Complementarity problems, Variational inequalities and Extended Lorentz cones

by

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Abstract

In this thesis, we introduced the concept of extended Lorentz cones. We discussed the solvability of variational inequalities and complementarity problems associated with an unrelated closed convex cone. This cone does not have to be an isotone projection cone. We showed that the solution of variational inequalities and complementarity problems can be reached as a limit of a sequence defined in an ordered space which is ordered by extended Lorentz cone. Moreover, we applied our results in game theory and conic optimization problems. We also discussed the positive operators. We showed necessary and sufficient conditions under which a linear operator is a positive operator of extended Lorentz cone. We also showed sufficient and necessary conditions under which a linear operator in a specific form is a positive operator.

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Acronyms

AVI	Affine variational inequality
CO	Conic optimization problem
CP	Complemenatarity problem
ICP	Implicit complemenatarity problem
KKT	Karush-Kuhn-Tucker
LCP	Linear complementarity problem
$MiCP$	Mixed complementarity problem
$MLCP$	Mixed linear complementarity problem
VI	Variational inequality

Notations

Spaces

$\mathbb{H}$	Hilbert space
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{R}^n_+$	nonnegative orthant of the n-dimensional Euclidean space
$\mathbb{R}^{n\times m}$	the set of $n \times m$ real matrices
$\mathbb{S}^n_+$	the cone of $n\times n$ symmetric positive semidefinite matrices
spF	Linear span of the set F

Matrices

$diag(y)$	the diagonal matrix whose diagonal entries are the entries of the vector y
I	Identity mapping(matrix)
JF	Jacobian determinant of function F
$\|M\|$	$\|M\| = \min\{c \geq 0 : \|Mx\| \leq c\|x\| \text{ for all } x \in \mathbb{R}^m\}$

Problem classes

$AVI(q, M, K)$	Affine variational inequality defined by a vector q, a matrix M and a polyhedron C
$VI(F, K)$	Variational inequality defined by a mapping F and a set K
$VI(q, M, K)$	$:= VI(F, K)$ with $F(x) = q + Mx$
$CP(F, K)$	Complementarity problem defined by a mapping F and a cone K
$FEA(F, K)$	Feasible set of the variational inequality defined by a mapping F and a set K
$Fix(D)$	the fixed point problem defined by D
$MiCP(G, H, C, n_1, n_2)$	n_1+n_2-dimensional mixed complementarity problem defined by mappings G, H and a cone C
$SOL(F, K)$	Solution set of the variational inequality defined by a mapping F and a set K

Vectors

$\nabla_i f$	the partial derivative of the function f with respect to the i-th variable
$\nabla\phi$	Gradient vector of the mapping ϕ
e	$= (1, \ldots, 1)^\top$ the vector with each entry 1
I	Identity mapping
JF	Jacobian determinant of function F
$\|x\|$	$:= \sqrt{\langle x, x\rangle}$ the Euclidean norm of vector x
$x^\top$	the tranpose of the vector x;
$\langle x, y\rangle$, $x^\top y$	inner product (scalar product) of the vectors x, y
$P_C(x)$	Projection mapping from the point x to the set C

Sets

∂S	Boudary set of the set S
$\operatorname{int} K$	The interior of the set K
$\mathrm{cone}\{u^1, \dots, u^m\}$	$:= \{\lambda_1 u^1 + \cdots + \lambda_m u^m : \lambda_1, \dots, \lambda_m \geq 0\}$
K°	polar cone of the cone K
K^*	dual cone of the cone K
$L(p, q)$	(p, q)-type extended Lorentz cone
$N_X(x)$	the normal cone of the set X at a point $x \in X$
$\lvert P\rvert$	the number of elements in the set P
$x^\perp$	The orthogonal complement set of the vector x

Others

$\lvert a\rvert$	the absolute value of the number a
$\nabla^2_{uv} f$	mixed second-order partial derivative of function f at variables u and v
$\mathrm{mid}(a, b, x)$	median value of a, b and x
$\mathcal{G}$	$:= [P, S_i, u_i]$ a game with P players, i-th player's strategy set S_i and utility function u_i
F_K^{nat}	the natural mapping associated with the pair (F, K).
$\mathrm{Aut}(K)$	the automorphism group of the cone K

Locations

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Chapter 1

Introduction

Variational inequalities and complementarity problems are models of various important problems in physics, engineering, economics and other sciences. The classical Nash equilibrium concept can also be reformulated by using variational inequalities (see [12]). They describe essential properties and features of objective functions and variables. "The systematic study of finite-dimensional NCP and VI began in the mid-1960s; in a span of four decades, the subject has developed into a very fruitful discipline in the field of mathematical programming" [12].

The Lorentz cone (second-order cone) is a very important cone in optimization problems. Many models in robust optimization, plant location problems and investment portfolio manangement can be formulated as a second-order cone program [5]. In this thesis, we generalized this cone to Extended Lorentz cones.

The investigation of complementarity problems and isotone projection mappings can be dated back to 1990s. "The pioneer of this approach for complementarity problems are G. Isac and A.B. Németh" [47]. In [20], G. Isac and A.B. Németh showed properties of isotone projection cones in Euclidean and Hilbert spaces. We use $(\mathbb{H}, \langle\cdot,\cdot\rangle)$ to denote a real Hibert space. Let $C \subseteq \mathbb{H}$ be a closed convex set and $P_C(\cdot)$ the metric projection onto C. More explicitly, it is defined by a solution of the following optimization problem with the constrained set $C \subseteq \mathbb{R}^m$

$$\mathbb{R}^m \ni x \mapsto P_C(x) = \operatorname{argmin}\{\|y - x\| : y \in C\}. \tag{1.1}$$

Let K be a pointed closed convex cone (see Chapter 2 for definition). We say that K is generating if $\mathbb{H} = K - K$ [22]. We recall that $x \leq_K y$ if $y - x \in K$. We say that K is an *isotone projection cone* if and only if, for every $x, y \in \mathbb{H}$, $x \leq_K y$ implies that, $P_K(x) \leq_K P_K(y)$. We call the set $K^* = \{x \in \mathbb{R}^m : \langle x, y\rangle \geq 0,\ \forall y \in K\}$ the *dual* of K.

Definition 1.0.1. *A complementarity problem (also called general complementarity problem or nonlinear complementarity problem) $CP(f, K)$ for $f : K \to \mathbb{H}$ is to find $x \in K$ such that $f(x) \in K^*$ and $\langle x, f(x)\rangle = 0$, where K^* denotes the dual cone of K.*

Definition 1.0.2. *The implicit complementarity problem $ICP(f, g, K)$ defined by f, g and K is to find $x \in K$ such that $g(x) \in K$, $f(x) \in K^*$ and $\langle g(x_0), f(x_0)\rangle = 0$.*

Definition 1.0.3. *A variational inequality $VI(f, C)$ associated to a mapping f and a set is to find a vector $x \in C$ such that $\langle x - y, f(x)\rangle \geq 0$ for any $y \in C$.*

In [20], with the aid of fixed point theory, the connection between complementarity problems and isotone projection cone was investigated as well. The authors proved the following theorem.

Theorem 1.0.1. *Let $(\mathbb{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space ordered by an* isotone projection cone *$K \subseteq \mathbb{H}$ and let $f : K \to \mathbb{H}$ be a continuous and monotone increasing mapping. Consider the following statement:*

(1) $\mathcal{D} = \{x \in K : f(x) \leq_K x\}$ is nonempty.

(2) $\mathcal{D}^* = \{x \in K : f(x) \leq_{K^*} x\}$ *is nonempty.*

(3) $\mathcal{F} = \{x \in K : P_K(f(x)) = x\}$ *is nonempty (which is equivalent to the fact that* $CP(I - f, K)$ *has a solution).*

(4) the sequence $\{x_n\}_{n\in\mathbb{N}}$ *defined by* $x_0 = 0$ *and* $x_{n+1} = P_K(f(x_n))$ *is convergent and its convergence point* $x^* \in \mathcal{F} \subseteq \mathcal{D}^*$ *and* x^* *is the least element of* $\mathcal{D}$*.*

Then (1) $\Longrightarrow$ *(4)* $\Longrightarrow$ *(3)* $\Longrightarrow$ *(2).*

This theorem showed that the solution of a complementarity problem can be found by an iteration with respect to a projection mapping. In chapters 5 and 6, we will see two similar types of theorems for some specified problems. G. Isac and A.B. Németh developed their result to solve complementarity problems (CP) and implicit complementarity problems (ICP) in Hilbert spaces by iterative methods in [21].

Definition 1.0.4. *Given* $\alpha \in \mathbb{R}$ *such that* $0 < \alpha < 1$ *and two mappings* $T_1, T_2 : \mathbb{H} \to \mathbb{H}$*, we say that* T_1 *is* α*-concave if for evey* $x \in \mathbb{H}$ *and every* λ *such that* $0 < \lambda < 1$ *we have* $\lambda^\alpha T_1(x) \leq T_1(\lambda x)$*;* T_2 *is* $-\alpha$*-convex if for every* $x \in \mathbb{H}$ *and every* λ *such that* $0 < \lambda < 1$ *we have* $T_2(\lambda x) \leq \lambda^\alpha T_2(x)$*.*

In [21], G. Isac and A.B. Németh showed the following theorems:

Theorem 1.0.2. *Let* $(\mathbb{H}, \langle\cdot,\cdot\rangle)$ *be a Hilbert space ordered by an isotone projection cone* $K \subseteq \mathbb{H}$ *and let* $f, h : K \to \mathbb{H}$ *be two continuous monotone decreasing mappings. Given* $x_0, y_0 \in K$ *with* $x_0 \leq y_0$ *consider the sequence* $\{x_n\}_{n\in\mathbb{N}}$*,* $\{y_n\}_{n\in\mathbb{N}}$ *defined by*

$$x^{n+1} = P_K(x^n - h(x^n) - f(x^n)) + h(y^n),$$
$$y^{n+1} = P_K(y^n - h(y^n) - f(y^n)) + h(x^n).$$

Suppose the following assumptions are satisfied:

(1) $x^0 \le x^1$ *and* $y^1 \le y^0$,

(2) if $\dim \mathbb{H} = \infty$, *the mapping* $\Phi(x) = h(x) + P_K(x - h(x) - f(x))$ *is nonexpansive or condensing.*

Then the problem $ICP(f \circ (I - h), K)$ *has a solution* $x^* \in K$ *such that for any* n, $x^n \le x^* \le y^n$. *Moreover, if* $\lim_{n\to\infty} \|y^n - x^n\| = 0$ *then* $\lim_{n\to\infty} x^n = x^*$.

Theorem 1.0.3. *Let* $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ *be a Hilbert space ordered by an isotone projection cone* $K \subseteq \mathbb{H}$. *Suppose that the mapping* $\Psi(x) = P_K(x) - f(P_K(x))$, *associated to the complementarity problem* $CP(f, K)$, *has a decompostion of the form* $\Psi(x) = T_1(x) + T_2(x)$, *where* T_1 *is increasing and* α*-concave , and* T_2 *is decreasing and* $-\alpha$*-convex. Given* $u_0 \in K$ *and* $\mu_0 > 1$ *such that* $\mu_0^{\alpha-1} u_0 \le T_1(u_0) + T_2(u_0) \le \mu_0^{1-\alpha} u_0$, *consider the sequence* $\{x^n\}_{n\in\mathbb{N}}$, $\{y^n\}_{n\in\mathbb{N}}$ *defined by*

$$x^n = T_1(x^{n-1}) + T_2(y^{n-1}),$$

$$y^n = T_1(y^{n-1}) + T_2(x^{n-1}),$$

where $x^0 = \mu_0^{-1} u_0$ *and* $y_0 = \mu_0 u_0$. *Then the following holds:*

(1) the sequence $\{x^n\}_{n\in\mathbb{N}}$, $\{y^n\}_{n\in\mathbb{N}}$ *are convergent,*

(2) $\lim_{n\to\infty} x^n = \lim_{n\to\infty} y^n$,

(3) the element $x^* = \lim_{n\to\infty} x^n = \lim_{n\to\infty} y^n$ *is a solution of the problem* $CP(f, K)$,

(4) $\|x^* - x^n\| \le \mu_0 (1 - 1/\mu_0^{2\alpha^n}) \|u_0\|$ *for all* $n \in \mathbb{N}$.

Following these two papers (i.e., [20, 21]), researchers focused on two areas: one is the properties of isotone projection cones in Hilbert spaces; the other is the relation between the projection method and complementarity problems and variational inequalities.

An order relation $\leq_K$ defined by K, is a reflexive, transitive and antisymmetric relation, which is compatible with the vector structure of $\mathbb{H}$. In this case we say that $(\mathbb{H}, K)$ is an ordered vector space and K is its *positive cone* [22].

If for any two elements $x, y \in \mathbb{H}$ there exists $\sup\{x, y\}$ (which will be denoted by $x \vee y$), then the ordered vector spaces is called a *vector lattice* and its positive cone K is said to be *latticial* [22]. In this case, $\inf\{x, y\}$ (denoted by $x \wedge y$) also exists for each $x, y \in \mathbb{H}$ and $x \wedge y = x + y - x \vee y$.

A closed half-space of $\mathbb{H}$ through 0 is a subset of $\mathbb{H}$ of the form $\{x \in \mathbb{H} : \langle x, p\rangle \leq 0\}$ where $p \in \mathbb{H}, p \neq 0$. A *polyhedral cone* in $\mathbb{H}$ is the intersection of finitely many closed half-spaces of $\mathbb{H}$ through 0 [22].

We say that a subset F of the cone K is a *face* if it is a cone that satisfies the condition: from $x \in F$, $y \in K$ and $y \leq_K x$, it follows that $y \in F$. The cone $K \subseteq \mathbb{H}$ is called *correct* if for each of its face F we have that $P_{spF}(K) \subseteq F$ where spF denotes the linear span of the set F [22].

