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ON THE INVOLUTION FIXITY OF
EXCEPTIONAL GROUPS OF LIE TYPE

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ABSTRACT. The involution fixity ifix(G) of a permutation group G of degree n is the maximum number of fixed points of an involution. In this paper we study the involution fixity of primitive almost simple exceptional groups of Lie type. We show that if T is the socle of such a group, then either ifix(T) $> n^{1/3}$, or ifix(T) = 1 and $T = {}^2B_2(q)$ is a Suzuki group in its natural 2-transitive action of degree $n = q^2 + 1$. This bound is best possible and we present more detailed results for each family of exceptional groups, which allows us to determine the groups with ifix(T) $\leqslant n^{4/9}$. This extends recent work of Liebeck and Shalev, who established the bound ifix(T) $> n^{1/6}$ for every almost simple primitive group of degree n with socle T (with a prescribed list of exceptions). Finally, by combining our results with the Lang-Weil estimates from algebraic geometry, we determine bounds on a natural analogue of involution fixity for primitive actions of exceptional algebraic groups over algebraically closed fields.

1. INTRODUCTION

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a permutation group on a finite set Ω of size n and recall that the *minimal degree* $\mu(G)$ of G is defined to be the minimal number of points moved by a non-identity element of G. This notion has been extensively studied since the nineteenth century, with many classical results in the context of lower bounds for primitive groups (see Bochert [7] and Jordan [25], for example). We refer the reader to [3, 24, 36] for more recent results and applications.

The related notion of *fixity* was first studied by Ronse in [55]. The fixity $f(G)$ of G is the maximal number of points fixed by a non-identity element, that is, $f(G) = n - \mu(G)$. In [56], Saxl and Shalev study primitive groups of bounded fixity. One of their main results, [56, Theorem 1.3], shows that if G is primitive with fixity f, then either G has a solvable subgroup whose index is bounded by a function of f, or G is almost simple with socle $\mathrm{L}_2(q)$ or ${}^2B_2(q)$ in their 2-transitive actions of degree $n = q + 1$ or $q^2 + 1$, respectively. More recently, this study has been extended by Liebeck and Shalev in [47]. For example, [47, Theorem 1] provides detailed information on the structure of a primitive permutation group G of degree n with $f(G) < n^{1/6}$. To do this, they first use the O'Nan-Scott theorem for primitive groups to establish a reduction to affine and almost simple groups, and they handle the affine groups by applying earlier work on the fixity of linear groups [46]. This leaves the almost simple groups. Here the main result is [47, Theorem 4], which shows that the socle of an almost simple primitive group contains an *involution* fixing at least $n^{1/6}$ points (with specified exceptions). This leads us naturally to the following notion.

Definition. Let $G \leqslant \mathrm{Sym}(\Omega)$ be a permutation group on a finite set Ω. The *involution fixity* ifix(G) of G is the maximal number of points fixed by an involution in G.

As noted in [47], the involution fixity of permutation groups has been studied in several papers, going back to work of Bender [6] in the early 1970s, who classified the transitive

groups G with $\mathrm{ifix}(G) = 1$ (these are the groups for which a point stabilizer is a so-called *strongly embedded* subgroup). In terms of applications, bounds on the involution fixity of almost simple primitive groups have recently been used in the context of finite geometry to study point-primitive generalized quadrangles (see [5]).

In this paper, we investigate the involution fixity of almost simple primitive permutation groups with socle an exceptional group of Lie type, with the aim of establishing accurate lower bounds. This complements recent work of Covato [22], who has studied the analogous problem for almost simple groups with socle a sporadic, alternating or classical group. A simplified version of our main result is the following.

Theorem 1. *Let G be an almost simple primitive permutation group of degree n with socle T, an exceptional group of Lie type over $\mathbb{F}_q$. Then either*

(i) $\mathrm{ifix}(T) > n^{1/3}$; *or*

(ii) $(T, n) = ({}^2B_2(q), q^2 + 1)$ *or* $({}^2G_2(3)', 9)$, *and* $\mathrm{ifix}(T) = 1$.

Note that ${}^2G_2(3)' \cong \mathrm{L}_2(8)$, so the second example in part (ii) can be viewed as a classical group. Also note that $\mathrm{ifix}(G) \geqslant \mathrm{ifix}(T)$, so the theorem gives a lower bound on the involution fixity of G.

Remark 1. The lower bound in part (i) is best possible in the sense that for any $\epsilon > 0$, there are infinitely many groups with $n^{1/3} < \mathrm{ifix}(T) < n^{1/3+\epsilon}$. For instance, if we consider the action of $G = {}^2G_2(q)'$ on the set of cosets of a Borel subgroup, then Theorem 3.2 states that $\mathrm{ifix}(T) > n^{1/3}$ and

$$\liminf_{q \to \infty} \frac{\log \mathrm{ifix}(T)}{\log n} = \frac{1}{3}.$$

The same conclusion holds when $T = G_2(q)$, $p = 3$ and G contains a graph or graph-field automorphism (again, with respect to the action of G on the cosets of a Borel subgroup).

With a view towards applications, it is desirable to strengthen the bound in part (i) of Theorem 1, at the possible expense of some additional (known) exceptions. In [22], Covato determines the almost simple primitive groups with socle T (a sporadic, alternating or classical group) such that $\mathrm{ifix}(T) \leqslant n^{4/9}$. Our next theorem establishes the analogous result for exceptional groups of Lie type.

Theorem 2. *Let G be an almost simple primitive permutation group of degree n with socle T, an exceptional group of Lie type over $\mathbb{F}_q$, and point stabilizer H. Set $H_0 = H \cap T$ and $q = p^f$ with p a prime. Then $\mathrm{ifix}(T) \leqslant n^{4/9}$ if and only if (T, H_0) appears in Table 1. In each of these cases, $\mathrm{ifix}(T) \geqslant n^\alpha$ for the given constant α.*

Remark 2. Let us make some comments on the statement of Theorem 2.

(a) The focus on the bound $\mathrm{ifix}(T) \leqslant n^{4/9}$ is somewhat arbitrary, although there are infinitely many examples where it is essentially best possible. For instance, if $G = {}^3D_4(q)$ and $H = P_2$ is a maximal parabolic subgroup, then $\mathrm{ifix}(T) > n^{4/9}$ and

$$\liminf_{q \to \infty} \frac{\log \mathrm{ifix}(T)}{\log n} = \frac{4}{9}$$

(see Theorem 3.2).

(b) More generally, there are many examples with $\mathrm{ifix}(T) < n^{1/2}$. In the special cases where H is a maximal parabolic, subfield or exotic local subgroup, the relevant groups are determined in Theorems 3.2, 6.1 and 7.1, respectively.

T	H_0	α	Conditions
$F_4(q)$	$(q^2+q+1)^2.(3 \times \mathrm{SL}_2(3))$	0.427	$q=2$, $G=\mathrm{Aut}(T)$
$G_2(q)'$	P_1	2/5	$q \geqslant 7$ odd
	P_2	2/5	$q \geqslant 7$
	$P_{1,2}$	1/3	$p=3$, $q \geqslant 9$
	$G_2(q_0)$	3/7	$q=q_0^k$ odd, k prime
	$^2G_2(q)$	2/5	$q=3^{2m+1}$, $m \geqslant 0$
	$2^3.\mathrm{L}_3(2)$	3/7	$q=p \geqslant 11$
	$\mathrm{U}_3(3){:}2$	3/7	$q=p \geqslant 11$
	$\mathrm{L}_2(13)$	3/7	$q \geqslant 17$, $p \neq 13$, $\mathbb{F}_q = \mathbb{F}_p[\sqrt{13}]$
	$\mathrm{L}_2(8)$	3/7	$q \geqslant 23$, $p \geqslant 5$, $\mathbb{F}_q = \mathbb{F}_p[\omega]$, $\omega^3 - 3\omega + 1 = 0$
	$3^{1+2}.8$	0.416	$q=2$
$^3D_4(q)$	$P_{1,3,4}$	4/11	q odd
	$^3D_4(q_0)$	3/7	$q=q_0^k$ odd, k prime
	$G_2(q)$	3/7	q odd
	$\mathrm{PGL}_3(q)$	2/5	q odd
	$\mathrm{PGU}_3(q)$	0.387	q odd
	$(q^4-q^2+1).4$	0.436	$q=2$
	$(q^2+q+1)^2.\mathrm{SL}_2(3)$	0.401	$q=2,3$
	$(q^2-q+1)^2.\mathrm{SL}_2(3)$	0.351	$q=2,3,4$
$^2G_2(q)'$	$q^{1+1+1}{:}(q-1)$	1/3	
	$^2G_2(q_0)$	0.426	$q=q_0^k$, k prime
	$(q-\sqrt{3q}+1){:}6$	0.442	$q=27$
	D_{18}	0.416	$q=3$
	D_{14}	0.386	$q=3$
	$2^3{:}7$	0	$q=3$
$^2B_2(q)$	$q^{1+1}{:}(q-1)$	0	
	$^2B_2(q_0)$	0.397	$q=q_0^k$, k prime
	$(q+\sqrt{2q}+1){:}4$	0.438	$q=8$
	$(q-\sqrt{2q}+1){:}4$	0.380	$q=8,32$

TABLE 1. The cases with $n^\alpha < \mathrm{ifix}(T) \leqslant n^{4/9}$ in Theorem 2

(c) For the cases in Table 1, the lower bound $\mathrm{ifix}(T) \geqslant n^\alpha$ is best possible in the sense that the given constant α is either accurate to 3 decimal places (as a lower bound which is valid for each value of q recorded in the final column of the table), or

$$\liminf_{q \to \infty} \frac{\log \mathrm{ifix}(T)}{\log n} = \alpha.$$

As noted in Section 9, if $T = {}^2G_2(q)$ and $H_0 = {}^2G_2(q_0)$ is a subfield subgroup then

$$\liminf_{q \to \infty} \frac{\log \mathrm{ifix}(T)}{\log n} = \beta \tag{1}$$

with $\beta = 3/7$. Similarly, (1) holds with $\beta = 2/5$ if $T = {}^3D_4(q)$ and $H_0 = \mathrm{PGU}_3(q)$ (with q odd), and also if $T = {}^2B_2(q)$ and $H_0 = {}^2B_2(q_0)$.

Finally, we turn to asymptotic results. Let G be an almost simple exceptional group of Lie type over $\mathbb{F}_q$ with socle $T = (\bar{G}_\sigma)'$, where $\bar{G}$ is a simple exceptional algebraic group

T	Type of H_0	β	Conditions
$E_8(q)$	$\Omega_5(q)$	$58/119$	$p \geqslant 5$
	$\mathrm{L}_2(q)$	$17/35$	3 classes; $p \geqslant 23, 29, 31$
$E_7(q)$	$\mathrm{L}_2(q)$	$31/65$	2 classes; $p \geqslant 17, 19$
$E_6^\epsilon(q)$	$\mathrm{L}_3^\epsilon(q^3)$	$13/27$	
$F_4(q)$	$G_2(q)$	$9/19$	$p = 7$
	$\mathrm{L}_2(q)$	$23/49$	$p \geqslant 13$
$G_2(q)$	P_1	$2/5$	$p \geqslant 3$
	P_2	$2/5$	
	$P_{1,2}$	$1/3$	$p = 3$
	$\mathrm{L}_2(q)$	$5/11$	$p \geqslant 7$
$^2F_4(q)$	$P_{2,3}$	$5/11$	
$^2G_2(q)'$	$q^{1+1+1}{:}(q-1)$	$1/3$	

TABLE 2. The cases with $\beta < 1/2$ in Theorem 3

(of type E_8, E_7, E_6, F_4 or G_2) over the algebraic closure of $\mathbb{F}_q$ and σ is an appropriate Steinberg endomorphism of $\bar{G}$. For the purposes of Theorem 3 below, let us say that a maximal subgroup H of G is *algebraic* if it is of the form $H = N_G(\bar{H}_\sigma)$, where $\bar{H}$ is a σ-stable closed positive dimensional subgroup of $\bar{G}$ (equivalently, H is algebraic if it is either a maximal parabolic subgroup, or a maximal rank subgroup as in [38], or if it is a non-maximal rank subgroup as in part (I) of Theorem 5.1). In the second column of Table 2 we refer to the *type* of H_0, which is simply the socle of H_0 for the non-parabolic subgroups in the table.

Theorem 3. *Let G be an almost simple primitive permutation group of degree n with socle T, an exceptional group of Lie type over $\mathbb{F}_q$, and point stabilizer H, an algebraic subgroup of G. Set $H_0 = H \cap T$ and $q = p^f$ with p a prime. Then*

$$\liminf_{q \to \infty} \frac{\log \mathrm{ifix}(T)}{\log n} = \beta$$

and either $\beta \geqslant 1/2$, or (T, H_0, β) is one of the cases listed in Table 2.

The proof of Theorem 3 follows by combining the asymptotic statements in Theorems 3.2, 4.1 and 5.3 on parabolic, maximal rank and lower rank algebraic subgroups, respectively (see Remark 5.4).

We can interpret Theorem 3 at the level of algebraic groups. Let $\bar{G}$ be a simple exceptional algebraic group over an algebraically closed field, let $\bar{H}$ be a closed subgroup of $\bar{G}$ and consider the natural action of $\bar{G}$ on the coset variety $\Omega = \bar{G}/\bar{H}$. For each $t \in \bar{G}$, the set of fixed points $C_\Omega(t)$ is a subvariety and

$$\mathrm{ifix}(\bar{G}) = \max\{\dim C_\Omega(t) \,:\, t \in \bar{G} \text{ is an involution}\}$$

is a natural definition for the involution fixity of $\bar{G}$ (note that $\bar{G}$ has only finitely many conjugacy classes of involutions, so this is well-defined). The algebraic group analogue of Theorem 3 is the following (in Table 3, it is convenient to set $p = \infty$ when the underlying field has characteristic zero).

Theorem 4. *Let $\bar{G}$ be a simple exceptional algebraic group over an algebraically closed field K of characteristic $p \geqslant 0$, let $\bar{H}$ be a maximal positive dimensional closed subgroup of $\bar{G}$ and consider the action of $\bar{G}$ on $\Omega = \bar{G}/\bar{H}$. Then either*

  (i) $\mathrm{ifix}(\bar{G}) \geqslant \frac{1}{2} \dim \Omega$*; or*

$\bar{G}$	$\bar{H}$	γ	Conditions
E_8	B_2	$58/119$	$p \geqslant 5$
	A_1	$17/35$	3 classes; $p \geqslant 23, 29, 31$
E_7	A_1	$31/65$	2 classes; $p \geqslant 17, 19$
F_4	G_2	$9/19$	$p = 7$
	A_1	$23/49$	$p \geqslant 13$
G_2	P_1	$2/5$	$p \neq 2$
	P_2	$2/5$	
	A_1	$5/11$	$p \geqslant 7$

TABLE 3. The constant γ in Theorem 4(ii)

(ii) $\mathrm{ifix}(\bar{G}) = \gamma \dim \Omega$ *and* $(\bar{G}, \bar{H}, \gamma)$ *is one of the cases listed in Table* 3.

The short proof combines Theorem 3 with the well known Lang-Weil estimate [30] from algebraic geometry (see Section 10 for the details). We refer the reader to [11] for some related results for simple algebraic groups.

Let us outline the main steps in the proof of Theorems 1, 2 and 3. Let G be an almost simple primitive permutation group with socle $T = (\bar{G}_\sigma)'$, an exceptional group of Lie type over $\mathbb{F}_q$ (where $q = p^f$ with p a prime), and point stabilizer H. By primitivity, H is a maximal subgroup of G with $G = HT$. The subgroup structure of finite groups of Lie type has been intensively studied in recent years, and through the work of many authors, the maximal subgroups of G can be partitioned into several naturally defined subgroup collections (in some sense, this is analogous to Aschbacher's celebrated subgroup structure theorem for classical groups [1]). This powerful reduction theorem provides the framework for the proofs of our main results, which are organised according to the type of H. Roughly speaking, one of the following holds (see Theorem 5.1):

(a) $H = N_G(\bar{H}_\sigma)$, where $\bar{H}$ is a σ-stable closed positive dimensional subgroup of $\bar{G}$;

(b) H is almost simple.

First we focus on the subgroups arising in (a), which we divide into three further subcollections, according to the structure of $\bar{H}$:

(i) Parabolic subgroups;

(ii) Reductive subgroups of maximal rank;

(iii) Reductive subgroups of smaller rank.

In Section 3, H is a maximal parabolic subgroup and we use character-theoretic methods (following [35]) to compute $\mathrm{fix}(t)$ for each involution $t \in T$ (the specific details depend heavily on the parity of the underlying characteristic p). Next, in Sections 4 and 5, we handle the reductive subgroups arising in (ii) and (iii) above. Typically, we identify a specific involution $t \in H_0$ (the possibilities for $H_0 = H \cap T$ are known) and then determine the T-class of t, which yields a lower bound

$$\mathrm{ifix}(T) \geqslant \mathrm{fix}(t) = \frac{|t^T \cap H_0|}{|t^T|} \cdot n \geqslant \frac{|t^{H_0}|}{|t^T|} \cdot n$$

(see Section 2). We can often identify the T-class of t by computing the Jordan form of t on a suitable $\bar{G}$-module V (to do this, we work with the restriction $V{\downarrow}\bar{H}^\circ$). When $p = 2$, earlier work of Lawther [32] on unipotent classes is also very useful. This approach relies

heavily on the extensive literature concerning conjugacy classes of involutions in finite groups of Lie type, some of which is briefly summarised in Section 2.

Next we handle subfield subgroups and exotic local subgroups (see [19]) in Sections 6 and 7, respectively (both cases are straightforward). Finally, to complete the proof, we turn to the remaining almost simple maximal subgroups H of G in Section 8. Let S be the socle of H and let $\mathrm{Lie}(p)$ be the set of finite simple groups of Lie type in characteristic p. Here the analysis naturally falls into two cases, according to whether or not $S \in \mathrm{Lie}(p)$. Although a complete list of maximal almost simple subgroups is not available, in general, there are strong restrictions on the possibilities that can arise (see Theorem 8.1). For each possible socle S, we choose an involution $t \in S$ so that $|t^S|$ is maximal. By assuming t is contained in the largest class of involutions in T, we obtain an upper bound on $|t^T|$ and this yields a lower bound on $\mathrm{ifix}(T)$ as above. Note that we may have very little information on the embedding of S in T, so it is difficult to determine the precise T-class of t, in general.

Finally, in Section 9 we handle a few extra cases that arise when T is a Suzuki, Ree or Steinberg triality group, and we give a proof of Theorem 4 in Section 10.

Let us say a few words on our notation. Let A and B be groups and let n and p be positive integers, with p a prime. We denote a cyclic group of order n by Z_n (or just n) and A^n is the direct product of n copies of A. In particular, p^n denotes an elementary abelian p-group of order p^n. We write $[n]$ for an unspecified solvable group of order n. A split extension of A by B will be denoted by writing $A{:}B$, while $A.B$ is an arbitrary extension. We write $A \circ B$ for a central product of A and B. If A is finite, then $\mathrm{soc}(A)$ denotes the socle of H (that is, the subgroup of A generated by its minimal normal subgroups) and $i_2(A)$ is the number of involutions in A. For matrices, it will be convenient to write $[A_1^{n_1}, \ldots, A_k^{n_k}]$ for a block-diagonal matrix with a block A_i occurring with multiplicity n_i. In addition, J_i will denote a standard unipotent Jordan block of size i.

For finite simple groups, we adopt the notation in [28]. For example, we will write $\mathrm{L}_n^+(q) = \mathrm{L}_n(q)$ and $\mathrm{L}_n^-(q) = \mathrm{U}_n(q)$ for $\mathrm{PSL}_n(q)$ and $\mathrm{PSU}_n(q)$, respectively, and similarly $E_6^+(q) = E_6(q)$ and $E_6^-(q) = {}^2E_6(q)$, etc. For a semisimple algebraic group $\bar{G}$ of rank r we write $\Phi(\bar{G})$, $\Phi^+(\bar{G})$ and $\Pi(\bar{G}) = \{\alpha_1, \ldots, \alpha_r\}$ for the set of roots, positive roots and simple roots of $\bar{G}$, with respect to a fixed Borel subgroup. We will often associate $\bar{G}$ with $\Phi(\bar{G})$ by writing $\bar{G} = A_r$, etc., and we will use $W(\bar{G})$ and $\mathcal{L}(\bar{G})$ for the Weyl group and Lie algebra of $\bar{G}$, respectively. We adopt the standard Bourbaki [10] labelling of simple roots and we will write $\{\lambda_1, \ldots, \lambda_r\}$ for the corresponding fundamental dominant weights of $\bar{G}$. If λ is a dominant weight, then $V_{\bar{G}}(\lambda)$ (or simply $V(\lambda)$) is the rational irreducible $\bar{G}$-module with highest weight λ (the trivial module will be denoted by 0 and for $\bar{G} = A_1$ we will write $V(n)$ rather than $V(n\lambda_1)$). Similarly, $W(\lambda)$ will denote the corresponding Weyl module for $\bar{G}$. Finally, if V is a $\bar{G}$-module then V^n will denote the direct sum of n copies of V. We will also use $V{\downarrow}\bar{H}$ for the restriction of V to a subgroup $\bar{H}$ of $\bar{G}$.

2. Involutions

We start by recalling some preliminary results which will be needed for the proofs of our main theorems. In particular, it will be convenient to record some of the information we need on involutions in simple exceptional groups of Lie type (in terms of conjugacy classes and centralizers); our main references are [23, Section 4.5] (in odd characteristic) and [2] (for even characteristic).

T	H_0	n	$\mathrm{ifix}(T)$	α
$G_2(2)'$	$3^{1+2}.8$	28	4	0.416
$^2G_2(3)'$	D_{14}	36	4	0.386
	D_{18}	28	4	0.416
	$2^3{:}7$	9	1	0

TABLE 4. The constant α in Proposition 2.1

Let G be an almost simple primitive permutation group of degree n with socle T and point stabilizer H (by primitivity, H is a maximal subgroup of G and $G = HT$). Set $H_0 = H \cap T$. We define

$$\mathrm{ifix}(G) = \max\{\mathrm{fix}(t) \,:\, t \in G \text{ is an involution}\}$$

to be the *involution fixity* of G, where $\mathrm{fix}(t)$ is the number of fixed points of t on Ω. It is easy to see that $n = |T : H_0|$ and

$$\mathrm{fix}(t) = \frac{|t^T \cap H_0|}{|t^T|} \cdot n \tag{2}$$

for all $t \in T$ (see [36, Lemma 2.5], for example). We will repeatedly make use of this observation throughout the paper, without further comment.

Let T be an exceptional group of Lie type over $\mathbb{F}_q$, where $q = p^f$ with p a prime. A convenient list of the relevant groups and their orders is given in [28, Table 5.1.B]. Before going any further, it will be convenient to establish Theorem 2 in the special cases where

$$T \in \{G_2(2)', {}^2G_2(3)', {}^2F_4(2)'\}. \tag{3}$$

Note that $G_2(2)' \cong \mathrm{U}_3(3)$ and $^2G_2(3)' \cong \mathrm{L}_2(8)$.

Proposition 2.1. *Let G be an almost simple primitive permutation group of degree n, with socle T and point stabilizer H. If $T \in \{G_2(2)', {}^2G_2(3)', {}^2F_4(2)'\}$ then either $\mathrm{ifix}(T) > n^{4/9}$, or (T, H_0) is one of the cases in Table 4 and $\mathrm{ifix}(T) \geqslant n^\alpha$ for the given constant α.*

Proof. This is a straightforward MAGMA [9] calculation, using a suitable permutation representation of G given in the Web Atlas [61] (indeed, it is easy to compute the exact value of $\mathrm{ifix}(T)$ in every case). $\qquad\square$

For the remainder of the paper, we may assume that T is not one of the groups in (3). Write $T = (\bar{G}_\sigma)'$, where $\bar{G}$ is a simple adjoint algebraic group over the algebraic closure $K = \bar{\mathbb{F}}_q$ and $\sigma : \bar{G} \to \bar{G}$ is an appropriate Steinberg endomorphism. Note that $\bar{G}_\sigma = \mathrm{Inndiag}(T)$ is the group of inner-diagonal automorphisms of T (see [23, Definition 2.5.10]). Given the formula in (2), we require information on the conjugacy classes of involutions in T and we will need to understand the fusion of H_0-classes in T. This analysis falls naturally into two cases, according to the parity of p (indeed, involutions are unipotent when $p = 2$, and semisimple when $p \neq 2$).

