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Asymptotic boundedness and stability of solutions to hybrid stochastic differential equations with jumps and the Euler-Maruyama approximation

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Abstract

In this paper, we are concerned with the asymptotic properties and numerical analysis of the solution to hybrid stochastic differential equations with jumps. Applying the theory of M-matrices, we will study the pth moment asymptotic boundedness and stability of the solution. Under the non-linear growth condition, we also show the convergence in probability of the Euler-Maruyama approximate solution to the true solution. Finally, some examples are provided to illustrate our new results.

Key words: Stochastic differential equations, asymptotic boundedness, asymptotic stability, numerical analysis, Markovian switching, Levy jumps.

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1 Introduction

Stochastic differential equations (SDEs) driven by jump type noises such as Levy jump have become extremely popular for modelling financial, physical and biological phenomena. In some circumstances, purely Brownian motion perturbation has its imperfections in capturing some large moves and unpredictable events. Levy-type perturbations come to the stage to reproduce the performance of those natural phenomena in some real world models. Hence, stochastic jump-diffusion systems have been studied intensively by many scholars (see, e.g., [1, 2, 3, 6, 9, 12, 18, 27, 28, 30, 32, 45]).

When these systems experience abrupt changes in their structure and parameters, continuous time Markov chains have been used to form hybrid SDEs with jumps

$$dx(t) = f(x(t^-), r(t))dt + g(x(t^-), r(t))dw(t) + \int_Z h(x(t^-), r(t), v)N(dt, dv), \quad (1.1)$$

on $t \geq 0$, where the Poisson random measures $N(dt, dv)$ is generated by a Poisson point processes $\bar{p}(t)$, $w(t)$ is a Brownian motion and $r(t)$ is a Markov chain taking values in $S = \{1, 2, \dots, N\}$ (see section 2 for more details). In recent years, hybrid SDEs with jumps have been received a great deal of attention. In particular, the study of stability problem regarding equation (1.1) has become an increasing interest (see, e.g., [15, 36, 37, 38, 40, 41, 42, 44]). However, in all of these existing papers, the coefficients of equation (1.1) are required to satisfy the local Lipschitz condition and the linear growth condition. However, the linear growth condition is often not met in many practical applications. For instance, we consider the nonlinear hybrid SDEs with pure jumps (5.9) in section 5, the coefficients of equation (5.9) do not satisfy linear growth condition. Therefore, it is very important to establish the stability theory of hybrid SDEs with jumps (1.1) under some weak conditions.

For the past few decades, many authors have devoted themselves to finding the alternative conditions to replace the linear growth condition for hybrid SDEs *driven by Brownian motions*. By using the method of Lyapunov functions, a lot of important stability results (see, e.g., [10, 14, 21, 22, 23, 24, 25, 31, 35]) have been obtained under the Khasminskii-type conditions. Meantime, in order to avoid constructing Lyapunov functions, some people studied the stochastic stability and stabilization of SDEs under the polynomial growth condition, (see, e.g., [11, 16, 33, 34, 43, 46]). However, under the polynomial growth condition, there is no literature concerned with the boundedness and stability of the solution to hybrid SDEs with *jumps*. In this paper, we will establish new moment boundedness and stability criteria for hybrid SDEs with jumps using the theory of M-matrices. Our new results show that we can examine the p-th moment asymptotic boundedness and stability of equation (1.1) without designing the Lyapunov function.

On the other hand, most of SDEs with jumps cannot be solved explicitly. Numerical approximation is an important tool in studying these equations. The classical convergence theory for numerical methods to SDEs with jumps requires the coefficients of the equations to satisfy the Lipschitz condition and the linear growth condition, (see, e.g., [5, 6, 7, 8, 9, 18]). However, as pointed out before, these conditions are somehow restrictive. Therefore, we want to know whether or not numerical solution to jump-diffusion SDEs with Markovian switching will converge to the solution under non-linear growth condition. The convergence we are concerned in this paper is the convergence in probability. In 2000, Marion et al. [19] made a first attempt to study the convergence in probability of the solution to a class of SDEs and they proved that the Euler-Maruyama (EM) approximate solution converges to the solution of SDEs in probability without the linear growth condition. Next, Mao [26] extended the convergence theory of [19] to the case of stochastic delay differential equations (SDDEs). Li et al. [17] and Yuan et al. [39] established the convergence in probability of the EM approximate solution to the solution of SDDEs with Markovian switching under the Khasminskii-type conditions. While Milosevic [20] showed the convergence in probability of the EM solution for a class of highly nonlinear pantograph stochastic differential equations under the nonlinear growth conditions. However, there is little known on the convergence of numerical solution in probability for hybrid SDEs with jumps under nonlinear growth condition. This work aims to fill this gap. Under the local Lipschitz condition and non-linear growth condition, we will investigate the convergence in probability of the EM approximate solution to the solution.

The rest of the paper is organized as follows. In Section 2, we introduce some notation and hypotheses and establish the existence and uniqueness of solutions to equation (1.1). In Section 3, we prove that equation (1.1) is asymptotically bounded in the pth moment and ultimately bounded with large probability, meanwhile, we show that equation (1.1) is asymptotically stable in the pth moment under the nonlinear growth condition. In Section 4, we show the convergence in probability of the numerical schemes (4.2) to equation (1.1) under non-linear growth condition. Finally, we give some examples to illustrate the theory in Section 5.

2 Preliminaries and the global solution

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Let $w(t) = (w_1(t), \cdots, w_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$. Let $|\cdot|$ be the Euclidean norm in R^n. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$ while its operator norm is denoted by $||A|| = \sup\{|Ax| : |x| = 1\}$.

Let $\{\bar{p} = \bar{p}(t), t \geq 0\}$ be a stationary $\mathcal{F}_t$-adapted and R^n-valued Poisson point process. Then, for $Z \in \mathcal{B}(R^n - \{0\})$, such that $0 \notin$ the closure of Z, we define the Poisson counting measure N associated with $\bar{p}$ by

$$N((0,t] \times Z) := \#\{0 < s \leq t, \bar{p}(s) \in Z\} = \sum_{t_0 < s \leq t} I_Z(\bar{p}(s)),$$

where $\#$ denotes the cardinality of set $\{.\}$. For simplicity, we denote $N(t, Z) := N((0,t] \times Z)$. It is known (see, e.g., [1]) that there exists a σ-finite Lévy measure π on $R^n - \{0\}$ such that

$$E[N(t, Z)] = \pi(Z)t, \quad P(N(t, Z) = n) = \frac{exp(-t\pi(Z))(\pi(Z)t)^n}{n!}.$$

Moreover, by Doob-Meyer's decomposition theorem, there exists a unique $\{\mathcal{F}_t\}$-adapted martingale $\tilde{N}(t, Z)$ and a unique $\{\mathcal{F}_t\}$-adapted natural increasing process $\hat{N}(t, Z)$ such that

$$N(t, Z) = \tilde{N}(t, Z) + \hat{N}(t, Z), \quad t \geq 0.$$

Here $\tilde{N}(t, Z)$ is called the compensated Poisson random measure and $\hat{N}(t, Z) = \pi(Z)t$ is called the compensator. For more details on the Poisson point process and Lévy jumps, see [1, 30].

Let $r(t), t \geq 0$ be a right-continuous Markov chain on the probability space $(\Omega, \mathcal{F}, P)$ taking values in a finite state space $S = \{1, 2 \dots N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by:

$$P(r(t + \Delta) = j | r(t) = i) = \begin{cases} \gamma_{ij}\Delta + \circ(\Delta), & if \quad i \neq j, \\ 1 + \gamma_{ij}\Delta + \circ(\Delta), & if \quad i = j. \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j, $i \neq j$, While $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$. As a standing hypothesis, we assume that the Markov chain $r(t)$ is irreducible. Under this condition, $r(t)$ has a unique stationary distribution $\tilde{\pi} = (\tilde{\pi}_1, \tilde{\pi}_2, \cdots, \tilde{\pi}_N) \in R^{1 \times N}$ satisfying the following linear equation $\tilde{\pi}\Gamma = 0$ subject to $\sum_{i=1}^N \tilde{\pi}_i = 1$ and $\tilde{\pi}_i > 0$, $\forall\ i \in S$. We assume that the Markov chain $r(.)$ is independent of the Brownian motion $w(.)$ and Poisson random measures $N(., Z)$.

Let us consider the nonlinear hybrid SDE with jumps

$$dx(t) \;=\; f(x(t^-), r(t))dt + g(x(t^-), r(t))dw(t) + \int_Z h(x(t^-), r(t), v)N(dt, dv) \quad (2.1)$$

on $t \geq 0$, with initial data $x(0) = x_0 \in R^n$ and $r(0) = r_0 \in S$, where $x(t^-) = \lim_{s \uparrow t} x(s)$,

$$f : R^n \times S \to R^n, \quad g : R^n \times S \to R^{n \times m} \quad \text{and} \quad h : R^n \times S \times Z \to R^n.$$

In this paper, the following hypotheses are imposed on the coefficients f, g, and h.

Assumption 2.1 *Let $p \geq 2$. For each integer $d > 0$, there exists a positive constant k_d such that*

$$|f(x,i) - f(y,i)|^2 \vee |g(x,i) - g(y,i)|^2 \quad \leq \quad k_d |x-y|^2,$$

and

$$\int_Z |h(x,i,v) - h(y,i,v)|^p \pi(dv) \leq k_d^{\frac{p}{2}} |x-y|^p$$

for all $i \in S$ and those $x, y \in R^n$ with $|x| \vee |y| \leq d$.

Assumption 2.2 *There exist real number α_{1i} and positive constants α_{0i}, α_{2i}, β_{li} ($l = 0,1,2$), γ_j ($j = 1,2$) as well as bounded functions $\bar{h}_i(\cdot)$ ($i \in S$) such that*

$$x^\top f(x,i) + \frac{p-1}{2}|g(x,i)|^2 \leq \alpha_{0i} + \alpha_{1i}|x|^2 - \alpha_{2i}|x|^{\gamma_1+2}$$

and

$$|x + h(x,i,v)|^2 \leq \bar{h}_i(v)(\beta_{0i} + \beta_{1i}|x|^2 + \beta_{2i}|x|^{\gamma_2+2})$$

for any $x \in R^n$, $i \in S$ and $v \in Z$.

Let $C(R^n \times S; R_+)$ denote the family of continuous functions from $R^n \times S$ to R_+. We will also denote by $C^{2,1}(R^n \times S; R_+)$ the family of all continuous non-negative functions $V(x,i)$ defined on $R^n \times S$ such that for each $i \in S$, they are continuously twice differentiable in x. Given $V \in C^{2,1}(R^n \times S; R_+)$, we define the function $LV : R^n \times S \to R$ by

$$\begin{aligned} LV(x,i) &= V_x(x,i)f(x,i) + \frac{1}{2}trace[g^\top(x,i)V_{xx}(x,i)g(x,i)] \\ &+ \int_Z [V(x+h(x,i,v),i) - V(x,i)]\pi(dv) + \sum_{j=1}^N \gamma_{ij}V(x,j), \end{aligned} \tag{2.2}$$

where

$$V_x(x,i) = \Big(\frac{\partial V(x,i)}{\partial x_1}, \cdots, \frac{\partial V(x,i)}{\partial x_n}\Big), \quad V_{xx}(x,i) = \Big(\frac{\partial^2 V(x,i)}{\partial x_i \partial x_j}\Big)_{n\times n}.$$

Theorem 2.3 *Let Assumptions 2.1 and 2.2 hold. Assume also that one of the following conditions holds:*

(a) $\gamma_1 > 0.5p\gamma_2$;

(b) $\gamma_1 = 0.5p\gamma_2$ *and* $p\alpha_{2i} > C_i^p \beta_{2i}^{\frac{p}{2}}$ *for all* $i \in S$, *where* $C_i^p = \int_Z (\bar{h}_i(v))^{p/2}\pi(dv) < \infty$.