In [19], G. Isac and A.B. Németh proved that if K is a generating isotone projection cone in $\mathbb{H}$ then it is latticial and correct. Moreover, they showed in [22] that if K is a closed generating cone in $\mathbb{R}^n$, then the following assertions are equivalent:

(i) K is an *isotone projection cone*,

(ii) K is correct and latticial,

(iii) K is polyhedral and correct,

(iv) there exists a set of linearly independent vectors $\{u^i | i = 1, \dots, n\}$ with the property that $\langle u^i, u^j\rangle \leq 0$ for any $i \neq j$ and such that $K = \{x \in \mathbb{R}^n | \langle x, u^i\rangle \leq 0\}$,

(v) K is latticial and $P_K(x) \leq x^+$ for every $x \in \mathbb{R}^n$, where $x^+ = x \vee 0$.

Define the following operations in $\mathbb{R}^n$ [15, 47]

$$x \sqcap y = P_{x-K}y,$$

$$x \sqcup y = P_{x+K}y,$$

$$x \sqcap_* y = P_{x-K^*}y,$$

$$x \sqcup_* y = P_{x+K^*}y.$$

The set $M \subseteq \mathbb{R}^n$ is said to be *invariant* with respect to the operation $\sqcap$ if $x, y \in M$ implies that $x \sqcap y \in M$. The invariance of M with respect to any of the operation $\sqcup$, $\sqcap_*$ and $\sqcup_*$ can be defined similarly.

In [47], A.B. Németh and S.Z. Németh proved that when $K \subseteq \mathbb{R}^n$ is a closed convex cone, if C is invariant with respect to one of the operations $\sqcup$, $\sqcup_*$ and one of the operations $\sqcap$, $\sqcap_*$, then C is invariant with respect to all the operations respect to all operations $\sqcap$, $\sqcup$, $\sqcap_*$, and $\sqcup_*$. We can simply call a set M which is invariant with respect to the operations $\sqcap$, $\sqcup$, $\sqcap_*$, and $\sqcup_*$ K-invariant .

When K is a nonzero closed convex cone, we say that a mapping $\rho : \mathbb{R}^m \to \mathbb{R}^m$ is a K-isotone (K^*-isotone) mapping if $x \leq_K y$ implies $\rho(x) \leq_K \rho(y)$ ($x \leq_{K^*} y$ implies $\rho(x) \leq_{K^*} \rho(y)$). Then the closed convex set $C \subseteq \mathbb{R}^m$ is called a *K-isotone* (*K^*-isotone*) *projection set* or simply *K-isotone* (*K^*-isotone*) if P_C is K-isotone (K^*-isotone). G. Isac, A.B. Németh and S.Z. Németh applied generalized lattice-like operations introduced in [15] and showed that when K is a closed convex cone, a closed convex set C is K-invariant if and only if P_C is K-isotone in [47].

The *convex conical hull* $\operatorname{cn} M$ of a set $M \subseteq \mathbb{R}^n$ is the convex cone defined by

$$\operatorname{cone} M = \{t_1 m^1 + \cdots + t_k m^k : k \in \mathbb{N}, m^i \in M, t_i \in \mathbb{R}_+; i = 1, \ldots, k\}$$

In this case we say that M generates the convex cone cone M. The convex cone K is called *simplicial* if it is the convex conical hull of n linearly independent vectors from $\mathbb{R}^n$, that is, if

$$K = \text{cone}\{e^1, \dots, e^n\} = \{t_1e^1 + \cdots + t_ne^n : t_i \in \mathbb{R}_+; i = 1, \dots, n\}$$

with $e^1, \dots, e^n$ linearly independent elements in $\mathbb{R}^n$. In $\mathbb{R}^n$, the simplicial cones are exactly the latticial ones [78].

Let K be a closed convex cone and $\rho : \mathbb{H} \to \mathbb{H}$ a mapping. Then ρ is called K^*-*subadditive* if $x \leq_K y$ implies $\rho(x + y) \leq_K \rho(x) + \rho(y)$ for any $x, y \in \mathbb{H}$. S.Z. Németh (see [50]) proved that when K and K^* are mutually dual closed convex cones in a Hilbert space $\mathbb{H}$, $P_K$is K-isotone if and only if P_{K^*} is K^*-subadditive.

A.B. Németh and S.Z. Németh [46] showed that if $\mathbb{H} = \mathbb{R}^n$, then the following are equivalent:

(i) P_K is K-isotone

(ii) P_{K^*} is K^*-subadditive

(iii) K^* is a simplicial cone generated by edges with mutually non-acute angles.

They also gave an algorithm to reduce a projection onto an isotone projection cone to a finite number of steps [45].

A number of papers [8, 23, 25, 33, 40, 49, 64, 66, 67, 69] considered the iterative methods to solve complementarity problems and variational inequalities from different iterative viewpoints. However, neither of these works used the ordering defined by a cone for showing the convergence of the corresponding iterative schemes. Instead, they used as a tool the Banach fixed point theorem and assumed Kachurovskii-Minty-Browder type monotonicity (see [10, 24, 38, 39]) and global Lipschitz properties.

Let $\mathbb{H}$ be a Hilbert space and $K \subseteq \mathbb{H}$ a closed convex cone. The mapping $f : K \to \mathbb{H}$ is called *pseudomonotone decreasing* if for every $x, y \in K$,

$$x \leq_K y \text{ and } 0 \leq_K f(y) \text{ implies } 0 \leq_K f(x).$$

If $l > 0$, the mapping f is called *projection order weakly l-Lipschitz* if the mapping $K \ni x \to P_K(lx - f(x))$ is monontone increasing where P_K is projection mapping onto K. S.Z. Németh showed an iterative method for complementarity problems on isotone projection cones in Hilbert space in [49]:

Theorem 1.0.4. *Let $\mathbb{H}$ be a Hilbert space, $K \subseteq \mathbb{H}$ be an isotone projection cone, $l > 0$ and let $f : K \to \mathbb{H}$ be a pseudomonotone decreasing, projection order weakly l-Lipschitz continuous mapping such that $K \cap f^{-1}(K) \neq \varnothing$. Let $\hat{x}$ be a solution of the complementarity problem $CP(f, K)$. Then for any $x_0 \in (\hat{x} + K) \cap f^{-1}(K)$, the recursion*

$$x^{n+1} = P_K\left(x^n - \frac{f(x^n)}{l}\right)$$

starting from x_0 is convergent and its limit x^* is a solution of the $CP(f, K)$ such that $\hat{x} \leq_K x^*$. In particular, if $\hat{x} \neq 0$, then the recursion is convergent to a nozero solution.

Let $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $K, L \subseteq \mathbb{H}$ be cones. The mapping $\zeta : \mathbb{H} \to \mathbb{H}$ is called (L, K)-isotone if $x \leq_L y$ implies that $\zeta(x) \leq_K \zeta(y)$. If $P_K : \mathbb{H} \to \mathbb{H}$ is (K^*, K)-isotone, then the cone K is called **-isotone projection cone*. M. Abbas and S.Z.Németh proved that the cone K is *-isotone projection cone, if and only if $P_K(u + v) \leq_K u$ for any $u \in K$ and any $v \in K^\circ$ where

$$K^\circ = \{x \in \mathbb{H} : \langle x, y \rangle \le 0, \forall y \in K\}$$

is the polar of K [1]. Moreover, they showed that in $\mathbb{R}^n$ a simplicial cone is *-isotone projection cone if and only if it is the polar of an isotone projection cone [1]. This result has been extended later to arbitrary cones (see Section 3 of [46] and [50]). The mapping $f : K \to \mathbb{H}$ is called *-*increasing* if f is (K, K^*)-isotone. The mapping f is called *-*decreasing* if $-f$ is *-increasing. The mapping $f : K \to \mathbb{H}$ is called a *-*pseudomonotone decreasing* if for every $x, y \in K$

$$y - x \in K \text{ and } f(y) \in K^* \text{ implies } f(x) \in K^*.$$

Let $f : K \to \mathbb{H}$ be a mapping and $l > 0$. The mapping f is called *-*order weekly l-Lipschitz* if

$$f(x) - f(y) \le_{K^*} l(x - y).$$

M.Abbas and S.Z. Németh proved the following theorems in [1]:

Theorem 1.0.5. *Let $\mathbb{H}$ be a Hilbert space, $K \subseteq \mathbb{H}$ be a regular *-isotone projection cone and $f : K \to \mathbb{H}$ be a continuous mapping such that $f^{-1}(K^*) \neq \varnothing$. Let $x^{n+1} = P_K(x^n - f(x^n))$ starting from $x^0 \in f^{-1}(K^*)$. If f is *-pseudomonotone decreasing, then the sequence $\{x^n\}_{n\in\mathbb{N}}$ is convergent and its limit x^* is a solution of the complementarity problem $CP(f, K)$.*

Theorem 1.0.6. *Let $\mathbb{H}$ be a Hilbert space, $K \subseteq \mathbb{H}$ be a regular *-isotone projection cone, $l > 0$ and $f : K \to \mathbb{H}$ be a *-pseudomonotone decreasing, projection order weekly l-Lipschitz continous mapping such that $f^{-1} \neq \varnothing$. Let $\hat{x}$ be a solution of the complemen-*

tarity problem $CP(f,K)$. Then, for any $x^0 \in (\hat{x}+K) \cup f^{-1}(K^*)$ the recursion

$$x^{n+1} = P_K\left(x^n - \frac{f(x^n)}{l}\right)$$

starting from x^0 is convergent and its limit x^* is a solution of the complementarity problem $CP(f,K)$ such that $\hat{x} \leq_K x^*$. In particular, if $\hat{x} \neq 0$, then the above recursion is convergent to a nonzero solution.

The set $\{x \in K : g(x) \leq_K x\}$ is called the *upper fixed point set* of g and is denoted $(UF)_g$. Let $K \subseteq \mathbb{R}^n$ be a simplicial cone and $f, g : K \to \mathbb{R}^n$ two mappings. The mapping f is called **-order Lipschitz with respect to* g if there is an $l > 0$ such that

$$f(x) - f(y) \leq_{K^*} l(g(x) - g(y))$$

for all $x, y \in \mathbb{R}^n$ with $y \leq_K x$. The mapping f is called *projection order Lipschitz with respect to* g if there is a constant $l > 0$ such that

$$P_K(lg(x) - f(x)) \leq_K P_K(lg(y) - f(y))$$

for all $x, y \in \mathbb{R}^n$ with $x \leq_K y$. The number l is called *a projection order Lipschitz constant* of f. M. Abbas and S. Z. Németh extended Theorem 1.0.4 and 1.0.5 and proved the following results in [2]:

Theorem 1.0.7. *Let $K \subseteq \mathbb{R}^n$ be a *-isotone projection cone and $f, g : K \to \mathbb{R}^n$ be continuous mappings such that $f^{-1}(K^*) \neq \varnothing$ and $f^{-1}(K^*) \subseteq (UF)_g \cap g^{-1}(K)$. Consider the recursion*

$$x^{n+1} = x^n - g(x^n) + P_K(g(x^n) - f(x^n)),\ n \in \mathbb{N}$$

starting from an $x^0 \in f^{-1}(K^*)$*. If* f *is* **-pseudomonotone decreasing, then the sequence* $\{x^n\}_{n\in\mathbb{N}}$ *is convergent and its limit* x^* *is a solution of the implicit complementarity problem* $ICP(f,g,K)$*.*

Theorem 1.0.8. *Let* $K \subseteq \mathbb{R}^n$ *be a* **-isotone projection cone and* $f,g : K \to \mathbb{R}^n$ *be continuous mappings such that* $f^{-1}(K^*) \neq \varnothing$ *and* $I-g$ *is* K*-isotone with* $f^{-1}(K^*) \subseteq (UF)_g \cap g^{-1}(K)$*. Suppose that* f *is* **-pseudomonotone decreasing, projection order Lipschitz map with respect to* g *with* $l>0$ *a projection order Lipschitz constant. Then, there exists a solution* $\hat{x}$ *of* $ICP(f,g,K)$*. Consider the following recursion:*

$$x^{n+1} = x^n - g(x_n) + P_K\left(g(x^n) - \frac{f(x^n)}{l}\right)$$

starting from $x_0 \in (\hat{x}+K) \cap f^{-1}(K^*)$*. Then the sequence* $\{x^n\}_{n\in\mathbb{N}}$ *is convergent and its limit* x^* *is a solution of the implicit complementarity problem* $ICP(f,g,K)$ *such that* $\hat{x} \leq_K x^*$*. In particular, if* $\hat{x} \neq 0$*, then the recursion is convergent to a nonzero solution.*

In [3] they generalized Theorem 1.0.7 and showed the following theorem:

Theorem 1.0.9. *Let* $\mathbb{H}$ *be a Hilbert space,* $K \subseteq \mathbb{H}$ *be an isotone projection cone and* $f,g : K \to \mathbb{R}^n$ *be continuous mappings such that* $f^{-1}(K^*) \neq \varnothing$ *and* $f^{-1}(K^*) \subseteq (UF)_g \cap g^{-1}(K)$*. Consider the recursion*

$$x^{n+1} = x^n - g(x^n) + P_K(g(x^n) - f(x^n)),\ n \in \mathbb{N}$$

starting from an $x_0 \in f^{-1}(K^*)$*. If* f *is pseudomonotone decreasing, then the sequence* $\{x^n\}_{n\in\mathbb{N}}$ *is convergent and its limit* x^* *is a solution of the implicit complementarity problem* $ICP(f,g,K)$*.*

An iteration similar to the above theorem will be applied in Chapters 5, 6, 7 and 8.

Gabay and Moulin showed the relationship between Nash equilibrium and variational inequalities in [14]. H.Nishimura and E.Ok provided a systematical development of the solvability of (general) variational inequalities on Hilbert lattices by applying the fixed point theory and isotonicity properties of the projection mapping in [55]. Since the Nash equilibrium is equivalent to variational inequalities, they proved the existence of Nash equilibrium in some special cases. Similar approach will be applied in Chapter 7.