There is an extensive literature on conjugacy classes in exceptional groups, both finite and algebraic. First assume $p = 2$. In [2, Sections 12-18], Aschbacher and Seitz provide detailed information on the classes of involutions in simple groups of Lie type, in terms of representatives and centralizer structure. For exceptional groups, we will use the notation of Liebeck and Seitz [44, Tables 22.2.1–22.2.7] for class representatives, which is consistent with the Bala-Carter labelling of unipotent classes (since we are interested in involutions, we need only consider the classes whose Bala-Carter label has only A_1 or $\tilde{A}_1$ components).

For instance, $F_4(q)$ has four classes of involutions, labelled as follows:

$$A_1, \ \tilde{A}_1, \ (\tilde{A}_1)_2, \ A_1\tilde{A}_1$$

(see [44, Table 22.2.4]). Note that the labels A_1 and $\tilde{A}_1$ indicate long and short root elements, respectively. The twisted groups ${}^3D_4(q)$ and ${}^2B_2(q)$ are not included in the tables in [44, Section 22], but it is well known that ${}^3D_4(q)$ has two classes of involutions and ${}^2B_2(q)$ has a unique class (see [58, 59], for example). For $p \neq 2$, we refer the reader to [23, Sections 4.3-4.6] for an in-depth treatment of involutions in groups of Lie type (both finite and algebraic). We will use the notation of [23, Table 4.5.1] for class representatives.

Remark 2.2. Let $t \in \mathrm{Inndiag}(T)$ be an involution. For $p = 2$ we observe that t lies in T and $t^T = t^{\mathrm{Inndiag}(T)}$. Similarly, [23, Theorem 4.2.2(j)] implies that $t^T = t^{\mathrm{Inndiag}(T)}$ when $p \neq 2$, but note that if $T = E_7(q)$ then there are involutions in $\mathrm{Inndiag}(T)$ that do not lie in T (see Remark 2.4 below). In all characteristics, we find that there is a 1-1 correspondence between conjugacy classes of involutions in $\bar{G}$ and those in T.

Most of the information we need on involutions in exceptional groups is summarised in the following proposition.

Proposition 2.3. *For each finite simple exceptional group of Lie type T, Table 5 lists a complete set of representatives of the T-classes of involutions $t \in T$, and each row of the table gives the order of the centralizer $C_{\mathrm{Inndiag}(T)}(t)$.*

Remark 2.4. In the third column of Table 5 we present a representative of each T-class of involutions in T. The case $T = E_7(q)$ with q odd requires special attention. Here $\mathrm{Inndiag}(T) = T.2$ and the ambient algebraic group $\bar{G} = E_7$ has three classes of involutions, with centralizers A_1D_6, $A_7.2$ and $T_1E_6.2$ (here T_1 denotes a 1-dimensional torus). The latter two classes each give rise to two $\mathrm{Inndiag}(T)$-classes of involutions, one in T and the other in $\mathrm{Inndiag}(T) \setminus T$, so in total there are five classes of involutions in $\mathrm{Inndiag}(T)$, but only three in T. More precisely, using the notation in Table 5, the involutions in T are as follows (see [23, Table 4.5.1]):

$$\begin{cases} t_1, t_4, t_7 & \text{if } q \equiv 1 \pmod 4 \\ t_1, t_4', t_7' & \text{if } q \equiv 3 \pmod 4 \end{cases}$$

In the final column of the table, we give the order of the centralizer of t in $\mathrm{Inndiag}(T)$. To avoid any possible confusion, note that this is always a monic polynomial in q, with the exception of the elements $t_4, t_4', t_7, t_7' \in E_7(q)$ with q odd, where the order is twice a monic polynomial.

We will also need some information on involutions in the low-rank classical groups. Let L be a classical group over $\mathbb{F}_q$ with natural module V of dimension n. First assume $p = 2$. Unipotent involutions in L are essentially determined up to conjugacy by their Jordan form on V, which is of the form $[J_2^s, J_1^{n-2s}]$ for some $s \geqslant 1$ (if s is even, there are two such classes when L is a symplectic or orthogonal group, denoted by the labels a_s and c_s in [2]). We will frequently use the notation from [2] for class representatives.

We will often be interested in the case where $x \in L$ is a long root element (that is, an element with Jordan form $[J_2^s, J_1^{n-2s}]$, where $s = 2$ if $L = \Omega_n^\epsilon(q)$ is an orthogonal group (an a_2 involution in the notation of [2]), and $s = 1$ in the other cases); the conjugacy class size of such an element is as follows (for example, see the proof of [12, Proposition 3.22]):

$$|x^L| = \begin{cases} (q^{n-1} - \epsilon^{n-1})(q^n - \epsilon^n)/(q - \epsilon) & L = \mathrm{L}_n^\epsilon(q) \\ q^n - 1 & L = \mathrm{Sp}_n(q) \\ (q^{n/2-2} + \epsilon)(q^{n-2} - 1)(q^{n/2} - \epsilon)/(q^2 - 1) & L = \Omega_n^\epsilon(q), \ n \geqslant 8 \text{ even} \end{cases}$$

We will refer to long root elements as A_1-involutions (in the notation of [2], these are b_1-involutions when $L = \mathrm{Sp}_n(q)$ and a_2-involutions when $L = \Omega_n^\epsilon(q)$), which is consistent with our usage for exceptional groups.

For p odd, the conjugacy class of an involution in L is essentially determined by its eigenvalues (in a suitable splitting field) on the natural module V. We refer the reader to [23, Table 4.5.1] for information on class representatives and centralizers (the relevant details are conveniently summarised in [13, Appendix B]).

3. Parabolic subgroups

Let G be an almost simple primitive permutation group of degree n, with socle T and point stabilizer H, where T is a group of exceptional Lie type over $\mathbb{F}_q$ with $q = p^f$ for a prime p. In this section, we will use character-theoretic methods to study the involution fixity of T in the special case where H is a maximal parabolic subgroup of G (this approach was first introduced in [35]). We adopt the standard notation $P_{i,j...}$ for the parabolic subgroup of T obtained by removing the nodes $i, j, \ldots$ from the corresponding Dynkin diagram of T, in terms of the standard Bourbaki labelling in [10]. Set $H_0 = H \cap T$.

Remark 3.1. In view of Proposition 2.1, throughout this section we will assume T is not one of the groups in (3).

Our main result for parabolic actions is the following, which establishes a strong form of Theorem 2 (and thus Theorem 1) in this special case. In Table 6, τ denotes a graph or graph-field automorphism of T.

Theorem 3.2. *Let G be an almost simple primitive permutation group of degree n with socle T, an exceptional group of Lie type over $\mathbb{F}_q$, and point stabilizer H, a maximal parabolic subgroup of G. Then one of the following holds:*

(i) $T = {}^2B_2(q)$, $H_0 = q^{1+1}{:}(q-1)$ *and* $\mathrm{ifix}(T) = 1$;

(ii) $\mathrm{ifix}(T) > n^{1/2}$ *and*

$$\liminf_{q\to\infty} \frac{\log \mathrm{ifix}(T)}{\log n} \geqslant \frac{1}{2};$$

(iii) (T, H_0) *is one of the cases in Table 6 and we have* $\mathrm{ifix}(T) > n^\gamma$ *and*

$$\liminf_{q\to\infty} \frac{\log \mathrm{ifix}(T)}{\log n} = \gamma,$$

for the given constant γ.

In particular, if (T, H_0) is in Table 6, then $\mathrm{ifix}(T) \leqslant n^{4/9}$ *unless*

(a) $(T, H_0) = ({}^3D_4(q), P_2)$, $({}^2F_4(q), P_{2,3})$; *or*

(b) $T = G_2(q)$ *and* $q < 7$.

Write $T = (\bar{G}_\sigma)'$, where $\bar{G}$ is a simple algebraic group over $K = \bar{\mathbb{F}}_q$ and σ is an appropriate Steinberg endomorphism of $\bar{G}$. Then $H = N_G(P_\sigma)$, where P is a σ-stable parabolic subgroup of $\bar{G}$, and we can consider the permutation character $\chi = 1_{P_\sigma}^{\bar{G}_\sigma}$, which gives $\chi(t) = \mathrm{fix}(t)$ for all $t \in T$. Let $W = W(\bar{G})$ be the Weyl group of $\bar{G}$. We partition the proof of Theorem 3.2 into two cases, according to the parity of p.

3.1. **Unipotent involutions.** First assume $p = 2$ and T is untwisted, so

$$T \in \{E_8(q), E_7(q), E_6(q), F_4(q), G_2(q)\}.$$

T	q	t	$|C_{\mathrm{Inndiag}(T)}(t)|$
$E_8(q)$	odd	t_1	$|\mathrm{SO}_{16}^+(q)|$
		t_8	$|\mathrm{SL}_2(q)||E_7(q)|$
	even	A_1	$q^{57}|E_7(q)|$
		A_1^2	$q^{78}|\mathrm{Sp}_{12}(q)|$
		A_1^3	$q^{81}|\mathrm{SL}_2(q)||F_4(q)|$
		A_1^4	$q^{84}|\mathrm{Sp}_8(q)|$
$E_7(q)$	odd	t_1	$|\mathrm{SL}_2(q)||\mathrm{SO}_{12}^+(q)|$
		t_4, t_4'	$2|\mathrm{SL}_8^\epsilon(q)|^\dagger$
		t_7, t_7'	$2(q-\epsilon)|E_6^\epsilon(q)|^\dagger$
	even	A_1	$q^{33}|\Omega_{12}^+(q)|$
		A_1^2	$q^{42}|\mathrm{SL}_2(q)||\mathrm{Sp}_8(q)|$
		$(A_1^3)^{(1)}$	$q^{27}|F_4(q)|$
		$(A_1^3)^{(2)}$	$q^{45}|\mathrm{SL}_2(q)||\mathrm{Sp}_6(q)|$
		A_1^4	$q^{42}|\mathrm{Sp}_6(q)|$
$E_6^\epsilon(q)$	odd	t_1	$(q-\epsilon)|\mathrm{SO}_{10}^\epsilon(q)|$
		t_2	$|\mathrm{SL}_2(q)||\mathrm{SL}_6^\epsilon(q)|$
	even	A_1	$q^{21}|\mathrm{SL}_6^\epsilon(q)|$
		A_1^2	$q^{24}(q-\epsilon)|\mathrm{Sp}_6(q)|$
		A_1^3	$q^{27}|\mathrm{SL}_2(q)||\mathrm{SL}_3^\epsilon(q)|$
$F_4(q)$	odd	t_1	$|\mathrm{SL}_2(q)||\mathrm{Sp}_6(q)|$
		t_4	$|\mathrm{SO}_9(q)|$
	even	A_1	$q^{15}|\mathrm{Sp}_6(q)|$
		$\tilde{A}_1$	$q^{15}|\mathrm{Sp}_6(q)|$
		$(\tilde{A}_1)_2$	$q^{20}|\mathrm{Sp}_4(q)|$
		$A_1\tilde{A}_1$	$q^{18}|\mathrm{SL}_2(q)|^2$
$G_2(q), q > 2$	odd	t_1	$|\mathrm{SL}_2(q)|^2$
	even	A_1	$q^5|\mathrm{SL}_2(q)|$
		$\tilde{A}_1$	$q^3|\mathrm{SL}_2(q)|$
$^3D_4(q)$	odd	t_2	$|\mathrm{SL}_2(q)||\mathrm{SL}_2(q^3)|$
	even	A_1	$q^9|\mathrm{SL}_2(q^3)|$
		A_1^3	$q^9|\mathrm{SL}_2(q)|$
$^2F_4(q), q > 2$	even	$(\tilde{A}_1)_2$	$q^{10}|^2B_2(q)|$
		$A_1\tilde{A}_1$	$q^9|\mathrm{SL}_2(q)|$
$^2G_2(q), q > 3$	odd	t_1	$|\mathrm{SL}_2(q)|$
$^2B_2(q)$	even	$(\tilde{A}_1)_2$	q^2

† Here $q \equiv \epsilon \pmod 4$; see Remark 2.4

TABLE 5. Involution classes in finite simple exceptional groups of Lie type

According to [35, Lemma 2.4], the permutation character χ can be decomposed as a sum

$$\chi = \sum_{\phi \in \widehat{W}} n_\phi R_\phi, \tag{4}$$

where $\widehat{W}$ is the set of complex irreducible characters of W. Here the R_ϕ are almost characters of $\bar{G}_\sigma$ and the coefficients are given by $n_\phi = \langle 1_{W_P}^W, \phi \rangle$, where W_P is the corresponding

T	H_0	γ	Conditions
$G_2(q)$	P_1	$2/5$	q odd
	P_2	$2/5$	
	$P_{1,2}$	$1/3$	$p = 3$, $q \geqslant 9$, $\tau \in G$
$^3D_4(q)$	P_2	$4/9$	
	$P_{1,3,4}$	$4/11$	q odd
$^2F_4(q)$	$P_{2,3}$	$5/11$	
$^2G_2(q)$	$q^{1+1+1}{:}(q-1)$	$1/3$	

TABLE 6. The parabolic actions with $1 < \mathrm{ifix}(T) \leqslant n^{1/2}$

parabolic subgroup of W. The restriction of the R_ϕ to unipotent elements yields the Green functions of $\bar{G}_\sigma$, which are polynomials in q with non-negative coefficients. Lübeck [48] has implemented an algorithm of Lusztig [49] to compute the relevant Green functions and this allows us to calculate $\mathrm{fix}(t)$ for every involution $t \in T$. We refer the reader to [35, Section 2] for further details.

In the following proposition, we assume T is untwisted and H_0 is a maximal subgroup of T. Note that the latter condition simply means that we exclude the cases

$$(T, H_0) \in \{(E_6(q), P_{1,6}), (E_6(q), P_{3,5}), (F_4(q), P_{1,4}), (F_4(q), P_{2,3})\},$$

which can arise when G contains a graph or graph-field automorphism of T.

Proposition 3.3. *Suppose $p = 2$, T is untwisted and H is a maximal parabolic subgroup of G such that H_0 is maximal in T. Then $\mathrm{ifix}(T) > n^\alpha$ and*

$$\lim_{q\to\infty,\, q \text{ even}} \frac{\log \mathrm{ifix}(T)}{\log n} = \alpha \tag{5}$$

where α is given in Table 7. In particular, $\mathrm{ifix}(T) \leqslant n^{1/2}$ if and only if $T = G_2(q)$ and $H_0 = P_2$, in which case $\mathrm{ifix}(T) \leqslant n^{4/9}$ if $q \geqslant 8$.

Proof. First observe that the decomposition in (4) is given in [35, pp.413–415] (with respect to Carter's labelling of irreducible characters in [18, Section 13.2]). Let $t \in T$ be an A_1-involution (see Table 5). As noted above, we can use [48] to compute $\chi(t) = \mathrm{fix}(t)$ and verify the lower bound on $\mathrm{ifix}(T)$. Alternatively, since t is a long root element, we can use [35, Proposition 2.1], which does not require [48].

For example, suppose $T = E_6(q)$ and $H_0 = P_2$. Here $|P_\sigma| = q^{21}|\mathrm{GL}_6(q)|$, so

$$n = |\bar{G}_\sigma : P_\sigma| = (q^6 + q^3 + 1)(q^6 + 1)(q^4 + 1)(q^3 + 1)(q^2 + q + 1)$$

and [35, p.414] gives

$$\chi = R_{\phi_{1,0}} + R_{\phi_{6,1}} + R_{\phi_{20,2}} + R_{\phi_{30,3}} + R_{\phi_{15,4}}.$$

Note that T has three classes of involutions, labelled A_1, A_1^2 and A_1^3 (see Table 5).

For an A_1-involution $t \in T$, using [48], we compute

$$\chi(t) = q^{15} + 2q^{14} + 3q^{13} + 4q^{12} + 5q^{11} + 6q^{10} + 5q^9 + 5q^8 + 5q^7 + 4q^6 + 3q^5 + 3q^4 + 2q^3 + q^2 + q + 1$$

and thus $\mathrm{ifix}(T) > n^\alpha$ with $\alpha = 5/7$. Alternatively, in the notation of [35, Proposition 2.1], we have $P_0 = P$, $lht(\alpha_0) = 11$, $L' = A_5$ and we get

$$\chi(t) = q^{11}(q^{-11} + q^{-10} + q^{-9} + 2q^{-8} + 3q^{-7} + 3q^{-6} + 4q^{-5} + 5q^{-4} + 5q^{-3} + 5q^{-2} + 6q^{-1}) + q^{10}(q^5 + 2q^4 + 3q^3 + 4q^2 + 5q),$$

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
$E_8(q)$	$\frac{10}{13}, \frac{43}{78}$	$\frac{35}{46}, \frac{51}{92}$	$\frac{75}{98}, \frac{27}{49}$	$\frac{81}{106}, \frac{29}{53}$	$\frac{79}{104}, \frac{57}{104}$	$\frac{75}{97}, \frac{53}{97}$	$\frac{65}{83}, \frac{47}{83}$	$\frac{15}{19}, \frac{11}{19}$
$E_7(q)$	$\frac{25}{33}, \frac{7}{11}$	$\frac{31}{42}, \frac{11}{21}$	$\frac{35}{47}, \frac{29}{47}$	$\frac{39}{53}, \frac{31}{53}$	$\frac{37}{50}, \frac{13}{25}$	$\frac{16}{21}, \frac{4}{7}$	$\frac{7}{9}, \frac{5}{9}$	
$E_6(q)$	$\frac{3}{4}, \frac{5}{8}$	$\frac{5}{7}, \frac{13}{21}$	$\frac{18}{25}, \frac{3}{5}$	$\frac{20}{29}, \frac{17}{29}$	$\frac{18}{25}, \frac{3}{5}$	$\frac{3}{4}, \frac{5}{8}$		
$F_4(q)$	$\frac{11}{15}, \frac{11}{15}$	$\frac{7}{10}, \frac{7}{10}$	$\frac{7}{10}, \frac{13}{20}$	$\frac{11}{15}, \frac{2}{3}$				
$G_2(q)$	$\frac{3}{5}, \frac{2}{5}$	$\frac{2}{5}, \frac{2}{5}$						

TABLE 7. The constants α, β in Propositions 3.3 and 3.6

which agrees with the expression above. By repeating the calculation for involutions in the classes labelled A_1^2 and A_1^3, we deduce that $\mathrm{ifix}(T) = \mathrm{fix}(t)$ when t is an A_1-involution and thus (5) holds. (In fact, one can show that if t is an A_1-involution, then $\mathrm{fix}(t) \geqslant \mathrm{fix}(u)$ for all non-trivial unipotent elements $u \in T$; see [35, p.413].)

The case $T = G_2(q)$ with $H_0 = P_2$ requires special attention. Here $q \geqslant 4$ (see Remark 3.1), $n = (q^4 + q^2 + 1)(q + 1)$ and T has two classes of involutions, labelled A_1 and $\tilde{A}_1$ in Table 5, corresponding to long and short root elements, respectively. For $t \in A_1$ we get $\mathrm{fix}(t) = q^2 + q + 1$, whereas $\mathrm{fix}(t) = q^2 + 2q + 1$ for $t \in \tilde{A}_1$. We conclude that $n^{2/5} < \mathrm{ifix}(T) < n^{1/2}$ and $\alpha = 2/5$ in (5). Note that if $q = 4$ then

$$\frac{\log \mathrm{ifix}(T)}{\log n} = \frac{\log 25}{\log 1365} > 0.445,$$

whereas $\mathrm{ifix}(T) < n^{4/9}$ for $q \geqslant 8$.

The remaining cases are similar and we omit the details. It is worth noting that in every case, we find that $\mathrm{fix}(t)$ is maximal when t is an A_1-involution, unless $T = F_4(q)$ and $H_0 = P_1$ or P_2, for which short root elements (that is, $\tilde{A}_1$-involutions) have the largest number of fixed points. □

Next we assume $T = E_6(q)$ or $F_4(q)$, and H_0 is a non-maximal subgroup of T.

Proposition 3.4. *Suppose $p = 2$ and (T, H_0) is one of the cases in Table 9. Then $\mathrm{ifix}(T) > n^\alpha$ and (5) holds, where α is given in the third column of the table.*

Proof. We proceed as in the proof of the previous proposition. First we express χ as a sum of almost characters, as in (4); see Table 8. Next we use [48] to compute $\chi(t)$ for an A_1-involution $t \in T$. For example, if $T = F_4(q)$ and $H_0 = P_{2,3}$ then

$$n = \frac{(q^{12} - 1)(q^8 - 1)(q^4 + q^2 + 1)}{(q - 1)^2}$$

and

$$\chi(t) = q^{15} + 3q^{14} + 6q^{13} + 10q^{12} + 14q^{11} + 17q^{10} + 19q^9 + 19q^8 + 18q^7 + 15q^6$$
$$+ 12q^5 + 9q^4 + 6q^3 + 4q^2 + 2q + 1,$$

which gives $\mathrm{ifix}(T) > n^{15/22}$.

Alternatively, as in the proof of the previous proposition, we can use [35, Proposition 2.1] to compute $\chi(t)$ for an A_1-involution t. In the notation of [35, Section 2], we have $P_0 = P_1$, $lht(\alpha_0) = 8$, $L' = A_1 \tilde{A}_1$ and we get

$$\chi(t) = \frac{|(P_0)_\sigma|}{|P_\sigma|} \cdot (q^8(q^{-8} + q^{-7} + q^{-6} + q^{-5} + 2q^{-4} + 2q^{-3} + 2q^{-2} + 2q^{-1}) + q^8)$$

T	H_0	$1^W_{W_P}$
$E_6(q)$	$P_{1,6}$	$\phi_{1,0} + 2\phi_{6,1} + \phi_{15,5} + 3\phi_{20,2} + \phi_{24,6} + \phi_{30,3} + 2\phi_{64,4}$
	$P_{3,5}$	$\phi_{1,0} + 2\phi_{6,1} + \phi_{10,9} + 3\phi_{15,4} + \phi_{15,5} + 5\phi_{20,2} + 2\phi_{24,6} + 4\phi_{30,3} + \phi_{60,11} + 6\phi_{60,5}$
		$+3\phi_{60,8} + 6\phi_{64,4} + 3\phi_{80,7} + 4\phi_{81,6} + \phi_{81,10} + 2\phi_{90,8}$
$F_4(q)$	$P_{1,4}$	$\phi_{1,0} + \phi'_{2,4} + \phi''_{2,4} + \phi_{4,8} + 2\phi_{4,1} + \phi'_{6,6} + 2\phi'_{8,3} + 2\phi''_{8,3} + \phi'_{9,6} + 3\phi_{9,2} + \phi''_{9,6}$
		$+\phi_{12,4} + 2\phi_{16,5}$
	$P_{2,3}$	$\phi_{1,0} + \phi'_{2,4} + \phi''_{2,4} + \phi'_{4,7} + \phi''_{4,7} + \phi_{4,8} + 2\phi_{4,1} + \phi''_{6,6} + 2\phi'_{6,6} + 3\phi'_{8,3} + \phi'_{8,9}$
		$+3\phi''_{8,3} + \phi''_{8,9} + 2\phi'_{9,6} + 4\phi_{9,2} + 2\phi''_{9,6} + \phi_{9,10} + 3\phi_{12,4} + 4\phi_{16,5}$

TABLE 8. The decomposition of χ in (4)

T	H_0	α	β
$E_6(q)$	$P_{1,6}$	$3/4$	$7/12$
	$P_{3,5}$	$22/31$	$18/31$
$F_4(q)$	$P_{1,4}$	$7/10$	$-$
	$P_{2,3}$	$15/22$	$-$
$G_2(q)$	$P_{1,2}$	$-$	$1/3$

TABLE 9. The constants α, β in Propositions 3.4 and 3.6

with

$$\frac{|(P_0)_\sigma|}{|P_\sigma|} = (q^5 + q^4 + q^3 + q^2 + q + 1)(q^2 + 1),$$

which is in agreement with the above expression. By carrying out a similar calculation for elements in the other three involution classes, one checks that $\mathrm{fix}(t)$ is maximal when t is an A_1-involution and the result follows.

The other cases are very similar and we omit the details. $\square$

To complete the proof of Theorem 3.2 for unipotent involutions, we handle the twisted groups $T \in \{{}^2B_2(q), {}^2G_2(q), {}^2F_4(q), {}^2E_6(q)\}$.

