Non-linear electrodynamics: A classical view of the quantum vacuum

Alexander J. Macleod* & Adam Noble

SILIS, SUPA, Department of Physics, University of Strathclyde, UK

*alexander.macleod@strath.ac.uk

The linear vacuum

In the linear vacuum an electromagnetic field,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

will propagate according to Maxwellian electrodynamics. From this field there are two Lorentz invariant quantities which can be obtained,

$$X = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2}\left(E^2 - B^2\right)$$
$$Y = -\frac{1}{4} \widetilde{F}^{\mu\nu} F_{\mu\nu} = (\mathbf{E}.\mathbf{B})$$

where $\widetilde{F}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}/2$ is the dual electromagnetic field.
The field equations can be obtained from the action,

$$S = \int d^4z \sqrt{-g} X,$$

and are given by,

$$\partial_\mu F^{\mu\nu} = 0$$
$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$$

These equations have been extensively tested and verified, and account for all the low intensity properties of electromagnetic fields in the vacuum.

Non-linear theories

Non-linear vacuum electrodynamics theories aim to describe high intensity processes which are not captured by the theory of Maxwellian electrodynamics. They are defined by a generalization of the action,

$$S = \int d^4z \sqrt{-g} \mathcal{L}(X, Y),$$

where $\mathcal{L}(X, Y)$ is an arbitrary function of the Lorentz invariants. Varying the action with respect to the gauge field A_μ gives a modification to Maxwell's equations, which allow for the non-linear interaction of electromagnetic fields,

$$\partial_\mu H^{\mu\nu} = 0,$$
$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$$

where we have defined the excitation tensor,

$$H^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial X} F^{\mu\nu} + \frac{\partial\mathcal{L}}{\partial Y} \widetilde{F}^{\mu\nu}.$$

The excitation tensor characterises the non-linear interaction between electromagnetic fields, which causes waves moving through a strong background to behave in an analogous way to light interacting with a medium (see e.g. [1] for review).
In the literature two particular non-linear theories have been extensively studied — Euler-Heisenberg [2] and Born-Infeld [3].

Euler-Heisenberg

Quantum field theory led to the discovery of fluctuations in the vacuum in the form of virtual particle-antiparticle pairs which can interact with an electromagnetic field. The theory of Euler-Heisenberg describes this interaction using an effective theory in which electromagnetic waves can interact with each other,

$$\mathcal{L}_{EH} = X + \frac{2\alpha^2}{45 m^4}\left(4X^2 + 7Y^2\right).$$

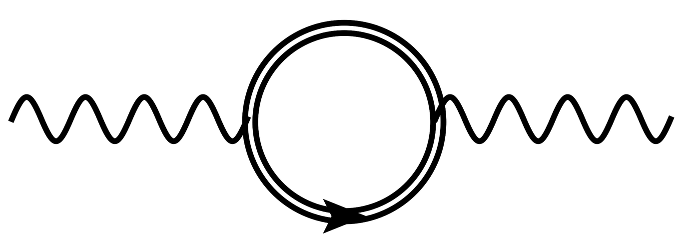

Fig. 1: Vacuum polarization.

Born-Infeld

The theory of Born-Infeld was an attempt to give a finite value for the classical electron self-energy and is defined by,

$$\mathcal{L}_{BI} = \frac{1}{\kappa^2}\left(1 - \sqrt{1 - 2\kappa^2 X - \kappa^4 Y^2}\right),$$

where κ is a new fundamental constant. The discovery of renormalization was eventually found to take care of the divergent self-energy, however interest in Born-Infeld has been renewed due to its rediscovery as a low energy limit in string theory.

Conformal invariance

Quantum electrodynamics is characterized by its Lorentz symmetry, however Maxwellian electrodynamics respects the the larger conformal symmetry group. The use of conformal invariance has been helpful in the study of many intractable problems — can it be used to help approximate quantum processes with a non-linear theory? The group is defined by the multiplicative invariance of the line element under coordinate transformations [4],

$$g_{\mu\nu} dx^\mu dx^\nu = \Omega(x)^2 g_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu.$$

Conformal invariance reduces the Lagrangian density to,

$$\mathcal{L}_{\text{conformal}} = X \mathcal{G}\left(Y/X\right).$$

The function $\mathcal{G}$ depends only on the combination Y/X, but is otherwise arbitrary. Any non-linear extension of electrodynamics must reproduce the results of Maxwell's theory in the low field limit to be physically realistic, i.e. we require,

$$\lim_{X,Y\to 0} \mathcal{G} = 1.$$

However, the restriction of the functional dependence being Y/X means that the only function satisfying this limit is unity. Maxwellian theory has a privileged status as the only physical conformal theory of electrodynamics. Conformal transformations can rescale a weak field into a strong field. A field described by Maxwell at low intensity must also be described by Maxwell at high intensity for conformally invariant theories.