Then for any given initial data x_0 and r_0, there exists a unique global solution $x(t)$ to equation (2.1) such that $x(t) \in L^p$ for all $t \geq 0$.

Corollary 2.4 *Let Assumptions 2.1 and 2.2 hold. Assume also that one of the following conditions holds:*

(a) $\gamma_1 > \gamma_2$*;*

(b) $\gamma_1 = \gamma_2$ *and* $2\alpha_{2i} > C_i\beta_{2i}$ *for all* $i \in S$*, where* $C_i = \int_Z \bar{h}_i(v)\pi(dv) < \infty$*.*

Then for any given initial data x_0 *and* r_0*, there exists a unique global solution* $x(t)$ *to equation (2.1) such that in* $x(t) \in L^2$ *for all* $t \geq 0$*.*

Remark 2.5 *The key of the proof of Theorem 2.3 is the boundedness of* $LV(x, i)$*. Under Assumptions 2.1 and 2.2, the conditions (a) and (b) play the important role to suppress potential explosion of the solution* $x(t)$*.*

To emphasize the main purpose of this paper, we shall leave the proof of the existence and uniqueness of the solution to the Appendix but concentrate on the establishment of new criteria on asymptotic properties and numerical analysis of solutions.

3 Asymptotic Boundedness and Stability of Solutions

In this section, we shall use the theory of M-matrices to discuss the asymptotic behavior of the solution, i.e., the asymptotic boundedness and stability in pth moment of the solution to equation (2.1).

For the convenience of the reader, let us cite some useful results on M-matrices. For more detailed information, please see e.g. [24]. We will need a few more notations. If B is a vector or matrix, by $B \gg 0$ we mean all elements of B are positive. If B_1 and B_2 are vectors or matrices with same dimensions we write $B_1 \gg B_2$ if and only if $B_1 - B_2 \gg 0$. Moreover, we also adopt here the traditional notation by letting

$$Z^{N\times N} = \{A = [a_{ij}]_{N\times N} : a_{ij} \leq 0,\ i \neq j\}.$$

Definition 3.1 *A square matrix* $A = (a_{ij})_{N\times N}$ *is called a nonsingular M-matrix if* A *can be expressed in the form* $A = sI - B$ *with some* $B \geq 0$ *and* $s > \rho(B)$*, where* I *is the identity matrix and* $\rho(B)$ *the spectral radius of* B*.*

Before we state our main results, we need the following useful lemma (see, e.g., [24]).

Lemma 3.2 *If $A \in Z^{N\times N}$, then the following statements are equivalent:*

(1) A is a nonsingular M-matrix.

(2) A is semi-positive; that is, there exists $x \gg 0$ in R^n such that $Ax \gg 0$.

(3) A^{-1} exists and its elements are all nonnegative.

(4) All the leading principal minors of A are positive; that is

$$\begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix} > 0 \quad \text{for every } k = 1, 2, \cdots, N.$$

Theorem 3.3 *Let Assumptions 2.1 and 2.2 hold. Assume that*

$$\mathcal{A}_p := -\text{diag}(p\alpha_{11} + C^p_{h_1}\beta_{11}^{\frac{p}{2}}, \cdots, p\alpha_{1N} + C^p_{h_N}\beta_{1N}^{\frac{p}{2}}) - \Gamma \tag{3.1}$$

is a nonsingular M-matrix and one of the following conditions holds:

(a) $\gamma_1 > 0.5p\gamma_2$;

(b) $\gamma_1 = 0.5p\gamma_2$ and $p\alpha_{2i} > C^p_{h_i}\beta_{2i}^{\frac{p}{2}}$ for all $i \in S$, where $C^p_{h_i} = 3^{\frac{p}{2}-1}C^p_i < \infty$.

Then there is a positive constant C (independent of the initial data) such that for any initial data x_0 and r_0, the solution of equation (2.1) has the property that

$$\limsup_{t\to\infty} E|x(t)|^p \le C. \tag{3.2}$$

In other words, the hybrid SDEs with jumps (2.1) is asymptotically bounded in pth moment.

Proof. As $\mathcal{A}_p$ is a nonsingular M-matrix, by lemma 3.2, we see that $\theta = (\theta_1, \cdots, \theta_N)^\top := \mathcal{A}_p^{-1}\overrightarrow{1} > 0$, that is,

$$-(p\alpha_{1i} + C^p_{h_i}\beta_{1i}^{\frac{p}{2}})\theta_i - \sum_{j=1}^{N}\gamma_{ij}\theta_j > 0 \tag{3.3}$$

for all $i \in S$, where $\overrightarrow{1} = (1, \cdots, 1)^\top$. Define the function $V(x, i) = \theta_i|x|^p$ and choose a constant

$$\varepsilon \in \Big(0,\ \min_{i\in S}\{-p\alpha_{1i} - C^p_{h_i}\beta_{1i}^{\frac{p}{2}} - \frac{1}{\theta_i}\sum_{j=1}^{N}\gamma_{ij}\theta_j\}\Big). \tag{3.4}$$

By the generalized Itô formula, we have

$$\begin{aligned} & e^{\varepsilon t}V(x(t),r(t)) - V(x_0,r_0) \\ = & \int_0^t e^{\varepsilon s}[LV(x(s^-),r(s)) + \varepsilon V(x(s^-),r(s))]ds \\ + & \int_0^t e^{\varepsilon s} p\theta_{r(s)}|x(s^-)|^{p-2}x(s^-)^\top g(x(s^-),r(s))dw(s) \\ + & \int_0^t \int_Z e^{\varepsilon s}[\theta_{r(s)}|x(s^-)+h(x(s^-),r(s),v))|^p - \theta_{r(s)}|x(s^-)|^p]\tilde{N}(ds,dv), \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} LV(x(s^-),r(s)) = & \; p\theta_{r(s)}|x(s^-)|^{p-2}x(s^-)^\top f(x(s^-),r(s)) + \frac{p}{2}\theta_{r(s)}|x(s^-)|^{p-2}|g(x(s^-),i)|^2 \\ + & \frac{p(p-2)}{2}\theta_{r(s)}|x(s^-)|^{p-4}|x(s^-)^\top g(x(s^-),r(s))|^2 + \sum_{j=1}^N \gamma_{r(s)j}\theta_j|x(s^-)|^p \\ + & \int_Z [\theta_{r(s)}|x(s^-)+h(x(s^-),r(s),v))|^p - \theta_{r(s)}|x(s^-)|^p]\pi(dv). \end{aligned} \tag{3.6}$$

By Assumption 2.2, it follows that

$$\begin{aligned} LV(x(s^-),i) \le & \; p\theta_i|x(s^-)|^{p-2}[x(s^-)^\top f(x(s^-),i) + \frac{p-1}{2}|g(x(s^-),i)|^2] \\ + & \int_Z [\theta_i|x(s^-)+h(x(s^-),i,v)|^p - \theta_i|x(s^-)|^p]\pi(dv) + \sum_{j=1}^N \gamma_{ij}\theta_j|x(s^-)|^p \\ \le & \; p\theta_i|x(s^-)|^{p-2}[\alpha_{0i} + \alpha_{1i}|x(s^-)|^2 - \alpha_{2i}|x(s^-)|^{\gamma_1+2}] \\ + & \int_Z [\theta_i[\bar{h}_i(v)(\beta_{0i} + \beta_{1i}|x(s^-)|^2 + \beta_{2i}|x(s^-)|^{\gamma_2+2})]^{\frac{p}{2}} - \theta_i|x(s^-)|^p]\pi(dv) \\ + & \sum_{j=1}^N \gamma_{ij}\theta_j|x(s^-)|^p. \end{aligned} \tag{3.7}$$

By the Young inequality

$$a^r b^{1-r} \le ar + b(1-r), \text{ for any } a,b \ge 0 \text{ and } r \in [0,1], \tag{3.8}$$

we have

$$\begin{aligned} p\alpha_{0i}|x(s^-)|^{p-2} = & \; p\Big(\alpha_{0i}^{\frac{p}{2}}\Big(\frac{p-2}{\pi(Z)}\Big)^{\frac{p-2}{2}}\Big)^{\frac{2}{p}}\Big(\frac{\pi(Z)}{p-2}|x(s^-)|^p\Big)^{\frac{p-2}{p}} \\ \le & \; 2\alpha_{0i}^{\frac{p}{2}}\Big(\frac{p-2}{\pi(Z)}\Big)^{\frac{p-2}{2}} + \pi(Z)|x(s^-)|^p. \end{aligned}$$

Using the basic inequality $|a+b+c|^{\frac{p}{2}} \le 3^{\frac{p}{2}-1}(|a|^{\frac{p}{2}}+|b|^{\frac{p}{2}}+|c|^{\frac{p}{2}})$, we have

$$\begin{aligned} LV(x(s^-),i) &\le \theta_i[2\alpha_{0i}^{\frac{p}{2}}\Big(\frac{p-2}{\pi(Z)}\Big)^{\frac{p-2}{2}} + \pi(Z)|x(s^-)|^p] + p\theta_i|x(s-)|^{p-2}[\alpha_{1i}|x(s^-)|^2 \\ &- \alpha_{2i}|x(s^-)|^{\gamma_1+2}] + \int_Z \Big\{\theta_i[3^{\frac{p}{2}-1}(\bar{h}(v))^{\frac{p}{2}}(\beta_{0i}^{\frac{p}{2}} + \beta_{1i}^{\frac{p}{2}}|x(s^-)|^p \\ &+ \beta_{2i}^{\frac{p}{2}}|x(s^-)|^{0.5p\gamma_2+p})] - \theta_i|x(s^-)|^p\Big\}\pi(dv) + \sum_{j=1}^N \gamma_{ij}\theta_j|x(s^-)|^p \\ &\le [2\Big(\frac{p-2}{\pi(Z)}\Big)^{\frac{p-2}{2}}\alpha_{0i}^{\frac{p}{2}} + C_{h_i}^p\beta_{0i}^{\frac{p}{2}}]\theta_i + [(p\alpha_{1i} + C_{h_i}^p\beta_{1i}^{\frac{p}{2}})\theta_i \\ &+ \sum_{j=1}^N \gamma_{ij}\theta_j]|x(s^-)|^p - p\theta_i\alpha_{2i}|x(s^-)|^{\gamma_1+p} + C_{h_i}^p\beta_{2i}^{\frac{p}{2}}\theta_i|x(s^-)|^{0.5p\gamma_2+p}. \end{aligned}$$

Thus

$$\begin{aligned} & LV(x(s^-),i) + \varepsilon V(x(s^-),i) \\ \le\ & [2\Big(\frac{p-2}{\pi(Z)}\Big)^{\frac{p-2}{2}}\alpha_{0i}^{\frac{p}{2}} + C_{h_i}^p\beta_{0i}^{\frac{p}{2}}]\theta_i + [(p\alpha_{1i} + C_{h_i}^p\beta_{1i}^{\frac{p}{2}} + \varepsilon)\theta_i + \sum_{j=1}^N \gamma_{ij}\theta_j]|x(s^-)|^p \\ -\ & p\alpha_{2i}\theta_i|x(s^-)|^{\gamma_1+p} + C_{h_i}^p\beta_{2i}^{\frac{p}{2}}\theta_i|x(s^-)|^{0.5p\gamma_2+p} \\ \le\ & [2\Big(\frac{p-2}{\pi(Z)}\Big)^{\frac{p-2}{2}}\alpha_{0i}^{\frac{p}{2}} + C_{h_i}^p\beta_{0i}^{\frac{p}{2}}]\theta_i - p\alpha_{2i}\theta_i|x(s^-)|^{\gamma_1+p} + C_{h_i}^p\beta_{2i}^{\frac{p}{2}}\theta_i|x(s^-)|^{0.5p\gamma_2+p}, \end{aligned}$$

where (3.4) has been used.