Proposition 3.5. *Suppose $p = 2$, T is twisted and H is a maximal parabolic subgroup of G. Then one of the following holds:*

(i) *$T = {}^2B_2(q)$ and $\mathrm{ifix}(T) = 1$;*

(ii) *$(T, H_0, \alpha) = ({}^3D_4(q), P_2, 4/9)$ or $({}^2F_4(q), P_{2,3}, 5/11)$, where $n^\alpha < \mathrm{ifix}(T) \leqslant n^{1/2}$ and α is the constant in (5);*

(iii) *$\mathrm{ifix}(T) > n^{1/2}$ and (5) holds with α given in Table 10.*

Proof. For $T = {}^2E_6(q)$ we proceed as in the proofs of the previous two propositions, using [48] and the decomposition of χ in (4) (see [15, p.125]). We omit the details. If $T = {}^2B_2(q)$ then $\chi = 1 + \psi$, where ψ is the Steinberg character of T. Since $\psi(x) = 0$ for all non-identity unipotent elements $x \in T$, we deduce that $\mathrm{ifix}(T) = 1$.

Next assume that $T = {}^2F_4(q)$ with $q \geqslant 8$ (recall that we are excluding the case $T = {}^2F_4(2)'$, which was handled in Proposition 2.1), so $H_0 = P_{1,4}$ or $P_{2,3}$, and we have

$$P_{1,4} = [q^{10}]{:}({}^2B_2(q) \times (q-1)), \quad P_{2,3} = [q^{11}]{:}\mathrm{GL}_2(q).$$

T	H_0	α	β
$^2E_6(q)$	$P_{1,6}$	3/4	7/12
	P_2	13/21	13/21
	$P_{3,5}$	17/31	17/31
	P_4	17/29	17/29
$^2F_4(q)$	$P_{1,4}$	3/5	−
	$P_{2,3}$	5/11	−
$^3D_4(q)$	$P_{1,3,4}$	7/11	4/11
	P_2	4/9	4/9
$^2G_2(q)$	$P_{1,2}$	−	1/3

TABLE 10. The constants α, β in Propositions 3.5 and 3.6

The decomposition of χ as a sum of unipotent characters is given in [35, p.416], so we can use [50] to compute $\mathrm{fix}(t)$ for any involution $t \in T$. For example, if $H_0 = P_{1,4}$ then

$$n = (q^6 + 1)(q^3 + 1)(q + 1)$$

and

$$\chi = \chi_1 + \chi_2 + \chi_9 + \chi_{10} + \chi_{11}$$

in the notation of [50, Tabelle 1]. There are two T-classes of involutions, with representatives labelled u_1 and u_2 in [50] (in terms of Table 5, $u_1 = (\tilde{A}_1)_2$ and $u_2 = A_1\tilde{A}_1$). We calculate

$$\chi(u_1) = 1 + q(q^3 - q + 1) + \frac{1}{4}q^2(q - \sqrt{2q} + 1)(2q^3 + \sqrt{2q^5} + \sqrt{2q^3} + q + 1)$$

$$+ \frac{1}{4}q^2(q + \sqrt{2q} + 1)(2q^3 - \sqrt{2q^5} - \sqrt{2q^3} + q + 1) + \frac{1}{2}q^2(q^2 + 1)$$

$$= q^6 + q^4 + q^3 + q + 1$$

and $\chi(u_2) = q^4 + q^3 + q + 1$. Therefore

$$\frac{\log \mathrm{ifix}(T)}{\log n} \geqslant \frac{\log 266761}{\log 1210323465} > 0.597$$

and $\alpha = 3/5$ in (5). Similarly, if $H_0 = P_{2,3}$ then $\mathrm{ifix}(T) = q^5 + q^4 + q^3 + q^2 + 1 > n^{5/11}$ and (5) holds with $\alpha = 5/11$.

Finally, let us turn to the case $T = {}^3D_4(q)$, so $H_0 = P_{1,3,4}$ or P_2, where

$$P_{1,3,4} = [q^{11}]{:}((q^3 - 1) \times \mathrm{SL}_2(q)), \quad P_2 = q^{1+8}{:}((q - 1) \times \mathrm{SL}_2(q^3)).$$

There are two classes of involutions in T, with representatives labelled u_1 and u_2 (these correspond to the classes labelled A_1 and A_1^3 in Table 5). First assume $H_0 = P_{1,3,4}$, so

$$n = (q^8 + q^4 + 1)(q^3 + 1)$$

In the notation of [58], we have $\chi = 1 + \rho_1 + \rho_2 + \epsilon_2$ (see [35, p.416]) and by inspecting [58, Table 2] we calculate

$$\chi(u_1) = q^7 + q^4 + q^3 + 1, \quad \chi(u_2) = q^4 + q^3 + 1$$

Therefore $\mathrm{ifix}(T) > n^{7/11}$ and $\alpha = 7/11$ in (5). Similarly, if $H_0 = P_2$ then $\chi = 1 + \rho_1 + \rho_2 + \epsilon_1$, $\mathrm{ifix}(T) = q^4 + q^3 + q + 1 > n^{4/9}$ and we have $\alpha = 4/9$ in (5). $\qquad\square$

3.2. **Semisimple involutions.** Now assume p is odd. Once again, we can use character-theoretic methods to compute $\mathrm{fix}(t)$ for every involution $t \in T$. The key tool to do this is [35, Corollary 3.2]. The following result completes the proof of Theorem 3.2.

Proposition 3.6. *Suppose $p \neq 2$ and H is a maximal parabolic subgroup of G. Then*

$$\liminf_{q \to \infty, q \text{ odd}} \frac{\log \mathrm{ifix}(T)}{\log n} = \beta \tag{6}$$

with β as in Tables 7, 9 and 10, and either

  (i) $\mathrm{ifix}(T) > n^{1/2}$; *or*

  (ii) (T, H_0) *is one of the cases in Table 6 and $\mathrm{ifix}(T) > n^{\gamma}$ for the given constant γ.*

Moreover, if (T, H_0) is in Table 6, then $\mathrm{ifix}(T) \leqslant n^{4/9}$ unless $(T, H_0) = (^3D_4(q), P_2)$, or $T = G_2(q)$ and $q \in \{3, 5\}$.

Proof. First assume $T = {}^2G_2(q)$, so $H_0 = q^{1+1+1}{:}(q-1)$, T has a unique class of involutions (with centralizer $2 \times \mathrm{L}_2(q)$) and $\chi = 1 + \psi$, where ψ is the Steinberg character. Therefore, for any involution $t \in T$ we have $\chi(t) = 1 + |C_T(t)|_p = 1 + q$ and we conclude that $n^{1/3} < \mathrm{ifix}(T) < n^{4/9}$ and $\beta = 1/3$ in (6).

Next assume $T = {}^3D_4(q)$. Here T has $q^8(q^8 + q^4 + 1)$ involutions, which form a single conjugacy class t^T. By arguing as in the proof of Proposition 3.5, using [58] (see the second column in the table on p.689) with the given decomposition of χ into unipotent characters, we calculate that

$$\mathrm{ifix}(T) = \begin{cases} q^4 + 2q^3 + q + 2 & \text{if } H_0 = P_{1,3,4} \\ q^4 + q^3 + 2q + 2 & \text{if } H_0 = P_2 \end{cases}$$

It follows that $n^{4/9} < \mathrm{ifix}(T) < n^{1/2}$ and $\beta = 4/9$ if $H_0 = P_2$. Similarly, if $H_0 = P_{1,3,4}$ then $n^{4/11} < \mathrm{ifix}(T) < n^{4/9}$ and $\beta = 4/11$.

For the remainder, we may assume that either T is untwisted or $T = {}^2E_6(q)$. Here we use the formula for $\chi(t)$ in [35, Corollary 3.2], which can be evaluated for any semisimple element $t \in T$ with the aid of MAGMA. This allows us to compute $\mathrm{ifix}(T)$ precisely.

As before, let W be the Weyl group of $\bar{G}$ and let $W_P \leqslant W$ be the Weyl group of the corresponding parabolic subgroup P of $\bar{G}$. Let w_0 be the longest word in W and set $w^* = w_0$ if $T = {}^2E_6(q)$, otherwise $w^* = 1$. Let $\Pi(\bar{G}) = \{\alpha_1, \ldots, \alpha_r\}$ be the simple roots of $\bar{G}$ and let $\alpha_0 \in \Phi(\bar{G})$ be the highest root. We recall that the semisimple classes in $\bar{G}_\sigma$ are parametrised by pairs $(J, [w])$, where J is a proper subset of $\Pi(\bar{G}) \cup \{\alpha_0\}$ (determined up to W-conjugacy), W_J is the subgroup of W generated by the reflections in the roots in J, and $[w] = W_J w$ is a conjugacy class representative of $N_W(W_J)/W_J$. Let us also recall that the $\bar{G}_\sigma$-classes of σ-stable maximal tori of $\bar{G}$ are parametrised by the σ-conjugacy classes in W (two elements $x, y \in W$ are σ-conjugate if there exists $z \in W$ such that $y = z^{-1}x\sigma(z)$). Let $C_1, \ldots, C_k$ be the conjugacy classes of W, so $C_i w^*$ is the i-th σ-conjugacy class and let T_i be a representative of the corresponding $\bar{G}_\sigma$-class of maximal tori of $\bar{G}_\sigma$. In addition, let r_i be the relative rank of T_i (that is, r_i is the multiplicity of $q - \epsilon$ as a divisor of $|T_i|$, where $\epsilon = -1$ if $T = {}^2E_6(q)$, otherwise $\epsilon = 1$).

To illustrate our approach, we will focus on the following three cases (the other cases are very similar and we omit the details):

  (a) $T = F_4(q)$ and $H_0 = P_1$.
  (b) $T = {}^2E_6(q)$ and $H_0 = P_{1,6}$.
  (c) $T = E_7(q)$ and $H_0 = P_2$.

First consider (a). Let $t \in T$ be a t_4-involution, so $|(C_{\bar{G}}(t)^\circ)_\sigma| = |\mathrm{SO}_9(q)|$ (see Table 5). In terms of the above notation, we have $J = \{\alpha_1, \alpha_2, \alpha_3, \alpha_0\}$ and $w = 1$. By applying [35,

Corollary 3.2] we get

$$\chi(t) = \sum_{i=1}^{k} \frac{|W|}{|C_i|} \cdot \frac{|W_P \cap C_i w^*|}{|W_P|} \cdot \frac{|W_J \cap C_i|}{|W_J|} \cdot \frac{(-1)^{r_i} |(C_{\bar{G}}(t)^\circ)_\sigma|_{p'}}{|T_i|} \qquad (7)$$

with $w^* = 1$. With the aid of MAGMA, we calculate

$$\chi(t) = (q^4 + q^2 + 1)(q^4 + 1)(q^2 + 1)(q + 1)$$

and we conclude that $\text{ifix}(T) > n^{11/15}$. Moreover, one checks that $\text{ifix}(t_4) > \text{ifix}(t_1)$, so $\beta = 11/15$ in (6). Similarly, in (b) we take a t_1-involution $t \in T$, so $|(C_{\bar{G}}(t)^\circ)_\sigma| = (q+1)|\mathrm{SO}_{10}^-(q)|$, $J = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_0\}$ and $w = 1$. Then (7) holds (with $w^* = w_0$) and we get

$$\chi(t) = (q^7 + q^4 + q^3 + q + 2)(q^5 + 1)(q^2 + 1),$$

which gives the desired result (with $\beta = 7/12$).

Finally, let us consider case (c). First assume $q \equiv 1 \pmod 4$ and let $t \in T$ be a t_7 involution, so $|(C_{\bar{G}}(t)^\circ)_\sigma| = (q-1)|E_6(q)|$, $J = \{a_1, \ldots, \alpha_6\}$, $w = 1$ and [35, Corollary 3.2] gives

$$\chi(t) = \sum_{i=1}^{k} \frac{|W|}{|C_i|} \cdot \frac{|W_P \cap C_i|}{|W_P|} \cdot \frac{|W_J \cap C_i|}{|W_J|} \cdot \frac{(-1)^{r_i+1} |(C_{\bar{G}}(t)^\circ)_\sigma|_{p'}}{|T_i|}$$
$$= 2(q^6 + q^3 + 1)(q^6 + 1)(q^4 + q^2 + 2)(q^4 + 1)(q^3 + 1)(q^2 + q + 1)$$

This implies that $\text{ifix}(T) > n^{25/42}$. Now assume $q \equiv 3 \pmod 4$, so the involutions in T are of type t_1, t_4' and t_7' (see Table 5). One checks that $\text{fix}(t) = 0$ when $t = t_4'$ or t_7', so let t be a t_1-involution. Then $J = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_0\}$, $w = 1$ and using [35, Corollary 3.2] we calculate

$$\chi(t) = (q^6 + 2q^4 + q^3 + 2q^2 + 3)(q^4 - q^3 + q^2 - q + 1)(q^4 + 1)(q^3 + 1)(q^2 + 1)(q + 1)^3$$

and this yields $\text{ifix}(T) > n^{11/21}$. In particular, $\beta = 11/21$ in (6). $\square$

4. MAXIMAL RANK SUBGROUPS

As in the previous section, let $T = (\bar{G}_\sigma)'$ be a simple exceptional group of Lie type over $\mathbb{F}_q$. Here we are interested in the case where $H = N_G(\bar{H}_\sigma)$ and $\bar{H}$ is a σ-stable non-parabolic maximal rank subgroup of $\bar{G}$ (in particular, $\bar{H}^\circ$ contains a σ-stable maximal torus of $\bar{G}$). The possibilities for H are determined in [38] and the analysis falls naturally into two cases, according to whether or not H is the normalizer of a maximal torus. Our main result is the following (note that throughout this section, we continue to exclude the groups in (3)).

Theorem 4.1. *Let G be an almost simple primitive permutation group of degree n with socle T, an exceptional group of Lie type over $\mathbb{F}_q$, and point stabilizer H, a maximal rank non-parabolic subgroup of G. Set $H_0 = H \cap T$. Then one of the following holds:*

   (i) $\text{ifix}(T) > n^{4/9}$;

   (ii) (T, H_0) *is one of the cases in Table 11 and $\text{ifix}(T) > n^\alpha$ for the given constant α.*

Moreover,

$$\liminf_{q \to \infty} \frac{\log \text{ifix}(T)}{\log n} = \beta$$

and either $\beta \geqslant 1/2$, or $T = E_6^\epsilon(q)$, $\text{soc}(H_0) = \mathrm{L}_3^\epsilon(q^3)$ and $\beta = 13/27$.

T	H_0	α	Conditions
$F_4(q)$	$(q^2+q+1)^2.(3 \times \mathrm{SL}_2(3))$	0.427	$q=2$, $G=\mathrm{Aut}(T)$
$^3D_4(q)$	$(q^4-q^2+1).4$	0.436	$q=2$
	$(q^2+q+1)^2.\mathrm{SL}_2(3)$	0.401	$q=2$
		0.442	$q=3$
	$(q^2-q+1)^2.\mathrm{SL}_2(3)$	0.351	$q=2$
		0.419	$q=3$
		0.439	$q=4$
$^2G_2(q)$	$(q-\sqrt{3q}+1){:}6$	0.442	$q=27$
$^2B_2(q)$	$(q+\sqrt{2q}+1){:}4$	0.438	$q=8$
	$(q-\sqrt{2q}+1){:}4$	0.380	$q=8$
		0.436	$q=32$

TABLE 11. The maximal rank actions with $n^\alpha < \mathrm{ifix}(T) \leqslant n^{4/9}$

Our basic strategy is to identify an involution $t \in H_0$ and then derive an upper bound $|C_{H_0}(t)| \leqslant f(q)$ for some function f. Then by (2) we have

$$\mathrm{ifix}(T) \geqslant \frac{|T|}{f(q)|t^T|} \quad \text{and} \quad \beta \geqslant \liminf_{q \to \infty} \frac{\log|T| - \log(f(q)|t^T|)}{\log n} \tag{8}$$

and we need to identify the T-class of t (recall that the possibilities are listed in Table 5). For the purposes of obtaining lower bounds on $\mathrm{ifix}(T)$ and β, we can assume that t belongs to the largest class of involutions in T and we find that this crude approach is effective in some cases. However, in general we need to do more to establish the desired bounds. Here it can be helpful to consider the restriction $V{\downarrow}\bar{H}^\circ$ of a $\bar{G}$-module V, such as the Lie algebra $\mathcal{L}(\bar{G})$ or the minimal module for $\bar{G}$, noting that in these cases the composition factors of $V{\downarrow}\bar{H}^\circ$ are readily available in the literature (for example, comprehensive tables are presented in [60]). If $p \neq 2$ then we can use the decomposition of $V{\downarrow}\bar{H}^\circ$ to compute the dimension of the 1-eigenspace of t on V, which is equal to $\dim C_{\bar{G}}(t)$ when $V = \mathcal{L}(\bar{G})$ (see [18, Section 1.14]). This usually allows us to identify the T-class of t. Similarly, when $p = 2$ we can work out the Jordan form of t on V and then use [32] to determine the T-class of t (alternatively, we can appeal to Lawther's results in [31] on the fusion of unipotent classes in connected reductive subgroups of $\bar{G}$).

4.1. **Normalizers of maximal tori.** To begin with, let us assume that $H = N_G(\bar{S}_\sigma)$ where $\bar{S}$ is a σ-stable maximal torus of $\bar{G}$ (the remaining maximal rank subgroups will be handled in Section 4.2). Set $H_1 = N_{\bar{G}_\sigma}(\bar{S}_\sigma)$. As explained in [38, Section 1], we have

$$H_1 = N_{\bar{G}_\sigma}(\bar{S}_\sigma) = \bar{S}_\sigma.N \tag{9}$$

and $H_0 = H_1 \cap T$, where N is isomorphic to a subgroup of the Weyl group $W = N_{\bar{G}}(\bar{S})/\bar{S}$. By the main theorem of [38] we have $H_1 = N_{\bar{G}_\sigma}(\bar{S})$ and the possibilities for H_1 are presented in [38, Table 5.2]. It will be convenient to set $S = \bar{S}_\sigma \cap T$.

Proposition 4.2. *Let G, T, H, H_0 and n be as in the statement of Theorem 4.1 and assume H is the normalizer of a maximal torus of G. Then*

$$\liminf_{q \to \infty} \frac{\log \mathrm{ifix}(T)}{\log n} \geqslant \frac{1}{2}$$

and either

(i) $\mathrm{ifix}(T) > n^{4/9}$; *or*

(ii) (T, H_0) is one of the cases in Table 11 and $\mathrm{ifix}(T) > n^\alpha$ for the given constant α.

We will prove Proposition 4.2 in Lemmas 4.5–4.7. We start by establishing an elementary result on the longest word in W.

Lemma 4.3. *Let $\bar{G}$ be a simple adjoint algebraic group of type B_2, G_2, D_4, F_4, E_7 or E_8 over an algebraically closed field K of characteristic $p \geqslant 0$. Let $\bar{S}$ be a maximal torus of $\bar{G}$ and let w_0 be the longest word in the Weyl group $W = N_{\bar{G}}(\bar{S})/\bar{S}$.*

  (i) *w_0 is an involution acting by inversion on $\bar{S}$.*
  (ii) *w_0 is the image of an involution $\hat{w}_0 \in N_{\bar{G}}(\bar{S})$ such that $\bar{S}\hat{w}_0 = \hat{w}_0^{\bar{S}}$.*
  (iii) *$\dim C_{\bar{G}}(\hat{w}_0) = \frac{1}{2}(\dim \bar{G} - r)$, where r is the rank of $\bar{G}$.*

Proof. Part (i) is well-known (for example, see [28, p.192]). By [54, Proposition 8.22], there is an involution $\hat{w}_0 \in N_{\bar{G}}(\bar{S})$ such that $w_0 = \bar{S}\hat{w}_0$, so $\langle \bar{S}, \hat{w}_0 \rangle = \bar{S}{:}\langle \hat{w}_0 \rangle$ is a split extension and every element in the coset $\bar{S}\hat{w}_0$ is an involution. Since $\hat{w}_0$ inverts $\bar{S}$, it follows that $s\hat{w}_0$ and $\hat{w}_0$ are $\bar{S}$-conjugate if and only if $s = t^2$ for some $t \in \bar{S}$. But every element in $\bar{S}$ is a square since K is algebraically closed, so (ii) follows. Finally, part (iii) follows from [14, Lemma 3.6]. $\square$

Corollary 4.4. *Let $\bar{G}$ be as in the statement of Lemma 4.3 and suppose $T = (\bar{G}_\sigma)'$ and*
$$H_1 = N_{\bar{G}_\sigma}(\bar{S}_\sigma) = \bar{S}_\sigma.N$$
for some σ-stable maximal torus $\bar{S}$ of $\bar{G}$. Recall that $S = \bar{S}_\sigma \cap T$. Then there exists an involution $t \in H_0$ such that

  (i) *t inverts S,*
  (ii) *every element in the coset St is T-conjugate to t, and*
  (iii) *t^T is the largest class of involutions in T.*

Proof. Let w_0 be the longest word in the Weyl group $W = N_{\bar{G}}(\bar{S})/\bar{S}$ and let $\hat{w}_0 \in N_{\bar{G}}(\bar{S})$ be the involution in Lemma 4.3(ii). Since w_0 is the only non-trivial element of $Z(W)$, it follows that σ fixes w_0, so $\bar{S}{:}\langle \hat{w}_0 \rangle$ is σ-stable and $(\bar{S}{:}\langle \hat{w}_0 \rangle)_\sigma \leqslant H_1$. Moreover, the conjugacy class $\hat{w}_0^{\bar{S}}$ is also stabilized by σ, so Lang's theorem implies that H_1 contains an involution $t \in \hat{w}_0^{\bar{S}}$.

We claim that $t \in H_0$. Apart from the case where $\bar{G} = E_7$ and $p \neq 2$, we have $\bar{G}_\sigma = T$, so $H_0 = H_1$ and the claim is clear. Now assume $\bar{G} = E_7$ and $p \neq 2$. Seeking a contradiction, let us assume that $t \notin H_0$, in which case $(\bar{S}_\sigma{:}\langle t \rangle) \cap T = \bar{S}_\sigma$. Let $\hat{G}$ be the simply connected algebraic group and by an abuse of notation we will write σ for the corresponding Steinberg endomorphism of $\hat{G}$. Note that $\bar{G} = \hat{G}/Z$ and $\bar{S} = \hat{S}/Z$, where $Z = Z(\hat{G}) = Z(\hat{G}_\sigma)$ (see [23, Table 2.2]). Now $Z \leqslant \hat{S}_\sigma$ and so $\hat{S}_\sigma/Z = \bar{S}_\sigma \cap T$. But $\hat{S}$ and $\bar{S}$ are connected isogenous groups, so $|\hat{S}_\sigma| = |\bar{S}_\sigma|$ and thus $|\bar{S}_\sigma \cap T| = |\bar{S}_\sigma|/2$, which implies that $\bar{S}_\sigma \nleqslant T$. We conclude that $t \in H_0$ as claimed.

Now t inverts S (since it inverts the algebraic torus $\bar{S}$), so (i) follows. Part (iii) is an immediate corollary of Lemma 4.3(iii). Finally, observe that $St \subset \bar{S}\hat{w}_0$ so all elements in St are $\bar{G}$-conjugate by Lemma 4.3(ii). Part (ii) now follows since $x^{\bar{G}} \cap T = x^T$ for all involutions $x \in T$ (see Remark 2.2). $\square$

We are now ready to begin the proof of Proposition 4.2. Set
$$\beta = \liminf_{q \to \infty} \frac{\log \mathrm{ifix}(T)}{\log n}.$$

Recall that if X is a finite group, then $i_2(X)$ denotes the number of involutions in X.

Lemma 4.5. *Proposition 4.2 holds if $T \in \{{}^2B_2(q), {}^2G_2(q), G_2(q)\}$.*

T	H_0	q
$E_8(q)$	$(q+1)^8.W$	2
$E_7(q)$	$(q+1)^7.W$	2
$F_4(q)$	$(q^2+q+1)^2.(3 \times \mathrm{SL}_2(3))$	2
$^3D_4(q)$	$(q^4-q^2+1).4$	2
	$(q^2+q+1)^2.\mathrm{SL}_2(3)$	2, 3
	$(q^2-q+1)^2.\mathrm{SL}_2(3)$	2, 3, 4

TABLE 12. Some normalizers of maximal tori

Proof. Write $H_1 = N_{\bar{G}_\sigma}(\bar{S}_\sigma) = S.N$ as in (9) and note that $H_0 = H_1$. Since $p = 3$ when $T = G_2(q)$ (see [38, Table 5.2]), in each case T has a unique class of involutions (see Table 5). Therefore

$$\mathrm{ifix}(T) = \frac{i_2(H_0)}{i_2(T)} \cdot n \tag{10}$$

and it suffices to compute $i_2(H_0)$. In the next paragraph, we write D_m for the dihedral group of order m.

