alpha\beta},$$

and in Abraham theory is,

$$\Omega^\mu{}_\nu = \Theta^\mu{}_\nu + \frac{1}{2}(f^{\mu\alpha}h_{\nu\alpha} - h^{\mu\alpha}f_{\nu\alpha}) - \frac{1}{2}\left[u^\mu u_\alpha(f_{\nu\beta}h^{\alpha\beta} - h_{\nu\beta}f^{\alpha\beta}) + u_\nu u^\alpha(f^{\mu\beta}h_{\alpha\beta} - h^{\mu\beta}f_{\alpha\beta})\right]$$

where u^μ ($u^2 = 1$) is a 4-velocity field.
A key result of this work is that the *exact* energy-momentum tensors of Abraham and Minkowski are equal. This then allows us to express the non-linear vacuum energy-momentum as [4],

$$T^\mu{}_\nu = \Theta^\mu{}_\nu + \mathscr{M}^\mu_\nu = \Omega^\mu{}_\nu + \mathscr{A}^\mu{}_\nu.$$

In non-linear vacuum electrodynamics we can obtain Abraham *or* Minkowski by appropriate partitioning of the total energy-momentum.

Abraham or Minkowski?

In the usual context of light-matter interactions, the velocity field u^μ in the Abraham formulation is the 4-velocity of the medium.

However, in non-linear vacuum electrodynamics, there is no physical medium present and no clear indication of how to interpret u^μ.

This means an arbitrary u^μ must be chosen, which introduces a preferred frame and breaks Lorentz invariance.

Minkowski energy-momentum is the correct description in non-linear vacuum electrodynamics.

[1] Pfeifer, R. N., Nieminen, T. A., Heckenberg, N. R., and Rubinsztein-Dunlop, H. *Rev. of Mod. Phys.* **79**(4), 1197 (2007).

[2] Barnett, S. M. and Loudon, R. *Philos. Trans. R. Soc. London, Ser. A* **368**(1914), 927–939 (2010).

[3] Di Piazza, A., Müller, C., Hatsagortsyan, K., and Keitel, C. *Rev. Mod. Phys.* **84**(3), 1177 (2012).

[4] Macleod, A. J., Noble, A., and Jaroszynski, D. A. *Proc.SPIE* **10234**, 0F (2017).