In either case (a) or (b), it is easy to see that there is a positive constant C_1 such that

$$LV(x(s^-),i) + \varepsilon V(x(s^-),i) \le C_1.$$

Taking the expectations on both sides of (3.5), we get

$$\theta_m E(e^{\varepsilon t}|x(t)|^p) \ \le\ \theta_M|x_0|^p + \frac{C_1 e^{\varepsilon t}}{\varepsilon},$$

where $\theta_m = \min_{i\in S}\theta_i$ and $\theta_M = \max_{i\in S}\theta_i$. Dividing both sides by $e^{\varepsilon t}$ and then letting $t\to\infty$, we obtain that

$$\limsup_{t\to\infty} E|x(t)|^p \le C := C_1/\varepsilon$$

as required. The proof is therefore complete.

Remark 3.4 *Theorem 3.3 shows that the hybrid SDEs with jumps (2.1) is asymptotically bounded in the pth moment. In particular, when $p=2$, Theorem 3.3 shows that, under Assumptions 2.1 and 2.2, if $\mathcal{A}_2 := -\mathrm{diag}(2\alpha_{11}+C_1\beta_{11},\cdots,2\alpha_{1N}+C_N\beta_{1N})-\Gamma$ is a nonsingular M-matrix while one of the following conditions holds:*

(a) $\gamma_1 > \gamma_2$;

(b) $\gamma_1 = \gamma_2$ *and* $2\alpha_{2i} > C_i\beta_{2i}$ *for all* $i \in S$,

then the solution $x(t)$ *of equation (2.1) is asymptotically bounded in mean square.*

As an application of Theorem 3.3 together with the Chebyshev inequality, we get the following results.

Theorem 3.5 *If the conditions of Theorem 3.3 hold, then equation (2.1) is stochastically ultimately bounded. That is, for any* $\varepsilon \in (0,1)$, *there is a positive number* $\bar{M}$ *independent of initial data* x_0 *and* r_0 *such that*

$$\limsup_{t\to\infty} P\{|x(t)| \le \bar{M}\} \ge 1-\varepsilon.$$

Theorem 3.5 shows that equation (2.1) will be ultimately bounded with large probability, while the following theorem estimates the limit of the average in the time of the pth moment.

Theorem 3.6 *Under the conditions of Theorem 3.3, the solution* $x(t)$ *of equation (2.1) satisfies*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t E|x(t)|^p dt \le \sum_{i=1}^N \tilde{\pi}_i K_i \tag{3.9}$$

for any initial data x_0 *and* r_0, *where*

$$K_i := \sup_{x\in R^n} G(x,i) < \infty,$$

in which

$$G(x,i) \;=\; [2\Big(\frac{p-2}{\pi(Z)}\Big)^{\frac{p-2}{2}}\alpha_{0i}^{\frac{p}{2}} + C_{h_i}^p\beta_{0i}^{\frac{p}{2}}] + (p\alpha_{1i} + C_{h_i}^p\beta_{1i}^{\frac{p}{2}} + 1)|x|^p - p\alpha_{2i}|x|^{\gamma_1+p} + C_{h_i}^p\beta_{2i}^{\frac{p}{2}}|x|^{0.5p\gamma_2+p}.$$

Proof. This theorem can be proved in the same way as Theorems 2.3 and 3.3 were proved so we omit its proof.

Let us now proceed to discuss the asymptotic stability in the qth moment of the solution to equation (2.1).

Assumption 3.7 *There exist positive constants* k_l, q_l, $l = 1,2,3$ *such that for all* $x \in R^n$, $i \in S$ *and* $v \in Z$

$$|f(x,i)|^2 \le k_1(1+|x|^{q_1+2}), \qquad |g(x,i)|^2 \le k_2(1+|x|^{q_2+2})$$

and

$$|h(x,i,v)|^2 \le k_3(1+|x|^{q_3+2})|v|^2.$$

Before we state the asymptotic stability result, we need one useful lemma.

Lemma 3.8 *([29]) Let $T_0 > 0$ be a sufficiently large number. If $\gamma(\cdot) \in L^1([T_0,\infty); R_+)$ and it is uniformly continuous on $[T_0,\infty)$, then*

$$\lim_{t\to\infty} \gamma(t) = 0.$$

Theorem 3.9 *Let all the conditions in Theorem 3.3 hold and $\alpha_{0,i} = \beta_{0,i} = 0$ for all $i \in S$. Let Assumption 3.7 hold. Assume moreover that for some $q \geq 2$ such that $p \geq (q+q_1) \vee (q+q_2) \vee [0.5q(q_3+2)]$,*

$$\bar{\mathcal{A}}_q := -\mathrm{diag}(q\alpha_{11} + \bar{C}^q_{h_1}\beta_{11}^{\frac{q}{2}}, \cdots, q\alpha_{1N} + \bar{C}^q_{h_N}\beta_{1N}^{\frac{q}{2}}) - \Gamma \tag{3.10}$$

is a nonsingular M-matrix and

$$\gamma_1 \geq 0.5p\gamma_2 \text{ and } (\pi(Z) \wedge q\alpha_{2i}) \geq \bar{C}^q_{h_i}\beta_{2i}^{\frac{q}{2}} \text{ for all } i \in S, \text{ where } \bar{C}^q_{h_i} = 2^{\frac{q}{2}-1}C_i^q < \infty. \tag{3.11}$$

Then for any initial data x_0 and r_0, the solution $x(t)$ of equation (2.1) has the property that

$$\lim_{t\to\infty} E|x(t)|^q = 0. \tag{3.12}$$

In other words, the hybrid SDEs with jumps (2.1) is asymptotically stable in the qth moment.

Proof. Fix any initial data x_0 and r_0. By Theorem 3.3, the solution is already asymptotically bounded in L^p. That is, there is a sufficiently large T_0 such that

$$E|x(t)|^p \leq C, \quad \forall t \geq T_0. \tag{3.13}$$

As $\bar{\mathcal{A}}_q$ is a nonsingular M-matrix, by Lemma 3.2, we see that $\bar{\theta} = (\bar{\theta}_1, \cdots, \bar{\theta}_N)^\top := \bar{\mathcal{A}}_q^{-1}\overrightarrow{1} > 0$. This also implies

$$-(q\alpha_{1i} + \bar{C}^q_{h_i}\beta_{1i}^{\frac{q}{2}})\bar{\theta}_i - \sum_{j=1}^N \gamma_{ij}\bar{\theta}_j = 1, \quad \forall i \in S. \tag{3.14}$$

Define the function $V(x,i) = \bar{\theta}_i|x|^q$. Applying the generalized Itô formula, we have

$$\begin{aligned} V(x(t), r(t)) &= \bar{\theta}_{r(0)}|x_0|^q + \int_0^t LV(x(s^-), r(s))ds \\ &+ \int_0^t q\bar{\theta}_{r(s)}|x(s^-)|^{q-2}x(s^-)^\top g(x(s^-), r(s))dw(s) \\ &+ \int_0^t \int_Z [\bar{\theta}_{r(s)}|x(s^-) + h(x(s^-), r(s), v))|^q - \bar{\theta}_{r(s)}|x(s^-)|^q]\tilde{N}(ds,dv). \end{aligned} \tag{3.15}$$

By Assumption 2.2 with $\alpha_{0,i} = \beta_{0,i} = 0$, it follows that

$$\begin{aligned} LV(x(s^-),i) &\leq q\bar\theta_i |x(s^-)|^{q-2}[\alpha_{1i}|x(s^-)|^2 - \alpha_{2i}|x(s^-)|^{\gamma_1+2}] + \int_Z \Big(\bar\theta_i [\bar h_i(v)(\beta_{1i}|x(s^-)|^2 \\ &+ \beta_{2i}|x(s^-)|^{\gamma_2+2})]^{\frac{q}{2}} - \bar\theta_i |x(s^-)|^q\Big)\pi(dv) + \sum_{j=1}^N \gamma_{ij}\bar\theta_j |x(s^-)|^q. \end{aligned}$$

Using the basic inequality $|a+b|^{\frac{q}{2}} \leq 2^{\frac{q}{2}-1}(|a|^{\frac{q}{2}} + |b|^{\frac{q}{2}})$, we have

$$\begin{aligned} LV(x(s^-),i) &\leq \Big([q\alpha_{1i} + \bar C^q_{h_i}\beta_{1i}^{\frac{p}{2}} - \pi(Z)]\bar\theta_i + \sum_{j=1}^N \gamma_{ij}\bar\theta_j\Big)|x(s^-)|^q \\ &- q\alpha_{2i}\bar\theta_i |x(s^-)|^{\gamma_1+q} + \bar C^q_{h_i}\beta_{2i}^{\frac{q}{2}}\bar\theta_i |x(s^-)|^{0.5p\gamma_2+q}. \end{aligned}$$

This, together with (3.14), implies

$$LV(x(s^-),i) \leq -|x(s^-)|^q + \bar\theta_i |x(s^-)|^q\Big(-\pi(Z) - q\alpha_{2i}|x(s^-)|^{\gamma_1} + \bar C^q_{h_i}\beta_{2i}^{\frac{q}{2}}|x(s^-)|^{0.5q\gamma_2}\Big).$$

But, by condition (3.11),

$$-\pi(Z) - q\alpha_{2i}|x(s^-)|^{\gamma_1} + \bar C^q_{h_i}\beta_{2i}^{\frac{p}{2}}|x(s^-)|^{0.5q\gamma_2} \leq 0.$$

Hence

$$LV(x(s^-),i) \leq -|x(s^-)|^q.$$

Taking the expectations on both sides of (3.15), we get

$$\bar\theta_m E|x(t)|^q \leq \bar\theta_M |x_0|^q - \int_0^t E|x(s^-)|^q ds, \tag{3.16}$$

where $\bar\theta_m = \min_{i\in S}\bar\theta_i$ and $\bar\theta_M = \max_{i\in S}\bar\theta_i$. Letting $t\to\infty$ and then using the Fubini theorem, we obtain

$$\int_0^\infty E|x(t)|^q dt < \infty.$$

This of course implies that $\int_{T_0}^\infty E|x(t)|^q dt < \infty$.