First assume $T = {}^2G_2(q)$ and $H_0 = S.N$, where $|S| = q+1$ and $N = Z_6$. By [26], we have $H_0 = (2^2 \times D_{(q+1)/2}){:}3$, so $i_2(H_0) = q+4$ and the result follows. Next assume $T = G_2(q)$ and $H_0 = (q-\epsilon)^2.D_{12}$, where $p = 3$ and $q \geqslant 9$ by maximality. Here Corollary 4.4 implies that $i_2(H_0) \geqslant (q-\epsilon)^2$ and we deduce that $\beta \geqslant 1/2$ and $\mathrm{ifix}(T) > n^{4/9}$ for $q > 9$. If $q = 9$, then using MAGMA we calculate that $i_2(H_0) = 115$ when $\epsilon = 1$, and $i_2(H_0) = 163$ when $\epsilon = -1$, whence $\mathrm{ifix}(T) > n^{4/9}$ as required.

In each of the remaining cases (see [38, Table 5.2]), we have $H_0 = S.N$ where $|S|$ is odd and N has a unique involution. By applying Corollary 4.4, we deduce that $i_2(H_0) = |S|$ and the result follows. $\square$

Lemma 4.6. *Proposition 4.2 holds if $T \in \{{}^3D_4(q), {}^2F_4(q), F_4(q), E_7(q), E_8(q)\}$.*

Proof. This is similar to the previous lemma. Fix an involution $t \in H_0$ such that $|t^T \cap H| \geqslant |S|$ and t belongs to the largest class of involutions in T (see Corollary 4.4). This gives lower bounds on $\mathrm{ifix}(T)$ and β as in (8), and we deduce that $\beta \geqslant 1/2$ in all cases. Moreover, the same bounds imply that $\mathrm{ifix}(T) > n^{4/9}$, with the exception of the cases in Table 12.

Suppose $T = {}^3D_4(q)$ or $F_4(q)$, as in Table 12, so $H_1 = H_0 = S.N$. Here $|S|$ is odd and N has a unique involution, so $i_2(H_0) = |t^T \cap H| = |S|$ and we can compute $\mathrm{ifix}(T)$ precisely. One can check that $\mathrm{ifix}(T) \leqslant n^{4/9}$ in each case, so these exceptions are recorded in Table 11. Finally, we claim that $\mathrm{ifix}(T) > n^{4/9}$ in the two remaining cases in Table 12. To see this, we use MAGMA to construct $H_0 = S.W$ as a subgroup of $E_7(4)$ or $E_8(4)$, respectively, and we find a class of involutions in H_0 that is contained in the A_1-class of T. For instance, if $T = E_8(2)$ and $H_0 = 3^8.W$, then there is an involution $y \in H_0$ in the A_1-class of T with $|y^{H_0}| = 360$, and we deduce that $\mathrm{fix}(y) > n^{4/9}$ as required. $\square$

Finally, we complete the proof of Proposition 4.2 by dealing with the case $T = E_6^\epsilon(q)$. Here $\bar{G} = E_6$ and every involution in $\bar{G}_\sigma$ is contained in T, so for the purposes of computing $\mathrm{ifix}(T)$ we may assume that $\bar{G}_\sigma \leqslant G$ and $H_1 \leqslant H$, where $H_1 = N_{\bar{G}_\sigma}(\bar{S}_\sigma)$ as above.

For the proof of Lemma 4.7 below, we need a more detailed understanding of the construction of H_1 (see [38, Sections 1–3] and [52, Section 25.1] for further details). Fix a σ-stable maximal torus $\bar{T}$ of $\bar{G}$ and set $W = N_{\bar{G}}(\bar{T})/\bar{T}$. Let

$$U_\alpha = \{x_\alpha(c) : c \in K\} \tag{11}$$

be the root subgroup of $\bar{G}$ corresponding to $\alpha \in \Phi(\bar{G})$. For $\epsilon = +$ we may assume that σ is the standard Frobenius morphism σ_q (acting as $x_\alpha(c) \mapsto x_\alpha(c^q)$ on root group elements), and for $\epsilon = -$ we take $\sigma = \sigma_q \tau$, where τ is the standard graph automorphism of $\bar{G}$ induced from the order two symmetry ρ of the Dynkin diagram (so $\sigma : x_\alpha(c) \mapsto x_{\rho(\alpha)}(c^q)$ for $\alpha \in \Pi(\bar{G})$). Without loss of generality, we may assume that

$$\bar{T} = \langle h_{\alpha_i}(c_i) \mid c_i \in K^*, \ i = 1, \dots, 6 \rangle,$$

where

$$h_\alpha(c) = x_\alpha(c) x_{-\alpha}(-c^{-1}) x_\alpha(c-1) x_{-\alpha}(1) x_\alpha(-1)$$

as in [16, Lemma 6.4.4]. As before, let w_0 be the longest word of W, so that $w_0 \tau$ acts as -1 on $\Phi(\bar{G})$ (see [28, p.192]). Set $w^* = w_0$ if $\epsilon = -$, otherwise $w^* = 1$.

Recall that there is a 1-1 correspondence between the $\bar{G}_\sigma$-classes of σ-stable maximal tori in $\bar{G}$ and the σ-conjugacy classes of W. More precisely, if we write $\bar{S} = \bar{T}^g$ for some $g \in \bar{G}$, then $w = \sigma(g)g^{-1} \in N_{\bar{G}}(\bar{T})$ and its image in W (which we also write as w) represents the σ-class corresponding to $\bar{S}^{\bar{G}_\sigma}$. As in [38, Section 1], it is convenient to replace σ by σw in order to simplify the description of H_1, where $(\sigma w)^g = \sigma$ and $(\bar{G}_{\sigma w})^g = \bar{G}_\sigma$. Then $T = (\bar{G}_{\sigma w})'$ and

$$H_1 = N_{\bar{G}}(\bar{T})_{\sigma w} = \bar{T}_{\sigma w}.N \tag{12}$$

with $N = \{x \in W : xw = w\sigma(x)\}$. It will be convenient to set $S = \bar{T}_{\sigma w}$.

Lemma 4.7. *Proposition 4.2 holds if $T = E_6^\epsilon(q)$.*

Proof. According to [38, Table 5.2], we have $H_1 = S.N$ as in (12), where

(a) $S = (q - \epsilon)^6$ and $N = W$, with $q \geqslant 5$ if $\epsilon = 1$; or
(b) $S = (q^2 + \epsilon q + 1)^3$ and $N = 3^{1+2}.\mathrm{SL}_2(3)$, with $(\epsilon, q) \neq (-1, 2)$.

We will use the notation for involutions presented in Table 5.

First consider (a). In the notation of (12), note that $w = w^*$. First assume p is odd. We claim that S contains a t_1-involution t, in which case the trivial bound $|C_{H_0}(t)| \leqslant |H_0|$ yields $\mathrm{ifix}(T) > n^{4/9}$ and $\beta \geqslant 5/9$. To prove the claim, let $\bar{M} = D_5 T_1$ be a σw-stable maximal subgroup of $\bar{G}$ containing $\bar{T}$, so $Z(D_5 T_1) < \bar{T} < \bar{M}$ is a chain of σw-stable subgroups. Now $Z(D_5 T_1)$ contains a unique involution (a t_1-involution), which must be fixed by σw, hence $S = \bar{T}_{\sigma w}$ contains a t_1-involution, as claimed.

Now assume $p = 2$, so $|S|$ is odd. We claim that H_1 contains an A_1-involution, which yields $\mathrm{ifix}(T) > n^{4/9}$ and $\beta \geqslant 1/2$. To justify the claim, let

$$t = x_{\alpha_2}(1) x_{-\alpha_2}(1) x_{\alpha_2}(1) \in N_{\bar{G}}(\bar{T}),$$

where $x_\alpha(1) \in U_\alpha$, as in (11). Then t is an involution that is fixed by σw, so $t \in H_1$. Moreover, t is contained in the A_1-type subgroup $\langle U_{\alpha_2}, U_{-\alpha_2} \rangle$, which has a unique conjugacy class of involutions. Thus t is $\bar{G}$-conjugate to $x_{\alpha_2}(1)$ and the claim follows.

Now consider (b). If H_1 contains a t_2-involution when $p \neq 2$, or an involution in one of the classes labelled A_1 or A_1^2 when $p = 2$, then we immediately deduce that $\beta \geqslant 1/2$ and $\mathrm{ifix}(T) > n^{4/9}$. Therefore, we may as well assume that every involution in H_1 is contained in the largest T-class of involutions (which can be read off from Table 5) and we need to determine an appropriate lower bound on $i_2(H_1)$.

In terms of the above correspondence between the $\bar{G}_\sigma$-classes of σ-stable maximal tori in $\bar{G}$ and the σ-classes of W, the given maximal torus S corresponds to the class of w with Carter diagram $3A_2$ (see [38, Table C]). By considering the classes in W, we can write $w = xw^*$, where

$$x = (1, 3, 43)(2, 72, 35)(4, 22, 62)(5, 6, 47) \dots \in W$$

as a permutation of $\Phi(\bar{G})$, where our ordering of the 72 roots in $\Phi(\bar{G})$ is consistent with MAGMA (note that we only give part of the permutation x, but this is enough to uniquely determine it). It follows that

$$S = \langle h_{\alpha_1}(c_1^{-\epsilon q})h_{\alpha_3}(c_1), h_{\alpha_2}(c_2^{-\epsilon q})h_{-\alpha_0}(c_2), h_{\alpha_5}(c_3^{-\epsilon q})h_{\alpha_6}(c_3) \mid c_i^{q^2+\epsilon q+1} = 1\rangle,$$

where α_0 is the highest root in $\Phi(\bar{G})$. Now $N = 3^{1+2}.\mathrm{SL}_2(3)$ has 9 involutions, which form a single conjugacy class, say $\{m_1, \ldots, m_9\}$. Since $|S|$ is odd, each subgroup $N_i = S.\langle m_i\rangle \leqslant H_1$ is a split extension and we note that $i_2(H_1) \geqslant \sum_i i_2(N_i)$ since $S\langle m_i\rangle \cap S\langle m_j\rangle = S$ for all $i \neq j$. We claim that $i_2(N_i) \geqslant (q^2 + \epsilon q + 1)^2$ for each i, so $i_2(H_1) \geqslant 9(q^2 + \epsilon q + 1)^2$. This implies that $\beta \geqslant 1/2$ and $\mathrm{ifix}(T) > n^{4/9}$, so it remains to justify the lower bound on $i_2(N_i)$.

The number of involutions in N_i is at least the number of elements of S inverted by m_i. Thus for each i, it suffices to find a subtorus of S of size $(q^2 + \epsilon q + 1)^2$ inverted by m_i. With the aid of MAGMA, it is easy to determine the action of m_i on S. For example, one of the m_i, say m_1, acts as follows

$$h_{\alpha_1}(c) \longleftrightarrow h_{\alpha_2}(c)$$
$$h_{\alpha_3}(c) \longleftrightarrow h_{-\alpha_0}(c)$$
$$h_{\alpha_5}(c) \longleftrightarrow h_{\alpha_5}(c^{-1}) = h_{\alpha_5}(c)^{-1}$$
$$h_{\alpha_6}(c) \longleftrightarrow h_{\alpha_6}(c^{-1}) = h_{\alpha_6}(c)^{-1}$$

which implies that the subtorus

$$\langle h_{\alpha_1}(c_1^{-\epsilon q})h_{\alpha_3}(c_1)h_{\alpha_2}(c_1^{\epsilon q})h_{-\alpha_0}(c_1^{-1}), h_{\alpha_5}(c_2^{-\epsilon q})h_{\alpha_6}(c_2) \mid c_i^{q^2+\epsilon q+1} = 1\rangle < S$$

is inverted by m_1. Similarly, each of the remaining m_i inverts a torus of size $(q^2 + \epsilon q + 1)^2$, which proves the claim. $\qquad\square$

4.2. The remaining maximal rank subgroups.
To complete the proof of Theorem 4.1, we may assume that $H = N_G(\bar{H}_\sigma)$, where $\bar{H}$ is a σ-stable maximal rank subgroup and the connected component $\bar{H}^\circ$ is not a maximal torus of $\bar{G}$. The possibilities for H are given in [38, Table 5.1].

Proposition 4.8. *Let G, T, H, H_0 and n be as in the statement of Theorem 4.1 and assume H is not the normalizer of a maximal torus of G. Then $\mathrm{ifix}(T) > n^{4/9}$. Moreover,*

$$\liminf_{q\to\infty} \frac{\log \mathrm{ifix}(T)}{\log n} = \beta$$

and either $\beta \geqslant 1/2$, or $T = E_6^\epsilon(q)$, $\mathrm{soc}(H_0) = \mathrm{L}_3^\epsilon(q^3)$ and $\beta = 13/27$.

As before, we will identify an involution $t \in H_0$, compute an upper bound $|C_{H_0}(t)| \leqslant f(q)$ and determine the T-class of t, which allows us to evaluate the bounds in (8). We present our results in Tables 13–15, which essentially reduces the proof of Proposition 4.8 to verifying the information in these tables.

We start by describing the structure of $\bar{H}_\sigma$, which is recorded in the second column of [38, Table 5.1] (see [38, Sections 1 and 2] for further details and some instructive examples, as well as [17, Sections 1–3]). Write $\bar{H} = \bar{H}^\circ.F$, where F is identified with a subgroup of the Weyl group $W = W(\bar{G})$. Then $\bar{H}^\circ = MR$ is a commuting product of σ-stable subgroups, where $M = (\bar{H}^\circ)'$ is semisimple and R is a torus (possibly trivial). Then M_σ is a subgroup of $\bar{H}_\sigma$ and we have

$$M_\sigma = (\hat{X}_1 \circ \hat{X}_2 \circ \cdots \circ \hat{X}_m).D = Z.(X_1 \times X_2 \times \cdots \times X_m).D \qquad (13)$$

for some positive integer m, where each X_i is a group of Lie type (simple, or isomorphic to $\mathrm{L}_2(2)$, $\mathrm{L}_2(3)$ or $\mathrm{U}_3(2)$), $\hat{X}_i$ is a cover of X_i, Z is a subgroup of the centre of the central

T	Type of H_0	t	$f(q)$	t^T	γ
$^2G_2(q)'$	$\mathrm{L}_2(q)$	t_1'	$2(q+1)$	t_1	$1/2$
$^3D_4(q)$	$\mathrm{L}_2(q) \times \mathrm{L}_2(q^3)$	(t_1, t_1)	$2(q-1)(q^3-1)$	t_2	$1/2$
		$(A_1, 1)$	$q|\mathrm{SL}_2(q^3)|$	A_1	$1/2$
	$\mathrm{L}_3^\epsilon(q)$	γ_1	$2|\mathrm{SO}_3(q)|$	t_2	$1/2$
		A_1	$2q^3(q^3-\epsilon)$	A_1	$2/3$
$G_2(q)$	$\mathrm{L}_2(q)^2$	(t_1, t_1)	$2(q-1)^2$	t_1	$1/2$
		$(A_1, 1)$	$q|\mathrm{SL}_2(q)|$	A_1	$1/2$
	$\mathrm{L}_3^\epsilon(q)$	t_1 and γ_1	$2|\mathrm{GL}_2^\epsilon(q)|$ and $2|\mathrm{SO}_3(q)|$	t_1	$1/2$
		A_1	$2q^3(q-\epsilon)$	A_1	$2/3$
$^2F_4(q)$	$\mathrm{U}_3(q)$	γ_1	$2|\mathrm{SL}_2(q)|$	$A_1\tilde{A}_1$	$1/2$
	$^2B_2(q)^2$	$((\tilde{A}_1)_2, 1)$	$q^2|{}^2B_2(q)|$	$(\tilde{A}_1)_2$	$1/2$
	$\mathrm{Sp}_4(q)$	c_2	$2q^4$	$A_1\tilde{A}_1$	$1/2$
$F_4(q)$	$\mathrm{L}_2(q) \times \mathrm{PSp}_6(q)$	$(1, t_1'')^\dagger$	$2|\mathrm{SL}_2(q)|^2|\mathrm{Sp}_4(q)|$	t_4	$5/7$
	$\Omega_9(q)$	t_4	$|\mathrm{SO}_8^+(q)|$	t_4	$1/2$
		A_1	$q^{11}|\mathrm{SL}_2(q)||\mathrm{Sp}_4(q)|$	A_1	$3/4$
	$\mathrm{P}\Omega_8^+(q)$	t_2	$6|\mathrm{SL}_2(q)|^4$	t_1	$1/2$
		A_1	$6q^9|\mathrm{SL}_2(q)|^3$	A_1	$3/4$
	$^3D_4(q)$	t_2	$3|\mathrm{SL}_2(q)||\mathrm{SL}_2(q^3)|$	t_1	$1/2$
		A_1	$3q^9|\mathrm{SL}_2(q^3)|$	A_1	$3/4$
	$\mathrm{L}_3^\epsilon(q)^2$	$(1, t_1)$	$2|\mathrm{SL}_2(q)||\mathrm{GL}_3^\epsilon(q)|$	t_4	$2/3$
		$(A_1, 1)$	$2q^3|\mathrm{GL}_3^\epsilon(q)|$	A_1	$2/3$
	$\mathrm{Sp}_4(q)^2$	$(A_1, 1)$	$2q^3|\mathrm{SL}_2(q)||\mathrm{Sp}_4(q)|$	A_1	$5/8$
	$\mathrm{Sp}_4(q^2)$	φ	$2|\mathrm{Sp}_4(q)|$	$(\tilde{A}_1)_2$	$5/8$
$E_6^\epsilon(q)$	$\mathrm{L}_2(q) \times \mathrm{L}_6^\epsilon(q)$	$(1, t_2'')^\dagger$	$2|\mathrm{SL}_2(q)|^2|\mathrm{GL}_4^\epsilon(q)|$	t_1	$3/5$
		$(1, A_1)$	$q^9|\mathrm{SL}_2(q)||\mathrm{GL}_4^\epsilon(q)|$	A_1	$7/10$
	$\mathrm{L}_3^\epsilon(q)^3$	$(t_1, t_1, 1)$	$6(q-\epsilon)|\mathrm{SL}_2(q)|^2|\mathrm{GL}_3^\epsilon(q)|$	t_1	$5/9$
		$(A_1, 1, 1)$	$6q^3(q-\epsilon)|\mathrm{SL}_3^\epsilon(q)|^2$	A_1	$2/3$
	$\mathrm{L}_3(q^2) \times \mathrm{L}_3^{-\epsilon}(q)$	$(t_1, 1)$	$2|\mathrm{GL}_2^\epsilon(q^2)||\mathrm{SL}_3^{-\epsilon}(q)|$	t_1	$5/9$
		$(1, A_1)$	$2q^3(q+\epsilon)|\mathrm{SL}_3(q^2)|$	A_1	$2/3$
	$\mathrm{L}_3^\epsilon(q^3)$	t_1	$3|\mathrm{GL}_2^\epsilon(q^3)|$	t_2	$13/27$
		A_1	$3q^9(q^3-\epsilon)$	A_1^3	$13/27$
	$\mathrm{P}\Omega_8^+(q)$	t_2	$6(q-\epsilon)^2|\mathrm{SL}_2(q)|^4$	t_2	$1/2$
		A_1	$6q^9(q-\epsilon)^2|\mathrm{SL}_2(q)|^3$	A_1	$3/4$
	$^3D_4(q)$	t_2	$3(q^2+\epsilon q+1)|\mathrm{SL}_2(q)||\mathrm{SL}_2(q^3)|$	t_2	$1/2$
		A_1	$3q^9(q^2+\epsilon q+1)|\mathrm{SL}_2(q^3)|$	A_1	$3/4$
	$\mathrm{P}\Omega_{10}^\epsilon(q)$	t_2	$|\mathrm{SL}_2(q)|^2|\mathrm{GL}_4^\epsilon(q)|$	t_2	$1/2$
		A_1	$q^{13}|\mathrm{SL}_2(q)||\mathrm{SL}_4^\epsilon(q)|$	A_1	$3/4$

$\dagger$ The notation t_1'', t_2'' is explained in Remark 4.15.

TABLE 13. Maximal rank subgroups, $T \neq E_7(q), E_8(q)$

Type of H_0	t	$f(q)$	t^T	γ
$\mathrm{L}_2(q) \times \mathrm{P\Omega}_{12}^+(q)$	$(1, t_2)$	$2|\mathrm{SL}_2(q)|^3|\mathrm{SO}_8^+(q)|$	t_1	$1/2$
	$(1, A_1)$	$q^{17}|\mathrm{SL}_2(q)|^2|\Omega_8^+(q)|$	A_1	$3/4$
$\mathrm{L}_8^\epsilon(q)$	t_4	$4(q-\epsilon)|\mathrm{SL}_4^\epsilon(q)|^2$	t_1	$19/35$
	A_1	$2q^{13}|\mathrm{GL}_6^\epsilon(q)|$	A_1	$5/7$
$\mathrm{L}_3^\epsilon(q) \times \mathrm{L}_6^\epsilon(q)$	$(1, t_3)$ or $(1, t_3')$	$2(q+\epsilon)|\mathrm{SL}_3^\epsilon(q)||\mathrm{SL}_3(q^2)|$	t_7 or t_7'	$3/5$
	$(1, A_1)$	$2q^9|\mathrm{SL}_3^\epsilon(q)||\mathrm{GL}_4^\epsilon(q)|^2$	A_1	$11/15$
$\mathrm{L}_2(q)^3 \times \mathrm{P\Omega}_8^+(q)$	$(1,1,1,t_2)$	$24|\mathrm{SL}_2(q)|^7$	t_1	$1/2$
	$(1,1,1,A_1)$	$6q^9|\mathrm{SL}_2(q)|^6$	A_1	$3/4$
$\mathrm{L}_2(q^3) \times {}^3D_4(q)$	$(1, t_2)$	$3|\mathrm{SL}_2(q)|^4|\mathrm{SL}_2(q^3)|$	t_1	$1/2$
	$(1, A_1)$	$3q^9|\mathrm{SL}_2(q)|^3|\mathrm{SL}_2(q^3)|$	A_1	$3/4$
$\mathrm{L}_2(q)^7$	$(t_1, t_1, t_1, -)$ or $(t_1', t_1', t_1', -)$	$1344(q+1)^3|\mathrm{SL}_2(q)|^4$	t_7 or t_7'	$4/7$
	$(A_1, -)$	$168q|\mathrm{SL}_2(q)|^6$	A_1	$5/7$
$\mathrm{L}_2(q^7)$	t_1 or t_1'	$7(q^7+1)$	t_4 or t_4'	$1/2$
	A_1	$7q^7$	A_1^4	$1/2$
$E_6^\epsilon(q)$	t_2	$2|\mathrm{SL}_2(q)||\mathrm{GL}_6^\epsilon(q)|$	t_1	$5/9$
	A_1	$2q^{21}|\mathrm{GL}_6^\epsilon(q)|$	A_1	$7/9$

TABLE 14. Maximal rank subgroups, $T = E_7(q)$

product of the $\hat{X}_i$, and D is a group inducing diagonal automorphisms on the X_i (possibly $Z = 1$ and $D = 1$). To simplify the notation, we will refer to

$$X_1 \times X_2 \times \cdots \times X_m$$

as the *type* of $H_0 = \bar{H}_\sigma \cap T$.

Example 4.9. Suppose $T = E_6(q)$ and set $d = (2, q-1)$ and $e = (3, q-1)$, so $\bar{G}_\sigma = T.e$. Consider a σ-stable maximal rank subgroup $\bar{H} = A_1 A_5$. Here $\bar{H}$ is connected and semisimple so $F = 1$ and $\bar{H} = M$. Moreover,

$$N_{\bar{G}_\sigma}(\bar{H}_\sigma) = M_\sigma = d.(\mathrm{L}_2(q) \times \mathrm{L}_6(q)).de = H_0.e$$

so in (13) we have $Z = Z_d$, $D = Z_d \times Z_e$ and H_0 has type $\mathrm{L}_2(q) \times \mathrm{L}_6(q)$. Similarly, if $\bar{H} = D_5 T_1$ then $M = D_5$ and $M_\sigma = h.\mathrm{P\Omega}_{10}^+(q)$ with $h = (4, q-1)$, so $Z = Z_h$, $D = 1$ and H_0 has type $\mathrm{P\Omega}_{10}^+(q)$.