Energy-momentum

The energy-momentum of light waves interacting with a real medium has two conflicting descriptions in the literature — Minkowski and Abraham (see [5] for review). Non-linear theories cause the vacuum to behave analogously to a dielectric material, but are simpler to work with. Can the problem be resolved in this context?
The energy-momentum tensor in non-linear electrodynamics is given by [6],

$$T^\mu{}_\nu = H^{\mu\alpha} F_{\alpha\nu} - \delta^\mu_\nu \mathcal{L}.$$

In the usual context, the energy-momentum is partitioned into wave and material parts. The analogous case is to separate the full electromagnetic field into a strong background and weak probe, $F^{\mu\nu} = \mathcal{F}^{\mu\nu} + f^{\mu\nu}$, such that, $H^\mu{}_\nu = \mathcal{H}^\mu{}_\nu + h^\mu{}_\nu + \mathscr{H}^\mu{}_\nu + \mathcal{O}(f^3)$. Expanding the non-linear energy-momentum,

$$T^\mu{}_\nu = \mathcal{T}^\mu{}_\nu + t^\mu{}_\nu + \underbrace{h^{\mu\alpha} f_{\nu\alpha} - \frac{1}{4}\delta^\mu_\nu h^{\alpha\beta} f_{\alpha\beta}}_{\text{Minkowski}} + \mathscr{H}^{\mu\alpha} \mathcal{F}_{\nu\alpha}$$

The $\mathcal{O}(f^2)$ contribution is naturally defined in terms of the Minkowski energy-momentum, which is not symmetric in general. However, the antisymmetric part of the final term in the expansion above exactly cancels the non-symmetric part of the Minkowski term, ensuring the total is symmetric. Appealing to Lorentz invariance of the theory also identifies Minkowski as the best description [7].

Non-linear generation of fields

Taking a background field $\mathcal{F}^{\mu\nu}$ for which the invariants X, Y vanish – such as a plane-wave – then *both*,

$$\partial_\mu \mathcal{F}^{\mu\nu} = 0 \qquad \text{and} \qquad \partial_\mu \mathcal{H}^{\mu\nu} = 0,$$

are satisfied. The superposition of two plane-waves is also a solution of Maxwell's equations. Defining two counter-propagating linearly polarized plane-waves,

$$\mathcal{F}^{\mu\nu} = \sum_{i=1,2} \mathsf{F}^{\mu\nu}_i(\phi_i), \qquad \mathsf{F}^{\mu\nu}_i(\phi_i) = (k_i^\mu \epsilon^\nu - k_i^\nu \epsilon^\mu)\mathcal{E}_i(\phi_i), \qquad \phi_i = k_i.x,$$

the non-linear interaction between these will generate a new field $f^{\mu\nu}$, satisfying,

$$\partial_\mu f^{\mu\nu} = -\partial_\mu \mathcal{H}^{\mu\nu} \equiv j^\nu.$$

This spreads out with the initial fields, so the total outgoing field in the direction of the pulse with initial momentum k_1 will be,

$$\mathscr{F}^{\mu\nu}_1(\phi_1) = ({k_1}^\mu \epsilon^\nu - {k_1}^\nu \epsilon^\mu)\left[\mathcal{E}_1(\phi_1) - \frac{1}{4}\beta \mathcal{E}_1'(\phi_1) \int_{-\infty}^{\infty} d\phi_2 \mathcal{E}_2(\phi_2)\mathcal{E}_2(\phi_2)\right]$$

with β a theory dependent constant. This essentially looks like the first term in a Taylor expansion of the initial field. The outgoing wave will be delayed by an amount proportional to the fluence of the k_2 direction wave.

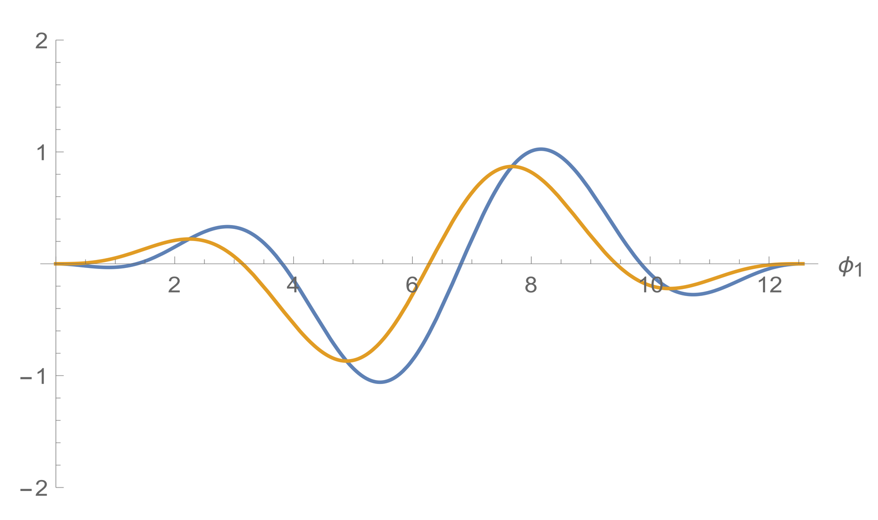

Fig. 2: Delay of field due to non-linear interaction for $N = 2$ cycle $\sin^2$ pulses.
Initial pulse $\mathsf{F}^{\mu\nu}_1(\phi_1)$ for Euler-Heisenberg theory.
Outgoing pulse $\mathscr{F}^{\mu\nu}_1(\phi_1)$ has been delayed relative to the initial pulse, but has same total energy.

Note: parameters adjusted to demonstrate effect.

[1] Di Piazza, A., Müller, C., Hatsagortsyan, K., and Keitel, C. *Rev. Mod. Phys.* **84**(3), 1177 (2012).

[2] Heisenberg, W. and Euler, H. *Z. Phys.* **98**(11-12), 714–732 (1936).

[3] Born, M. and Infeld, L. *Proc. R. Soc. London, Ser. A* **144**(852), 425–451 (1934).

[4] Codirla, C. and Osborn, H. *Ann. Phys.* **260**(1), 91–116 (1997).

[5] Barnett, S. M. and Loudon, R. *Philos. Trans. R. Soc. London, Ser. A* **368**(1914), 927–939 (2010).

[6] Macleod, A. J., Noble, A., and Jaroszynski, D. A. *Proc.SPIE* **10234**, 0F (2017).

[7] Macleod, A. J. and Noble, A., "In preparation," . (2017).