We now claim that $E|x(t)|^q$ is uniformly continuous on $t \in [T_0,\infty)$. By the generalized Itô formula, we have that for any $t > s > T_0$,

$$\begin{aligned} E|x(t)|^q &= E|x(s)|^q + qE\int_s^t |x(\sigma^-)|^{q-2}x(\sigma^-)^\top f(x(\sigma^-),r(\sigma))d\sigma \\ &+ \frac{q(q-1)}{2}E\int_s^t |x(\sigma^-)|^{q-2}|g(x(\sigma^-),r(\sigma))|^2 ds \\ &+ E\int_s^t\int_Z [|x(\sigma^-) + h(x(\sigma^-),r(\sigma),v))|^q - |x(\sigma^-)|^q]\pi(dv)d\sigma. \end{aligned}$$

Then, by Assumption 3.7, we have

$$\begin{aligned}|E|x(t)|^q - E|x(s)|^q| &\le \frac{q}{2}E\int_s^t |x(\sigma^-)|^q d\sigma + \frac{q}{2}k_1 E\int_s^t (|x(\sigma^-)|^{q-2} + |x(\sigma^-)|^{q+q_1})d\sigma \\ &+ \frac{q(q-1)}{2}k_2 E\int_s^t (|x(\sigma^-)|^{q-2} + |x(\sigma^-)|^{q+q_2})d\sigma \\ &+ E\int_s^t\int_Z \Big||x(\sigma^-)+h(x(\sigma^-),r(\sigma),v)|^q - |x(\sigma^-)|^q\Big|\pi(dv)d\sigma. \qquad (3.17)\end{aligned}$$

By the Young inequality (3.8), we show that

$$k_1|x(\sigma^-)|^{q-2} \le k_1[1^{\frac{2}{q}}\Big(|x(\sigma^-)|^q\Big)^{1-\frac{2}{q}}] \le \frac{2}{q}k_1 + (1-\frac{2}{q})k_1|x(\sigma^-)|^q, \qquad (3.18)$$

and

$$k_2|x(\sigma^-)|^{q-2} \le \frac{2}{q}k_2 + (1-\frac{2}{q})k_2|x(\sigma^-)|^q. \qquad (3.19)$$

By the Hölder inequality, there exists an $\delta > 0$ such that

$$\begin{aligned}|x(\sigma^-) + h(x(\sigma^-), r(\sigma), v)|^q &= \Big|x(\sigma^-) + \delta^{\frac{1}{q}}\frac{h(x(\sigma^-), r(\sigma), v)}{\delta^{\frac{1}{q}}}\Big|^q \\ &\le (1+\delta^{\frac{1}{q-1}})^{q-1}\Big(\frac{1}{\delta}|h(x(\sigma^-), r(\sigma), v)|^q + |x(\sigma^-)|^q\Big).\end{aligned}$$

Then, Assumption 3.7 implies that

$$\begin{aligned}&|x(\sigma^-) + h(x(\sigma^-), r(\sigma), v)|^q \\ \le\ &(1+\delta^{\frac{1}{q-1}})^{q-1}\Big(\frac{1}{\delta}[(2k_3)^{\frac{q}{2}}|v|^q(1+|x(\sigma^-)|^{0.5q(q_3+2)})] + |x(\sigma^-)|^q\Big).\end{aligned}$$

Letting $\delta = (2k_3)^{\frac{p-1}{2}}|v|^{q-1}$ yields that

$$|x(\sigma^-) + h(x(\sigma^-), r(\sigma), v)|^q \le (1+\sqrt{2k_3}|v|)^q\Big(1+|x(\sigma^-)|^q+|x(\sigma^-)|^{0.5q(q_3+2)}\Big). \qquad (3.20)$$

Inserting (3.18)-(3.20) into (3.17), it follows that

$$\begin{aligned}|E|x(t)|^q - E|x(s)|^q| &\le C_3(t-s) + C_4E\int_s^t |x(\sigma^-)|^q d\sigma + \frac{q}{2}k_1E\int_s^t |x(\sigma^-)|^{q+q_1}d\sigma \\ &+ \frac{q(q-1)}{2}k_2E\int_s^t |x(\sigma^-)|^{q+q_2}d\sigma + C_vE\int_s^t |x(\sigma^-)|^{0.5q(q_3+2)}d\sigma,\end{aligned}$$

where

$$C_v = \int_Z (1+\sqrt{2k_3}|v|)^q\pi(dv),\ C_3 = C_v + k_1 + (q-1)k_2,\ C_4 = C_v + \frac{q-2}{2}k_1 + \frac{(q-1)(q-2)}{2}k_2.$$

Recalling that $p \ge (q+q_1) \vee (q+q_2) \vee [0.5q(q_3+2)]$ and using (3.13), we get

$$|E|x(t)|^q - E|x(s)|^q| \le C(t-s).$$

This implies that $E|x(t)|^q$ is uniformly continuous on $[T_0, \infty)$. Finally, by Lemma 3.8, we can obtain that $\lim_{t\to\infty} E|x(t)|^q = 0$ as required. The proof is therefore complete.

Remark 3.10 *In this work, we consider the asymptotic boundedness and stability of the solution to the hybrid SDE with jumps (2.1) under a nonlinear growth condition. As the linear growth condition is a special case of our case, some known results [15, 36, 37, 40, 41, 42, 44] are improved and generalized in this paper.*

Remark 3.11 *If $h = 0$ or $N = 0$, then equation (2.1) is reduced to the hybrid SDEs without jumps. Consequently, our results can be reduced to some results in e.g. [24, 41]. Moreover, when there is no Markovian switching (i.e., delete $r(t)$), equation (2.1) has been studied by many authors including Applebaum [1, 2], Albeverio [3], Baran [6], Wee [32] and Zhu [45]. Therefore, we improve the existing results to cover a class of more general hybrid SDEs with jumps. Moreover, unlike Applebaum [1, 2], Albeverio [3] and Zhu [45], we need not require the coefficients f, g, h satisfying the linear growth conditions.*

4 Convergence analysis of the EM approximate solutions

In this section, we will study the convergence of the EM approximate solutions for hybrid SDEs with jumps (2.1) under the local Lipschitz condition and nonlinear growth condition.

Before we define the EM approximate solution for equation (2.1), we need the property of the embedded discrete Markov chain. The following lemma describes this property.

Lemma 4.1 *[4] Given $h > 0$, we define $r_h^n = r(nh)$ for $n = 0, 1, 2, \cdots$. Then $\{r_n^h, n = 0, 1, 2, \cdots\}$ is a discrete Markov chain with the one-step transition probability matrix*

$$P(h) = (P_{ij}(h))_{N\times N} = e^{h\Gamma}.$$

According to [24], we can simulate the discrete Markov chain $\{r_n^h, n = 0, 1, 2, \cdots\}$. Now, we shall define the EM approximate solution of the hybrid SDEs with jumps (2.1).

For a given constant stepsize $h > 0$, we define the EM method for equation (2.1) as follows

$$y_{n+1} = y_n + f(y_n, r_n^h)h + g(y_n, r_n^h)\Delta w_n + \int_Z h(y_n, r_n^h, v)N(h, dv), \tag{4.1}$$

with initial value $y_0 = x_0$ and y_n denotes the numerical approximation of $x(t)$ with $t_n = nh$. Moreover, $\Delta w_n = w(t_{n+1}) - w(t_n)$ and $N(h, dv) = N(t_{n+1}, dv) - N(t_n, dv)$ are independent increments of the Brownian motion and Poisson random measures, respectively.

To define the continuous-time approximate solution, let us introduce two step processes

$$z(t) = y_n, \quad \bar{r}(t) = r_n^h,$$

for $t \in [t_n, t_{n+1})$. Hence we define the continuous version $y(t)$ as follows

$$y(t) = y(0) + \int_0^t f(z(s), \bar{r}(s))ds + \int_0^t g(z(s), \bar{r}(s))dw(s) + \int_0^t \int_Z h(z(s), \bar{r}(s), v)N(ds, dv). \tag{4.2}$$

It is not hard to verify that $y(t_n) = y_n$, that is, $y(t)$ coincides with the discrete solutions at the grid-points.

Let us define three stopping times

$$\alpha_d = \inf\{t \in [0, T] : |x(t)| \geq d\}, \quad \beta_d = \inf\{t \in [0, T] : |y(t)| \geq d\},$$

and $\gamma_d = \alpha_d \wedge \beta_d$, where $\inf \emptyset$ is set as ∞.

Lemma 4.2 *[13] Let $\phi : R_+ \times Z \to R^n$ and assume that*

$$\int_0^t \int_Z |\phi(s, v)|^p \pi(dv)ds < \infty, \quad p \geq 2.$$

Then, there exists $D_p > 0$ such that

$$\begin{aligned} E\Big(\sup_{0\leq t\leq u} |\int_0^t \int_Z \phi(s, v)\tilde{N}(ds, dv)|^p\Big) &\leq D_p\Big(E(\int_0^u \int_Z |\phi(s, v)|^2 \pi(dv)ds)^{\frac{p}{2}} \\ &\quad + E\int_0^u \int_Z |\phi(s, v)|^p \pi(dv)ds\Big). \end{aligned}$$

Lemma 4.3 *Under Assumption 2.1,*

$$\begin{aligned} E\int_0^T |a(z(s \wedge \gamma_d), r(s \wedge \gamma_d)) - a(z(s \wedge \gamma_d), \bar{r}(s \wedge \gamma_d))|^p ds &\leq M_d h, \\ E\int_0^T \int_Z |h(z(s \wedge \gamma_d), r(s \wedge \gamma_d), v) - h(z(s \wedge \gamma_d), \bar{r}(s \wedge \gamma_d), v)|^p \pi(dv)ds &\leq M_d h, \end{aligned}$$

where $a = f, g$ and M_d is a constant independent of h.

Proof. We omit the proof because it is similar to that of Theorem 4.1 in [24].

Lemma 4.4 *Under Assumption 2.1,*

$$E[\sup_{0\leq t\leq T} |x(t \wedge \gamma_d) - y(t \wedge \gamma_d)|^p] \leq \bar{C}_d h, \tag{4.3}$$

where $\bar{C}_d$ is a constant independent of h.