Example 4.10. For a slightly more complicated example, suppose $T = E_7(q)$ with $p \ne 2$, so $\bar{G}_\sigma = T.2$. Let $\bar{H} = A_1^7.\mathrm{L}_2(3)$, so $M = A_1^7$ and $F = \mathrm{L}_2(3)$. According to [38, Table 5.1], there are precisely two $\bar{G}_\sigma$-classes of σ-stable conjugates of $\bar{H}$ that give rise to maximal subgroups of $\bar{G}_\sigma$. One of these classes comprises subgroups of the form

$$\bar{H}_\sigma = \mathrm{PGL}_2(q^7).7,$$

in which case $M_\sigma = \mathrm{PGL}_2(q^7)$ and H_0 has type $\mathrm{L}_2(q^7)$. To determine the precise structure of $H_0 = \bar{H}_\sigma \cap T$, write $\bar{G} = \hat{G}/Z$, where $\hat{G}$ is the simply connected version of $\bar{G} = E_7$ and $Z = Z(\hat{G})$. Consider the subsystem subgroup $\hat{H} \leqslant \hat{G}$ corresponding to $\bar{H}$. Then $Z \leqslant \hat{H}^\circ$ and $\hat{H}^\circ/Z = \bar{H}^\circ$. Since $|(\hat{H}^\circ)_\sigma| = |(\bar{H}^\circ)_\sigma|$ it follows that $(\hat{H}^\circ)_\sigma/Z_\sigma = T \cap (\bar{H}^\circ)_\sigma$ has index two in $(\bar{H}^\circ)_\sigma$. Therefore $(\bar{H}^\circ)_\sigma = \mathrm{PGL}_2(q^7)$ and $T \cap (\bar{H}^\circ)_\sigma = \mathrm{L}_2(q^7)$, so $H_0 = \mathrm{L}_2(q^7).7$.

Remark 4.11. By inspecting [38, Table 5.1], we see that there are just two instances where the type of H_0 does not uniquely determine H_0 up to conjugacy; namely, the subgroups of type $\mathrm{U}_3(q)$ and $\mathrm{U}_5(q^2)$ in ${}^2F_4(q)$ and $E_8(q)$, respectively. In both cases, there are two

Type of H_0	t	$f(q)$	t^T	γ
$\mathrm{P}\Omega_{16}^+(q)$	t_4	$2\lvert\mathrm{SO}_8^+(q)\rvert^2$	t_1	$1/2$
	A_1	$q^{23}\lvert\mathrm{SL}_2(q)\rvert\lvert\Omega_{12}^+(q)\rvert$	A_1	$49/64$
$\mathrm{L}_2(q)\times E_7(q)$	$(1,t_1'')^\dagger$	$2\lvert\mathrm{SL}_2(q)\rvert^2\lvert\mathrm{SO}_{12}^+(q)\rvert$	t_8	$4/7$
	$(1,A_1)$	$q^{33}\lvert\mathrm{SL}_2(q)\rvert\lvert\Omega_{12}^+(q)\rvert$	A_1	$13/16$
$\mathrm{L}_9^\epsilon(q)$	t_3	$2\lvert\mathrm{GL}_3^\epsilon(q)\rvert\lvert\mathrm{SL}_6^\epsilon(q)\rvert$	t_8	$23/42$
	A_1	$2q^{15}\lvert\mathrm{GL}_7^\epsilon(q)\rvert$	A_1	$3/4$
$\mathrm{L}_3^\epsilon(q)\times E_6^\epsilon(q)$	$(1,t_2)$	$2\lvert\mathrm{SL}_2(q)\rvert\lvert\mathrm{SL}_3^\epsilon(q)\rvert\lvert\mathrm{SL}_6^\epsilon(q)\rvert$	t_8	$5/9$
	$(1,A_1)$	$2q^{21}\lvert\mathrm{SL}_3^\epsilon(q)\rvert\lvert\mathrm{SL}_6^\epsilon(q)\rvert$	A_1	$7/9$
$\mathrm{L}_5^\epsilon(q)^2$	(t_1,t_2)	$4\lvert\mathrm{SL}_2(q)\rvert\lvert\mathrm{GL}_3^\epsilon(q)\rvert\lvert\mathrm{GL}_4^\epsilon(q)\rvert$	t_8	$27/50$
	$(A_1,1)$	$4q^7\lvert\mathrm{GL}_3^\epsilon(q)\rvert\lvert\mathrm{SL}_5^\epsilon(q)\rvert$	A_1	$3/4$
$\mathrm{U}_5(q^2)$	γ_1	$4\lvert\mathrm{SO}_5(q^2)\rvert$	t_1	$1/2$
	A_1	$4q^{14}(q^2+1)\lvert\mathrm{SU}_3(q^2)\rvert$	A_1^2	$31/50$
$\mathrm{P}\Omega_8^+(q)^2$	(t_2,t_2)	$48\lvert\mathrm{SL}_2(q)\rvert^8$	t_1	$1/2$
	$(A_1,1)$	$12q^6\lvert\mathrm{SL}_4(q)\rvert\lvert\Omega_8^+(q)\rvert$	A_1	$47/64$
$\mathrm{P}\Omega_8^+(q^2)$	t_2	$12\lvert\mathrm{SL}_2(q^2)\rvert^4$	t_1	$1/2$
	A_1	$12q^{12}\lvert\mathrm{SL}_4(q^2)\rvert$	A_1^2	$19/32$
${}^3D_4(q)^2$	(t_2,t_2)	$6\lvert\mathrm{SL}_2(q)\rvert^2\lvert\mathrm{SL}_2(q^3)\rvert^2$	t_1	$1/2$
	$(A_1,1)$	$6q^9\lvert\mathrm{SL}_2(q^3)\rvert\lvert{}^3D_4(q)\rvert$	A_1	$3/4$
${}^3D_4(q^2)$	t_2	$6\lvert\mathrm{SL}_2(q^2)\rvert\lvert\mathrm{SL}_2(q^6)\rvert$	t_1	$1/2$
	A_1	$6q^{18}\lvert\mathrm{SL}_2(q^6)\rvert$	A_1^2	$5/8$
$\mathrm{L}_3^\epsilon(q)^4$	$(t_1,t_1,t_1,1)$	$48\lvert\mathrm{GL}_2(q)\rvert^3\lvert\mathrm{SL}_3^\epsilon(q)\rvert$	t_8	$29/54$
	$(A_1,1,1,1)$	$48q^3(q-\epsilon)\lvert\mathrm{SL}_3^\epsilon(q)\rvert^3$	A_1	$3/4$
$\mathrm{U}_3(q^2)^2$	(γ_1,γ_1)	$8\lvert\mathrm{SO}_3(q^2)\rvert^2$	t_1	$1/2$
	$(A_1,1)$	$8q^6\lvert\mathrm{GU}_3(q^2)\rvert$	A_1^2	$11/18$
$\mathrm{U}_3(q^4)$	γ_1	$8\lvert\mathrm{SO}_3(q^4)\rvert$	t_1	$1/2$
	γ_1	$8\lvert\mathrm{SL}_2(q^4)\rvert$	A_1^4	$1/2$
$\mathrm{L}_2(q)^8$	$(t_1,t_1,t_1,t_1,-)$	$21504(q-1)^4\lvert\mathrm{SL}_2(q)\rvert^4$	t_8	$15/28$
	$(A_1,-)$	$1344q\lvert\mathrm{SL}_2(q)\rvert^7$	A_1	$3/4$

$\dagger$ The notation t_1'' is explained in Remark 4.15.

TABLE 15. Maximal rank subgroups, $T = E_8(q)$

classes of subgroups of the same type, but the information given in Tables 13 and 15 is identical for subgroups in both classes, so this potential ambiguity is not an issue.

Next we describe the information presented in Tables 13-15. First notice that the cases are listed in the same order as they appear in [38, Table 5.1] and we implicitly assume all the conditions on G and q given in this table. The fourth column in Table 13 (and the third in Tables 14 and 15) provides an upper bound $\lvert C_{H_0}(t)\rvert \leqslant f(q)$ and the next column records the T-class of t (in terms of the notation of Table 5), where $t \in H_0$ is the involution specified in the third column. The final column gives a lower bound γ for β. Notice that in most cases there are two rows for each possibility for H_0; here $p \neq 2$ in the first row and $p = 2$ in the second.

Let us explain our notation for specifying an involution $t \in H_0$. Typically, we choose

$$t = (x_1, \ldots, x_m) \in X,$$

where

$$X = \hat{X}_1 \circ \hat{X}_2 \circ \cdots \circ \hat{X}_m \leqslant M_\sigma \tag{14}$$

is a central product of quasisimple groups of Lie type. We extend the notation from Section 2 to involution classes in $\hat{X}_i$ (recall that this notation is consistent with [23] and [44] for $p \neq 2$ and $p = 2$, respectively). Sometimes it will be convenient to write

$$t = (x_1, \ldots, x_l, -)$$

if $x_i = 1$ for $i > l$.

To obtain the upper bounds $|C_{H_0}(t)| \leqslant f(q)$, we will often use the following easy lemma. Write $X = \hat{X}/Y$, where $\hat{X} = \hat{X}_1 \times \hat{X}_2 \times \cdots \times \hat{X}_m$ and $Y \leqslant Z(\hat{X})$. Since $X \leqslant H_0$ we have

$$|C_{H_0}(t)| \leqslant |C_X(t)||H_0 : X|. \tag{15}$$

Lemma 4.12. *Let* $t = (x_1, \ldots, x_m) \in X$ *be an involution and write* $t = Y\hat{t}$ *with* $\hat{t} = (\hat{x}_1, \ldots, \hat{x}_m) \in \hat{X}$. *Then*

$$|C_X(t)| \leqslant |Y|_{2'}^{-1} \prod_i |C_{\hat{X}_i}(\hat{x}_i)|.$$

Proof. Without loss of generality, we may assume that $\hat{t}$ is a 2-element. Let $\pi : \hat{X} \to \hat{X}/Y$ be the quotient map and note that

$$\hat{t}^{\hat{X}} \subseteq \pi^{-1}(t^X) = \bigsqcup_i s_i Y$$

for some $s_i \in \hat{X}$. Each coset $s_i Y$ contains at most $|Y|_2$ 2-elements, so $|\hat{t}^{\hat{X}}| \leqslant |Y|_2 |t^X|$ and the result follows. $\square$

Remark 4.13. As indicated in the tables, we sometimes choose an involution $t \in H_0$ which induces a graph automorphism γ_1 on one or more of the $\hat{X}_i$ factors. In the special case where $T = F_4(q)$ and $H_0 = \mathrm{Sp}_4(q^2){:}2$ with $p = 2$, we take t to be an involutory field automorphism φ of $\mathrm{Sp}_4(q^2)$.

Remark 4.14. In the cases $(\bar{G}, \bar{H}^\circ) = (F_4, A_2\tilde{A}_2)$, (E_7, A_1^7) and (E_8, A_1^8), the ordering of the X_i in (13) is important. In the first case, the notation indicates that X_1 and X_2 are generated by long and short root subgroups of T, respectively. In the latter two cases, we assume that the final four A_1 factors of $\bar{H}^\circ$ are contained in a D_4 subsystem subgroup of $\bar{G}$ (this is consistent with [60], where the composition factors of $\mathcal{L}(\bar{G})\downarrow\bar{H}^\circ$ are given in Tables 59 and 60).

Remark 4.15. Suppose $p \neq 2$, $T = F_4(q)$, $E_6^\epsilon(q)$ or $E_8(q)$ and H_0 is of type $\mathrm{L}_2(q) \times \mathrm{PSp}_6(q)$, $\mathrm{L}_2(q) \times \mathrm{L}_6^\epsilon(q)$ or $\mathrm{L}_2(q) \times E_7(q)$, respectively. In these cases we need to explain the notation $(1, t_i'')$ appearing in Tables 13 and 15. This denotes the image in $X = \hat{X}_1 \circ \hat{X}_2$ of an involution in the quasisimple group $\hat{X}_2$. Now $Z(\hat{X}_2)$ has order 2 and t_i', t_i'' are the preimages of a fixed involution of type t_i in the simple group $\hat{X}_2/Z(\hat{X}_2)$ (in the notation of [23, Table 4.5.1]). One can show that t_i' and t_i'' are not T-conjugate and we take t_i'' to be in the smaller T-class. For example, if $T = F_4(q)$ and $H_0 = (\mathrm{SL}_2(q) \circ \mathrm{Sp}_6(q)).2$, then $t_1', t_1'' \in \mathrm{Sp}_6(q)$ have respective Jordan forms $[-I_4, I_2]$ and $[-I_2, I_4]$. See the proof of Lemma 4.19 for further details in the other two cases of interest.

We begin the proof of Proposition 4.8 by handling the special case where $T = E_6^\epsilon(q)$ and H_0 is of type $\mathrm{L}_3^\epsilon(q^3)$. This is the only case with $\beta < 1/2$.

Lemma 4.16. *Suppose* $T = E_6^\epsilon(q)$ *and* H_0 *is of type* $\mathrm{L}_3^\epsilon(q^3)$. *Then* $\mathrm{ifix}(T) > n^{4/9}$ *and* $\beta = 13/27$.

Proof. Here $H_0 = \mathrm{L}_3^\epsilon(q^3).\langle\varphi\rangle = \mathrm{L}_3^\epsilon(q^3).3$, where φ is a field automorphism of order 3. Let $t \in \mathrm{L}_3^\epsilon(q^3)$ be a representative of the unique class of involutions in H_0 (as indicated in Table 13, t is a t_1-involution if $p \neq 2$ and an A_1-involution when $p = 2$). In particular,

$$|C_{H_0}(t)| = \begin{cases} 3|\mathrm{GL}_2^\epsilon(q^3)| & p \neq 2 \\ 3(q^3 - \epsilon)q^9 & p = 2 \end{cases}$$

and the bound $\mathrm{ifix}(T) > n^{4/9}$ follows. In fact, we claim that t belongs to the largest class of involutions in T (that is, the T-class labelled t_2 for $p \neq 2$, and A_1^3 for $p = 2$), which implies that $\beta = 13/27$. To see this, first observe that we may assume

$$t \in A = \{(x, x, x) \,:\, x \in A_2\} < A_2^3 = \bar{H}^\circ$$

at the level of algebraic groups (this follows from the construction of H_0). Now t is a representative of the unique class of involutions in $A \cong A_2$ and it is sufficient to determine the $\bar{G}$-class of t. To do this, we consider the action of $\bar{H}^\circ = A_2^3$ on the 27-dimensional module $V = V_{\bar{G}}(\lambda_1)$:

$$V{\downarrow}A_2^3 = (V_1 \otimes V_2^* \otimes 0) \oplus (0 \otimes V_2 \otimes V_3^*) \oplus (V_1^* \otimes 0 \otimes V_3),$$

where V_i denotes the natural module for the i-th factor in $\bar{H}^\circ$ and 0 is the trivial module. For $p \neq 2$ we calculate that t has Jordan form $[-I_4, I_5]$ on each of three summands, so t has Jordan form $[-I_{12}, I_{15}]$ on V and by inspecting [40, Proposition 1.2], we conclude that t is a t_2-involution. Similarly, if $p = 2$ then t has Jordan form $[J_2^{12}, J_1^3]$ on V and [32, Table 5] implies that t is in the A_1^3-class (the same conclusion also follows from [31, Section 4.9]). $\qquad\square$

We partition the analysis of the remaining subgroups into two cases, according to the parity of p. The next two lemmas settle the case $p = 2$.

Lemma 4.17. *Proposition 4.8 holds if $p = 2$ and $T \in \{{}^3D_4(q), {}^2F_4(q)\}$.*

Proof. First assume $T = {}^2F_4(q)$ and recall that $q > 2$ (see Proposition 2.1 for the case $q = 2$). The maximal subgroups of G have been determined by Malle [51].

Suppose $H_0 = \mathrm{SU}_3(q){:}2$ or $\mathrm{PGU}_3(q){:}2$. In both cases, choose an involutory graph automorphism $t \in H_0$, so $|C_{H_0}(t)| = 2|\mathrm{SL}_2(q)|$. According to Table 5, we have $|C_T(t)| = q^{10}|^2B_2(q)|$ or $q^9|\mathrm{SL}_2(q)|$. But Lagrange's theorem implies that $|C_{H_0}(t)|$ divides $|C_T(t)|$, so $|C_T(t)| = q^9|\mathrm{SL}_2(q)|$ is the only option and thus t is in the T-class $A_1\tilde{A}_1$. The case $H_0 = {}^2B_2(q)^2.2$ is similar. Here we take t to be an involution in one of the ${}^2B_2(q)$ factors, so $|C_{H_0}(t)| = q^2|^2B_2(q)|$ and using Lagrange's theorem we see that t is in the $(\tilde{A}_1)_2$-class of T. Now assume $H_0 = \mathrm{Sp}_4(q){:}2$. Let $t \in \mathrm{Sp}_4(q)$ be a c_2-involution in the notation of [2]. By considering the construction of H_0, we may assume

$$t \in \{(x, x) \,:\, x \in C_2\} < C_2^2 < C_4 < F_4 = \bar{G}$$

and t is a c_4-involution in C_4. This is the class labelled $2A_1^{(2)}$ in [31, Table 6], so [31, Table 14] implies that t is an $A_1\tilde{A}_1$-involution, as recorded in Table 13.

Now assume $T = {}^3D_4(q)$ and note that the maximal subgroups of G are given in [27]. For $H_0 = \mathrm{L}_2(q) \times \mathrm{L}_2(q^3)$ we choose $t = (A_1, 1) \in H_0$, which is a long root element in the first factor, so $|C_{H_0}(t)| = q|\mathrm{SL}_2(q^3)|$ and using Lagrange's theorem we see that t is in the A_1-class of T. Finally, let us assume H_0 is of type $\mathrm{L}_3^\epsilon(q)$, so $\bar{H} = A_2T_2.2$ and $X = \mathrm{SL}_3^\epsilon(q)$ (see (14)). Let $t \in X$ be an involution. Then $|C_X(t)| = q^3(q - \epsilon)$ and by applying the bound in (15) we deduce that

$$|C_{H_0}(t)| \leqslant q^3(q - \epsilon) \cdot 2(q^2 + \epsilon q + 1) = 2q^3(q^3 - \epsilon)$$

as recorded in Table 13. Finally, Lagrange's theorem implies that t is an A_1-involution. $\quad\square$

Lemma 4.18. *Proposition 4.8 holds if $p = 2$.*

Proof. We may assume that $T \in \{G_2(q), F_4(q), E_6^\epsilon(q), E_7(q), E_8(q)\}$. We will postpone the analysis of the following two cases to the end of the proof:

(a) $T = F_4(q)$ and $H_0 = \mathrm{Sp}_4(q^2){:}2$;
(b) $T = E_8(q)$ and $H_0 = \mathrm{U}_3(q^4).8$.

In many cases, if $t \in H_0$ is a long root element in one of the $\hat{X}_i$ factors then t is an A_1-involution in T and the result quickly follows. For example, suppose $T = E_7(q)$ and $H_0 = \mathrm{L}_2(q) \times \Omega_{12}^+(q)$. Let $x \in \Omega_{12}^+(q)$ be a long root element and set $t = (1, x) \in H_0$. Then t is an A_1-involution in T and

$$|C_{H_0}(t)| = |\mathrm{SL}_2(q)||C_{\Omega_{12}^+(q)}(x)| = q^{17}|\mathrm{SL}_2(q)|^2|\Omega_8^+(q)|.$$

This yields $\mathrm{ifix}(T) > n^{4/9}$ and $\beta \geqslant 3/4$.

In the remaining cases (excluding (a) and (b) above), we choose t to be a long root element in one of the quasisimple factors as above and we use the embedding of H_0 in $\bar{G}$ to determine the T-class of t and verify the required bounds (the difference here is that t is no longer an A_1-involution in T). For example, suppose $T = E_7(q)$ and $H_0 = \mathrm{L}_2(q^7).\mathrm{L}_3(2)$, so $X = \mathrm{L}_2(q^7)$ and by considering the construction of H_0, we may assume that

$$t \in A = \{(x, x, x, x, x, x, x) : x \in A_1\} < A_1^7 = \bar{H}^\circ. \tag{16}$$

From the restriction of $V = V_{\bar{G}}(\lambda_7)$ to A_1^7 we deduce that

$$V{\downarrow}A = (V(1) \otimes V(1) \otimes V(1))^7.$$

Now $A \cong A_1$ has a unique class of involutions and we calculate that t has Jordan form $[J_2^{28}]$ on V. By inspecting [32, Table 7], we conclude that t is in the A_1^4-class of T.

Finally, let us turn to the cases (a) and (b) above. First consider (a). Let $t \in H_0$ be an involutory field automorphism of $\mathrm{Sp}_4(q^2)$, so $|C_{H_0}(t)| = 2|\mathrm{Sp}_4(q)|$. We claim that t is in the $(\tilde{A}_1)_2$-class of T. To see this, we may assume that

$$t \in C_2^2.2 < C_4 < F_4$$

with t interchanging the two C_2 factors (this follows from the construction of H_0). Then t is an a_4-involution in C_4, which is labelled $2A_1$ in [31, Table 6], and by inspecting [31, Table 14] we see that t is in the T-class $(\tilde{A}_1)_2$. The result follows.

Now consider (b). Let $t \in H_0$ be an involutory graph automorphism of $\mathrm{U}_3(q^4)$, so we have $|C_{H_0}(t)| = 8|\mathrm{SL}_2(q^4)|$. At the level of algebraic groups, t acts as a graph automorphism on each A_2 factor of $\bar{H}^\circ = A_2^4$, so t inverts a maximal torus of $\bar{G} = E_8$ and thus t is in the A_1^4-class by Lemma 4.3(iii). □

Lemma 4.19. *Proposition 4.8 holds if $p \neq 2$.*

Proof. As before, it is sufficient to justify the relevant information presented in Tables 13-15. In particular, we need to explain the choice of involution $t \in H_0$, establish the upper bound on $|C_{H_0}(t)|$ and determine the T-class of t.

First suppose $T = G_2(q)$, so H_0 is of type $\mathrm{L}_2(q)^2$ or $\mathrm{L}_3^\epsilon(q)$ (see [38, Table 5.1]). Note that T has $q^4(q^4 + q^2 + 1)$ involutions, which form a single conjugacy class, so (10) holds. In the first case,

$$H_0 = (\mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)).2$$

and we take $t \in H_0$ to be the image of $(A, A) \in \mathrm{GL}_2(q)^2$, where $A = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(q)$ is a t_1-type involution (note that $t \in \mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)$ if and only if $q \equiv 1 \pmod 4$). In Table 13, we denote this choice of t by writing $t = (t_1, t_1)$. Note that $|C_{\mathrm{SL}_2(q)}(A)| = q - 1$ and thus $|C_{H_0}(t)| \leqslant 2(q-1)^2$ by Lemma 4.12. It is now easy to check that the desired bounds hold.

Finally, if $H_0 = \mathrm{SL}_3^\epsilon(q){:}2$ and $t \in H_0$ is an involutory graph automorphism, then $|C_{H_0}(t)| = 2|\mathrm{SO}_3(q)|$ and the result follows if $q \geqslant 5$. For $q = 3$ we need to compute $i_2(H_0)$ in order to get $\mathrm{ifix}(T) > n^{4/9}$, so we write "$t_1$ and γ_1" in Table 13, which are representatives for the two classes of involutions in H_0. Note that $i_2(H_0) = 351$ if $\epsilon = +$, otherwise $i_2(H_0) = 315$.

To complete the proof, we will focus on the following cases to illustrate the main ideas (the other cases are similar and we omit the details):

(a) $T = F_4(q)$ and H_0 is of type $\mathrm{P\Omega}_8^+(q)$ or $\mathrm{L}_3^\epsilon(q)^2$.

(b) $T = E_6^\epsilon(q)$ and H_0 is of type $\mathrm{L}_2(q) \times \mathrm{L}_6^\epsilon(q)$.

(c) $T = E_7(q)$ and H_0 is of type $\mathrm{L}_2(q^7)$.

(d) $T = E_8(q)$ and H_0 is of type $\mathrm{P\Omega}_{16}^+(q)$, $\mathrm{L}_2(q) \times E_7(q)$ or $\mathrm{L}_2(q)^8$.