Note that in the above papers, the limits of the corresponding iterations, which are solutions of $CP(f,K)$, are based on isotonicity properties of the projection onto the cones K. In Chapters 5 and 6, we will study the solvability of $VI(f,K)$ and $CP(f,K)$ associated to a pointed closed convex cone L where L is not necessarily related to the closed convex set K and the closed convex cone K, respectively. Let $\{x^n\}_{n\in\mathbb{N}}$ is defined by

$$x^{n+1} = P_K(x^n - F(x^n)). \tag{1.2}$$

We say the set $\Omega \subset \mathbb{R}^m$ is called *K-bounded from below* (*K-bounded from above*) if there exists a vector $y \in \mathbb{R}^m$ such that $y \leq_K x$ ($x \leq_K y$), for all $x \in \Omega$. In this case y is called a *lower K-bound* (*upper K-bound*) of Ω. If $y \in \Omega$, then y is called the K-least element (K-greatest element) of Ω. We will prove the following propositions:

Proposition 1.0.1. *Let L be a pointed closed convex cone, $K \subset \mathbb{R}^m$ be a closed convex cone such that $K \cap L \neq \varnothing$ and K^* be its dual, and $F : \mathbb{R}^m \to \mathbb{R}^m$ be a continuous mapping. Consider the sequence $\{x^n\}_{n\in\mathbb{N}}$ defined by* (1.2)*. Suppose that the mappings P_K and $I - F$ are L-isotone and $x^0 = 0 \leq_L x^1$. Denote by I the identity mapping. Let*

$$\Omega = K \cap L \cap F^{-1}(L) = \{x \in K \cap L : F(x) \in L\}$$

and

$$\Gamma = \{x \in K \cap L : P_K(x - F(x)) \leq_L x\}.$$

Consider the following assertions:

(i) $\Omega \neq \varnothing$.

(ii) $\Gamma \neq \varnothing$.

(iii) The sequence $\{x^n\}_{n\in\mathbb{N}}$ *is convergent and its limit* x^* *is a solution of* $CP(F, K)$. *Moreover,* x^* *is the L-least element of* Γ *and a lower L-bound of* Ω.

Then, $\Omega \subset \Gamma$ *and (i)* $\Longrightarrow$ *(ii)* $\Longrightarrow$ *(iii).*

Proposition 1.0.2. *Let* $K \subset \mathbb{R}^m$ *be a closed convex set,* $F : \mathbb{R}^m \to \mathbb{R}^m$ *be a continuous mapping and* L *be a cone. Consider the sequence* $\{x^n\}_{n\in\mathbb{N}}$ *defined by* (1.2). *Suppose that the mappings* P_K *and* $I - F$ *are L-isotone and* $x^0 \leq_L x^1$. *Denote by* I *the identity mapping. Let*

$$\Omega = \{x \in K \cap (x^0 + L) : F(x) \in L\},$$

$$\Gamma = \{x \in K \cap (x^0 + L) : P_K(x - F(x)) \leq_L x\}.$$

Consider the following assertions:

(i) $\Omega \neq \varnothing$.

(ii) $\Gamma \neq \varnothing$.

(iii) The sequence $\{x^n\}_{n\in\mathbb{N}}$ *is convergent and its limit* x^* *is a solution of* $VI(F, K)$. *Moreover,* x^* *is the L-least element of* Γ.

Then, $\Omega \subset \Gamma$ *and (i)* $\Longrightarrow$ *(ii)* $\Longrightarrow$ *(iii).*

Then we will set L to be the extended Lorentz cone $L(p, q)$. Then we get the following theorems:

Theorem 1.0.10. *Let $K = \mathbb{R}^p \times C$, where C is a closed convex cone, K^* be the dual of K, $G : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p$ and $H : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^q$ be continuous mappings, $F = (G, H) : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p \times \mathbb{R}^q$, and $L = L(p, q)$ be the extended Lorentz cone defined by (2.9). Let $x^0 = 0 \in \mathbb{R}^p$, $u^0 = 0 \in \mathbb{R}^q$ and consider the sequence $\{(x^n, u^n)\}_{n\in\mathbb{N}}$ defined by (1.2). Let $x, y \in \mathbb{R}^p$ and $u, v \in \mathbb{R}^q$. Suppose that $y - x \geq \|v - u\|e$ implies*

$$y - x - G(y, v) + G(x, u) \geq \|v - u - H(y, v) + H(x, u)\|e,$$

and $x^1 \geq \|u^1\|e$ (in particular this holds when $-G(0, 0) \geq \|H(0, 0)\|e$).

Let

$$\Omega = \{(x, u) \in \mathbb{R}^p \times C : x \geq \|u\|e,\ G(x, u) \geq \|H(x, u)\|e\},$$

and

$$\Gamma = \{(x, u) \in \mathbb{R}^p \times C : x \geq \|u\|e,\ G(x, u) \geq \|u - P_C(u - H(x, u))\|e\}.$$

Consider the following assertions:

(i) $\Omega \neq \varnothing$.

(ii) $\Gamma \neq \varnothing$.

(iii) The sequence $\{(x^n, u^n)\}_{n\in\mathbb{N}}$ is convergent and its limit (x^*, u^*) is a solution of $MiCP(G, H, C, p, q)$. Moreover, (x^*, u^*) is a lower $L(p, q)$-bound of Ω and the $L(p, q)$-least element of Γ.

Then, $\Omega \subset \Gamma$ and (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

Theorem 1.0.11. *Let $K = \mathbb{R}^p \times C$, where C is a nonempty, closed and convex subset of $\mathbb{R}^q$. Let $G : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p$, $H : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^q$ be continuous mappings, $F = (G, H) : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p \times \mathbb{R}^q$. Let $(x^0, u^0) \in \mathbb{R}^p \times C$ and consider the sequence $(x^n, u^n)_{n\in\mathbb{N}}$ defined by (1.2). Let $x, y \in \mathbb{R}^p$ and $u, v \in \mathbb{R}^q$. Suppose that $x^1 - x^0 \geq \|u^1 - u^0\|e$ (in particular, by Remark 6.2.2, this holds if $u^0 \in C$ and $-G(x^0, u^0) \geq \|H(x^0, u^0)\|e$) and that $y - x \geq \|v - u\|e$ implies*

$$y - x - G(y, v) + G(x, u) \geq \|v - u - H(y, v) + H(x, u)\|e.$$

Let

$$\Omega = \{(x, u) \in \mathbb{R}^p \times C : x - x^0 \geq \|u - u^0\|e \ , \ G(x, u) - x^0 \geq \|H(x, u) - u^0\|e\}$$

and

$$\Gamma = \{(x, u) \in \mathbb{R}^p \times C : x - x^0 \geq \|u - u^0\|e, \\ G(x, u) - x^0 \geq \|u - u^0 - P_C(u - H(x, u))\|e\}.$$

Consider the following assertions

(I) $\Omega \neq \varnothing$.

(II) $\Gamma \neq \varnothing$.

(III) The sequence $\{(x^n, u^n)\}_{n\in\mathbb{N}}$ is convergent and its limit (x^*, u^*) is a solution of $VI(F, K)$. Moreover, (x^*, u^*) is the smallest element of Γ with respect to the partial order defined by the extended Lorentz cone $L(p, q)$ defined by (2.9).

Then, $\Omega \subset \Gamma$ and (I) $\Longrightarrow$ (II) $\Longrightarrow$ (III).

It is easy to see that the theorems in the papers mentioned earlier required some "extra" conditions. Namely the cone K is required to be a isotone projection cone or a *-isotone projection cone. In Theorems 1.0.10 and 1.0.9, C is just required to be closed and convex. In Chapter 3, we will see that any complementarity problem can be formulated as a mixed complementarity problem. That means, we can solve $CP(f, C)$ by formulating it as a $MiCP(G, H, C, p, q)$ and then applying the above theorem. Hence our results can be used for a much wider class of problems.

We define the set of *complementarity pair* as

$$C(K) = \{(x, s) : x \in K, s \in K^*, \langle x, s \rangle = 0\}.$$

A matrix A is said to be *Lyapunov-like* on K if

$$\langle Ax, s \rangle = 0 \text{ for all } (x, s) \in C(K).$$

The set of the Lyapunov-like matrices on K forms a vector space denoted by $LL(K)$, whose dimension $\beta(K)$ is called the *Lyapunov rank* of K. Following our paper, R. Sznajder (see [70]) proved that the Lyapunov rank of extended Lorentz cone is

$$\beta(L(p, q)) = \frac{q^2 - q + 2}{2}.$$

In addition, he showed that $L(p, q)$ is irreducible. These results can be considered as further properties of the extended Lorentz cones and complement the results of Chapter 8.

The thesis is structured on chapters and sections. The main purpose of the thesis is to present isotonicity results based on the order defined by a cone and use them for showing the convergence of the corresponding iterative schemes.

Chapter 2 is devoted to convex analysis and ordered Euclidean space. We will introduce terminologies and notations used throughout the thesis. In Section 2.3, we will define the notion of K-isotone mappings with resppect to a pointed closed convex cone K. In Section 2.3, we will extend the notion of Lorentz cones (also called "second order cones" or "ice cream cones" in the literature) and show that the projection mapping P_K onto $K = \mathbb{R}^p \times C$, where C is a closed convex set (in particular any closed convex cone) is L-isotone with respect to the extended Lorentz cone L. Morevover, we will determine all sets K for which P_K is L-isotone. The L-isotonicity of P_K, $K = \mathbb{R}^p \times C$, will be crucial for Section 5.1 to generate an iterative sequence, which is convergent to a solution of a general mixed complementarity problem.

Chapter 3 deals with the definitions and elementary properties of variational inequalities and complementarity problems. It is mainly based on concepts defined in [12]. The definition of CP and $MiCP$ extend those considered in [12] from the nonnegative orthant to a general closed convex cone.

Chapter 4 is aiming to present the duality between optimization problems and complementarity problems in a more clear-cut way than usually found in the literature and is based on our preprint [52]. Although the Karush-Kuhn-Tucker (KKT) conditions suggest a connection between constrained optimization and complementarity problems, it is difficult to find this connection explained in a perspicuous way, easily accessible to beginners of the field as well. The connection is more in the domain of the mathematical folklore, assuming that it should be clear that the complementary slackness condition corresponds to a complementarity problem (see [52]). Due to the recent development of conic optimization and the applications of cone-complementarity problems, it is desirable to make this connection for more general cones, while still keeping it accessible to a wider audience. Especially because apparently all applications of cone-complementarity problems defined by cones essentially different from the nonnegative orthant are based on this correspondence. There are several such applications in physics, mechanics, economics, game theory, robotics [4, 6, 11, 26, 31, 56, 57, 77, 80, 82].

Chapters 5 and 6 include our main results from [51] and [53], my joint work with S. Z. Németh. We showed that a convergence point of an isotone projection mappings, as stated above, is a solution of some variational inequalities. Section 5.1 plays a transitional role from the complementarity problems to the mixed complementarity problems, in the sense that the isotonicity properties of Section 2.3 will be directly used for nonlinear complementarity problems on which the mixed complementarity problems are based. In Section 5.3, we will show a numerical example corresponding to previous sections. Section 5.4 is aiming to convince the reader that the family of K-isotone mappings is very wide.

In Section 6.1, we will find solutions of a variational inequality by analyzing the monotone convergence with respect to a cone of the Picard iteration corresponding to the equivalent fixed point problem. In Section 6.2, we will specialize these results to variational inequalities defined on cylinders, by using the extended Lorentz cone for the corresponding monotone convergence above. In this case we can drop the condition of Proposition 5.1.1 that the projection onto the closed and convex set in the definition of the variational inequality is isotone with respect to the extended Lorentz cone, because this condition is automatically satisfied, obtaining the more explicit result of Theorem 5.2.1. The latter result extends the results of Nemeth and Zhang [51] for mixed complementarity problems. In Section 6.3, a large class of affine mappings and cylinders which satisfy the conditions of Theorem 5.2.1 is presented. In Section 6.5, we further specialize the results for unbounded box constrained variational inequalities. In Section 6.6, we test the numerical examples of Chapter 5 from the viewpoint of variational inequalities and show the iteration processes.

Next, we will apply the results of Chapter 5 and Chapter 6 to game theory and conic optimization in Chapter 7. Game theory is now not only an attractive research field for mathematicians but also a powerful tool in many other areas such as economics [32, 75], politics [37] and even biology [65]. The original systematic study in game theory dates back to the year of 1945 [76]. Many extraordinary mathematicians and economists made significant contribution in both theoretical and applied game theory [41–44, 62, 63, 68]. Many economists also used the game theory as a crucial tool in the study in many related and important area [18,71]. Their works were recognised to be essential and some of them such as John Nash (1994), Leonid Hurwicz (2007), Lloyd Shapley (2012) and Jean Tirole (2014), were awarded the Nobel prize. In Section 7.1 we will derive a new results in Nash equilibrium. Following this, in Section 7.2, we will apply the results of Chapters 4 and 5.

Chapter 8 is based on [54]. The research on positive operators of extended Lorentz cone was motivated by [30] and the isotonicity conditions of Section 6.4. In Section 8.1, we will review the existing results of positive operators of a Lorentz cone. Then, we will introduce some notations used in this chapter. Section will illustrate the main results of this chapter. We will show necessary conditions and sufficient conditions for positive operators of extended Lorentz cones. Moreover, we will show the necessary and sufficient conditions in a special class of the positive operators and will show the reason why the conditions of [30] do not work for extended Lorentz cones.

The final chapter will summarize the thesis.