Proof. For simplicity, denote $e(t)=x(t)-y(t)$. For any $t_1\in[0,T]$, by the basic inequality $|a+b+c|^p\le 3^{p-1}(|a|^p+|b|^p+|c|^p)$, it follows that

$$\begin{aligned}
&E\sup_{0\le t\le t_1}|e(t\wedge\gamma_d)|^p\\
\le\;& 3^{p-1}E\sup_{0\le t\le t_1}\Big|\int_0^{t\wedge\gamma_d}[f(x(s^-),r(s))-f(z(s),\bar r(s))]ds\Big|^p\\
+\;& 3^{p-1}E\sup_{0\le t\le t_1}\Big|\int_0^{t\wedge\gamma_d}[g(x(s^-),r(s))-g(z(s),\bar r(s))]dw(s)\Big|^p\\
+\;& 3^{p-1}E\sup_{0\le t\le t_1}\Big|\int_0^{t\wedge\gamma_d}\int_Z[h(x(s^-),r(s),v)-h(z(s),\bar r(s),v)]N(ds,dv)\Big|^p\\
=\;& 3^{p-1}(H_1+H_2+H_3).
\end{aligned}\tag{4.4}$$

Using the Hölder inequality, we obtain

$$H_1\;\le\;T^{p-1}E\int_0^{t_1\wedge\gamma_d}|f(x(t^-),r(t))-f(z(t),\bar r(t))|^pdt.$$

By Assumption 2.1, Lemma 4.3 and the basic inequality

$$|a+b|^p\le(1+\epsilon^{\frac{1}{p-1}})^{p-1}(|a|^p+\frac{|b|^p}{\epsilon}),\quad a,b\in R^n,\;p\ge 2\;\text{ and }\epsilon>0,$$

it follows that

$$\begin{aligned}
H_1\;&\le\;C(d)E\int_0^{t_1}|x(t\wedge\gamma_d)^- - z(t\wedge\gamma_d)|^pds+C(d)h\\
&\le\;C(d)E\int_0^{t_1}\Big(|x(t\wedge\gamma_d)^- - y(t\wedge\gamma_d)|^p+|y(t\wedge\gamma_d)-z(t\wedge\gamma_d)|^p\Big)dt+C(d)h,
\end{aligned}$$

where $C(d)$ is a constant independent of h, and in the computation below $C(d)$ varies line-by-line. In the same way as Mao did in [24], we can show using lemma 4.2 that

$$E[\sup_{0\le t\le T}|y(t\wedge\gamma_d)-z(t\wedge\gamma_d)|^p]\le C(d)h.\tag{4.5}$$

Hence,

$$H_1\;\le\;C(d)\int_0^{t_1}E\sup_{0\le s\le t}|e(s\wedge\gamma_d)|^pdt+C(d)h.\tag{4.6}$$

Using the Burkholder-Davis-Gundy inequality and the Hölder inequality, we can derive that

$$\begin{aligned}
H_2\;&\le\;C_pE\Big(\int_0^{t_1\wedge\gamma_d}|g(x(t^-),r(t))-g(z(t),\bar r(t))|^2dt\Big)^{\frac{p}{2}}.\\
&\le\;C_pT^{\frac{p}{2}-1}E\int_0^{t_1\wedge\gamma_d}|g(x(t^-),r(t))-g(z(t),\bar r(t))|^pdt.
\end{aligned}$$

In the same way as H_1 was estimated, we can then show

$$H_2 \;\leq\; C(d)\int_0^{t_1} E\sup_{0\leq s\leq t}|e(s\wedge\gamma_d)|^p dt + C(d)h. \tag{4.7}$$

To estimate H_3, we first apply the basic inequality $|a+b|^p \leq 2^{p-1}(|a|^p+|b|^p)$ to get

$$\begin{aligned} H_3 \;\leq\; & 2^{p-1}E\sup_{0\leq t\leq t_1}\Big|\int_0^{t\wedge\gamma_d}\int_Z [h(x(s^-),r(s),v)-h(z(s),\bar{r}(s),v)]\tilde{N}(ds,dv)\Big|^p \\ & +2^{p-1}E\sup_{0\leq t\leq t_1}\Big|\int_0^{t\wedge\gamma_d}\int_Z [h(x(s^-),r(s),v)-h(z(s),\bar{r}(s),v)]\pi(dv)ds\Big|^p. \end{aligned}$$

By Lemma 4.2 and the Hölder inequality, we obtain

$$\begin{aligned} & E\sup_{0\leq t\leq t_1}\Big|\int_0^{t\wedge\gamma_d}\int_Z [h(x(s^-),r(s),v)-h(z(s),\bar{r}(s),v)]\tilde{N}(ds,dv)\Big|^p \\ \leq\; & D_p\Big[E\Big(\int_0^{t_1\wedge\gamma_d}\int_Z |h(x(t^-),r(t),v)-h(z(t),\bar{r}(t),v)|^2\pi(dv)dt\Big)^{\frac{p}{2}} \\ & +E\int_0^{t_1\wedge\gamma_d}\int_Z |h(x(t^-),r(t),v)-h(z(t),\bar{r}(t),v)|^p\pi(dv)dt\Big] \end{aligned}$$

and

$$\begin{aligned} & E\sup_{0\leq t\leq t_1}\Big|\int_0^{t\wedge\gamma_d}\int_Z [h(x(s^-),r(s),v)-h(z(s),\bar{r}(s),v)]\pi(dv)ds\Big|^p \\ \leq\; & [\pi(Z)T]^{p-1}E\int_0^{t_1\wedge\gamma_d}\int_Z |h(x(t^-),r(t),v)-h(z(t),\bar{r}(t),v)|^p\pi(dv)dt. \end{aligned}$$

In the same way as H_1 was estimated, we can then obtain

$$H_3 \;\leq\; C(d)\int_0^{t_1} E\sup_{0\leq s\leq t}|e(s\wedge\gamma_d)|^p dt + C(d)h. \tag{4.8}$$

Substituting (4.6), (4.7) and (4.8) into (4.4), we obtain that

$$E\sup_{0\leq t\leq t_1}|x(t\wedge\gamma_d)-y(t\wedge\gamma_d)|^p \leq C(d)\int_0^{t_1}\sup_{0\leq s\leq t}E|x(s\wedge\gamma_d)-y(s\wedge\gamma_d)|^2 ds + C(d)h.$$

The Gronwall inequality implies that

$$E\sup_{0\leq t\leq T}|x(t\wedge\gamma_d)-y(t\wedge\gamma_d)|^p \;\leq\; C(d)e^{C(d)T}h.$$

Therefore the proof is complete.

Now, we will show the convergence in probability of the EM approximate solution $y(t)$ to the true solution $x(t)$ of equation (2.1).

Theorem 4.5 *If the conditions of Theorem 2.3 hold, then the EM approximate solution* $y(t)$ *converges to the solution* $x(t)$ *of equation (2.1) in the sense that*

$$\lim_{h\to 0}\Big(\sup_{0\le t\le T}|x(t)-y(t)|\Big)=0 \quad \textit{in} \quad \textit{probability}.$$

Proof. We divide the whole proof into three steps.

Step 1. It is easy to see from the proof of Theorem 2.3 that

$$E|x(\alpha_d\wedge T)|^p \le C. \tag{4.9}$$

(Recall that C is independent d). Noting that $|x(\alpha_d)|\ge d$ when $\alpha_d\le T$, we get

$$d^p P(\alpha_d\le T)\le E|x(\alpha_d\wedge T)|^p\le C.$$

That is

$$P(\alpha_d\le T)\le \frac{C}{d^p}.$$

Hence, given $\varepsilon(0,1)$, there exists a sufficiently large d^* such that

$$P(\alpha_d\le T)\le \frac{\varepsilon}{3}, \qquad \forall\ d\ge d^*. \tag{4.10}$$

Step 2. An application of the generalized Itô formula to $V(y(t),r(t))=|y(t)|^2$ gives

$$\begin{aligned} dV(y(t),r(t)) &= V_x(y(t),r(t))f(z(t),\bar r(t))dt+V_x(y(t),r(t))g(z(t),\bar r(t))dw(t)\\ &+ \frac{1}{2}trace[g^\top(z(t),\bar r(t))V_{xx}(y(t),r(t))g(z(t),\bar r(t))]dt\\ &+ \int_Z\Big(V(y(t)+h(z(t),\bar r(t),u),r(t))-V(y(t),r(t))\Big)N(dt,dv)\\ &+ \sum_{j=1}^N \gamma_{r(t)j}V(y(t),j)dt. \end{aligned}$$

Integrating from 0 to $\beta_d\wedge t$ and taking expectations gives,

$$\begin{aligned} E|y(\beta_d\wedge t)|^2 &\le E|y(0)|^2+E\int_0^{\beta_d\wedge t} LV(y(s),r(s)))ds\\ &+ 2E\int_0^{\beta_d\wedge t}|y(s)||f(z(s),\bar r(s))-f(y(s),r(s))|ds\\ &+ E\int_0^{\beta_d\wedge t}\Big(|g(z(s),\bar r(s))|^2-|g(y(s),r(s))|^2\Big)ds\\ &+ E\int_0^{\beta_d\wedge t}\int_Z\Big(|y(s)+h(z(s),\bar r(s),v)|^2-|y(s)+h(y(s),r(s),v)|^2\Big)\pi(dv)ds\\ &\le E|y(0)|^2+E\int_0^{\beta_d\wedge t}LV(y(s),r(s))ds+\sum_{i=1}^3 I_i. \end{aligned} \tag{4.11}$$

Let us estimate I_1. By Assumption 2.1, the Jensen inequality and (4.5), we have

$$\begin{aligned} I_1 &\le 2d\int_0^t \Big(E|f(z(s\wedge\beta_d),\bar{r}(s\wedge\beta_d)) - f(y(s\wedge\beta_d),r(s\wedge\beta_d))|^2\Big)^{\frac{1}{2}}ds \\ &\le 2d\int_0^t \Big(E|f(z(s\wedge\beta_d),\bar{r}(s\wedge\beta_d)) - f(z(s\wedge\beta_d),r(s\wedge\beta_d))|^2\Big)^{\frac{1}{2}}ds \\ &+ 2d\int_0^t \Big(E|f(z(s\wedge\beta_d),r(s\wedge\beta_d)) - f(y(s\wedge\beta_d),r(s\wedge\beta_d))|^2\Big)^{\frac{1}{2}}ds \\ &\le 2dJ_1 + 2d\sqrt{k_d}\sqrt{C(d)}Th^{\frac{1}{2}}. \end{aligned}$$

Let $N=[T/h]$ be the integer part of T/h and let I_G be the indicator function of the set G. Then for any $t\in[0,T]$, we have

$$\begin{aligned} J_1 &\le \sum_{n=0}^{N}\int_{t_n}^{t_{n+1}} \Big(E|f(y_n,r_n^h) - f(y_n,r(s\wedge\beta_d))|^2\Big)^{\frac{1}{2}}ds \\ &\le 2\sum_{n=0}^{N}\int_{t_n}^{t_{n+1}} \Big[E(|f(y_n,r_n^h)|^2 + |f(y_n,r(s\wedge\beta_d))|^2)I_{\{r(s\wedge\beta_d)\neq r(t_n)\}}\Big]^{\frac{1}{2}}ds. \end{aligned}$$

Using the basic inequality $|a+b|^2\le 2(|a|^2+|b|^2)$ and Assumption 2.1, we derive that

$$|f(y_n,r_n^h)|^2 \vee |f(y_n,r(s))|^2 \le 2(k_d|y_n|^2+\bar{K}_1^2).$$

where $\bar{K}_1=\max_{i\in S}\{|f(0,i)|\}$. Hence,

$$\begin{aligned} J_1 &\le 4\sqrt{k_d}\sum_{n=0}^{N}\int_{t_n}^{t_{n+1}} \Big[E(E(|y_n|^2|r(t_n))E(I_{\{r(s\wedge\beta_d)\neq r(t_n)\}}|r(t_n)))\Big]^{\frac{1}{2}}ds \\ &+ 4\bar{K}_1\sum_{n=0}^{N}\int_{t_n}^{t_{n+1}} \Big[E(E(I_{\{r(s\wedge\beta_d)\neq r(t_n)\}}|r(t_n)))\Big]^{\frac{1}{2}}ds \end{aligned}$$

where in the last step we use the fact that for $s\in[t_n,t_{n+1}\wedge\beta_d)$, $I_{\{r(s)\neq r(t_n)\}}$ are conditionlly independent with respect to the σ-algebra generated by $r(t_n)$. Using the Markov property,

$$E(I_{\{r(s\wedge\beta_d)\neq r(t_n)\}}|r(t_n)) \le \Big(\max_{1\le i\le N}(-\gamma_{ii})h + \circ(h)\Big)\sum_{i\in S}I_{\{r(t_n)=i\}} \le Ch.$$