First assume $T = F_4(q)$ and H_0 is of type $\mathrm{P\Omega}_8^+(q)$, so

$$H_0 = \mathrm{Spin}_8^+(q).\mathrm{Sym}_3 = 2^2.\mathrm{P\Omega}_8^+(q).\mathrm{Sym}_3.$$

Let $t \in \mathrm{Spin}_8^+(q)$ be a t_2-involution in the notation of [23, Table 4.5.2]. This element has centralizer $\mathrm{SL}_2(q)^4$ in $\mathrm{Spin}_8^+(q)$, so $|C_{H_0}(t)| \leqslant 6|\mathrm{SL}_2(q)|^4$ and we claim that t is a t_1-involution in T. To see this, we consider the action of $\bar{H}^\circ = D_4$ on the Lie algebra $V = \mathcal{L}(\bar{G})$:

$$V{\downarrow}D_4 = V(\lambda_1) \oplus V(\lambda_2) \oplus V(\lambda_3) \oplus V(\lambda_4).$$

We calculate that t has Jordan form $[-I_{16}, I_{12}]$ on $V(\lambda_2)$, and $[-I_4, I_4]$ on each of the other summands, so t has Jordan form $[-I_{28}, I_{24}]$ on V. Therefore, $\dim C_{\bar{G}}(t) = 24$, so $C_{\bar{G}}(t) = A_1 C_3$ and the claim follows.

Now suppose H_0 is of type $\mathrm{L}_3^\epsilon(q)^2$, so $\bar{H} = (A_2\tilde{A}_2).2$ and $H_0 = M_\sigma.2$ with

$$M_\sigma = e.\mathrm{L}_3^\epsilon(q)^2.e = (\mathrm{SL}_3^\epsilon(q) \circ \mathrm{SL}_3^\epsilon(q)).e$$

and $e = (3, q - \epsilon)$. Here $\bar{H}^\circ = A_2\tilde{A}_2$ and we may assume that the first $\mathrm{SL}_3^\epsilon(q)$ factor contains long root elements of T and the second short root elements. Let $A \in \mathrm{SL}_3^\epsilon(q)$ be an involution (labelled t_1 in [23, Table 4.5.2]) and let $t \in \mathrm{SL}_3^\epsilon(q) \circ \mathrm{SL}_3^\epsilon(q)$ be the image of $(1, A) \in \mathrm{SL}_3^\epsilon(q) \times \mathrm{SL}_3^\epsilon(q)$, which is denoted $t = (1, t_1)$ in Table 13. By applying Lemma 4.12 (with $|Y|_{2'} = e$) we get

$$|C_{H_0}(t)| \leqslant 2(q - \epsilon)|\mathrm{SL}_2(q)||\mathrm{SL}_3^\epsilon(q)|$$

and it remains to show that t is a t_4-involution of T. Once again, to do this we consider the action of $\bar{H}^\circ$ on $V = \mathcal{L}(\bar{G})$:

$$V{\downarrow}A_2\tilde{A}_2 = \mathcal{L}(A_2\tilde{A}_2) \oplus (V(\lambda_1) \otimes V(2\lambda_2)) \oplus (V(\lambda_2) \otimes V(2\lambda_1)).$$

We calculate that t has Jordan form $[-I_4, I_{12}]$ on $\mathcal{L}(A_2\tilde{A}_2)$, and $[-I_6, I_{12}]$ on the two other summands, whence t has Jordan form $[-I_{16}, I_{36}]$ on V. Therefore, $C_{\bar{G}}(t) = B_4$ and this justifies the claim.

Next assume $T = E_6^\epsilon(q)$ and H_0 is of type $\mathrm{L}_2(q) \times \mathrm{L}_6^\epsilon(q)$. Here $\bar{H} = A_1 A_5$ and $H_0 = (\mathrm{SL}_2(q) \circ \mathrm{SL}_6^\epsilon(q)).2$. As explained in Remark 4.15, an involution in the t_2-class of $\mathrm{L}_6^\epsilon(q)$ has preimages $t_2', t_2'' \in \mathrm{SL}_6^\epsilon(q)$, with respective Jordan forms $[-I_2, I_4]$ and $[-I_4, I_2]$. Consider the images of $(1, t_2')$ and $(1, t_2'')$ in $\mathrm{SL}_2(q) \circ \mathrm{SL}_6^\epsilon(q)$. We claim that $(1, t_2'')$ is a t_1-involution in T, while $(1, t_2')$ is of type t_2. To see this, let $V = \mathcal{L}(\bar{G})$ and observe that

$$V{\downarrow}A_1 A_5 = \mathcal{L}(A_1 A_5) \oplus (V(1) \otimes V(\lambda_3)).$$

Both $(1, t_2')$ and $(1, t_2'')$ have Jordan form $[-I_{16}, I_{22}]$ on $\mathcal{L}(A_1 A_5)$ (indeed, both elements have centralizer $A_1^2 A_3 T_1$ in $A_1 A_5$, which is 22-dimensional), whereas the Jordan forms on $V(1) \otimes V(\lambda_3)$ are $[-I_{24}, I_{16}]$ and $[-I_{16}, I_{24}]$, respectively. This justifies the claim, and we

choose $t = (1, t_2'')$ as in Table 13. Finally, we note that $|C_{H_0}(t)| \leqslant 2|\mathrm{SL}_2(q)|^2|\mathrm{GL}_4^\epsilon(q)|$ since $|C_{\mathrm{SL}_6^\epsilon(q)}(t_2')| = |\mathrm{SL}_2(q)||\mathrm{GL}_4^\epsilon(q)|$.

Now assume $T = E_7(q)$ and H_0 is of type $\mathrm{L}_2(q^7)$. Here $\bar{H} = A_1^7.\mathrm{L}_3(2)$, $\bar{H}_\sigma = \mathrm{PGL}_2(q^7).7$ and $H_0 = \mathrm{L}_2(q^7).7$ (see Example 4.10). Now H_0 has a unique class of involutions t^{H_0}, labelled by t_1 if $q \equiv 1 \pmod 4$ and t_1' if $q \equiv 3 \pmod 4$ (see [23, Table 4.5.1]); this explains why we write "t_1 or t_1'" in Table 14. Note that $|C_{H_0}(t)| \leqslant 7(q^7 + 1)$. To identify the T-class of t, we may assume that t is contained in a diagonal A_1-type subgroup $A < \bar{G}$ as in the proof of Lemma 4.18 (see (16)), which has a unique class of involutions. If we set $V = \mathcal{L}(\bar{G})$, then using the restriction of V to A_1^7, we calculate that

$$V{\downarrow}A = (V(2) \otimes V(2))^7 \oplus V(2)^{21} \oplus 0^7$$

and thus t has Jordan form $[-I_{70}, I_{63}]$. Therefore $\dim C_{\bar{G}}(t) = 63$, so $C_{\bar{G}}(t) = A_7$ and we deduce that t is a t_4-involution if $q \equiv 1 \pmod 4$, otherwise t is in the t_4'-class.

To complete the proof, let us assume $T = E_8(q)$ and H_0 is one of the subgroups in (d) above. First assume H_0 is of type $\mathrm{P}\Omega_{16}^+(q)$. Here $\bar{H}$ is a half-spin group of type D_8 and we have

$$H_0 = \mathrm{HSpin}_{16}^+(q).2 = 2.\mathrm{P}\Omega_{16}^+(q).2.$$

Let $t \in \mathrm{HSpin}_{16}^+(q)$ be the image of an involution in the t_4-class of $\mathrm{Spin}_{16}^+(q) = 2.\mathrm{HSpin}_{16}^+(q)$ (in the notation of [23, Table 4.5.2]), so $|C_{H_0}(t)| \leqslant 2|\mathrm{SO}_8^+(q)|^2$. Then $C_{\bar{H}}(t)^\circ = D_4 D_4$ and thus t is a t_1-involution in T.

Next assume H_0 is of type $\mathrm{L}_2(q) \times E_7(q)$, so

$$H_0 = 2.(\mathrm{L}_2(q) \times E_7(q)).2 = (\mathrm{SL}_2(q) \circ \hat{E}_7(q)).2$$

where $\hat{E}_7(q) = 2.E_7(q)$ is the simply connected group. An involution in the t_1-class of $E_7(q)$ has preimages t_1' and t_1'' in $\hat{E}_7(q)$, with trace -8 and 8 on $V_{E_7}(\lambda_1)$, respectively (see [40, Proposition 1.2]). Consider the images of $(1, t_1')$ and $(1, t_1'')$ in $\mathrm{SL}_2(q) \circ \hat{E}_7(q)$. We claim that $(1, t_1')$ is a t_1-involution in T, while $(1, t_1'')$ is of type t_8. To see this, let $V = \mathcal{L}(\bar{G})$ and observe that

$$V{\downarrow}A_1 E_7 = \mathcal{L}(A_1 E_7) \oplus (V(1) \otimes V(\lambda_7)).$$

Both $(1, t_1')$ and $(1, t_1'')$ have Jordan form $[-I_{64}, I_{72}]$ on $\mathcal{L}(A_1 E_7)$ (indeed, both elements have centralizer $A_1^2 D_6$ in $A_1 E_7$, which is 72-dimensional), whereas the Jordan forms on $V(1) \otimes V(\lambda_7)$ are $[-I_{64}, I_{48}]$ and $[-I_{48}, I_{64}]$, respectively. This justifies the claim, and we choose $t = (1, t_1'')$ as in Table 15. Finally, we note that $|C_{H_0}(t)| \leqslant 2|\mathrm{SL}_2(q)|^2|\mathrm{SO}_{12}^+(q)|$ since $|C_{\hat{E}_7(q)}(t_1'')| = |\mathrm{SL}_2(q)||\mathrm{SO}_{12}^+(q)|$.

Finally, let us assume that H_0 is of type $\mathrm{L}_2(q)^8$, so

$$H_0 = M_\sigma.\mathrm{AGL}_3(2) = 2^4.\mathrm{L}_2(q)^8.2^4.\mathrm{AGL}_3(2).$$

As in [19, Lemma 2.14], write $\bar{H}^\circ = A_1^8 = J_1 \cdots J_8$ and $Z(J_i) = \langle e_i \rangle$. Recall that we may assume $J_5 J_6 J_7 J_8$ is contained in a D_4 subsystem subgroup of $\bar{G}$ (see Remark 4.14), which implies that $J_1 J_2 J_3 J_4$ is also contained in such a subsystem subgroup (for instance, we can take the A_1 subgroups $\langle X_\alpha, X_{-\alpha} \rangle$ with $\alpha \in \{\alpha_2, \alpha_3, \alpha_5, -\alpha_2 - \alpha_3 - 2\alpha_4 - \alpha_5\}$). As before, let $A = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(q)$, a t_1-type involution, and let t be the image in H_0 of $(A, A, A, A, 1, 1, 1, 1) \in \mathrm{GL}_2(q)^8$. Our choice of ordering for the J_i implies that $e_1 e_2 e_3 e_4 = 1$ by [19, Lemma 2.14(ii)] and hence t is indeed an involution. Moreover, we calculate that

$$|C_{H_0}(t)| \leqslant 2^4|\mathrm{AGL}_3(2)|(q-1)^4|\mathrm{SL}_2(q)|^4 = 21504(q-1)^4|\mathrm{SL}_2(q)|^4.$$

By considering the restriction of $\mathcal{L}(\bar{G})$ to $J_1 J_2 J_3 J_4 = A_1^4$ (via the restriction to A_1^8), or by inspecting Step 2 in the proof of [19, Lemma 2.17], we deduce that t is a t_8-involution in T. The result follows. $\qquad\square$

This completes the proof of Theorem 4.1.

5. Proof of Theorem 2, Part I

By our earlier work in Sections 3 and 4, we have established Theorem 2 in the case where H_0 contains a maximal torus of T and we now move on to consider the remaining possibilities for H. It will be convenient to postpone the analysis of the twisted groups $^2B_2(q)$, $^2G_2(q)$, $^2F_4(q)$ and $^3D_4(q)$ to Section 9, so in the next four sections we will assume

$$T \in \{E_8(q), E_7(q), E_6^\epsilon(q), F_4(q), G_2(q)\}, \tag{17}$$

where $q = p^f$ with p a prime. The following fundamental result (see [37, Theorem 2]) partitions the remaining subgroups into various types (this allows us to refer to *type* (I) *actions* and *type* (II) *subgroups* of G, and so on).

Theorem 5.1. *Let G be an almost simple group with socle $T = (\bar{G}_\sigma)'$ as in* (17). *Let H be a maximal subgroup of G with $G = HT$ and assume H does not contain a maximal torus of T. Set $H_0 = H \cap T$. Then one of the following holds:*

  (I) *$H = N_G(\bar{H}_\sigma)$, where $\bar{H}$ is a maximal closed σ-stable positive dimensional subgroup of G (not parabolic or of maximal rank);*

 (II) *H is of the same type as G (possibly twisted) over a subfield of $\mathbb{F}_q$;*

(III) *H is an exotic local subgroup (determined in [19]);*

(IV) *H is almost simple, and not of type (I) or (II);*

 (V) *$T = E_8(q)$, $p \geqslant 7$ and $H_0 = (\mathrm{Alt}_5 \times \mathrm{Alt}_6).2^2$.*

It is easy to check that $\mathrm{ifix}(T) > n^{4/9}$ in (V), so it remains to handle the subgroups of type (I) - (IV). In this section, we will focus on the cases arising in part (I) of Theorem 5.1 (the subgroups in (II), (III) and (IV) will be handled in Sections 6, 7 and 8, respectively).

Following [41, Theorem 8], we partition the type (I) subgroups into three cases:

  (a) $T = E_7(q)$, $p \geqslant 3$ and $\bar{H}_\sigma = (2^2 \times \mathrm{P}\Omega_8^+(q).2^2).\mathrm{Sym}_3$ or $^3D_4(q).3$;

  (b) $T = E_8(q)$, $p \geqslant 7$ and $H_0 = \mathrm{PGL}_2(q) \times \mathrm{Sym}_5$; \hfill (18)

  (c) $(T, \mathrm{soc}(H_0))$ is one of the cases listed in Table 16 (see [41, Table 3]).

Remark 5.2. We take the opportunity to clarify a couple of potential ambiguities in [41, Table 3] that arise when $T = E_6^\epsilon(q)$. Set $\bar{G} = E_6$.

  (i) First we observe that the maximal subgroups $A_2.2$ and G_2 of $\bar{G}$ are not σ-stable when σ induces a graph automorphism of $\bar{G}$. Indeed, σ-stability would imply that such a subgroup is centralized by a graph automorphism of $\bar{G}$ and is therefore contained in either F_4 or C_4, contradicting maximality. This explains the condition $\epsilon = +$ in Table 16 for $\mathrm{soc}(H_0) = \mathrm{L}_3^\pm(q)$ and $G_2(q)$.

 (ii) We also note that $\bar{H} = A_2G_2$ is a maximal subgroup of $\bar{G}$, with the property that a graph automorphism of $\bar{G}$ induces a graph automorphism on the A_2 factor. In particular, $N_{\bar{G}}(\bar{H}) = \bar{H}$, which means that the value $t = 2$ given in the fourth column of [42, Table 10.1] should be $t = 1$. Indeed, the restriction of $V_{\bar{G}}(\lambda_1)$ to $\bar{H}$ given in the third column of [42, Table 10.2] shows that no element in $\bar{G}$ induces a graph automorphism on the A_2 factor. In terms of Table 16, this means that $\mathrm{soc}(H_0) = \mathrm{L}_3^\epsilon(q) \times G_2(q)$ for the corresponding subgroup of T.

Theorem 5.3. *Let G be an almost simple primitive permutation group of degree n, with socle T and point stabilizer H. Assume T is one of the groups in* (17) *and H is of type*

T	$\mathrm{soc}(H_0)$
$E_8(q)$	$\mathrm{L}_2(q)$ (3 classes, $p \geqslant 23, 29, 31$), $\Omega_5(q)$ ($p \geqslant 5$), $\mathrm{L}_2(q) \times \mathrm{L}_3^\epsilon(q)$ ($p \geqslant 5$),
	$G_2(q) \times F_4(q)$, $\mathrm{L}_2(q) \times G_2(q)^2$ ($p \geqslant 3, q \geqslant 5$), $\mathrm{L}_2(q) \times G_2(q^2)$ ($p \geqslant 3, q \geqslant 5$)
$E_7(q)$	$\mathrm{L}_2(q)$ (2 classes, $p \geqslant 17, 19$), $\mathrm{L}_3^\epsilon(q)$ ($p \geqslant 5$), $\mathrm{L}_2(q)^2$ ($p \geqslant 5$),
	$\mathrm{L}_2(q) \times G_2(q)$ ($p \geqslant 3, q \geqslant 5$), $\mathrm{L}_2(q) \times F_4(q)$ ($q \geqslant 4$), $G_2(q) \times \mathrm{PSp}_6(q)$
$E_6^\epsilon(q)$	$\mathrm{L}_3^\pm(q)$ ($\epsilon = +, p \geqslant 5$), $G_2(q)$ ($\epsilon = +, p \neq 7$), $\mathrm{PSp}_8(q)$ ($p \geqslant 3$), $F_4(q)$,
	$\mathrm{L}_3^\epsilon(q) \times G_2(q)$ ($(q, \epsilon) \neq (2, -)$)
$F_4(q)$	$\mathrm{L}_2(q)$ ($p \geqslant 13$), $G_2(q)$ ($p = 7$), $\mathrm{L}_2(q) \times G_2(q)$ ($p \geqslant 3, q \geqslant 5$)
$G_2(q)$	$\mathrm{L}_2(q)$ ($p \geqslant 7$)

TABLE 16. Some possibilities for $\mathrm{soc}(H_0)$ in Theorem 5.1(I)

T	$\mathrm{soc}(H_0)$	β	Conditions
$E_8(q)$	$\Omega_5(q)$	$58/119$	$p \geqslant 5$
	$\mathrm{L}_2(q)$	$17/35$	3 classes; $p \geqslant 23, 29, 31$
$E_7(q)$	$\mathrm{L}_2(q)$	$31/65$	2 classes; $p \geqslant 17, 19$
$F_4(q)$	$G_2(q)$	$9/19$	$p = 7$
	$\mathrm{L}_2(q)$	$23/49$	$p \geqslant 13$
$G_2(q)$	$\mathrm{L}_2(q)$	$5/11$	$p \geqslant 7$

TABLE 17. Type (I) actions with $\beta < 1/2$ in (19)

(I) *in Theorem 5.1. Then* $\mathrm{ifix}(T) > n^{4/9}$. *Moreover,*

$$\liminf_{q \to \infty} \frac{\log \mathrm{ifix}(T)}{\log n} = \beta \tag{19}$$

and either $\beta \geqslant 1/2$, *or* (T, H_0, β) *is one of the cases in Table* 17.

Remark 5.4. By combining the asymptotic statement in Theorem 5.3 with the analogous statements in Theorems 3.2 and 4.1, this completes the proof of Theorem 3.

We start by handling the special cases in (a) and (b) above (see (18)).

Lemma 5.5. *Theorem 5.3 holds if* $T = E_7(q)$, $p \geqslant 3$ *and* $\bar{H}_\sigma = (2^2 \times \mathrm{P}\Omega_8^+(q).2^2).\mathrm{Sym}_3$ *or* $^3D_4(q).3$.

Proof. In both cases, $\bar{H} = (2^2 \times D_4).\mathrm{Sym}_3$. Set $V = \mathcal{L}(\bar{G})$ and note that

$$V{\downarrow}D_4 = \mathcal{L}(D_4) \oplus V(2\lambda_1) \oplus V(2\lambda_3) \oplus V(2\lambda_4).$$

Let $t \in \bar{H}^\circ = D_4$ be a $\bar{t}_2$-involution (in the notation of [23, Table 4.3.1]). Now t has Jordan form $[-I_{16}, I_{12}]$ on $\mathcal{L}(D_4)$, and it acts as $[-I_{16}, I_{19}]$ on each of the other three summands. Therefore, t has Jordan form $[-I_{64}, I_{69}]$ on V and thus t is a $\bar{t}_1$-involution in $\bar{G}$.

First assume $\bar{H}_\sigma = (2^2 \times \mathrm{P}\Omega_8^+(q).2^2).\mathrm{Sym}_3$. Let t be a t_2-involution (in the notation of [23, Table 4.5.1]) in the $\mathrm{P}\Omega_8^+(q)$ subgroup of H_0. Then $|C_{H_0}(t)| \leqslant 96|\mathrm{L}_2(q)|^4$ and the above calculation shows that t is in the t_1-class of T. This yields $\mathrm{ifix}(T) > n^{4/9}$ and $\beta \geqslant 1/2$.

Finally, suppose $\bar{H}_\sigma = {}^3D_4(q).3$. Now $\bar{H}_\sigma$ contains a unique class of involutions, say $t^{\bar{H}_\sigma}$, which is labelled t_2 in Table 5. Then $|C_{H_0}(t)| = 3|\mathrm{SL}_2(q)||\mathrm{SL}_2(q^3)|$ and once again the above argument shows that t is a t_1-involution of T. The desired result follows. $\square$

Lemma 5.6. *Theorem 5.3 holds if* $T = E_8(q)$, $p \geqslant 7$ *and* $H_0 = \mathrm{PGL}_2(q) \times \mathrm{Sym}_5$.

Proof. Here $\bar{H} = X \times \mathrm{Sym}_5$ and X is an adjoint group of type A_1. We claim that X is contained in a maximal rank subgroup $M = A_1 E_7$ of $\bar{G}$. Given the claim, it follows that $Z(M) \leqslant C_{\bar{G}}(X) = \mathrm{Sym}_5$ and thus the Sym_5 factor of H_0 contains a t_8-involution of T. This immediately implies that $\mathrm{ifix}(T) > n^{4/9}$ and $\beta \geqslant 1/2$.

The claim can be deduced by carefully inspecting [60, Tables 13 and 13A]. To do this, first note that X is constructed in [37, Lemma 1.5] as a diagonal subgroup of $A_1 A_1 < A_4 A_4$, with each A_1 acting as $V(4)$ on $V_{A_4}(\lambda_1)$. In the notation of [60], we have $X = E_8(\#17^{\{0\}})$, which is a subgroup of D_8 with

$$V_{D_8}(\lambda_1){\downarrow}X = V(6) \oplus V(4) \oplus V(2) \oplus 0.$$

It follows that X is contained in a subgroup $A_1^2 D_6 < D_8$ (this can also be seen by inspecting [60, Table 13A]) and we conclude by noting that $A_1^2 D_6 < A_1 E_7$. This justifies the claim and the result follows. $\qquad\square$

To complete the proof of Theorem 5.3, we may assume that $(T, \mathrm{soc}(H_0))$ is one of the cases listed in Table 16. Our approach is very similar to the proof of Proposition 4.8. As recorded in Table 18, we identify an involution $t \in H_0$, we compute an upper bound $|C_{H_0}(t)| \leqslant f(q)$ and we determine the T-class of t, which allows us to use the bounds in (8). In this way, we deduce that $\mathrm{ifix}(T) > n^{4/9}$ and we establish the bound $\beta \geqslant \gamma$, where γ is given in the final column of Table 18. In the table, we list the socle of H_0 – in some cases, this does not uniquely determine H_0 (up to conjugacy), but this potential ambiguity has no effect on the analysis of these cases.

Remark 5.7. Consider the case in Table 16 with $T = E_7(q)$ and $\mathrm{soc}(H_0) = \mathrm{L}_2(q)^2$, so $\bar{G} = E_7$, $p \geqslant 5$ and $\bar{H} = J_1 J_2$ is a direct product of two adjoint groups of type A_1. Write $\bar{G} = \hat{G}/Z$ and $\bar{H} = \hat{H}/Z$, where $\hat{G}$ is the simply connected group and $Z = Z(\hat{G})$. To be consistent with [60], we will order the factors of $\bar{H}$ so that $\hat{H} = \hat{J}_1 \hat{J}_2$ and $\hat{J}_1$ is simply connected (note that the composition factors of $\mathcal{L}(\bar{G}){\downarrow}\bar{H}$ are given in [60, Table 59]). It follows that

$$H_0 = (\hat{J}_1 \hat{J}_2)_\sigma / Z = (\mathrm{SL}_2(q) \times \mathrm{PGL}_2(q))/Z = \mathrm{L}_2(q) \times \mathrm{PGL}_2(q)$$

and this explains why we can choose an involution of the form $(1, t_1)$ in H_0.