Chapter 2

Convex analysis and Ordered Euclidean space

2.1 Convex and Nonlinear analysis

Denote by $\mathbb{N}$ the set of nonnegative integers. Let $m \in \mathbb{N}$. Identify $\mathbb{R}^m$ with the set of column vectors with m real components. The canonical scalar product in $\mathbb{R}^m$ is defined by $\langle x, y\rangle = x^\top y$, for any $x, y \in \mathbb{R}^m$. Let $\|\cdot\|$ be the norm corresponding to the scalar product $\langle\cdot,\cdot\rangle$, that is, $\|x\| = \sqrt{\langle x, x\rangle}$, for any $x \in \mathbb{R}^m$. For any $m \in \mathbb{N}$ denote

$$\mathbb{R}^m_+ = \{x = (x_1, \ldots, x_m)^\top \in \mathbb{R}^m : x_1 \geq 0, \ldots, x_m \geq 0\}$$

and call it the *nonnegative orthant* of $\mathbb{R}^m$. Let $p, q \in \mathbb{N}$. Define the Cartesian product $\mathbb{R}^p \times \mathbb{R}^q$ as the set of the pair of vectors (x, u), where $x \in \mathbb{R}^p$ and $u \in \mathbb{R}^q$. Any vector $(x, u) \in \mathbb{R}^p \times \mathbb{R}^q$ can be identified with the vector $\left(x^\top, y^\top\right)^\top \in \mathbb{R}^{p+q}$. The scalar product in $\mathbb{R}^p \times \mathbb{R}^q$ is given by

$$\langle (x,u),(y,v)\rangle = \left\langle \left(x^\top, u^\top\right)^\top, \left(y^\top, v^\top\right)^\top \right\rangle = \langle x,y\rangle + \langle u,v\rangle .$$

A set $M \subseteq \mathbb{R}^m$ is called *affine* if $(1-\lambda)x + \lambda y \in M$ for every $x \in M$, $y \in M$ and $\lambda \in \mathbb{R}$ [58]. The smallest affine set containing M is called the *affine hull* of M and is denoted by $\operatorname{aff} M$. The *relative interior* of a convex set $M \subseteq \mathbb{R}^m$ denoted by $\operatorname{relint} M$ is:

$$\operatorname{relint} M = \{x \in \operatorname{aff} M : \exists \epsilon > 0, \bar{B}(x,\epsilon) \cap (\operatorname{aff} M) \subseteq M\},$$

where $\bar{B}(x,r) = \{y : \|y-x\| \leq r\}$. The *affine hyperplane* with the normal $u \in \mathbb{R}^m \setminus \{0\}$ and through $a \in \mathbb{R}^m$ is the set defined by

$$\mathcal{H}(u,a) = \{x \in \mathbb{R}^m :\ \langle x-a, u\rangle = 0\}. \tag{2.1}$$

An affine hyperplane $\mathcal{H}(u,a)$ determines two *closed halfspaces* $\mathcal{H}_-(a,u)$ and $\mathcal{H}_+(u,a)$ of $\mathbb{R}^m$, defined by

$$\mathcal{H}_-(u,a) = \{x \in \mathbb{R}^m : \langle x-a,u\rangle \leq 0\},$$

and

$$\mathcal{H}_+(u,a) = \{x \in \mathbb{R}^m : \langle x-a,u\rangle \geq 0\}.$$

An affine hyperplane through the origin will be simply called *hyperplane*. A *supporting halfspace* to C is a closed halfspace which contains C and has a point of C in its boundary. A *supporting hyperplane* to C, is a hyperplane which is the boundary of a supporting halfspace to C [58].

Let $\mathbb{V}$ be a real vector space. A set $K \subset \mathbb{V}$ is a cone if for any $x \in K$, $\lambda x \in K$ for any

$\lambda \geq 0$. A cone $K \subset \mathbb{V}$ is called a *convex cone* if $x + y \in K$, whenever $x, y \in K$. It is easy to show that every convex cone is a convex set.

A convex cone $K \subset \mathbb{R}^m$ which is a closed set is called a closed convex cone. A closed convex cone $K \subset \mathbb{R}^m$ is called *pointed* if $K \cap (-K) = \{0\}$, where 0 is the origin of $\mathbb{R}^m$.

For two vectors $x, y \in \mathbb{R}^m$, we say that $x \perp y$ if $x^\top y = 0$. Let $K \subset \mathbb{R}^m$ be a cone. K^* consists of the zero vector and all non-zero vectors that make a non-obtuse angle with every vector in K. Then, the set

$$K^* = \{x \in \mathbb{R}^m : \langle x, y \rangle \geq 0,\ \forall y \in K\}$$

is called the *dual cone* of K and it is easy to see that it is a closed convex cone. In Figure 3.1, we will show an example for K and K^* when $m = 2$. Meanwhile, the set

$$K^\circ = \{a \in \mathbb{R}^m : \langle a, v \rangle \leq 0, \forall v \in K\}.$$

is called the *polar cone* of K. Clearly, $K^\circ = -K^*$. K° consists of the zero vector and all non-zero vectors that make a non-acute angle with all vector in K. It is known that $(K^*)^* = K$. In Figure 3.2, we will show an example for K and K° when $m = 2$. It is easy to prove that the dual cone of $\mathbb{R}^m_+$ is itself.

Next, let us introduce the projection mapping:

Definition 2.1.1. *Given a closed convex set C, P_C denotes the metric projection onto C. More explicitly, it is defined by a solution of the following optimization problem with the constrained set $C \subseteq \mathbb{R}^m$*

$$\mathbb{R}^m \ni x \mapsto P_C(x) = \operatorname{argmin}\{\|y - x\| : y \in C\}. \tag{2.2}$$

Since C is convex, the point $P_C(x)$ is unique. Indeed if not, then there exists at least two different points $z^{(1)} \in C$ and $z^{(2)} \in C$ such that $\|z^{(1)} - x\| = \|z^{(2)} - x\|$ is the shortest distance from x to C. Then the point $0.5z^{(1)} + 0.5z^{(2)} \in C$ by the convexity, and by the triangle inequality :

$$\|0.5z^{(1)} + 0.5z^{(2)} - x\| \leq \|0.5z^{(1)} - 0.5x\| + \|0.5z^{(2)} - 0.5x\| \leq \|z^{(2)} - x\|.$$

According to the definition of the projection point, the point $0.5z^{(1)} + 0.5z^{(2)}$ is also a projection point and the equality will be held in the above inequality. The equality holds if and only if $z^{(1)} = z^{(2)}$, then this proves the uniqueness of the projection point. The following proposition and lemma illustrate some important properties of the projection mapping:

Proposition 2.1.1 (The characterization of projection mapping)**.** *Given $x \in \mathbb{R}^m$ and $x \notin C$, a vector $z \in C$ is equal to $P_C(x)$ if and only if:*

$$(y - z)^\top (x - z) \leq 0 \tag{2.3}$$

for any $y \in C$.

The following lemma illustrates the nonexpansivity of projection mapping(see [79]):

Lemma 2.1.1. *Let P_C and $\| \cdot \|$ be the projection mapping and induced norm function defined above, respectively. Then, for any $x, y \in \mathbb{R}^m$, we have:*

$$\|P_C(x) - P_C(y)\| \leq \|x - y\|. \tag{2.4}$$

From the definition above it follows that

$$\begin{aligned} P_{y+C}(x) &= \operatorname{argmin}_z\{\|z-x\| : z \in y+C\} \\ &= y + \operatorname{argmin}_t\{\|x-y-t\| : t \in C\} \\ &= y + P_C(x-y) \end{aligned} \tag{2.5}$$

for any $x, y \in \mathbb{R}^m$. The projection mapping in Euclidean space will also be discussed in next chapters.

So far the definitions of cone, dual cone, closed convex cone and projection mapping are given, then we can introduce the *Moreau Theorem* which is widely used in optimization:

Theorem 2.1.1 (Moreau Theorem)**.** *Given a closed convex cone K in the Hilbert space $(\mathbb{R}^m, \langle\cdot,\cdot\rangle)$ and K° be its polar cone. Then for $x, y, z \in \mathbb{R}^m$, the following two assertions are equivalent:*

1. $z = x + y$, $x \in K$, $y \in K^\circ$ and $\langle x, y\rangle = 0$,

2. $x = P_K(z)$ and $y = P_{K^\circ}(z)$.

Then let us define the normal cone

Definition 2.1.2. *Consider a closed and convex set $X \subseteq \mathbb{R}^n$ and a point $x \in X$. The set*

$$N_X(x) \equiv \{d \in \mathbb{R}^n : d^T(y-x) \le 0, \forall y \in X\}. \tag{2.6}$$

is called the normal cone *of X at x.*

It is easy to conclude from the definition that $v \in N_X(x)$ if and only if

$$\langle v, y-x\rangle \le 0 \tag{2.7}$$

for all $y \in X$. Moreover, if X is a closed convex cone in $\mathbb{R}^n$, then

$$N_X(x) = (-X^*) \cap x^{\perp} \tag{2.8}$$

where $x^{\perp} = \{y \in \mathbb{R}^n : y \perp x\}$ denotes the *orthogonal complement* of x. Indeed, if $v \in (-X^*) \cap x^{\perp}$, for any $y \in X$, then $\langle v, y - x\rangle = \langle v, y\rangle - \langle v, x\rangle = \langle v, y\rangle \leq 0$, hence $v \in N_X(x)$.

Conversely, if $v \in N_X(x)$, then by taking $y = (1/2)x \in X$ and $y = 2x \in X$, we get $\langle v, x\rangle \leq 0 \leq \langle v, x\rangle$, so $v \perp x$. Thus, for any $y \in X$,$\langle v, y - x\rangle = \langle v, y\rangle \leq 0$, hence $v \in -X^*$. In conclusion, $v \in (-X^*) \cap x^{\perp}$.

2.2 Introduction of ordered space and isotonicity

Let $K \subset \mathbb{R}^m$ be a pointed closed convex cone. Denote $\leq_K$ the relation defined by $x \leq_K y \iff y - x \in K$ and call it *the order relation defined by* K. The relation $\leq_K$ is reflexive, transitive, antisymmetric and compatible with the linear structure of $\mathbb{R}^m$ in the sense that $x \leq_K y$ implies that $tx + z \leq_K ty + z$, for any $z \in \mathbb{R}^m$ and any $t \in \mathbb{R}_+$. Moreover, $\leq_K$ is continuous at 0 in the sense that if $x^n \to x$ when $n \to \infty$ and $0 \leq_K x^n$ for any $n \in \mathbb{N}$, then $0 \leq_K x$. Conversely any reflexive, transitive and antisymmetric relation $\leq$ which is compatible with the linear structure of $\mathbb{R}^m$ and continuous at 0 is defined by a pointed closed convex cone. More specifically, $\leq$ is equivalent to $\leq_K$, when $K = \{x \in \mathbb{R}^m : 0 \leq x\}$ is a pointed closed convex cone.
Let $K \subset \mathbb{R}^m$ be a pointed closed convex cone. The mapping $F : \mathbb{R}^m \to \mathbb{R}^m$ is called *K-isotone* if $x \leq_K y$ implies $F(x) \leq_K F(y)$.

The nonempty closed convex set $C \subseteq \mathbb{R}^m$ is called *K-isotone projection set* if P_C is K-isotone.

The set $\Omega \subset \mathbb{R}^m$ is called *K-bounded from below* (*K-bounded from above*) if there exists a vector $y \in \mathbb{R}^m$ such that $y \leq_K x$ ($x \leq_K y$), for all $x \in \Omega$. In this case y is called a *lower K-bound* (*upper K-bound*) of Ω. If $y \in \Omega$, then y is called the K-least element (K-greatest element) of Ω.

Let $\mathcal{I} \subset \mathbb{N}$ be an unbounded set of nonnegative integers. The sequence $\{x^n\}_{n\in\mathcal{I}}$ is called K-increasing (K-decreasing) if $x^{n_1} \leq_K x^{n_2}$ ($x^{n_2} \leq_K x^{n_1}$), whenever $n_1 \leq n_2$.

The sequence $\{x^n\}_{n\in\mathcal{I}}$ is called *K-bounded from below* (K-bounded from above) if the set $\{x^n : n \in \mathcal{I}\}$ is K-bounded from below (K-bounded from above).

A closed convex cone K is called regular if any K-increasing (K-decreasing) sequence which is K-bounded from above is convergent. It is easy to show that this is equivalent to the convergence of any K-decreasing sequence which is K-bounded from below. It is known (see [36]) that any pointed closed convex cone in $\mathbb{R}^m$ is regular.

2.3 Isotonicity of the projection with respect to extended Lorentz cones

For $a, b \in \mathbb{R}^m$ denote $a \geq b$ if all components of a are at least as large as the corresponding components of b, or equivalently $b \leq_{\mathbb{R}^m_+} a$. Let p, q be positive integers. Denote by $e \in \mathbb{R}^p$ the vector whose all components are 1. Let

$$L(p,q) = \{(x,u) \in \mathbb{R}^p \times \mathbb{R}^q : x \geq \|u\|e\} \tag{2.9}$$

and

$$M(p,q) = \{(x,u) \in \mathbb{R}^p \times \mathbb{R}^q : \langle x, e\rangle \geq \|u\|, x \geq 0\}. \tag{2.10}$$

Proposition 2.3.1. *$M(p,q) = L(p,q)^*$.*

Proof. Let $(x,u) \in L(p,q)$ and $(y,v) \in M(p,q)$ be arbitrary. Then, by using the Cauchy inequality, we get

$$\begin{aligned}\langle (x,u),(y,v)\rangle &= \langle x,y\rangle + \langle u,v\rangle \geq \langle \|u\|e,y\rangle + \langle u,v\rangle \\ &= \|u\|\,\langle e,y\rangle + \langle u,v\rangle \geq \|u\|\|v\| + \langle u,v\rangle \geq 0.\end{aligned}$$

Hence, $M(p,q) \subset L(p,q)^*$. Conversely, let $(x,u) \in L(p,q)^*$ be arbitrary. We have $(e^i,0) \in L(p,q)$. Hence, $0 \leq \langle (x,u),(e^i,0)\rangle = \langle x,e^i\rangle + \langle u,0\rangle = x_i$. Thus, $x \geq 0$. We also have $(e,-u/\|u\|) \in L(p,q)$. Hence, $0 \leq \langle (x,u),(e,-u/\|u\|)\rangle = \langle x,e\rangle - \|u\|$. Thus, $\langle x,e\rangle \geq \|u\|$. Therefore, $(x,u) \in M(p,q)$ which implies $L(p,q)^* \subset M(p,q)$. □

Remark 2.3.1. *The mutually dual $(p+q)$-dimensional extended Lorentz cone $L(p,q)$ and $M(p,q)$ defined by* (2.9) *and* (2.10) *are pointed closed convex (and hence regular) cones. The cone $L(p,q)$ is a polyhedral cone if and only if $q=1$. If $q=1$, then the minimal number of generators of L is $(p+2)(1-\delta_{p1})+2\delta_{p1}$, where δ denotes the Kronecker symbol . If $q=1$, $p=1$, then a minimal set of generators of $L(p,q)$ is $\{(1,1),(1,-1)\}$, and if $q=1$, $p>1$, then a minimal set of generators of $L(p,q)$ is $\{(e,1),(e,-1),(e^i,0) : i=1,\dots,p\}$. If $q=1$, then $M(p,q)=L(p,q)^*$ is a $p+1$ dimensional polyhedral cone with the minimal number of generators $2p$ and a minimal set of generators of $L(p,q)^*$ is $\{(e^i,1),(e^i,-1) : i=1,\dots,p\}$. If $q=1$ and $p>1$, then note that the number of generators of $L(p,q)$ and $M(p,q)$ coincide if and only if they are* 2 *or* 3*-dimensional cones. The cone $L(p,q)$ is a subdual cone and $L(p,q)$ is self-dual if and only if $p=1$, that is, $L(1,q)$ is the $(q+1)$-dimensional Lorentz cone. $L(p,q)$ is a self-dual polyhedral cone if and only if $p=q=1$. At the end of this chapter, we will show figures of extended Lorentz cones $L(1,2)$, $L(2,1)$ and $M(2,1)$.*

We will prove only two of the properties of Remark 2.3.1 in the next proposition. The rest are left to the reader.