Then we have

$$I_1 \le 2d\Big(4(d\sqrt{k_d}+\bar{K}_1)\sqrt{C}+\sqrt{k_d}\sqrt{C(d)}\Big)Th^{\frac{1}{2}}. \tag{4.12}$$

Rearranging I_2 by plus-and minus technique, we obtain that

$$\begin{aligned} I_2 &\le E\int_0^{\beta_d\wedge t} \Big[|g(z(s),\bar{r}(s))||g(z(s),\bar{r}(s)) - g(y(s),r(s))| \\ &+ |g(y(s),r(s))||g(z(s),\bar{r}(s)) - g(y(s),r(s))|\Big]ds. \end{aligned}$$

Using the Hölder inequality and the basic inequality

$$(a+b)^p \le a^p + b^p, \quad \forall a, b \ge 0,\ 0 < p \le 1, \tag{4.13}$$

we get

$$\begin{aligned} I_2 \ \le\ & \int_0^t \Big(E|g(z(s\wedge\beta_d),\bar{r}(s\wedge\beta_d))|^2\Big)^{\frac{1}{2}} \Big(E|g(z(s\wedge\beta_d),\bar{r}(s\wedge\beta_d)) - g(y(s\wedge\beta_d),r(s\wedge\beta_d))|^2\Big)^{\frac{1}{2}} ds \\ + & \int_0^t \Big(E|g(y(s\wedge\beta_d),r(s\wedge\beta_d))|^2\Big)^{\frac{1}{2}} \Big(E|g(z(s\wedge\beta_d),\bar{r}(s\wedge\beta_d)) - g(y(s\wedge\beta_d),r(s\wedge\beta_d))|^2\Big)^{\frac{1}{2}} ds. \end{aligned}$$

Similarly, we have

$$I_2 \ \le\ 2\sqrt{2}(d\sqrt{k_d} + \bar{K}_2)\Big(4(d\sqrt{k_d}+\bar{K}_2)\sqrt{C} + \sqrt{k_d}\sqrt{C(d)}\Big)Th^{\frac{1}{2}}, \tag{4.14}$$

where $\bar{K}_2 = \max_{i\in S}\{|g(0,i)|\}$. Now, let us estimate I_3. Rearranging I_3 by plus-and minus technique again, we obtain that

$$\begin{aligned} I_3 \ \le\ & 2E\int_0^{\beta_d\wedge t}\int_Z \Big[|y(s)||h(z(s),\bar{r}(s),v) - h(y(s),r(s),v)|\Big]\pi(dv)ds \\ + \ & E\int_0^{\beta_d\wedge t}\int_Z \Big[|h(z(s),\bar{r}(s),v)||h(z(s),\bar{r}(s),v) - h(y(s),r(s),v)|\Big]\pi(dv)ds \\ + \ & E\int_0^{\beta_d\wedge t}\int_Z \Big[|h(y(s),r(s),v)||h(z(s),\bar{r}(s),v) - h(y(s),r(s),v)|\Big]\pi(dv)ds. \end{aligned}$$

Using the Hölder inequality and the basic inequality (4.13) again, it follows that

$$\begin{aligned} I_3 \ \le\ & d\sqrt{\pi(Z)}\int_0^t \Big[E\int_Z |h(z(s\wedge\beta_d),\bar{r}(s\wedge\beta_d),v) - h(y(s\wedge\beta_d),r(s\wedge\beta_d),v)|^2\pi(dv)\Big]^{\frac{1}{2}} ds \\ + \ & \int_0^t \Big[\Big(E\int_Z |h(z(s\wedge\beta_d),\bar{r}(s\wedge\beta_d),v)|^2\pi(dv)\Big)^{\frac{1}{2}} \\ & \qquad \times\Big(E\int_Z |h(z(s\wedge\beta_d),\bar{r}(s\wedge\beta_d),v) - h(y(s\wedge\beta_d),r(s\wedge\beta_d),v)|^2\pi(dv)\Big)^{\frac{1}{2}}\Big] ds \\ + \ & \int_0^t \Big[\Big(E\int_Z |h(y(s\wedge\beta_d),r(s\wedge\beta_d),v)|^2\pi(dv)\Big)^{\frac{1}{2}} \\ & \qquad \times\Big(E\int_Z |h(z(s\wedge\beta_d),\bar{r}(s\wedge\beta_d),v) - h(y(s\wedge\beta_d),r(s\wedge\beta_d),v)|^2\pi(dv)\Big)^{\frac{1}{2}}\Big] ds. \end{aligned}$$

Similar to the estimation of I_1, we obtain that

$$I_3 \ \le\ (d\sqrt{\pi(Z)} + 2d\sqrt{2k_d} + 2\sqrt{\bar{K}_3})\Big(4(d\sqrt{k_d} + \sqrt{\bar{K}_3})\sqrt{C} + \sqrt{k_d}\sqrt{C(d)}\Big)Th^{\frac{1}{2}}, \tag{4.15}$$

where $\bar{K}_3 = \max_{i\in S}\{\int_Z |h(0,i,v)|^2\pi(dv)\}$. Inserting (4.12), (4.14) and (4.15) into (4.11), we have

$$E|y(\beta_d\wedge t)|^2 \ \le\ E|y(0)|^2 + C(d)h^{\frac{1}{p}} + E\int_0^{\beta_d\wedge t} LV(y(s),r(s))ds.$$

Repeating the procedure from Theorem 2.3, we can prove that

$$E|y(\beta_d \wedge T)|^2 \leq E|y(0)|^2 + CT + C(d)h^{\frac{1}{p}}. \tag{4.16}$$

Since $|y(\beta_d)| \geq d$, as $\beta_d < T$, we derive from (4.16) that

$$\begin{aligned} E|y(0)|^2 + CT + C(d)h^{\frac{1}{p}} &\geq E|y(\beta_d \wedge t)|^2 I_{\{\beta_d < T\}}(w)] \\ &\geq d^2 P(\beta_d \leq T). \end{aligned}$$

So we have

$$P(\beta_d \leq T) \quad \leq \quad \frac{E|y(0)|^2 + CT + C(d)h^{\frac{1}{p}}}{d^2}. \tag{4.17}$$

Now, for any $\varepsilon \in (0,1)$, choose $d = d^*$ sufficiently large for $\frac{E|y(0)|^2+CT}{d^{*2}} < \frac{\varepsilon}{6}$, and then choose h^* sufficiently small for $\frac{C(d)h^{*\frac{1}{p}}}{d^{*2}} < \frac{\varepsilon}{6}$. It then follows from (4.17) that

$$P(\beta_d < T) \leq \frac{\varepsilon}{3}, \quad \forall h \leq h^*. \tag{4.18}$$

Step 3. Let $\epsilon, \delta \in (0,1)$ be arbitrarily small, set

$$\bar{\Omega} = \{w : \sup_{0 \leq t \leq T} |x(t) - y(t)| \geq \delta\},$$

we have

$$\begin{aligned} P(\bar{\Omega}) &\leq P(\bar{\Omega} \cap \{\gamma_d > T\}) + P(\gamma_d < T) \\ &\leq P(\bar{\Omega} \cap \{\gamma_d > T\}) + P(\alpha_d < T) + P(\beta_d < T). \end{aligned}$$

By (4.10) and (4.18), we get

$$P(\bar{\Omega}) \quad \leq \quad P(\bar{\Omega} \cap \{\gamma_d > T\}) + \frac{2\varepsilon}{3}. \tag{4.19}$$

Using lemma 4.4, we have

$$\begin{aligned} \bar{C}_d h &\geq E[\sup_{0 \leq t \leq T} |x(t) - y(t)|^p I_{\{\gamma_d > T\}}(w)] \\ &\geq E[\sup_{0 \leq t \leq T} |x(t) - y(t)|^p I_{\{\gamma_d > T\}}(w) I_{\bar{\Omega}}(w)] \\ &\geq \delta P(\bar{\Omega} \cap \{\gamma_d > T\}). \end{aligned} \tag{4.20}$$

Inserting (4.20) into (4.19), we obtain that

$$P(\bar{\Omega}) \quad \leq \quad \frac{\bar{C}_d}{\delta} h + \frac{2\varepsilon}{3}.$$

Consequently, we can choose h sufficiently small for $\frac{\bar{C}_d h}{\delta} h < \frac{\varepsilon}{3}$ to obtain

$$P(\sup_{0 \leq t \leq T} |x(t) - y(t)| \geq \delta) < \varepsilon.$$

The proof is therefore complete.

5 Examples

In this section, we show some examples to illustrate the asymptotic boundedness and stability results.

Example 5.1 *Let $w(t)$ is a scalar Brownian motion. Let $r(t)$ be a right-continuous Markov chain taking values in $S = \{1, 2\}$ with the generator*

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}.$$

Let $N(dt, dv)$ be a Poisson random measures and σ-finite measure $\pi(dv)$ is given by $\pi(dv) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv,\ -\infty < v < +\infty$. Of course, $w(t)$, $N(dt, dv)$ and $r(t)$ are assumed to be independent.