Remark 5.8. Let us explain the entries γ_1 and $(1, \gamma_1)$ appearing in the third column of Table 18. In both cases, γ_1 denotes an involutory graph automorphism on the $\mathrm{L}_3^\delta(q)$ factor of $\mathrm{soc}(H_0)$. To justify the existence of these involutions in H_0, let $(\bar{G}, \bar{H}) = (E_6, A_2.2)$ and $(E_8, A_1 A_2.2)$ be the corresponding algebraic groups, where $\bar{H}$ is σ-stable. From the structure and maximality of $\bar{H}$, it follows that there exists an involution in $\bar{H} \setminus \bar{H}^\circ$ inducing a graph automorphism on the A_2 factor of $\bar{H}^\circ$. Moreover, there is a unique conjugacy class of such involutions in $\bar{H} \setminus \bar{H}^\circ$ (see [23, Table 4.3.1]), so this class is σ-stable and thus Lang's theorem implies that $\bar{H}_\sigma$ (and thus H_0) contains an involution inducing a graph automorphism on the $\mathrm{L}_3^\delta(q)$ factor.

Proof of Theorem 5.3. We need to justify the information presented in Table 18. The arguments are similar (and easier) than those in the previous section for maximal rank subgroups, so we only provide a brief sketch. Let us write

$$\mathrm{soc}(H_0) = X_1 \times \cdots \times X_m,$$

where each X_i is a simple group of Lie type (note that $m \leqslant 3$). The possibilities for $\bar{H}$ are given in [42, Table 1] and we note that $N_{\bar{G}_\sigma}(\bar{H}_\sigma) = \bar{H}_\sigma$, so either $H_0 = \bar{H}_\sigma$, or $T = E_6^\epsilon(q)$ or $E_7(q)$ and $|\bar{H}_\sigma : H_0| \leqslant (3, q - \epsilon)$ or $(2, q - 1)$, respectively. In fact, it is not too difficult to determine the precise structure of H_0, but the above information will be sufficient for our purposes.

T	$\mathrm{soc}(H_0)$	t	$f(q)$	t^T	γ
$G_2(q)$	$\mathrm{L}_2(q)$	t_1 and t_1'	$2(q-1)$ and $2(q+1)$	t_1	$5/11$
$F_4(q)$	$\mathrm{L}_2(q) \times G_2(q)$	$(t_1,1)$	$2(q-1)|G_2(q)|$	t_4	$3/5$
	$G_2(q)$	t_1	$|\mathrm{SL}_2(q)|^2$	t_1	$9/19$
	$\mathrm{L}_2(q)$	t_1	$2(q-1)$	t_1	$23/49$
$E_6^\epsilon(q)$	$F_4(q)$	t_1	$|\mathrm{SL}_2(q)||\mathrm{Sp}_6(q)|$	t_2	$7/13$
		A_1	$q^{15}|\mathrm{Sp}_6(q)|$	A_1	$10/13$
	$\mathrm{PSp}_8(q)$	t_2	$2|\mathrm{Sp}_4(q)|^2$	t_1	$13/21$
	$\mathrm{L}_3^\epsilon(q) \times G_2(q)$	$(t_1,1)$	$|\mathrm{GL}_2^\epsilon(q)||G_2(q)|$	t_1	$1/2$
		$(1,A_1)$	$q^5|\mathrm{SL}_2(q)||\mathrm{SL}_3^\epsilon(q)|$	A_1	$5/7$
	$\mathrm{L}_3^\pm(q), \epsilon = +$	γ_1	$2|\mathrm{SO}_3(q)|$	t_2	$1/2$
	$G_2(q), \epsilon = +$	t_1	$|\mathrm{SL}_2(q)|^2$	t_1	$1/2$
		$\tilde{A}_1$	$q^3|\mathrm{SL}_2(q)|$	A_1^3	$1/2$
$E_7(q)$	$G_2(q) \times \mathrm{PSp}_6(q)$	(t_1,t_1)	$|\mathrm{SL}_2(q)|^3|\mathrm{Sp}_4(q)|$	t_1	$25/49$
		$(A_1,1)$	$q^5|\mathrm{SL}_2(q)||\mathrm{Sp}_6(q)|$	A_1	$40/49$
	$\mathrm{L}_2(q) \times F_4(q)$	$(1,t_1)$	$|\mathrm{SL}_2(q)|^2|\mathrm{Sp}_6(q)|$	t_1	$7/13$
		$(1,A_1)$	$q^{15}|\mathrm{SL}_2(q)||\mathrm{Sp}_6(q)|$	A_1	$10/13$
	$\mathrm{L}_2(q) \times G_2(q)$	$(1,t_1)$	$|\mathrm{SL}_2(q)|^3$	t_1	$15/29$
	$\mathrm{L}_2(q)^2$	$(1,t_1)$	$2|\mathrm{GL}_2(q)|$	t_1	$65/127$
	$\mathrm{L}_3^\epsilon(q)$	t_1	$2|\mathrm{GL}_2^\epsilon(q)|$	t_1	$13/25$
	$\mathrm{L}_2(q)$	t_1 or t_1'	$2(q+1)$	t_4 or t_4'	$31/65$
$E_8(q)$	$G_2(q) \times F_4(q)$	(t_1,t_4)	$|\mathrm{SL}_2(q)|^2|\mathrm{SO}_9(q)|$	t_8	$47/91$
		$(1,A_1)$	$q^{15}|G_2(q)||\mathrm{Sp}_6(q)|$	A_1	$10/13$
	$\mathrm{L}_2(q) \times G_2(q)^2$	$(1,t_1,t_1)$	$2|\mathrm{SL}_2(q)|^5$	t_8	$121/217$
	$\mathrm{L}_2(q) \times G_2(q^2)$	$(1,t_1)$	$2|\mathrm{SL}_2(q)||\mathrm{SL}_2(q^2)|^2$	t_8	$121/217$
	$\mathrm{L}_2(q) \times \mathrm{L}_3^\epsilon(q)$	$(1,\gamma_1)$	$2|\mathrm{SL}_2(q)||\mathrm{SO}_3(q)|$	t_8	$130/237$
	$\Omega_5(q)$	t_1	$2|\mathrm{GL}_2(q)|$	t_1	$58/119$
	$\mathrm{L}_2(q)$	t_1	$2(q-1)$	t_1	$17/35$

TABLE 18. The algebraic subgroups in Theorem 5.3

First assume $p = 2$. Here the only possibilities for T and $\mathrm{soc}(H_0)$ are as follows:

$$T = E_8(q): \quad G_2(q) \times F_4(q)$$
$$T = E_7(q): \quad \mathrm{L}_2(q) \times F_4(q) \, (q \geqslant 4), \, G_2(q) \times \mathrm{Sp}_6(q)$$
$$T = E_6^\epsilon(q): \quad G_2(q) \, (\epsilon = +), \, F_4(q), \, \mathrm{L}_3^\epsilon(q) \times G_2(q) \, ((q,\epsilon) \neq (2,-))$$

If $T = E_6(q)$ and $\mathrm{soc}(H_0) = G_2(q)$ then let $t \in H_0$ be an involution in the class $\tilde{A}_1$, so $|C_{H_0}(t)| = q^3|\mathrm{SL}_2(q)|$ (see Table 5). By inspecting [31, Table 31], we see that t is in the T-class A_1^3 and it is easy to check that the desired bounds hold. In each of the remaining cases, we can choose an involution $t \in X_i$ that is in the A_1-class of T (we choose i so that $|t^{X_i}|$ is maximal) and the result quickly follows.

For the remainder, let us assume p is odd. First consider the cases in Table 17, where we claim that $\beta < 1/2$. Here it is straightforward to find a representative of each class of

involutions in H_0 (or in $\bar{H}_\sigma$ if $T = E_7(q)$), using [23, Table 4.5.1], and we can determine the corresponding T-class by studying the restriction of $\mathcal{L}(\bar{G})$ to $\bar{H}$ (see [42, Table 10.1]). It turns out that every involution in H_0 is contained in the largest T-class and the result follows.

For example, suppose $T = F_4(q)$ and $\mathrm{soc}(H_0) = \mathrm{L}_2(q)$. Here $H_0 = \mathrm{PGL}_2(q)$ has two classes of involutions, say x_1 and x_2, with labels t_1 and t_1', respectively. Now $\bar{H} = A_1$ has a unique class of involutions, say $y^{\bar{H}}$, so x_1 and x_2 are $\bar{H}$-conjugate and thus $\bar{G}$-conjugate. In view of Remark 2.2, all the involutions in H_0 are T-conjugate. To identify the T-class, we calculate the dimension of the 1-eigenspace of y on $V = \mathcal{L}(\bar{G})$. By [42, Table 10.1], the restriction $V{\downarrow}\bar{H}$ has the same composition factors as the $\bar{H}$-module

$$W = W(22) \oplus W(14) \oplus W(10) \oplus W(2),$$

where $W(m)$ is the Weyl module for $\bar{H}$ of highest weight $m\lambda_1$. Since $V{\downarrow}\langle y \rangle$ is completely reducible, we just need to calculate the Jordan form of y on W, which is easily seen to be $[-I_{28}, I_{24}]$ (note that y has Jordan form $[-I_{m+1}, I_m]$ on $W(2m)$ with m odd). We conclude that the involutions in T are in the t_1-class of T. The result follows.

Similar arguments apply in each of the remaining cases. We highlight the following three cases:

  (a) $T = G_2(q)$ and $\mathrm{soc}(H_0) = \mathrm{L}_2(q)$;
  (b) $T = E_6^\epsilon(q)$ and $\mathrm{soc}(H_0) = \mathrm{L}_3^\epsilon(q)$;
  (c) $T = E_8(q)$ and $\mathrm{soc}(H_0) = \mathrm{L}_2(q) \times \mathrm{L}_3^\epsilon(q)$.

Consider (a). This is the only case where we need to compute $|t^T \cap H_0|$ in order to verify the bound $\mathrm{ifix}(T) > n^{4/9}$. Here $H_0 = \mathrm{PGL}_2(q)$ and $t^T \cap H_0 = (t_1)^{H_0} \cup (t_1')^{H_0}$, so $|t^T \cap H_0| = i_2(H_0) = q^2$ (this explains why we write "t_1 and t_1'" in the first row of Table 18). Note that T has a unique class of involutions.

Now consider case (b). Here $p \geqslant 5$ and H_0 contains an involutory graph automorphism $t = \gamma_1$ of $\mathrm{L}_3^\epsilon(q)$, as explained in Remark 5.8. We claim that t is in the T-class t_2. To see this, let $V = \mathcal{L}(\bar{G})$ and observe that

$$V{\downarrow}A_2 = \mathcal{L}(A_2) \oplus V(4\lambda_1 + \lambda_2) \oplus V(\lambda_1 + 4\lambda_2).$$

Since t interchanges the final two summands, we deduce that t has Jordan form $[-I_{40}, I_{38}]$ on V and the claim follows. Similarly, in (c) we choose $t = (1, \gamma_1) \in H_0$, where γ_1 induces a graph automorphism on the $\mathrm{L}_3^\epsilon(q)$ factor of $\mathrm{soc}(H_0)$ (again, see Remark 5.8 for the existence of t). To determine the T-class of t, we appeal to [60, 9.8], which shows that $C_{\bar{G}}(t) = A_1 E_7$ and thus t is a t_8-involution. The result follows. $\square$

6. Proof of Theorem 2, Part II

In this section we turn to the subgroups arising in part (II) of Theorem 5.1; our main result is Theorem 6.1 below. We continue to assume that T is one of the groups in (17). The possibilities for H are as follows:

  (a) H is a subfield subgroup of G (defined over a subfield $\mathbb{F}_{q_0}$ of $\mathbb{F}_q$ of prime index);
  (b) $T = E_6(q)$, $H_0 = {}^2E_6(q_0)$ and $q = q_0^2$;
  (c) $T = F_4(q)$, $H_0 = {}^2F_4(q)$ and $q = 2^{2m+1}$ (with $m \geqslant 0$);
  (d) $T = G_2(q)$, $H_0 = {}^2G_2(q)$ and $q = 3^{2m+1}$ (with $m \geqslant 0$).

Theorem 6.1. *Let G be an almost simple primitive permutation group of degree n, with socle T and point stabilizer H. Assume T is one of the groups in (17) and H is of type (II) in Theorem 5.1. Then either $\mathrm{ifix}(T) > n^{1/2}$, or $n^\alpha < \mathrm{ifix}(T) < n^{4/9}$ and (T, H_0, α) is one of the cases in Table 19.*

T	H_0	α	Conditions
$G_2(q)$	$G_2(q_0)$	$3/7$	q odd
	${}^2G_2(q)$	$2/5$	$q = 3^{2m+1}$, $m \geqslant 0$

TABLE 19. Type (II) actions with $\mathrm{ifix}(T) \leqslant n^{1/2}$

Proof. First assume H is a subfield subgroup of G. A similar argument applies in each case, so for brevity we illustrate the general approach with an example. Suppose $T = E_8(q)$ and $H_0 = E_8(q_0)$, where $q = q_0^k$ with k prime. First assume $p \neq 2$ and let $t \in H_0$ be an involution, so t is of type t_1 or t_8 as an element of H_0 (see Table 5), and we note that the T-class of t has the same label (that is, $t^T \cap H_0 = t^{H_0}$). In particular, if t is a t_8-involution, then the expression for $\mathrm{fix}(t)$ in (2) implies that $\mathrm{fix}(t) > n^{1/2}$ if and only if

$$\frac{|\mathrm{SL}_2(q)||E_7(q)|}{|E_8(q)|^{1/2}} > \frac{|\mathrm{SL}_2(q_0)||E_7(q_0)|}{|E_8(q_0)|^{1/2}}.$$

One checks that the expression on the left hand side is increasing as a function of q and the inequality is easily verified since $q \geqslant q_0^2$. Similarly, if $p = 2$ and $t \in H_0$ is an A_1-involution, then $|C_T(t)| = q^{57}|E_7(q)|$, $|C_{H_0}(t)| = q_0^{57}|E_7(q_0)|$ and once again it is straightforward to check that $\mathrm{fix}(t) > n^{1/2}$.

To complete the proof, it remains to consider the cases labelled (b), (c) and (d) above. The argument in (b) is almost identical to the subfield case $E_6(q^{1/2}) < E_6(q)$ and we omit the details. In (c), if we take an involution $t \in H_0$ in the class $(\tilde{A}_1)_2$, then the T-class of t has the same label (see [57, Table II]) and the result quickly follows. Finally, in (d) we note that both T and H_0 have a unique class of involutions and it is simple to check that $\mathrm{ifix}(T) = q(q^2 - 1)$ and $n = q^3(q^3 - 1)(q + 1)$. The result follows. $\square$

Remark 6.2. It is straightforward to improve the bound $\mathrm{ifix}(T) > n^{1/2}$ in Theorem 6.1, if needed. For example, suppose $T = E_8(q)$, $H_0 = E_8(q_0)$ and $q = q_0^k$ is even. If we take an A_1-involution $t \in H_0$ as above, then

$$\mathrm{fix}(t) = \frac{q^{57}|E_7(q)|}{q_0^{57}|E_7(q_0)|}, \quad n = \frac{|E_8(q)|}{|E_8(q_0)|}$$

and one checks that $\mathrm{ifix}(T) > n^{95/124}$. Similarly, if q is odd then $\mathrm{ifix}(T) > n^{17/31}$.

7. PROOF OF THEOREM 2, PART III

Next we handle the exotic local subgroups arising in part (III) of Theorem 5.1. The relevant subgroups are listed in Table 20 (see [19, Table 1]). Note that the case $T = E_7(q)$ with $H_0 = (2^2 \times \mathrm{P}\Omega_8^+(q).2).\mathrm{Sym}_3$ has already been handled in Section 5, so it is omitted in Table 20.

Theorem 7.1. *Let G be an almost simple primitive permutation group of degree n, with socle T and point stabilizer H. Assume T is one of the groups in (17) and H is of type (III) in Theorem 5.1. Then $\mathrm{ifix}(T) > n^{\alpha}$ and either $\alpha \geqslant 1/2$, or (T, H_0, α) is one of the cases in Table 21. In particular, $\mathrm{ifix}(T) > n^{4/9}$ unless $T = G_2(q)$, $H_0 = 2^3.\mathrm{SL}_3(2)$ and $q = p \geqslant 11$.*

Proof. First consider case (a) in Table 20. Here $H_0 = \mathrm{AGL}_3(2)$ and (10) holds. Since $i_2(H_0) = 91$ it follows that $n^{3/7} < \mathrm{ifix}(T) < n^{1/2}$ and one checks that $\mathrm{ifix}(T) > n^{4/9}$ if $q \in \{3, 5, 7\}$.

	T	H_0	Conditions
(a)	$G_2(q)$	$2^3.\mathrm{SL}_3(2)$	$q = p \geqslant 3$
(b)	$F_4(q)$	$3^3.\mathrm{SL}_3(3)$	$q = p \geqslant 5$
(c)	$E_6^\epsilon(q)$	$3^{3+3}.\mathrm{SL}_3(3)$	$q = p \geqslant 5,\ q \equiv \epsilon \pmod 3$
(d)	$E_8(q)$	$2^{5+10}.\mathrm{SL}_5(2)$	$q = p \geqslant 3$
(e)	$E_8(q)$	$5^3.\mathrm{SL}_3(5)$	$p \neq 2, 5,\ q = \begin{cases} p & p \equiv \pm 1 \pmod 5 \\ p^2 & p \equiv \pm 2 \pmod 5 \end{cases}$
(f)	$^2E_6(2)$	$\mathrm{U}_3(2) \times G_2(2)$	
(g)	$E_7(3)$	$\mathrm{L}_2(3) \times F_4(3)$	

TABLE 20. Exotic local maximal subgroups

T	H_0	α	Conditions
$G_2(q)$	$2^3.\mathrm{SL}_3(2)$	$3/7$	$q = p \geqslant 3$
$F_4(q)$	$3^3.\mathrm{SL}_3(3)$	$6/13$	$q = p \geqslant 5$
$E_6^\epsilon(q)$	$3^{3+3}.\mathrm{SL}_3(3)$	$19/39$	$q = p \geqslant 5,\ q \equiv \epsilon \pmod 3$
$E_8(q)$	$5^3.\mathrm{SL}_3(5)$	$15/31$	$p \neq 2, 5,\ q = \begin{cases} p & p \equiv \pm 1 \pmod 5 \\ p^2 & p \equiv \pm 2 \pmod 5 \end{cases}$

TABLE 21. Type (III) actions with $\mathrm{ifix}(T) \leqslant n^{1/2}$

Next let us turn to (b). Here $H_0 = \mathrm{ASL}_3(3)$ has a unique class of involutions, say t^{H_0}, with $|t^{H_0}| = 1053$. By a theorem of Borovik [8] (also see [43, Theorem 1]), H_0 acts irreducibly on the Lie algebra $V = \mathcal{L}(\bar{G})$. By inspecting the character table of H_0 (with the aid of MAGMA, for example), we deduce that t has trace -4 on V, so t has Jordan form $[-I_{28}, I_{24}]$ and thus t is a t_1-involution in T. It follows that $n^{6/13} < \mathrm{ifix}(T) < n^{1/2}$.

In (c), first note that $H_0 = 3^{3+3}.\mathrm{SL}_3(3)$ is a maximal parabolic subgroup of $\Omega_7(3)$ (see the final paragraph in the proof of [19, Lemma 2.11]). One checks that H_0 has a unique class of involutions t^{H_0} (of size 9477) and by inspecting the character table of H_0 (constructed via MAGMA), using the fact that H_0 acts irreducibly on $\mathcal{L}(\bar{G})$, we deduce that t is a t_2-involution. Therefore, $n^{19/39} < \mathrm{ifix}(T) < n^{1/2}$. Case (e) is very similar. Here $H_0 = \mathrm{ASL}_3(5)$ has a unique class of involutions t^{H_0} of size 19375. Again, H_0 acts irreducibly on $\mathcal{L}(\bar{G})$ and from the character table of H_0 we deduce that t is a t_1-involution. This gives $n^{15/31} < \mathrm{ifix}(T) < n^{1/2}$.

Now let us turn to (d). Here $H_0 = N_T(E)$, where E is elementary abelian of order 2^5. By inspecting the proof of [19, Lemma 2.17] we see that there is an involution $t \in H_0 \setminus E$ which is contained in the t_8-class of T (in the notation of [19], we can take $t = e_1$, a generator of the centre of a fundamental SL_2 subgroup of $\bar{G} = E_8$). The bound $\mathrm{ifix}(T) > n^{1/2}$ quickly follows.

Finally, let us consider cases (f) and (g) in Table 20. In (f), set $t = (1, A_1) \in H_0$ (that is, we take t to be an A_1-involution in the $G_2(2)$ factor) and note that $|t^{H_0}| = 63$. Then t is an A_1-involution in T and we quickly deduce that $\mathrm{ifix}(T) > n^{1/2}$. Now let us turn to case (g). Let $t \in H_0$ be a t_1-involution in the $F_4(3)$ factor (that is, $t = (1, t_1) \in H_0$). We claim that t is a t_1-involution in T. To see this, let $V = \mathcal{L}(\bar{G})$ be the Lie algebra of $\bar{G}$ and let $\bar{H} = A_1 F_4$ be a σ-stable subgroup of $\bar{G}$ such that $H_0 = N_T(\bar{H}_\sigma)$. Then

$$V \downarrow A_1 F_4 = \mathcal{L}(A_1 F_4) \oplus (W(2) \otimes W(\lambda_4)),$$

T	H_0	Conditions
$G_2(q)$	$L_2(13)$	$q \geqslant 17$, $p \neq 13$, $\mathbb{F}_q = \mathbb{F}_p[\sqrt{13}]$
	$L_2(8)$	$q \geqslant 23$, $p \geqslant 5$, $\mathbb{F}_q = \mathbb{F}_p[\omega]$, $\omega^3 - 3\omega + 1 = 0$
	$U_3(3){:}2$	$q = p \geqslant 11$

TABLE 22. Type (IV) actions with $\mathrm{ifix}(T) \leqslant n^{4/9}$

where $W(\mu)$ is the Weyl module of highest weight μ. Now $\dim C_{\bar{H}}(t) = 27$, so t has Jordan form $[-I_{28}, I_{27}]$ on $\mathcal{L}(A_1 F_4)$. Similarly, by applying [40, Proposition 1.2], we deduce that t acts as $[-I_{12}, I_{14}]$ on $W(\lambda_4)$, so $[-I_{36}, I_{42}]$ is the Jordan form on $W(2) \otimes W(\lambda_4)$. Therefore, t has Jordan form $[-I_{64}, I_{69}]$ on V, so $\dim C_{\bar{G}}(t) = 69$ and the claim follows. One now checks that $\mathrm{ifix}(T) > n^{1/2}$, as required. $\qquad\square$

8. PROOF OF THEOREM 2, PART IV

In this section we study the subgroups in part (IV) of Theorem 5.1, which allows us to complete the proof of Theorem 2 when T is one of the groups in (17). Here H is almost simple and there are two cases to consider, according to the nature of the socle S of H. Recall that T is defined over $\mathbb{F}_q$, where $q = p^f$ and p is a prime. Let $\mathrm{rk}(T)$ be the (untwisted) Lie rank of T (and similarly $\mathrm{rk}(S)$ if S is a group of Lie type). Let $\mathrm{Lie}(p)$ be the set of finite simple groups of Lie type defined over fields of characteristic p. The following result is part of [41, Theorem 8] (the value of $u(E_8(q))$ in part (ii)(c) is taken from [33]).

Theorem 8.1. *Let G be an almost simple group with socle T, where T is one of the groups in (17). Let H be a maximal almost simple subgroup of G as in part (IV) of Theorem 5.1, with socle S. Then one of the following holds:*

- (i) $S \notin \mathrm{Lie}(p)$ *and the possibilities for S are described in* [40];
- (ii) $S = H(q_0) \in \mathrm{Lie}(p)$, $\mathrm{rk}(S) \leqslant \frac{1}{2}\mathrm{rk}(T)$ *and one of the following holds:*
  - (a) $q_0 \leqslant 9$;
  - (b) $S = L_3^\epsilon(16)$;
  - (c) $S = L_2(q_0)$, ${}^2B_2(q_0)$ *or* ${}^2G_2(q_0)$, *where $q_0 \leqslant (2, q-1)u(T)$ and $u(T)$ is defined as follows:*

T	$G_2(q)$	$F_4(q)$	$E_6^\epsilon(q)$	$E_7(q)$	$E_8(q)$
$u(T)$	12	68	124	388	1312

Our main result is the following.