Proposition 2.3.2. *$L(p,q)$ is subdual and $L(p,q)$ is self-dual if and only if $p = 1$.*

Proof. Let $(x,u) \in L(p,q)$. It is easy to see that $x \geq 0$. Equation (2.9) multiplied scalarly by e gives $\langle x, e\rangle \geq p\|u\| \geq \|u\|$, which implies that $(x,u) \in M(p,q)$, where $M(p,q)$ is the cone given by (2.10). Hence, by Proposition 2.3.1, it follows that $(x,u) \in L(p,q)^*$. In conclusion, $L(p,q)$ is subdual. If $p = 1$, then $L(p,q)$ is the $(q+1)$-dimensional Lorentz cone and hence it is self-dual. Suppose that $p > 1$. Let $u \in \mathbb{R}^q$ such that $1 < \|u\| < p$. Then, Proposition 2.3.1 and the equation (2.10) imply that $(e,u) \in L(p,q)^*$. On the other hand, the equation (2.9) shows that $(e,u) \notin L(p,q)$. Hence, $L(p,q)$ is self-dual if and only if $p = 1$. $\square$

Consider $L(p,q)$ defined by (2.9). It is easy to see that $L(p,q)$ is a pointed closed convex cone. Due to the fact that $L(p,q)$ and $M(p,q)$ coincides with the $(q+1)$-dimensional Lorentz cone for $p = 1$ (see Remark 2.3.1), we will call them mutually dual *extended Lorentz cones.*

Recall that an affine hyperplane H is called tangent to a closed convex set $C \subset \mathbb{R}^m$ at a point $x \in C$ if it is the unique supporting affine hyperplane to C at x (see pages 100 and 169 of [58]).

The following result has been showed in [47].

Theorem 2.3.1. *The closed convex set $C \subset \mathbb{R}^m$ with nonempty interior is a K-isotone projection set if and only if it is of the form*

$$C = \cap_{i\in\mathbb{N}} \mathcal{H}_-(u_i, a_i),$$

where each affine hyperplane $H(u_i, a_i)$ is tangent to C and it is a K-isotone projection *set.*

Lemma 2.3.1. *Let $H \subset \mathbb{R}^m$ be a hyperplane with a normal vector $a \in \mathbb{R}^m \setminus \{0\}$. Then, H is a K-isotone projection set if and only if*

$$\|a\|^2 \langle x, y \rangle \geq \langle a, x \rangle \langle a, y \rangle ,$$

for any $x \in K$ and $y \in K^*$.

Proof. Since P_H is linear, it follows that P_H is isotone if and only if

$$P_H x = x - \frac{\langle a, x \rangle}{\|a\|^2} a \in K, \tag{2.11}$$

for any $x \in K$. By the definition of the dual cone, it follows that relation (2.11) is equivalent to

$$\|a\|^2 \langle x, y \rangle = \langle a, x \rangle \langle a, y \rangle + \|a\|^2 \left\langle x - \frac{\langle a, x \rangle}{\|a\|^2} a, y \right\rangle \geq \langle a, x \rangle \langle a, y \rangle ,$$

for any $x \in K$ and $y \in K^*$. □

The next lemma follows easily from (2.5):

Lemma 2.3.2. *Let $z \in \mathbb{R}^m$ be a vector, $K \subset \mathbb{R}^m$ be a closed convex cone and $C \subset \mathbb{R}^m$ be a nonempty closed convex set. Then, C is a K-isotone projection set if and only if $C + z$ is a K-isotone projection set.*

Theorem 2.3.2.

1. *Let $K = \mathbb{R}^p \times C$, where C is an arbitrary nonempty closed convex set in $\mathbb{R}^q$ and $L(p, q)$ be the extended Lorentz cone defined by (2.9). Then, K is an $L(p, q)$-isotone projection set.*

2. *Let $p = 1$, $q > 1$ and $K \subset \mathbb{R}^p \times \mathbb{R}^q$ be a nonempty closed convex set. Then, K is an $L(p, q)$-isotone projection set if and only if $K = \mathbb{R}^p \times C$, for some $C \subset \mathbb{R}^q$ nonempty closed convex set.*

3. *Let $p, q > 1$, and*

$$K = \cap_{\ell \in \mathbb{N}} \mathcal{H}_-(\beta^\ell, \gamma^\ell) \subset \mathbb{R}^p \times \mathbb{R}^q,$$

where $\gamma^\ell = (a^\ell, u^\ell)$ is a unit vector. Then, K is an $L(p, q)$-isotone projection set if and only if for each ℓ one of the following conditions hold:

   (a) The vector $a^\ell = 0$.

   (b) The vector $u^\ell = 0$, and there exists $i \neq j$ such that $a_i^\ell = \sqrt{2}/2$, $a_j^\ell = -\sqrt{2}/2$ and $a_k^\ell = 0$, for any $k \notin \{i, j\}$.

Proof. 1. Suppose that $K = \mathbb{R}^p \times C$, where C is a closed convex set in $\mathbb{R}^q$. Let $(x, u), (y, v) \in \mathbb{R}^p \times \mathbb{R}^q$ such that $(x, u) \leq_L (y, v)$. Then, the nonexpansitivity of the projection (2.4) implies

$$y - x \geq \|v - u\|e \geq \|P_C v - P_C u\|e.$$

Thus, $(y, P_C v) - (x, P_C u) \in L$. Hence, $P_K(x, u) = (x, P_C u) \leq_L (y, P_C v) = P_K(y, v)$.

2. The cone becomes a Lorentz cone of dimension at least 3. This item was proved in [46, 47].

3. By Theorem 2.3.1 and Lemma 2.3.2, we can suppose without loss of generality that K is a hyperplane. Let $\gamma = (a, u)$ be the unit normal vector of K. Suppose that one of the following conditions hold

   (a) The vector $a = 0$.

(b) The vector $u = 0$, and there exists $i \neq j$ such that $a_i = \sqrt{2}/2$, $a_j = -\sqrt{2}/2$ and $a_k = 0$, for any $k \notin \{i, j\}$.

We need to show that K is an $L(p, q)$-isotone projection set. If (a) holds, then this follows easily from item 1. Hence, suppose that (b) holds. By Lemma 2.3.1 we need to show that

$$\langle \zeta, \xi \rangle \geq \langle \gamma, \zeta \rangle \langle \gamma, \xi \rangle, \tag{2.12}$$

for any $\zeta := (x, v) \in L(p, q)$ and $\xi := (y, w) \in M(p, q)$. Condition (2.12) is equivalent to

$$\langle x, y \rangle + \langle v, w \rangle \geq \frac{1}{2}(x_i - x_j)(y_i - y_j),$$

or to

$$\frac{1}{2}(x_i + x_j)(y_i + y_j) + \sum_{k \notin \{i,j\}} x_k y_k + \langle v, w \rangle \geq 0. \tag{2.13}$$

Hence, it is enough to show (2.13). By $(x, v) \in L(p, q)$, $(y, w) \in M(p, q)$ and the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \frac{1}{2}(x_i + x_j)(y_i + y_j) + \sum_{k \notin \{i,j\}} x_k y_k + \langle v, w \rangle \geq \frac{1}{2}(\|v\| + \|v\|)(y_i + y_j) \\ + \sum_{k \notin \{i,j\}} \|v\| y_k + \langle v, w \rangle = \|v\| \langle y, e \rangle + \langle v, w \rangle \geq \|v\| \|w\| + \langle v, w \rangle \geq 0. \end{aligned}$$

Conversely, suppose that K is an $L(p, q)$-isotone projection set. By Lemma 2.3.1, condition (2.12) holds. Let $x \in \mathbb{R}^p_+$ and $v \in \mathbb{R}^q$. Then, by (2.9), (2.10) and Proposition 2.3.1, it is easy to check that $\zeta := (\|v\|e, v) \in L(p, q)$, $\xi := (\|v\|x, -\langle e, x \rangle v) \in$

$M(p,q)$ and $\langle \zeta, \xi \rangle = 0$. Hence, condition (2.12) implies

$$0 \geq (\langle a, e \rangle \|v\| + \langle u, v \rangle)(\langle a, x \rangle \|v\| - \langle e, x \rangle \langle u, v \rangle). \tag{2.14}$$

If in (2.14) $x = e$ and we choose $v \neq 0$ such that $\langle u, v \rangle = 0$, we get $0 \geq \langle a, e \rangle \|v\|^2$, and hence $\langle a, e \rangle = 0$. Hence, (2.14) becomes

$$0 \geq \langle u, v \rangle (\langle a, x \rangle \|v\| - \langle e, x \rangle \langle u, v \rangle). \tag{2.15}$$

First, suppose that $u \neq 0$. Let $v^n \in \mathbb{R}^q$ be a sequence of points such that $\|v^n\| = 1$, $\langle u, v^n \rangle > 0$ and $\lim_{n \to +\infty} \langle u, v^n \rangle = 0$. Let n be an arbitrary positive integer. If in (2.15) we choose $\lambda > 0$ sufficiently large such that $x := a + \lambda e \geq 0$ and $v = v^n$, then we get $0 \geq \langle u, v^n \rangle (\|a\|^2 - \lambda p \langle u, v^n \rangle)$, or equivalently $\|a\|^2 \leq \lambda p \langle u, v^n \rangle$. By letting $n \to +\infty$ in the last inequality, we obtain $\|a\|^2 \leq 0$, or equivalently $a = 0$.

Next, suppose that $u = 0$. Let $x, y \in \mathbb{R}^p_+$ and $w \in \mathbb{R}^q$ such that $\langle x, y \rangle = 0$, $\langle y, e \rangle \geq \|w\|$. Then, by (2.9), (2.10) and Proposition 2.3.1, it is easy to check that $\zeta := (x, 0) \in L(p,q)$, $\xi := (y, w) \in M(p,q)$ and $\langle \zeta, \xi \rangle = 0$. Hence, equation (2.12) implies

$$0 \geq \langle a, x \rangle \langle a, y \rangle, \tag{2.16}$$

for any $x, y \in \mathbb{R}^p_+$ with $\langle x, y \rangle = 0$. Let $x = e^r$ and $y = e^s$, where $r \neq s$. Then, (2.16) becomes $a_r a_s \leq 0$. This together with $\langle e, a \rangle = 0$ and $1 = \|\gamma\|^2 = \|a\|^2$ gives that $\exists i \neq j$ such that $a_i = \sqrt{2}/2$, $a_j = -\sqrt{2}/2$ and $a_k = 0$, $\forall k \notin \{i, j\}$.

$\square$

2.4 Notes and comments

In this chapter, we introduced the mutually dual extended Lorentz cone $L(p,q)$ and $M(p,q)$. More properties of these cones were shown by R. Sznajder in [70].

The set $\Gamma(C)$ of *positive operators* of a cone C is defined in [30]

$$\Gamma(C) = \{A \in \mathbb{R}^{(p+q)\times(p+q)} : AC \subseteq C\}.$$

The set of positive operator is a cone in $\mathbb{R}^{n\times n}$ [30]. It can be easily checked that A is a positive operator of C if and only if $A^\top$ is a positive operator of C^*. In Chapter 8 we will discuss the positive operators for the mutually dual extended Lorentz cones. We will show necessary conditions and sufficient conditions for the positive operator of the mutually dual extended Lorentz cones.

Related to the papers mentioned in Chapter 1, we investigated the properties of isotone projection sets. We focused on the extended Lorentz cone. We proved Theorem 2.3.2 which provides the necessary and sufficient conditions for a set K to be $L(p,q)$-isotone projection set. These results will be fundemental for Chapters 5 and 6.