Consider the following scalar hybrid SDEs with jumps

$$dx(t) = f(x(t^-), r(t))dt + g(x(t^-), r(t))dw(t) + \int_0^\infty vh(x(t^-), r(t))N(dt, dv). \tag{5.1}$$

Here

$$f(x,1) = -3x - x^3,\ f(x,2) = -2x - 3x^3,\ g(x,1) = \sqrt{2}(1+x),$$
$$g(x,2) = x,\ h(x,1) = 0.1(sinx + x^2),\ h(x,2) = 0.2x^2,$$

for $x \in R$. Obviously, the coefficient g satisfies the global Lipschitz condition and the linear growth condition, while f, h satisfy the local Lipschitz condition but they do not satisfy the linear growth condition. In fact, the coefficients f and g also satisfy the weak linear growth conditions. Through a straight computation, we can have

$$x^\top f(x,1) + \frac{1}{2}|g(x,1)|^2 \le 3 - 1.5|x|^2 - |x|^4, \tag{5.2}$$
$$x^\top f(x,2) + \frac{1}{2}|g(x,2)|^2 \le 2 - 2|x|^2 - 2.5|x|^4, \tag{5.3}$$
$$|x + vh(x,1)|^2 \le (1 + 0.04v^2)(1 + |x|^2 + |x|^4), \tag{5.4}$$
$$|x + vh(x,2)|^2 \le (1 + 0.2v^2)(1 + |x|^2 + 0.2|x|^4), \tag{5.5}$$

where

$$\alpha_{01} = 3,\ \alpha_{02} = 2, \beta_{01} = 1,\ \beta_{02} = 1,\ \alpha_{11} = -1.5,\ \alpha_{12} = -2,$$
$$\alpha_{21} = 1,\ \alpha_{22} = 2.5,\ \beta_{11} = 1,\ \beta_{12} = 1,\ \beta_{21} = 1,\ \beta_{22} = 0.2 \tag{5.6}$$

and

$$\gamma_1 = 2,\ \gamma_2 = 2,\ \bar{h}_1(v) = 1 + 0.04v^2,\ \bar{h}_2(v) = 1 + 0.2v^2. \tag{5.7}$$

So the inequalities (5.2)-(5.5) show that Assumption 2.2 holds. Moreover, by the property of normal distribute, we can obtain that $\pi(Z) = \frac{1}{2}$, *and*

$$\begin{aligned} C_1 &= \int_0^\infty (1 + 0.04v^2)\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}dv = 0.52, \\ C_2 &= \int_0^\infty (1 + 0.2v^2)\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}dv = 0.6. \end{aligned} \tag{5.8}$$

On the one hand, the matrix $\mathcal{A}_2$ *defined by (3.1) is*

$$\begin{aligned} \mathcal{A}_2 &= \text{diag}(-2\alpha_{11} - C_1\beta_{11}, -2\alpha_{12} - C_2\beta_{12}) - \Gamma \\ &= \begin{pmatrix} 3.48 & -1 \\ -2 & 5.4 \end{pmatrix}. \end{aligned}$$

It is easy to compute

$$\mathcal{A}_2^{-1} = \begin{pmatrix} 0.32158 & 0.05955 \\ 0.11910 & 0.20724 \end{pmatrix}.$$

By lemma 3.2, we see that $\mathcal{A}_2$ *is a nonsingular M-matrix. Compute*

$$(\theta_1, \theta_2)^\top = \mathcal{A}_2^{-1}\overrightarrow{1} = (0.38113, 0.32635)^\top,$$

and

$$-2\alpha_{11} - C_1\beta_{11} - \frac{1}{\theta_1}\sum_{j=1}^N \gamma_{1j}\theta_j = 2.33627, \quad -2\alpha_{12} - C_2\beta_{12} - \frac{1}{\theta_2}\sum_{j=1}^N \gamma_{2j}\theta_j = 3.0643.$$

By Theorem 3.3, we can conclude that equation (5.1) is asymptotically bounded in mean square. That is,

$$\limsup_{t\to\infty} E|x(t)|^2 \le \frac{0.6}{\varepsilon}$$

where $0 < \varepsilon < 2.33627$.

On the other hand, similar to (4.2), we can obtain the EM approximate solution $y(t)$ *of equation (5.1). By (5.6), (5.7) and (5.8), we have*

$$\gamma_1 = \gamma_2, \quad 2\alpha_{2i} > C_i\beta_{2i}, \; i = 1, 2,$$

then Theorem 4.5 implies that the convergence in probability of numerical solution $y(t)$ *and the solution* $x(t)$ *to equation (5.1).*

Example 5.2 *Let $r(t)$ be a right-continuous Markov chain taking values in $S = \{1, 2\}$ with the generator*

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}.$$

Let $N(dt, dv)$ be a Poisson random measures and σ-finite measure $\pi(dv)$ is given by $\pi(dv) = \lambda f(v)dv$, where $\lambda = 2$ is the jump rate and $f(v) = \frac{1}{\sqrt{2\pi}v}e^{-\frac{(lnv)^2}{2}}$, $0 \le v < \infty$ is the density function of a lognormal random variable. Of course $N(dt, dv)$ and $r(t)$ are assumed to be independent.

Consider the following scalar hybrid SDEs with pure jumps

$$dx(t) = f(x(t^-), r(t))dt + \int_0^1 h(x(t^-), r(t), v)N(dt, dv), \tag{5.9}$$

with initial data $x(0) = x_0$ and $r(0) = 1$. Here

$$\begin{aligned} &f(x, 1) = -2x - 1.5x^5, \quad f(x, 2) = -x - x^5, \\ &h(x, 1, v) = 0.05v(x + x^3), \quad h(x, 2, v) = 0.1vx^3, \end{aligned}$$

for $x \in R$. Similarly, the coefficients f, h satisfy the local Lipschitz condition but they do not satisfy the linear growth condition. Through a straight computation, we can have

$$x^\top f(x, 1) \le -2|x|^2 - 1.5|x|^6, \quad x^\top f(x, 2) \le -|x|^2 - |x|^6, \tag{5.10}$$

$$|x + h(x, 1, v)|^2 \le (1 + 0.05v)^2(3|x|^2 + 1.5|x|^6), \tag{5.11}$$

$$|x + h(x, 2, v)|^2 \le (1 + 0.01v^2)(|x|^2 + |x|^6), \tag{5.12}$$

where

$$\alpha_{11} = -2,\ \alpha_{12} = -1,\ \alpha_{21} = 1.5,\ \alpha_{22} = 1,\ \beta_{11} = 3,\ \beta_{12} = 1,\ \beta_{21} = 1.5,\ \beta_{22} = 1 \tag{5.13}$$

and

$$\gamma_1 = 4,\ \gamma_2 = 4,\ \bar{h}_1(v) = (1 + 0.05v)^2,\ \bar{h}_2(v) = 1 + 0.01v^2. \tag{5.14}$$

So the inequalities (5.10), (5.11) and (5.12) show that Assumption 2.2 holds. Moreover, by the property of log-normal distribute $f(v)$, we can obtain that $\pi(Z) = 1$, and

$$\begin{aligned} C_1 = \int_Z \bar{h}_1(v)\pi(dv) &= 2\int_0^1 (1 + 0.05v)^2 \frac{1}{\sqrt{2\pi}v} e^{-\frac{(lnv)^2}{2}} dv \\ &\le 1 + 0.2\sqrt{e} + 0.005e^2, \end{aligned} \tag{5.15}$$

$$\begin{aligned} C_2 = \int_Z \bar{h}_2(v)\pi(dv) &= 2\int_0^1 (1 + 0.01v^2) \frac{1}{\sqrt{2\pi}v} e^{-\frac{(lnv)^2}{2}} dv \\ &\le 1 + 0.02e^2. \end{aligned} \tag{5.16}$$

It is easy to see that the Markov chain has its stationary probability distribution $\tilde{\pi} = (\tilde{\pi}_1, \tilde{\pi}_2)$ *given by*

$$\tilde{\pi}_1 = \frac{\gamma}{1+\gamma} = 0.8, \quad \tilde{\pi}_2 = \frac{1}{1+\gamma} = 0.2.$$

Note that $G(x,i)$ *defined in Theorem 3.6 has the form*

$$G(x,i) \;=\; (2\alpha_{1i} + \beta_{1i}C_i + 1)|x|^2 - 2\alpha_{2i}|x|^{\gamma_1+2} + \beta_{2i}C_i|x|^{\gamma_2+2},$$

for any $x \in R$ *and* $i \in S$. *By the conditions (5.13)-(5.16), we have*

$$\begin{aligned} G(x,1) &= (2\alpha_{11} + \beta_{11}C_1 + 1)|x|^2 - 2\alpha_{21}|x|^6 + \beta_{21}C_1|x|^6 \\ &\le 1.1|x|^2 - 0.95|x|^6 \le 0.456 \end{aligned}$$

and

$$\begin{aligned} G(x,2) &= (2\alpha_{12} + \beta_{12}C_2 + 1)|x|^2 - 2\alpha_{22}|x|^6 + \beta_{22}C_2|x|^6 \\ &\le 0.14776|x|^2 - 0.85224|x|^6 \le 0.024. \end{aligned}$$

The above conditions (5.13)-(5.16) imply that

$$\gamma_1 = \gamma_2, \quad 2\alpha_{21} - C_1\beta_{21} > 0 \text{ and } 2\alpha_{22} - C_2\beta_{22} > 0.$$

Hence, by Theorem 3.6, we can conclude that for any initial value x_0, *the solution* $x(t)$ *of equation (5.9) has the property that*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t E|x(s)|^2 ds \le 0.3696.$$

That is to say, the limit of the average in the time of the 2th moment is not greater than 0.3696.

Example 5.3 *Let* $r(t)$ *be a right-continuous Markov chain on the state space* $S = \{1, 2\}$ *with the generator*

$$\Gamma = \begin{pmatrix} -2 & 2 \\ \gamma & -\gamma \end{pmatrix},$$

where $\gamma > 0$. *Let* $N(dt, dv)$ *be a Poisson random measures and* σ*-finite measure* $\pi(dv)$ *is given by* $\pi(dv) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv$, $-\infty < v < +\infty$. *Assume that* $N(dt, dv)$ *and* $r(t)$ *are independent.*

Consider the following scalar hybrid SDEs with pure jumps

$$dx(t) = f(x(t^-), r(t))dt + \int_0^\infty h(x(t^-), r(t), v)N(dt, dv), \tag{5.17}$$

with initial data $x(0) = x_0$ and $r(0) = 1$. Here

$$f(x,1) = -2x - 1.6x^5, \quad f(x,2) = 0.5x - 3x^5,$$
$$h(x,1,v) = 0.5vx^2, \quad h(x,2,v) = \frac{\sqrt{3}}{3}vx^2,$$

for any $x \in R$. We note that equation (5.17) can be regarded as the result of the two equations

$$dx(t) = [-2x(t^-) - 1.6x^5(t^-)]dt + 0.5\int_0^\infty vx^2(t^-)N(dt,dv) \tag{5.18}$$

and

$$dx(t) = [0.5x(t^-) - 3x^5(t^-)]dt + \frac{\sqrt{3}}{3}\int_0^\infty vx^2(t^-)N(dt,dv) \tag{5.19}$$

switching among each other according to the movement of the Markov chain $r(t)$. It is easy to see that equation (5.18) is asymptotically stable but equation (5.19) is unstable. However, we shall see that due to the Markovian switching, the overall system (5.17) will be asymptotically stable in 4th moment for certain γ. In fact, the coefficients f, g satisfy the local Lipschitz condition but they do not satisfy the linear growth condition. Through a straight computation, we can have

$$x^\top f(x,1) \le -2|x|^2 - 1.6|x|^6, \quad x^\top f(x,2) \le 0.5|x|^2 - 3|x|^6, \tag{5.20}$$
$$|x + h(x,1,v)|^2 \le (1 + 0.5v^2)(|x|^2 + 0.5|x|^4), \tag{5.21}$$
$$|x + h(x,2,v)|^2 \le (1 + \frac{1}{6}v^2)(|x|^2 + 2|x|^4), \tag{5.22}$$

where

$$\alpha_{11} = -2,\ \alpha_{12} = 0.5,\ \alpha_{21} = 1.6,\ \alpha_{22} = 3,\ \beta_{11} = 1,\ \beta_{12} = 1,\ \beta_{21} = 0.5,\ \beta_{22} = 2 \tag{5.23}$$

and

$$\gamma_1 = 4,\ \gamma_2 = 2,\ \bar{h}_1(v) = 1 + 0.5v^2,\ \bar{h}_2(v) = 1 + \frac{1}{6}v^2. \tag{5.24}$$

So the inequalities (5.20), (5.21) and (5.22) show that Assumption 2.2 holds. Moreover, by the property of normal distribute, we can obtain that

$$C_1^4 = \int_0^\infty (1 + 0.5v^2)^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv = 1.375,$$
$$C_2^4 = \int_0^\infty (1 + \frac{1}{6}v^2)^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv = 0.708. \tag{5.25}$$

The above conditions (5.23)-(5.25) imply that

$$\gamma_1 = 2\gamma_2, \quad 4\alpha_{21} > \tilde{C}_{h_1}^4 \beta_{21}^2 \ \text{and}\ \ 4\alpha_{22} > \tilde{C}_{h_2}^4 \beta_{22}^2.$$

Hence, by (3.10), we get the matrix $\bar{\mathcal{A}}_4$

$$\begin{aligned}\bar{\mathcal{A}}_4 &= -\text{diag}(4\alpha_{11} + \bar{C}_{h_1}^4\beta_{11}^2, 4\alpha_{12} + \bar{C}_{h_2}^4\beta_{12}^2) - \Gamma \\ &= \begin{pmatrix} 7.25 & -2 \\ -\gamma & -3.416 + \gamma \end{pmatrix}.\end{aligned}$$

Since $\gamma > 0$, $\bar{\mathcal{A}}_4$ *is a nonsingular M-matrix if and only if* $\gamma > 4.72$. *By Theorem 3.9, we can conclude that equation (5.17) is asymptotically stable in 4th moment if* $\gamma > 4.72$.