Theorem 8.2. *Let G be an almost simple primitive permutation group of degree n, with socle T and point stabilizer H. Assume T is one of the groups in (17) and H is of type (IV) in Theorem 5.1. Then $\mathrm{ifix}(T) > n^\alpha$ and either $\alpha \geqslant 4/9$, or $\alpha = 3/7$ and (T, H_0) is one of the cases in Table 22.*

8.1. **The case $S \notin \mathrm{Lie}(p)$.** Let us start by considering the case where S is not a simple group of Lie type over a field of characteristic p (in the literature, H is said to be *non-generic*). The possibilities for T and S (up to isomorphism) are determined in [40], with the relevant cases recorded in a sequence of tables (see [40, Section 10]).

Lemma 8.3. *Theorem 8.2 holds if $T = G_2(q)$ and $S \notin \mathrm{Lie}(p)$.*

H_0	Conditions
$\mathrm{L}_2(13)$	$p \neq 13$, $\mathbb{F}_q = \mathbb{F}_p[\sqrt{13}]$
$\mathrm{L}_2(8)$	$p \geqslant 5$, $\mathbb{F}_q = \mathbb{F}_p[\omega]$, $\omega^3 - 3\omega + 1 = 0$
$\mathrm{U}_3(3){:}2$	$q = p \geqslant 5$
J_1	$q = 11$
J_2	$q = 4$

TABLE 23. Maximal almost simple subgroups, $T = G_2(q)$, $S \notin \mathrm{Lie}(p)$

Proof. The maximal subgroups of G are determined up to conjugacy in [21] (q even) and [26] (q odd) and the relevant cases are listed in Table 23 (as usual, $H_0 = H \cap T$). The result for $q \leqslant 4$ can be checked directly, using MAGMA, so let us assume $q \geqslant 5$. Then (10) holds (since q is odd) and the result follows by computing $i_2(H_0)$. □

For the remaining groups, we adopt the following approach. First we inspect [40] to determine the (finite) list of possibilities for S. Given such a subgroup S, set

$$
\begin{aligned}
a &= \max\{|t^S| \,:\, t \in S \text{ is an involution}\} \\
b &= \max\{|t^T| \,:\, t \in T \text{ is an involution}\} \\
c &= |\mathrm{Aut}(S)| \\
d &= |T|
\end{aligned}
\tag{20}
$$

If we choose an involution $t \in S$ with $|t^S| = a$, then $|t^T| \leqslant b$ and thus (2) implies that $\mathrm{ifix}(T) \geqslant \mathrm{fix}(t) \geqslant na/b$. Therefore,

$$
\frac{\log \mathrm{ifix}(T)}{\log n} \geqslant 1 - \frac{\log b - \log a}{\log n} \geqslant 1 - \frac{\log b - \log a}{\log d - \log c}
\tag{21}
$$

since $n \geqslant d/c$. It is easy to compute this lower bound and this quickly reduces the problem to a handful of cases that require closer attention.

Proposition 8.4. *Theorem 8.2 holds if $S \notin \mathrm{Lie}(p)$.*

Proof. By the previous lemma, we may assume $T \neq G_2(q)$. To illustrate the approach outlined above, suppose $T = F_4(q)$ and note that

$$
b = \begin{cases} q^{14}(q^4 + q^2 + 1)(q^4 + 1)(q^6 + 1) & q \text{ odd} \\ q^4(q^4 + q^2 + 1)(q^8 - 1)(q^{12} - 1) & q \text{ even} \end{cases}
\tag{22}
$$

(see Table 5). From [40], we deduce that

$$S \in \{\mathrm{Alt}_\ell, \mathrm{L}_2(r), \mathrm{L}_3^\epsilon(3), \mathrm{L}_4(3), {}^3D_4(2), \mathrm{M}_{11}, \mathrm{J}_1, \mathrm{J}_2\}$$

with $\ell \in \{5, \dots, 10\}$ and $r \in \{7, 8, 13, 17, 25, 27\}$. We now check that the lower bound in (21) yields $\mathrm{ifix}(T) > n^{4/9}$, unless $q = 2$ and S is one of the following:

$$\mathrm{Alt}_6, \ \mathrm{Alt}_8, \ \mathrm{Alt}_9, \ \mathrm{L}_2(7), \ \mathrm{L}_2(27), \ \mathrm{L}_3^\epsilon(3), \ \mathrm{L}_4(3).$$

In [53], Norton and Wilson determine the maximal subgroups of $F_4(2)$ and its automorphism group, and we see that $H_0 = \mathrm{L}_4(3){:}2$ is the only option. There are two classes of involutions in S and we choose $t \in S$ in the smaller class, so $|t^S| = 2106$. Now S has a 26-dimensional irreducible module V over $\mathbb{F}_2$, which we can identify with the minimal module for T (as noted in [43, Corollary 2], S acts irreducibly on V). With the aid of MAGMA, we calculate that t has Jordan form $[J_2^{10}, J_1^6]$ on V and by inspecting [32, Table 3] we deduce that t belongs to the T-class labelled $\tilde{A}_1$ or $(\tilde{A}_1)_2$ in Table 5. In particular, $|t^T| \leqslant (q^4 + 1)(q^6 - 1)(q^{12} - 1)$ and this implies that $\mathrm{ifix}(T) > n^{3/5}$.

Very similar reasoning applies when $T \in \{E_6^\epsilon(q), E_7(q), E_8(q)\}$ and we omit the details. (Note that for $T = E_6(2)$, $^2E_6(2)$ and $E_7(2)$, the maximal subgroups of G are given in [29], [20] and [4], respectively.) $\square$

8.2. The case $S \in \mathrm{Lie}(p)$.

To complete the proof of Theorem 8.2, we may assume that S is a simple group of Lie type over a field $\mathbb{F}_{q_0}$ of characteristic p (H is said to be *generic*). We will use the restrictions on both q_0 and $\mathrm{rk}(S)$ given in part (ii) of Theorem 8.1.

As in the previous section, for $T = G_2(q)$ we can inspect [21, 26] to determine the possibilities for S. In fact, one checks there are no maximal subgroups that satisfy the conditions in Theorem 8.1(ii) which are not of type (I) and (II) in Theorem 5.1. (Note that the maximal subgroup $H_0 = \mathrm{PGL}_2(q) < G_2(q)$ in [26] (with $p \geqslant 7$) is of the form $N_G(\bar{H}_\sigma)$ as in Theorem 5.1(I), with $\bar{H} = A_1$, so it has been handled in Section 5.)

The following lemma will be useful in the proof of Theorem 8.2 for $T = F_4(q)$ (we thank David Craven for suggesting this approach).

Lemma 8.5. *Let G be an almost simple group with socle $T = F_4(q)$ and let H be a maximal almost simple subgroup of G with socle $S = \mathrm{L}_3^\epsilon(q_0)$. Then*

$$(q, q_0) \notin \{(3,9), (4,16), (8,16)\}.$$

Proof. Write $T = \bar{G}_\sigma$, where σ is a Steinberg endomorphism of $\bar{G} = F_4$ and let V be the Lie algebra of $\bar{G}$. Seeking a contradiction, let us assume (q, q_0) is one of the three possibilities above. We proceed by arguing as in the proof of [39, Corollary 3].

Fix a semisimple element $x \in S$ of order $n > u(T) = 68$ (for example, if $q = 3$ we can take $n = 91$ if $\epsilon = +$, and $n = 80$ if $\epsilon = -$). By [39, Proposition 2], there exists a positive dimensional closed subgroup $\bar{L}$ of $\bar{G}$ containing x such that

(a) Some non-trivial power of x, say x^k, lies in $\bar{L}^\circ$; and

(b) Every x-invariant subspace of V is also $\bar{L}$-invariant.

Set $\bar{J} = \langle S, \bar{L} \rangle^\circ$. Then $1 \neq x^k \in S \cap \bar{J}$, so $S \cap \bar{J}$ is a non-trivial normal subgroup of S and thus $S < \bar{J}$ since S is simple.

Let $\mathcal{M}$ be the collection of all S-invariant subspaces of all $K\bar{G}$-composition factors of V and set

$$\bar{Y} = \bigcap_{U \in \mathcal{M}} \bar{G}_U,$$

where $\bar{G}_U$ denotes the setwise stabilizer of U in $\bar{G}$. By [39, Theorem 4], $\bar{Y}$ is a proper closed subgroup of $\bar{G}$ (it is also σ-stable by [39, Proposition 1.12(iii)]). Clearly, each $U \in \mathcal{M}$ is x-invariant, and thus $\bar{L}$-invariant by (b) above. In particular, each $U \in \mathcal{M}$ is $\bar{J}$-invariant and thus

$$S < \bar{J} \leqslant \bar{Y} < \bar{G}.$$

Finally, by arguing as in the proof of [39, Theorem 6], we deduce that $H \leqslant N_G(\bar{H}_\sigma) < G$ for some maximal closed connected reductive σ-stable subgroup $\bar{H}$ of $\bar{G}$ with $\bar{Y}^\circ \leqslant \bar{H}$ (see the remark following the statement of Theorem 6 in [39]). Therefore, $H = N_G(\bar{H}_\sigma)$ by the maximality of H. In terms of Theorem 5.1, this means that H is a subgroup of type (I), but these subgroups are known and by inspection there are no examples with socle S (see Table 16). This final contradiction completes the proof of the lemma. $\square$

Lemma 8.6. *Theorem 8.2 holds if $T = F_4(q)$ and $S \in \mathrm{Lie}(p)$.*

Proof. According to Theorem 8.1(ii), the possibilities for S are recorded in Table 24 (where q_0 is a power of p). To handle these cases, we start by considering the lower bound in

S	Conditions
$\mathrm{L}_2(q_0)$	$4 \leqslant q_0 \leqslant 68(2, q-1)$
$^2B_2(q_0)$	$q_0 \in \{8, 32\}$
$^2G_2(q_0)$	$q_0 = 27$
$\mathrm{L}_3^\epsilon(q_0)$	$q_0 \leqslant 16$
$G_2(q_0), \mathrm{PSp}_4(q_0), \Omega_5(q_0)$	$q_0 \leqslant 9$

TABLE 24. Possible maximal almost simple subgroups, $T = F_4(q)$, $S \in \mathrm{Lie}(p)$

(21). If we define a, b, c, d as in (20), then $\mathrm{ifix}(T) > n^{4/9}$ if

$$\frac{d^{5/9}}{b} > \frac{c^{5/9}}{a} \tag{23}$$

where b is given in (22) and

$$d = q^{24}(q^2 - 1)(q^6 - 1)(q^8 - 1)(q^{12} - 1).$$

For example, suppose $S = \mathrm{L}_2(q_0)$ where $4 \leqslant q_0 \leqslant 136$ is odd. Note that $q^5 \geqslant q_0$. In terms of (20) we have

$$a \geqslant \frac{1}{2}q_0(q_0 - 1), \quad c \leqslant 4q_0(q_0^2 - 1)$$

and we deduce that the bound in (23) holds unless $(q, q_0) = (3, 9)$ (it is helpful to observe that the expression on the left in (23) is increasing as a function of q). Here $(a, c) = (45, 1440)$ and (23) holds. Similarly, if q_0 is even then we can quickly reduce to the case $(q, q_0) = (2, 4)$. However, we can rule this case out by inspecting the complete list of maximal subgroups in [53].

The cases $S \in \{^2B_2(q_0), {}^2G_2(q_0), G_2(q_0), \Omega_5(q_0)\}$ are all straightforward and we omit the details. Next assume $S = \mathrm{PSp}_4(q_0)$, so $q_0 \leqslant 9$ and $q^3 \leqslant q_0$. If $q = 2$ then [53] implies that $H_0 = \mathrm{Sp}_4(4){:}2$, but this is one of the maximal rank subgroups we handled in Section 4, so we may assume $q \geqslant 3$. Now

$$a \geqslant \frac{1}{2}q_0^3(q_0 - 1)(q_0^2 + 1), \quad c \leqslant 6q_0^4(q_0^2 - 1)(q_0^4 - 1)$$

and thus (23) holds unless $(q, q_0) = (4, 2), (3, 3), (4, 4)$ or $(3, 9)$. In fact, in each of these cases we can verify (23) by taking precise values for a and c (for example, if $q_0 = 3$ then $a = 270$ and $c = 51840$).

To complete the proof of the lemma, we may assume $S = \mathrm{L}_3^\epsilon(q_0)$ and either $q_0 \leqslant 9$ or $q_0 = 16$ (see Theorem 8.1(ii)), so $q^4 \geqslant q_0$. Note that $\mathrm{U}_3(2)$ is not simple, so $q_0 \geqslant 3$ if $\epsilon = -$. In view of [53], we also observe that $q \geqslant 3$. Now

$$a = \begin{cases} q_0^2(q_0^2 + \epsilon q_0 + 1) & q_0 \text{ odd} \\ (q_0 + \epsilon)(q_0^3 - \epsilon) & q_0 \text{ even,} \end{cases} \quad c = 2\log_p q_0 \cdot q_0^3(q_0^2 - 1)(q_0^3 - \epsilon)$$

and one checks that (23) holds unless (q, q_0) is one of the following:

$$(3, 3), (4, 4), (4, 8), (8, 8), (3, 9), (4, 16), (8, 16), \tag{24}$$

where $\epsilon = -$ if $(q, q_0) = (3, 3)$ or $(8, 8)$. Note that we can immediately eliminate the cases $(q, q_0) = (3, 9), (4, 16)$ and $(8, 16)$ by applying Lemma 8.5.

First assume $\epsilon = +$. If $(q, q_0) = (4, 4)$ and $H_0 < \mathrm{Aut}(S)$ then we can take $c = 2q_0^3(q_0^2 - 1)(q_0^3 - 1)$ and we deduce that (23) holds. On the other hand, if $H_0 = \mathrm{Aut}(S)$ then there is an involution $t \in H_0$ with $|t^{H_0}| = 1008$, so we can take $a = 1008$ and once again we see that (23) holds. The case $(q, q_0) = (4, 8)$ is similar. Indeed, if $t \in S$ is an involution

then $|C_S(t)|$ is divisible by 7, so Lagrange's theorem implies that t is an involution of type A_1 or $\tilde{A}_1$ (as an element of T). Therefore, we can take $b = (q^4 + 1)(q^{12} - 1)$ and one easily checks that (23) holds.

Now assume $\epsilon = -$. Note that $(q, q_0) \neq (4, 8)$ by Lagrange's theorem. Suppose $(q, q_0) = (3, 3)$. If $H_0 = \mathrm{Aut}(H_0)$ then there is an involution $t \in H_0$ with $|t^{H_0}| = 252$, so we can take $a = 252$ and one checks that (23) holds. Similarly, if $H_0 = S$ then we can take $c = |S|$ and once again (23) holds. The cases $(q, q_0) = (4, 4)$ and $(8, 8)$ are entirely similar (if $H_0 \neq S$, we can take $a = 1040$ and 32832, respectively). $\qquad\square$

Proposition 8.7. *Theorem 8.2 holds if $S \in \mathrm{Lie}(p)$.*

Proof. We may assume that $T = E_6^\epsilon(q)$, $E_7(q)$ or $E_8(q)$. We proceed as in the proof of Lemma 8.6, with the aim of verifying the bound in (23). As before, the possibilities for S are restricted by the conditions in part (ii) of Theorem 8.1. Also note that some of the cases that arise will have already been handled in Sections 4 and 5. For example, see Section 4 for the case where $T = E_6^\epsilon(q)$ and $S = \mathrm{L}_3^\epsilon(q^3)$. In particular, by inspecting the complete list of maximal subgroups in [4, 20, 29], we observe that the relevant almost simple maximal subgroups in the cases where $T = E_6^\epsilon(2)$ or $E_7(2)$ have already been handled in Sections 4 and 5.

Although there are a few more possibilities for S to consider, the arguments are straightforward and no special complications arise (in particular, there is no need to establish the non-maximality of certain subgroups, as in Lemma 8.5). We ask the reader to check the details. $\qquad\square$

This completes the proof of Theorem 8.2.

9. COMPLETION OF PROOF OF THEOREM 2

In this section we complete the proof of Theorem 2 by handling the groups with

$$T \in \{{}^2B_2(q), {}^2G_2(q), {}^2F_4(q), {}^3D_4(q)\}. \tag{25}$$

In each case, the maximal subgroups of G have been determined (see Suzuki [59] for ${}^2B_2(q)$, Malle [51] for ${}^2F_4(q)$ and Kleidman [26, 27] for ${}^2G_2(q)$ and ${}^3D_4(q)$). In view of Proposition 2.1, we may assume that $T \neq {}^2G_2(3)'$, ${}^2F_4(2)'$. Moreover, since the relevant maximal rank subgroups (including parabolic subgroups) were handled in Sections 3 and 4, the remaining possibilities for $H_0 = H \cap T$ are as follows:

$$
\begin{array}{ll}
T = {}^2B_2(q): & {}^2B_2(q_0) \text{ with } q_0 \geqslant 8 \\
T = {}^2G_2(q): & {}^2G_2(q_0) \\
T = {}^2F_4(q): & {}^2F_4(q_0) \\
T = {}^3D_4(q): & {}^3D_4(q_0), G_2(q), \mathrm{PGL}_3^\epsilon(q)
\end{array}
$$

where $q = q_0^k$ for a prime k.

Proposition 9.1. *Let G be an almost simple primitive permutation group of degree n with socle T and point stabilizer H. Suppose T is one of the groups in (25) and H is a subfield subgroup. Then $\mathrm{ifix}(T) > n^\alpha$ and*

$$\liminf_{q \to \infty} \frac{\log \mathrm{ifix}(T)}{\log n} = \beta,$$

where α, β are given in Table 25. In particular, $\mathrm{ifix}(T) > n^{4/9}$ if and only if $T = {}^2F_4(q)$ or ${}^3D_4(q)$ (with q even).

T	H_0	α	β
$^2B_2(q)$	$^2B_2(q_0)$	0.397	$2/5$
$^2G_2(q)$	$^2G_2(q_0)$	0.426	$3/7$
$^2F_4(q)$	$^2F_4(q_0)$	$15/26$	$15/26$
$^3D_4(q)$	$^3D_4(q_0)$	$\begin{cases} 3/7 & q \text{ odd} \\ 0.637 & q \text{ even} \end{cases}$	$3/7$

TABLE 25. The constants α, β in Proposition 9.1

H_0	α	β
$G_2(q)$	$\begin{cases} 3/7 & q \text{ odd} \\ 5/7 & q \text{ even} \end{cases}$	$3/7$
$\mathrm{PGL}_3(q)$	$\begin{cases} 2/5 & q \text{ odd} \\ 7/10 & q \text{ even} \end{cases}$	$2/5$
$\mathrm{PGU}_3(q)$	$\begin{cases} 0.387 & q \text{ odd} \\ 0.672 & q \text{ even} \end{cases}$	$2/5$

TABLE 26. The constants α, β in Proposition 9.2

Proof. If $T = {}^2B_2(q)$ or $^2G_2(q)$ then both T and H_0 have a unique class of involutions, so (10) holds and the result quickly follows since

$$i_2({}^2B_2(q)) = (q^2+1)(q-1), \quad i_2({}^2G_2(q)) = q^2(q^2-q+1).$$

Similarly, if $T = {}^2F_4(q)$ and $t \in H_0$ is an $(\tilde{A}_1)_2$-involution, then

$$\mathrm{ifix}(T) = \mathrm{ifix}(t) = \frac{|t^{H_0}|}{|t^T|} \cdot n = \frac{(q_0^3+1)(q_0^2-1)(q_0^6+1)}{(q^3+1)(q^2-1)(q^6+1)} \cdot n$$

and it is easy to verify the desired bounds. The case $T = {}^3D_4(q)$ is entirely similar and we omit the details (note that T and H_0 have a unique class of involutions when q is odd). $\square$

Finally, we handle the two remaining cases that arise when $T = {}^3D_4(q)$.

Proposition 9.2. *Let G be an almost simple primitive permutation group of degree n, with socle $T = {}^3D_4(q)$ and point stabilizer H. Set $H_0 = H \cap T$ and assume that $H_0 = G_2(q)$ or $\mathrm{PGL}_3^\epsilon(q)$. Then $\mathrm{ifix}(T) > n^\alpha$ and*

$$\liminf_{q \to \infty} \frac{\log \mathrm{ifix}(T)}{\log n} = \beta,$$

where α, β are given in Table 26. In particular, $\mathrm{ifix}(T) > n^{4/9}$ if and only if q is even.

Proof. First assume $H_0 = G_2(q)$. If q is odd then both T and H_0 have a unique class of involutions, hence (10) holds and the result follows. If q is even and $t \in H_0$ is an A_1-involution, then t is in the A_1-class of T (see the proof of [35, Lemma 6.3], for example) and we deduce that

$$\mathrm{ifix}(T) = \frac{|t^{H_0}|}{|t^T|} \cdot n = \frac{q^6-1}{(q^2-1)(q^8+q^4+1)} \cdot n,$$

which gives the desired result. A very similar argument applies when $H_0 = \mathrm{PGL}_3^\epsilon(q)$. Indeed, we note that H_0 has a unique class of involutions and

$$i_2(H_0) = \begin{cases} (q + \epsilon)(q^3 - \epsilon) & \text{if } q \text{ even} \\ q^2(q^2 + \epsilon q + 1) & \text{if } q \text{ odd.} \end{cases}$$

Moreover, if q is even then the involutions in H_0 are contained in the A_1-class of T. The result quickly follows. $\qquad\square$

This completes the proof of Theorem 2.

10. ALGEBRAIC GROUPS

In this final section we prove Theorem 4. Let $\bar{G}$ be a simple exceptional algebraic group over an algebraically closed field K of characteristic $p \geqslant 0$ and let $\bar{H}$ be a maximal positive dimensional closed subgroup of $\bar{G}$. Recall that there is a natural action of $\bar{G}$ on the coset variety $\Omega = \bar{G}/\bar{H}$ and for each $t \in \bar{G}$, the set of fixed points $C_\Omega(t)$ is a subvariety. Set

$$\mathrm{ifix}(\bar{G}) = \max\{\dim C_\Omega(t) \; : \; t \in \bar{G} \text{ is an involution}\}.$$

The possibilities for $\bar{H}$ are determined up to conjugacy by Liebeck and Seitz in [42, Theorem 1] and the subgroups that arise are as follows:

  (a) Parabolic subgroups (one class for each node of the Dynkin diagram of $\bar{G}$):
  (b) Reductive subgroups of maximal rank (see [42, Table 10.3]);
  (c) Reductive subgroups of smaller rank (see [42, Table 10.1]).

Note that these are precisely the infinite analogues of the subgroups arising in the statement of Theorem 3.

Proof of Theorem 4. Set $\Omega = \bar{G}/\bar{H}$ and let $t \in \bar{H}$ be an involution. Then

$$\dim C_\Omega(t) = \dim \Omega - \dim t^{\bar{G}} + \dim(t^{\bar{G}} \cap \bar{H})$$

by [34, Proposition 1.14]. In particular, the dimension of $C_\Omega(t)$ (and also $\dim \Omega = \dim \bar{G} - \dim \bar{H}$) is independent of the choice of algebraically closed field of fixed characteristic p. In addition, it is well known that these dimensions in characteristic zero are equal to the dimensions over any algebraically closed field of prime characteristic r, for all large primes r. Therefore, we may assume $p > 0$ and $K = \bar{\mathbb{F}}_p$.

Let σ be a Steinberg endomorphism of $\bar{G}$ so that $\bar{G}(q) := \bar{G}_\sigma$ is defined over $\mathbb{F}_q$ for some p-power q. We may assume that $\bar{H}$ is also σ-stable, so we can consider $\bar{H}(q) := \bar{H}_\sigma$ and the action of $\bar{G}(q)$ on the set of cosets $\Omega(q) := \bar{G}(q)/\bar{H}(q)$. Let $t \in \bar{H}(q)$ be an involution and let e be the number of irreducible components of $C_\Omega(t)$ of maximal dimension. By [45, Lemma 7.1], which follows from the Lang-Weil estimates in [30], we deduce that

$$|C_{\Omega(q)}(t)| = (e + o(1))q^{\dim C_\Omega(t)}, \;\; |\Omega(q)| = (1 + o(1))q^{\dim \Omega}$$

for infinitely many values of q. Therefore,

$$\frac{\log |C_{\Omega(q)}(t)|}{\log |\Omega(q)|} = \frac{\dim C_\Omega(t)}{\dim \Omega} + o(1)$$

and the result now follows from Theorem 3. $\qquad\square$

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