The following are examples of a cone, its dual cone and its polar cone:

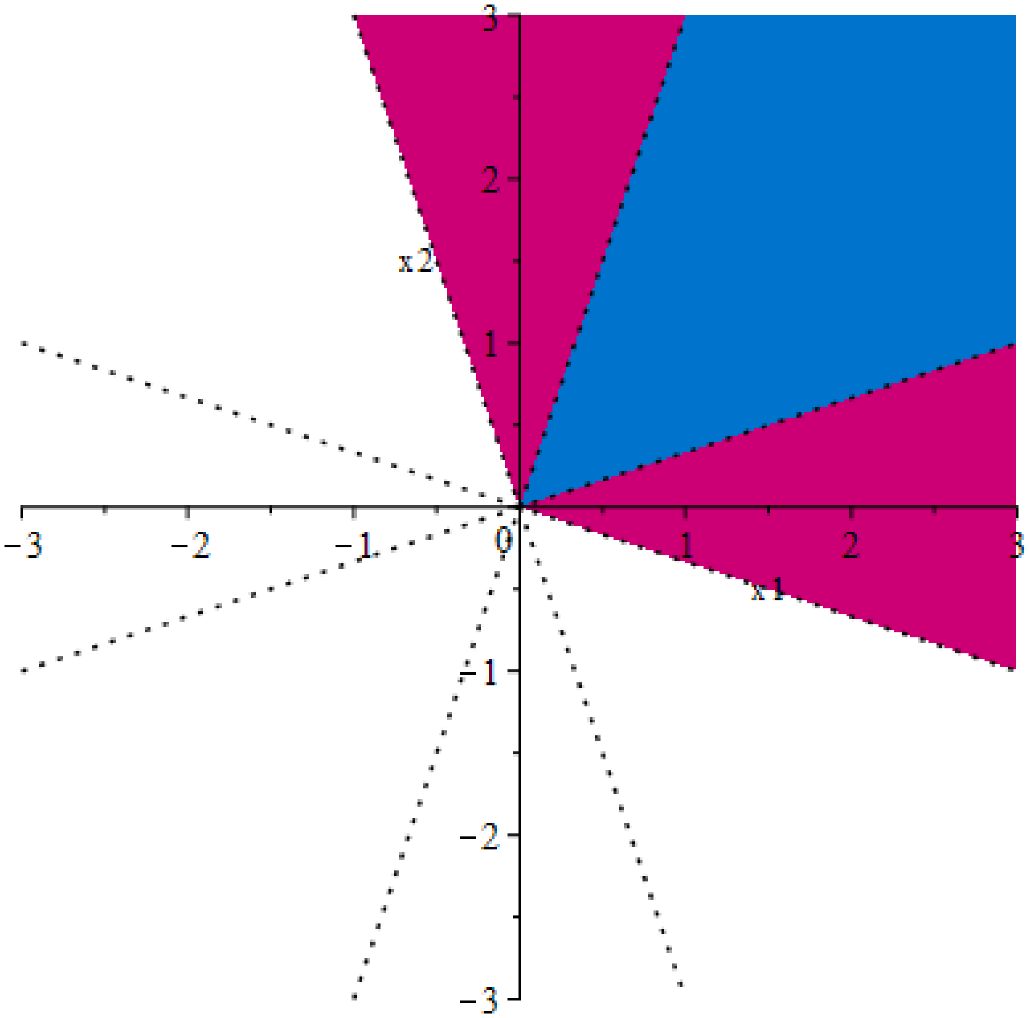

Figure 2.1: $K = \{x \in \mathbb{R}^2_+ : \frac{1}{3}x_1 \leq x_2 \leq 3x_1\}$ (blue area)
$K^* = \{x \in \mathbb{R}^2_+ : x_2 \geq \frac{1}{3}x_1, x_2 \geq -3x_1\}$ (red area and blue area)

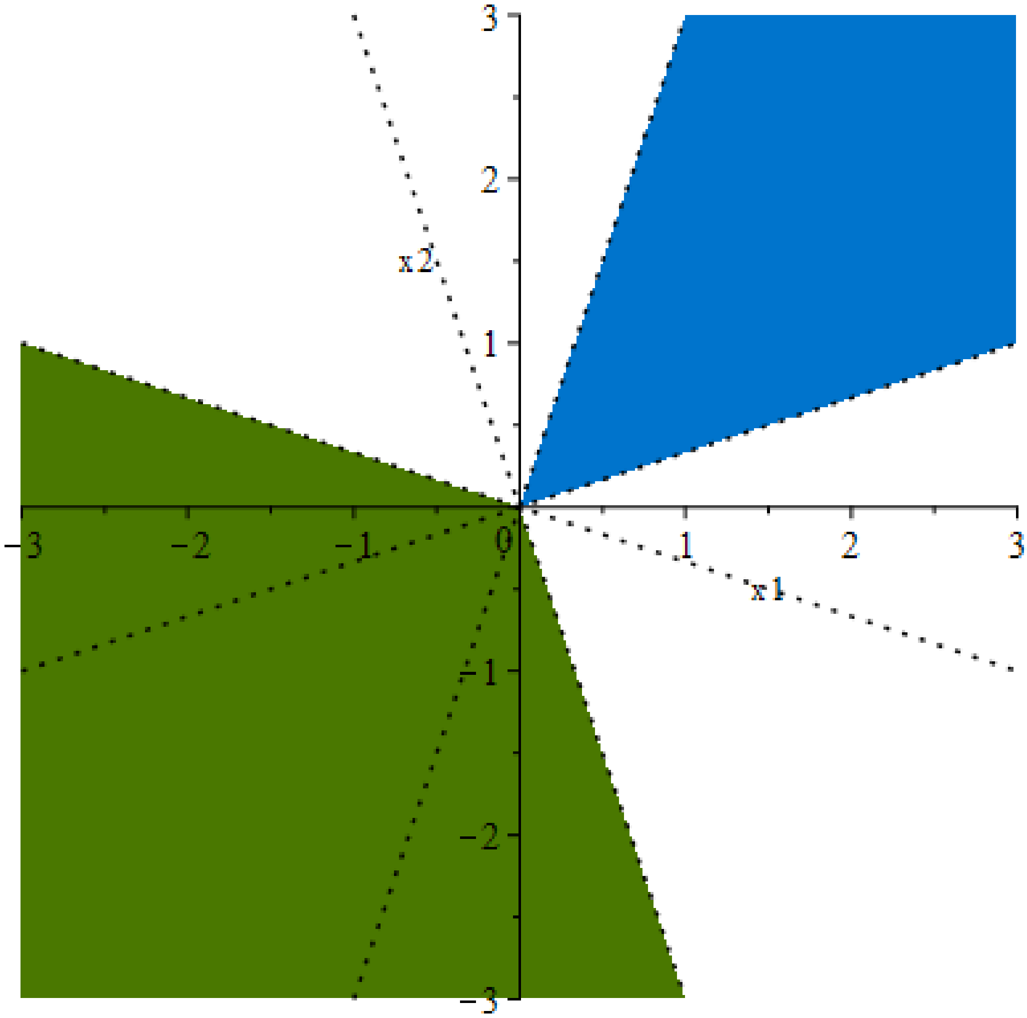

Figure 2.2: $K = \{x \in \mathbb{R}^2_+ : -\frac{1}{3}x_1 \leq x_2 \leq 3x_1\}$ (blue area)
$K^\circ = \{x \in \mathbb{R}^2_+ : x_2 \leq -\frac{1}{3}x_1, x_2 \leq -3x_1\}$ (green)

The following are figures of $L(1,2)$ and $L(2,1)$:

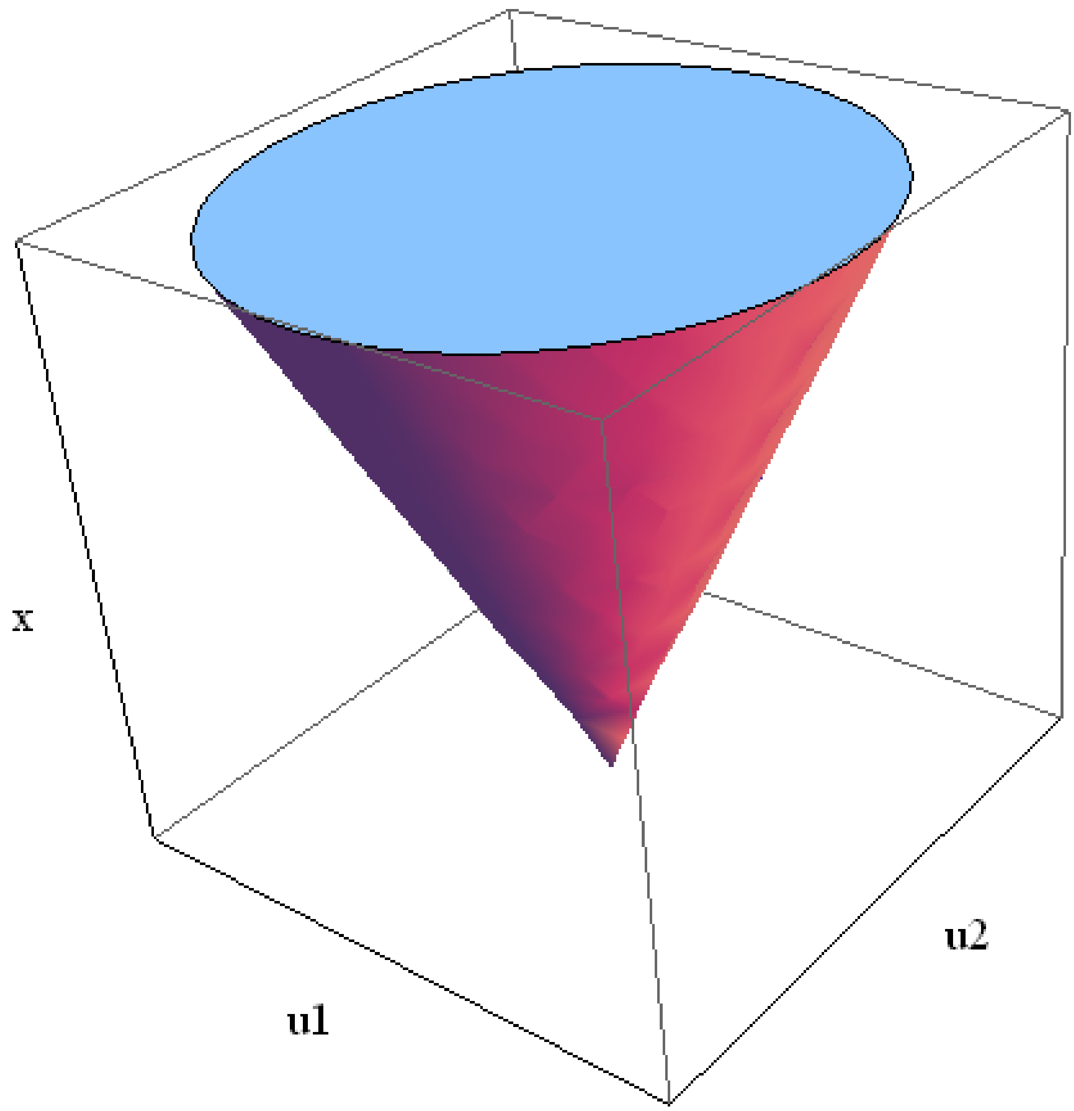

Figure 2.3: (Extended) Lorentz cone $L(1,2) = \{(x,u) \in \mathbb{R} \times \mathbb{R}^2 : x \geq \sqrt{u_1^2 + u_2^2}\}$

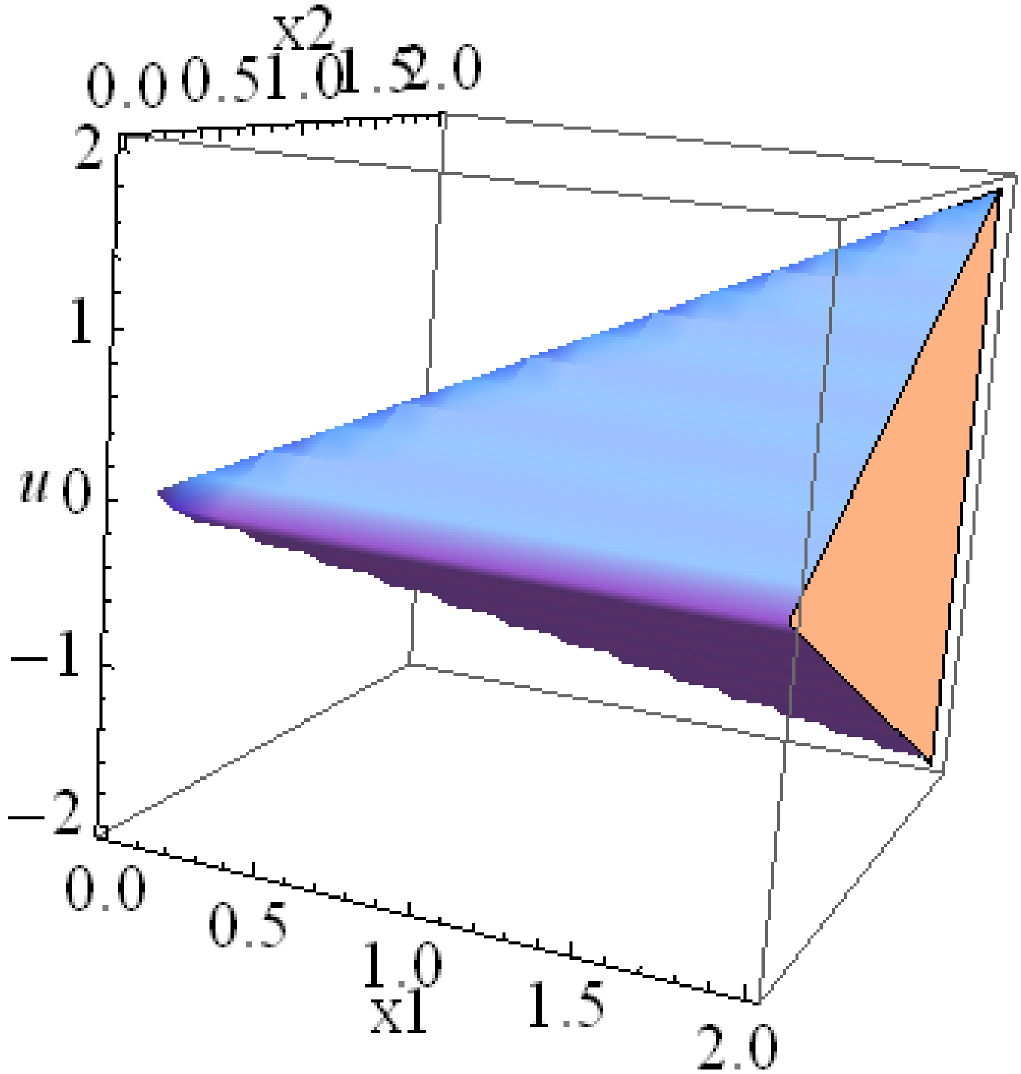

Figure 2.4: Extended Lorentz cone $L(2,1) = \{(x,u) \in \mathbb{R}^2 \times \mathbb{R} : x_1 \geq |u|, x_2 \geq |u|\}$