Appendix

In this appendix, we shall prove Theorem 2.3.

Proof of Theorem 2.3. Since the coefficients of equation (2.1) are locally Lipschitz continuous, for any given initial data x_0 and r_0, there is a maximal local solution $x(t)$ in L^p on $t \in [0, \sigma_\infty)$, where σ_∞ is the explosion time (see, e.g., [40]). Fix any initial data x_0 and r_0 and find a sufficiently large k_0 for $|x_0| < k_0$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\{t \in [0, \sigma_e) : |x(t)| \geq k\}$$

where, throughout this paper, we set $\inf \phi = \infty$ (as usual ϕ denote the empty set). Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, whence $\tau_\infty \leq \sigma_\infty$ a.s. Note if we can show that $\tau_\infty = \infty$ a.s., then $\sigma_\infty = \infty$ a.s. So we just need to show that $\tau_\infty = \infty$ a.s. Define $V(x, i) = |x|^p$. By the generalized Itô formula, we have that, for $k \geq k_0$ and $t \geq 0$,

$$\begin{aligned}& V(x(t\wedge\tau_k), r(t\wedge\tau_k)) \\ = \;& V(x(0), r(0)) + \int_0^{t\wedge\tau_k} LV(x(s^-), r(s))ds + \int_0^{t\wedge\tau_k} V_x(x(s^-), r(s))g(x(s^-), r(s))dw(s) \\ + \;& \int_0^{t\wedge\tau_k}\int_Z [V(x(s^-) + h(x(s^-), r(s), v), r(s)) - V(x(s^-), r(s))]\tilde{N}(ds, dv). \qquad (5.26)\end{aligned}$$

Taking the expectations on both side of (5.26), we obtain that

$$\begin{aligned}& E|x(t\wedge\tau_k)|^p \\ \leq \;& |x_0|^p + E\int_0^{t\wedge\tau_k} p|x(s^-)|^{p-2}[x(s^-)^\top f(x(s^-), r(s)) + \frac{p-1}{2}|g(x(s^-), r(s))|^2]ds \\ + \;& E\int_0^{t\wedge\tau_k}\int_Z [|x(s^-) + h(x(s^-), r(s), v)|^p - |x(s^-)|^p]\pi(dv)ds.\end{aligned}$$

By Assumption 2.2, we get

$$\begin{aligned}
& E|x(t\wedge\tau_k)|^p \\
\le\ & |x_0|^p + E\int_0^{t\wedge\tau_k}\Big(p|x(s^-)|^{p-2}\big[\alpha_{0,r(s)}+\alpha_{1,r(s)}|x(s^-)|^2-\alpha_{2,r(s)}|x(s^-)|^{\gamma_1+2}\big] \\
+\ & \int_Z\big[[\bar h_{r(s)}(v)(\beta_{0,r(s)}+\beta_{1,r(s)}|x(s^-)|^2+\beta_{2,r(s)}|x(s^-)|^{\gamma_2+2})]^{\frac{p}{2}}-|x(s^-)|^p\big]\pi(dv)\Big)ds \\
\le\ & |x_0|^p + E\int_0^{t\wedge\tau_k}\Big(p|x(s^-)|^{p-2}\big[\alpha_{0,r(s)}+\alpha_{1,r(s)}|x(s^-)|^2-\alpha_{2,r(s)}|x(s^-)|^{\gamma_1+2}\big] \\
+\ & \int_Z(\bar h_{r(s)}(v))^{p/2}\pi(dv)(\beta_{0,r(s)}+\beta_{1,r(s)}|x(s^-)|^2+\beta_{2,r(s)}|x(s^-)|^{\gamma_2+2})^{\frac{p}{2}}\Big)ds. \qquad (5.27)
\end{aligned}$$

Let us consider two cases specified in Theorem 2.3.

Case (a). In this case, we have $\gamma_1 > 0.5p\gamma_2$. It is easy to see that there is a positive constant C such that

$$\max_{i\in S}\Big(p|x|^{p-2}\big[\alpha_{0i}+\alpha_{1i}|x|^2-\alpha_{2i}|x|^{\gamma_1+2}\big]+\int_Z(\bar h_i(v))^{p/2}\pi(dv)(\beta_{0i}+\beta_{1i}|x|^2+\beta_{2i}|x|^{\gamma_2+2})^{\frac{p}{2}}\Big)\le C$$

for all $x\in R^n$. It then follows from (5.27) that

$$E|x(t\wedge\tau_k)|^p \le |x_0|^p + Ct. \qquad (5.28)$$

Noting that $|x(\tau_k)|\ge k$ whenever $\tau_k\le t$, we then drive that

$$|x_0|^p + Ct \ge E[|x(t\wedge\tau_k)|^p I_{\{\tau_k\le t\}}] \ge k^p P(\tau_k\le t).$$

Letting $k\to\infty$, we get $P(\tau_\infty\le t)=0$, i.e., $P(\tau_\infty>t)=1$. Since $t>0$ is arbitrary, we must have that $\tau_\infty=\infty$ a.s. That is to say, for any given initial data x_0 and r_0, the hybrid equation (2.1) has a unique global solution $x(t)$ on $t\in[0,\infty)$. Moreover, letting $k\to\infty$ in (5.28) yields $E|x(t)|^p\le|x_0|^p+Ct$. That is, $x(t)\in L^p$ for all $t\ge 0$.

Case (b). In this case, we have $\gamma_1=0.5p\gamma_2$ and $p\alpha_{2i}>C_i^p\beta_{2i}^{\frac{p}{2}}$ for all $i\in S$. By the Hölder inequality

$$|a+b+c|^{\frac{p}{2}}\le\Big(2+\frac{1}{\delta^{\frac{p}{p-2}}}\Big)^{\frac{p}{2}-1}(|a|^{\frac{p}{2}}+|b|^{\frac{p}{2}})+\Big(2\delta^{\frac{p}{p-2}}+1\Big)^{\frac{p}{2}-1}|c|^{\frac{p}{2}},$$

for any $a, b, c > 0$ and $\delta > 0$. we have

$$\begin{aligned} & E|x(t\wedge\tau_k)|^p \\ \leq\ & |x_0|^p + E\int_0^{t\wedge\tau_k} \Big\{ p|x(s^-)|^{p-2}\big[\alpha_{0,r(s)} + \alpha_{1,r(s)}|x(s^-)|^2 - \alpha_{2,r(s)}|x(s^-)|^{\gamma_1+2}\big] \\ +\ & C^p_{r(s)}\Big[\Big(2+\frac{1}{\delta^{\frac{p}{p-2}}}\Big)^{\frac{p}{2}-1}(\beta^{\frac{p}{2}}_{0,r(s)} + \beta^{\frac{p}{2}}_{1,r(s)}|x(s^-)|^p) + \Big(2\delta^{\frac{p}{p-2}}+1\Big)^{\frac{p}{2}-1}\beta^{\frac{p}{2}}_{2,r(s)}|x(s^-)|^{0.5p\gamma_2+p}\Big]\Big\}ds \\ =\ & |x_0|^p + E\int_0^{t\wedge\tau_k} \Big\{ C^p_{r(s)}\Big(2+\frac{1}{\delta^{\frac{p}{p-2}}}\Big)^{\frac{p}{2}-1}\beta^{\frac{p}{2}}_{0,r(s)} + p\alpha_{0,r(s)}|x(s^-)|^{p-2} \\ +\ & \Big[p\alpha_{1,r(s)} + C^p_{r(s)}\Big(2+\frac{1}{\delta^{\frac{p}{p-2}}}\Big)^{\frac{p}{2}-1}\beta^{\frac{p}{2}}_{1,r(s)}\Big]|x(s^-)|^p \\ -\ & \Big[p\alpha_{2,r(s)} - C^p_{r(s)}\Big(2\delta^{\frac{p}{p-2}}+1\Big)^{\frac{p}{2}-1}\beta^{\frac{p}{2}}_{2,r(s)}\Big]|x(s^-)|^{\gamma_1+p}\Big\}ds. \end{aligned} \tag{5.29}$$

Recalling $p\alpha_{2i} > C^p_i\beta^{\frac{p}{2}}_{2i}$, we can choose sufficiently small $\delta > 0$ such that $p\alpha_{2i} > C^p_i\Big(2\delta^{\frac{p}{p-2}} + 1\Big)^{\frac{p}{2}-1}\beta^{\frac{p}{2}}_{2i}$. Hence, there exists a constant C such that

$$\max_{i\in S}\Big\{p\alpha_{0,i}|x|^{p-2} + \Big[p\alpha_{1,i} + C^p_i\Big(2+\frac{1}{\delta^{\frac{p}{p-2}}}\Big)^{\frac{p}{2}-1}\beta^{\frac{p}{2}}_{1,i}\Big]|x|^p - [p\alpha_{2,i} - C^p_i\Big(2+\frac{1}{\delta^{\frac{p}{p-2}}}\Big)^{\frac{p}{2}-1}\beta^{\frac{p}{2}}_{2,i}]|x|^{\gamma_1+p})\Big\} \leq C$$

for all $x \in R^n$. It then follows from (5.29) that

$$E|x(t\wedge\tau_k)|^p \leq |x_0|^p + \max_{i\in S}\Big[C^p_{r(s)}\Big(2+\frac{1}{\delta^{\frac{p}{p-2}}}\Big)^{\frac{p}{2}-1}\beta^{\frac{p}{2}}_{0,r(s)} + C\Big]t. \tag{5.30}$$

Similar to the Case (a), we obtain that for any given initial data x_0 and r_0, the hybrid equation (2.1) has a unique global solution $x(t)$ on $t \in [0, \infty)$. Moreover, letting $k \to \infty$ in (5.30) yields $E|x(t)|^p \leq |x_0|^p + \max_{i\in S}\Big[C^p_{r(s)}\Big(2+\frac{1}{\delta^{\frac{p}{p-2}}}\Big)^{\frac{p}{2}-1}\beta^{\frac{p}{2}}_{0,r(s)} + C\Big]t$. That is, $x(t) \in L^p$ for all $t \geq 0$.

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