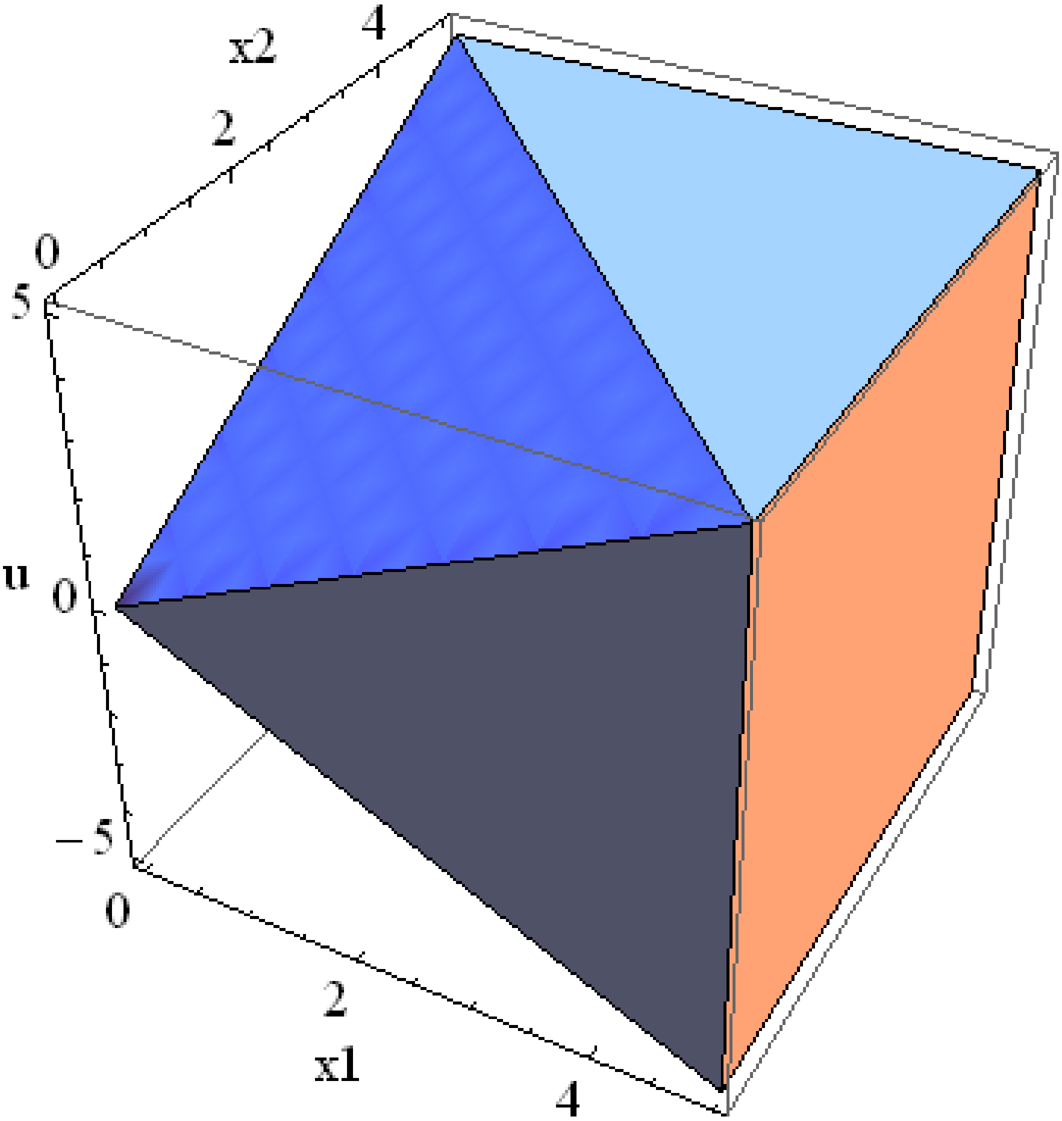

Figure 2.5: Extended Lorentz cone $M(2,1) = \{(x,u) \in \mathbb{R}^2 \times \mathbb{R} : x_1 + x_2 \geq |u|, x_1 \geq 0, x_2 \geq 0\}$

Chapter 3

Variational inequalities and related problems

The study of the finite dimensional variational inequalities and its related problems started in the mid-1960s. During these years, there was a significant development in this subject which made a great contribution in the field of mathematical programming (see [27, 81]). As a result, the study of VI also benifits from contribution of the associated area made by mathematicians, computer scientists, engineers and economists of diverse expertise (see [72–74]). In this chapter, we will introduce the definitions and some applications of the variational inequalities and the other related problems based on [12] and the joint papers [51] and [53].

3.1 The definition of Variational Inequalities and Complementarity problems

First, let us define the Variational inequalities.

Definition 3.1.1. *For a given subset K of $\mathbb{R}^n$ and a given mapping $F : K \to \mathbb{R}^n$, the* variational inequality*, denoted $VI(F, K)$, is to find a vector $x \in K$ such that*

$$(y - x)^\top F(x) \geq 0 \tag{3.1}$$

for any $y \in K$. Its solution set is denoted by $SOL(F, K)$.

Throughout this dissertation, the set K is always closed and convex and the function F is continuous. Because of this, we can conclude that the $SOL(F, K)$ is closed. A simple geometric interpretation of a VI is that a point $x \in K$ is the solution of VI if and only if $F(x) = 0$ or $F(x)$ forms non-obtuse angles with every vector of the form $y - x$ for all $y \in K \backslash \{x\}$.

Then we can formalize the above using the concept of normal cone in Definition 2.1.2 at $x \in K$ which is defined by:

$$N_K(x) = \{d \in \mathbb{R}^n : d^\top (y - x) \leq 0, \forall y \in K\}. \tag{3.2}$$

Note that $N_K(x)$ is a closed convex cone. The vectors in this set are called *normal vectors* to the set K at x. Hence, x is a solution of (3.1) if and only if $-F(x)$ is a normal vector to K, or equivalently

$$0 \in F(x) + N_K(x). \tag{3.3}$$

As a very important class of equilibrium problems, the VI includes many different subclasses. Indeed, if $x \in K$ and $F(x) = 0$, obviously $x \in SOL(K, F)$. Thus $F^{-1}(0) \cap K \subseteq SOL(K, F)$ always holds. Hence the simplest class is the nonlinear equations, when $K = \mathbb{R}^n$. If $K = \mathbb{R}^n$, $x \in SOL(F, \mathbb{R}^n)$ if and only if $F(x) = 0$, that is, $SOL(F, \mathbb{R}^n) = F^{-1}(0)$.

Indeed, when $K = \mathbb{R}^n$, if $x \in SOL(F, \mathbb{R}^n)$ we have

$$F(x)^\top d \geq 0, \forall d \in \mathbb{R}^n \ .$$

Let $d = -F(x)$, then we deduce that $F(x) = 0$. Therefore $SOL(F, \mathbb{R}^n) = F^{-1}(0)$.

A more general condition to derive $F(x) = 0$ is $SOL(F, K) \cap \operatorname{int} K \neq \varnothing$. Let $x \in SOL(K, F) \cap \operatorname{int} K$. Since x is an interior point of K, there exists a sufficiently small scalar $\tau \geq 0$ such that $y = x - \tau F(x)$ is in K. Substituting this into (3.1), we get $-F(x)^\top F(x) \geq 0$ which implies $F(x) = 0$.

If K is a cone, $VI(F, K)$ is equivalent to the *complementarity problem* $CP(F, K)$ defined as follows:

Definition 3.1.2. *For a given cone K of $\mathbb{R}^n$ and a given mapping $F : K \to \mathbb{R}^n$, the* complementarity problem, *denoted $CP(F, K)$, is to find a vector $x \in \mathbb{R}^n$ satisfying the following conditions:*

$$K \ni x \perp F(x) \in K^*, \tag{3.4}$$

where K^* is the dual cone *of K, that is,*

$$K^* = \{x \in \mathbb{R}^n : \langle x, y \rangle \geq 0, \ \forall y \in K\}.$$

Now we need to prove the equivalence of (3.4) and (3.1) when K is a cone. This is based on the following proposition which is a classical result in CP theory and can be found in [12].

Proposition 3.1.1. *Let K be a cone in $\mathbb{R}^n$. A vector x solves $VI(F, K)$ if and only if x solves $CP(F, K)$.*

Proof. Suppose that x solves $VI(F, K)$. Clearly $x \in K$. As K is a cone, $0 \in K$ and $2x \in K$. Then, by taking $y = 0$ and $y = 2x$ in (3.1), we get

$$x^\top F(x) \leq 0,$$

$$x^\top F(x) \geq 0.$$

Combining these two inequalities, we conclude that $x^\top F(x) = 0$. This implies $y^\top F(x) \geq 0$ for all $y \in K$, which means $F(x) \in K^*$. So x solves the $CP(K, F)$. Conversely, if x is a solution of $CP(K, F)$, it is trivial to show that x solves $VI(K, F)$. $\square$

Note that in this thesis when speaking of $CP(F, K)$, the set K is always supposed to be a cone. From Definition 3.4, the solution of the complementarity problem $CP(F, K)$ must satisfy three conditions: $x \in K$, $F(x) \in K^*$ and $x^\top F(x) = 0$. Then the feasibility can be defined by the first two conditions, that is, an n-dimensional vector x is *feasible* to the $CP(F, K)$ if

$$x \in K \text{ and } F(x) \in K^* \ . \tag{3.5}$$

Suppose $\operatorname{int} K^* \neq \varnothing$. The vector x is called *strictly feasible* if

$$x \in K \text{ and } F(x) \in \operatorname{int}(K^*) \ . \tag{3.6}$$

Note that in this definition the nonemptiness of $\operatorname{int}K$ is not required. If we add that $\operatorname{int}K \neq \varnothing$ and F is continuous, the complementarity problem is strictly feasible if and only if there exists a vector x such that

$$x \in \operatorname{int} K \text{ and } F(x) \in \operatorname{int}(K^*) \ . \tag{3.7}$$

We say the $CP(F, K)$ is (strictly) feasible if it has a (strictly) feasible vector. The set of feasible vectors is called *feasible region* of $CP(F, K)$ and it is denoted by $FEA(F, K)$. It can be easily seen that

$$SOL(F, K) \subseteq FEA(F, K) = K \cap F^{-1}(K^*) \ .$$

So the feasibility is a necessary condition of solvability. When F is an affine function and K is a polyhedral cone, the feasibility of $CP(K, F)$ can be determined by solving a linear programming problem. Further discussion about this can be found in the next subsection.

If $K = \mathbb{R}^n_+$, then $K = K^* = \mathbb{R}^n_+$. Since every entry of the two vectors x and $F(x)$ is nonnegative, the perpendicularity can be expressed more explicitly as follows:

$$x_i \geq 0, \ \ F_i(x) \geq 0 \text{ and } x_i F_i(x) = 0 \ ,$$

where $F_i(x)$ represents the ith entry of the vector $F(x)$. The term complementarity is motivated by the following observation: If one of x_i and $F_i(x)$ is positive, the other must be zero.

A class of complementarity problems is the *mixed complementarity problems*($MiCP$). In $MiCP$, the cone K is a special subset in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with $n_1 + n_2 = n$.

Definition 3.1.3. *Let $K = \mathbb{R}^{n_1} \times C$ where C is an arbitrary nonempty closed convex* *cone in* $\mathbb{R}^{n_2}$. *Let* $F = (G, H) : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ *where* $G : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ *and* $H : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ *are two mappings. The* $MiCP(G, H, C, n_1, n_2)$ *is to find a vector* $(u, v) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ *such that*

$$G(u, v) = 0, \ u \text{ free and } C \ni v \perp H(u, v) \in C^* \ . \tag{3.8}$$

Remark 3.1.1. *The relation of the* $MiCP$ *with* CP *can be explained as follows. If a vector* w *is in the dual cone of* $\mathbb{R}^{n_1}$ *in* $\mathbb{R}^{n_1}$, *then by the definition of a dual cone, for any nonzero vector* $v \in \mathbb{R}^{n_1}$, $v^\top w \geq 0$. *If* $v \in \mathbb{R}^{n_1}$, *then* $-v \in \mathbb{R}^{n_1}$. *Hence,* $-v^\top w \geq 0$. *Since* v *is arbitrary,* w *must be* $0 \in \mathbb{R}^{n_1}$. *Therefore the dual cone of* $\mathbb{R}^{n_1}$ *in* $\mathbb{R}^{n_1}$ *is* $\{0\} \subseteq \mathbb{R}^{n_1}$. *It is easy to see that* $(K_1 \times K_2)^* = K_1^* \times K_2^*$ *for any cones* $K_1 \subseteq \mathbb{R}^{n_1}$ *and* $K_2 \subseteq \mathbb{R}^{n_2}$, *where* K_1^* *denotes the dual cone of* K_1 *in* $\mathbb{R}^{n_1}$, K_2^* *denotes the dual cone of* K_2 *in* $\mathbb{R}^{n_2}$ *and* $(K_1 \times K_2)^*$ *denotes the dual cone of* $K_1 \times K_2$ *in* $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, *then the dual cone of* $\mathbb{R}^{n_1} \times C$ *is* $\{0\} \times C^*$. *It is easy to verify that the* $MiCP(G, H, C, n_1, n_2)$ *is a special case of* $CP(F, K)$ *with* $K = \mathbb{R}^{n_1} \times C$. *On the other hand, let* $x \in \mathbb{R}^m$ *be an arbitray vector and* $E : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ *be a zero function, that is,* $E(x, u) = 0$ *for any* $x \in \mathbb{R}^m$. *Suppose* $\hat{F} : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ *such that* $\hat{F}(x, u) = F(u)$. *Then the* $CP(F, K)$ *is equivalent to* $MiCP(E, \hat{F}, K, m, n)$.

3.2 Related problems

In the following, several special cases of $VI(K, F)$ where either K or F has some particular properties will be introduced.

To start with, let F be an affine function defined as:

$$F(x) \equiv q + Mx, \forall\ x \in \mathbb{R}^n\ , \tag{3.9}$$

where $q \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times n}$ is a square matrix. In this situation, we use the notation $VI(q, M, K)$ instead of $VI(F, K)$. If K is a polyhedral set at the same time, then $VI(F, K)$ is called *affine variational inequality* and is denoted by $AVI(q, M, K)$. Moreover, if the set K is polyhedral but F is not necessarily affine, the $VI(F, K)$ is called linearly constrained.

A simple (maybe the simplest) and important class of linearly constrained VI is the *box constrained* $VI(F, K)$, where the constrained set K is a "box" defined as

$$K \equiv \{x \in \mathbb{R}^n : a_i \leq x_i \leq b_i, i = 1, \ldots, n\}, \tag{3.10}$$

where a_i and b_i are allowed to be $\pm\infty$ respectively, that is,

$$-\infty \leq a_i < b_i \leq \infty,\ \forall i. \tag{3.11}$$

We will discuss the box constrained VI in detail later. We can also define the linear case for $MiCP$ and CP. If G and H are both affine functions in $MiCP$, the $MiCP$ is called *mixed linear complementarity problem*($MLCP$). A CP which is with an affine function F is called *linear complementarity problem*(LCP). When F is an affine function defined by a vector q and a matrix M, the $CP(F, K)$ will be denoted $LCP(q, M, K)$:

$$\mathbb{R}^n_+ \ni x \perp q + Mx \in \mathbb{R}^n_+.$$

The solution set of $LCP(q, M, K)$ is naturally written as $SOL(q, M, K)$. The linear case plays a very fundemental role in the study of CP and VI. In some of the most efficient algorithms, the linearization is applied for solving complicated CPs and VIs. The variational inequalities and complementarity problems are equivalent to specific fixed point problems. Let us formally define what a fix point problem is:

Definition 3.2.1. *Given a set $\mathcal{B}$ and a mapping $D : \mathcal{B} \mapsto \mathcal{B}$, The fixed point problem $Fix(D)$ defined by D is to find a point $x \in \mathcal{B}$ such that:*

$$x = D(x)$$

Next we are able to prove an equivalence between the variational inequality $VI(F, K)$ (or the complementarity problem $CP(F, K)$) and a fix point problem.

Lemma 3.2.1. *(Proposition 1.5.8 [12]). Let $K \subseteq \mathbb{R}^m$ be closed and convex and $F : \mathbb{R}^m \to \mathbb{R}^m$ be arbitrary. Then, $x \in SOL(F, K)$ if and only if x is a fixed point of the mapping $P_K \circ (I - F)$ where $I : \mathbb{R}^m \to \mathbb{R}^m$ is the identity mapping defined by $I(x) = x$ and P_K is the projection mapping to the set K.*

We now provide a proof for a particular case of the above lemma which is stated as Lemma 3.2.2. The general proof for the above lemma will be given in the proof of Proposition 3.2.1 which is a reformulation of Lemma 3.2.1.

Lemma 3.2.2. *Consider a closed convex cone K in the Euclidean space $(\mathbb{R}^m, \langle \cdot, \cdot \rangle)$ and a mapping $F : \mathbb{R}^m \to \mathbb{R}^m$. Then the $CP(F, K)$ is equivalent to the $Fix(P_K \circ (I - F))$ where $I : \mathbb{R}^m \to \mathbb{R}^m$ is the identity mapping defined by $I(x) = x$ and P_K is the projection mapping to the set K.*

Proof. For any $x \in \mathbb{R}^m$, let $z = (I - F)(x) = x - F(x)$ and $y = -F(x)$. Then $z = x + y$.

So if x is a solution of $CP(F,K)$, then by the definition of $CP(F,K)$, we get $x \in K$ and $y \in K^\circ$ (as $F(x) \in K^*$) and $\langle x, y\rangle = 0$. Thus, by Moreau Theorem (Theorem 2.1.1), we get $x = P_K(z)$. Therefore, x is a solution of $Fix(P_K \circ (I - F))$.

Conversely, if x is a solution of $Fix(P_K \circ (I - F))$, then, $x \in K$ and by Theorem 2.1.1 we can get

$$z = P_K(z) + P_{K^\circ}(z) = x + P_{K^\circ}(z).$$

Thus, $z - x = P_{K^\circ}(z) = y$ and $\langle x, y\rangle = 0$. So $y \in K^\circ$, $F(x) = -y \in K^*$. All in all, we get that $x \in K$, $F(x) \in K^*$ and $\langle x, F(x)\rangle = 0$ which implies that x is a solution of $CP(F,K)$. $\square$

By this lemma, if the sequence $\{x^n\}_{n\in\mathbb{N}}$ of the Picard iteration

$$x^{n+1} = P_K(x^n - F^n), \tag{3.12}$$

where $F^n = F(x^n)$ is convergent to $x^* \in K$ and the mapping F is continuous, then a simple limiting process in (3.12) yields that x^* is a fixed point of the mapping $K \ni x \to P_K(x - F(x))$, or equivalently a solution of the complementarity problem defined by K and F.

Here based on the above we have the following proposition:

Proposition 3.2.1. *([12] Proposition 1.5.9 Page 83) Given a closed convex set $K \subseteq \mathbb{R}^n$ and an arbitrary mapping $F : K \mapsto \mathbb{R}^n$, Then $x \in SOL(F,K)$ is equivalent to $F_K^{nat} = 0$ where the* natural mapping *$F_K^{nat} : \mathbb{R}^n \mapsto \mathbb{R}^n$ with respect to the set K and the mapping F*

is defined as:

$$F_K^{nat}(z) = x - P_K(x - F(x)) \tag{3.13}$$

Proof. Recall the Definition 3.1.1, $x \in SOL(F, K) \subseteq K$ if and only if

$$(y - x)^\top F(x) \geq 0, \forall y \in K$$

This can be rewritten as:

$$(y - x)^\top ((x - F(x)) - x) \leq 0, \forall y \in K$$

So by the equivalence given in the Proposition 2.1.1, the above inequality holds if and only if x is the projection point from $x - F(x)$ to K, that is, $x = P_K(x - F(x))$. This is equivalent to:

$$F_K^{nat}(x) = 0$$

□

Moreover, we get the following theorem:

Theorem 3.2.1. *([12] Theorem 2.2.3 Page 146) Consider a closed convex subset $K \subset \mathbb{R}^n$ and a continuous mapping $F : K \mapsto \mathbb{R}^n$. Consider the following three assertions*

(a) There exists a vector $x^b \in K$ which makes the following set:

$$S_< \equiv \{x \in K : (x - x^b)^\top F(x) < 0\}$$

bounded.

(b) There exists a set Λ which is bounded and open and a vector $x^b \in K \cap \Lambda$ such that :

$$(x - x^b)^\top F(x) \geq 0, \forall x \in K \cap \partial\Lambda$$

where $\partial\Lambda$ denotes the boundary of the set Λ.

(c) $SOL(K, F) \neq \varnothing$

Then (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c). Furthermore, similar to the set $S_<$, define the:

$$S_\leq \equiv \{x \in K : (x - x^b)^T F(x) \leq 0\}$$

If the $S_\leq \neq \varnothing$ and is bounded, then $SOL(F, K) \neq \varnothing$ and is compact.

A corollary can be directly deduced:

Corollary 3.2.1. *([12] Corollary 2.2.5 Page 148) Given a compact convex set $K \subseteq \mathbb{R}^n$ and a continuous mapping $F : K \to \mathbb{R}^n$, the set $SOL(K, F)$ is nonempty and compact.*

In addition, we can introduce an extension of optimization, *saddle problems*, which are also related to the VIs and CPs. A remarkable feature of the saddle problems is that the saddle problems are defined by a scalar function of two arguments. The initial study in this problems was strongly suggested by the programming duality theory. Meanwhile, the saddle problems is important in modelling some extensions of optimization problems. Let us first define the saddle problem:

Definition 3.2.2. *Let $L : \mathbb{R}^{n+m} \to \mathbb{R}$ denote a scalar funtion; The given sets $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ are closed. Then the saddle problems defined by (L, X, Y) is to find vector $(x, y) \in X \times Y$ which is defined as* saddle points*, such that*

$$L(u, y) \leq L(x, y) \leq L(x, v), \ \forall (u, v) \in X \times Y. \tag{3.14}$$

It is easily seen in (3.14) that x is the maximum point for the function $L(\cdot, y)$ for a given y and y is the minimum point for a given x. Particularly, if $L(\cdot, y)$ is concave for any given but arbitrary $y \in Y$ and $L(x, \cdot)$ is convex for any given but arbitrary $x \in X$, we call $L(x, y)$ is *concave-convex*. The saddle problem can be formulated as a variational inequality when L is a continuously differentiable and concave-convex. From the above, we can obtain that if L is concave-convex and X and Y are closed and convex, (x, y) is a saddle point if and only if it solves the $VI(F, X \times Y)$ where:

$$F(u,v) \equiv \begin{pmatrix} -\nabla_u L(u,v) \\ \nabla_v L(u,v) \end{pmatrix}, \; (u,v) \in \mathbb{R}^{n+m}. \tag{3.15}$$

A crucial case of the saddle point problem is when both X and Y are polyhedral and L is quadratic:

$$L(x,y) = a^\top x + b^\top y - x^\top M_1 x + y^\top M_2 y + x^\top A y, \; (x,y) \in \mathbb{R}^{n+m} \tag{3.16}$$

for some vectors $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, symmetric positive semidefinite matices $M_1 \in \mathbb{R}^{m \times m}$ and $M_2 \in \mathbb{R}^{n \times n}$ and a matrix $A \in \mathbb{R}^{m \times n}$. Generally, if $L(u, v)$ is twice continuously differentiable (its twice mixed differentiation is equal, that is, $\partial^2 L(u,v)/\partial u \partial v = \partial^2 L(u,v)/\partial v \partial u$), then $F(u, v)$ defined by (3.15) is a continuously differentiable function, the Jacobian matrix is :

$$JF = \begin{bmatrix} -\nabla^2_{uu} L(u,v) & -\nabla^2_{uv} L(u,v) \\ \nabla^2_{uv} L(u,v) & \nabla^2_{vv} L(u,v) \end{bmatrix}. \tag{3.17}$$

By the definition of L, we know that $JF(u, v)$ is antisymmetric. This terminology comes from the LCP theory and can be used to describe a partitioned matrix with the structure like JF. Moreover, by this antisymmetric, rather than symmetric property, the saddle point is not the stationary point of the optimization problem on $X \times Y$.

Following the above (L, X, Y) is a pair of dual optimization problems

$$\sup_{x \in X} \phi(x) \text{ and } \inf_{y \in Y} \psi(y), \tag{3.18}$$

where

$$\phi(x) \equiv \inf\{L(x, v), : v \in Y\} \text{ and } \psi(y) \equiv \sup\{L(u, y) : u \in X\}.$$

Note that it is allowed to have that $\phi(x) = +\infty$ or $\psi(y) = -\infty$. We could write the pair of problems in the form of a *maximin* problem and a *minimax* problem respectively:

$$\sup_{y \in Y} \inf_{x \in X} L(x, y) \text{ and } \inf_{y \in Y} \sup_{x \in X} L(x, y).$$

Then we could get the following theorem:

Theorem 3.2.2. *([12] 1.4.1 Page 22) Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$. Given a scalar funtion $L : X \times Y \mapsto \mathbb{R}$, we have:*

$$\sup_{x \in X} \inf_{y \in Y} L(x, y) \leq \inf_{y \in Y} \sup_{x \in X} L(x, y) \tag{3.19}$$

Furthermore, given $(x^*, y^*) \in X \times Y)$, the following three assertions are equivalent:

(i) (x^*, y^*) is a saddle point of the function L in $X \times Y$,

(ii) x^* is a maximizer of $\phi(x)$ on X, y^* is a minimizer of $\psi(y)$ on Y and then the equality of (3.19) holds,

(iii) $\phi(x^*) = \psi(y^*) = L(x^*, y^*)$.

Note that in this theorem, the continuity and differentiability of L will not influence the equivalence of the assertions. It doesn't tell us whether the saddle point exists or not.

Chapter 4

Conic optimization and complementarity problems

4.1 Preliminaries

The research of variational inequality and complementarity problems can be applied in optimization. Recall the Definition 3.1.1 , it can be reformulated to

$$y^\top F(x) \geq x^\top F(x),\ \forall y \in K.$$

Clearly, this problem is equivalent to the following optimization problem [12]

$$\min_{y \in K} y^\top F(x)$$

if we suppose x is fixed. Moreover, we have the following proposition which plays a very fundemental role in optimization problems.

Proposition 4.1.1 ([60], Theorem 2.67, Page 51)**.** *Let m be a positive integer. Provided* *that a function $f \in C^1(\mathbb{R}^m, \mathbb{R})$, then f is convex if and only if*

$$f(x) \geq f(y) + \nabla f(y)^\top (x - y),\ \forall x, y \in \mathbb{R}^m. \tag{4.1}$$

Moreover, if f is convex and x is the minimum point, then

$$\nabla f(y)^\top (x - y) \leq 0,\ \forall y \in \mathbb{R}^m. \tag{4.2}$$

Replace $\mathbb{R}^m$ by a general closed convex set $K \subseteq U \subseteq \mathbb{R}^m$ where U is an open set and suppose that $\phi : \mathbb{R}^m \to \mathbb{R}$ is a continuously differentiable function, and consider the following constrained optimization problem:

$$\min_{x \in K} \phi(x). \tag{4.3}$$

By the minimum principle in nonlinear programming([60]), we have that the local minimizer x will satisfy:

$$(y - x)^\top \nabla \phi(x) \geq 0,\ \forall y \in K. \tag{4.4}$$

Obviously, this is $VI(\nabla \phi, K)$. The inequality (4.4) is called *stationary point problem* associated with the optimization problem (4.3).

Remark 4.1.1. *Based on Proposition 4.1.1 and* (4.1) *and* (4.2)*, we can observe that if ϕ is differentiable and x^* is the solution of the above optimization problem, then x^* is a solution of $VI(\nabla \phi, K)$ or $CP(\nabla \phi, K)$, where K is a closed convex set or a closed convex cone, respectively. If ϕ is convex and differentiable, x^* is the solution of the optimization problem 4.3 if and only if it is a solution of $VI(\nabla \phi, K)$ or $CP(\nabla \phi, K)$, where K is a closed convex set or a closed convex cone, respectively.*

Next, we show how complementarity is related to optimization problems. Let's consider a simple optimization problem:

$$\min f(x) \text{ s. t. } x \geq 0$$

where $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function all over the real line. If this problem is solvable and the solution is $x^* \geq 0$, then we will have

$$x^* f'(x^*) = 0.$$

Generally, consider the following optimization problem:

$$\begin{aligned} &\min F(x) \\ &x \in K \ (\text{or } x \succcurlyeq_K 0) \end{aligned},$$

where $x \succcurlyeq_K 0$ is a standard notation in conic optimzation for $0 \leq_K x$, $F : \mathbb{R}^m \to \mathbb{R}$ is a differentiable function and $K \subseteq \mathbb{R}^m$ is a convex cone. We can easily prove that if $x^* \in \mathbb{R}^m$ is a solution of this problem, then we will have ${x^*}^T \nabla F(x^*) = 0$, and

$$K \ni x^* \perp \nabla F(x^*) \in K^*.$$

By Definition 3.1.2, this means that x^* a solution of $CP(\nabla F, K)$. In the following, we will give several examples where complementarity occurs. Suppose a linear programming problem is

$$\begin{aligned} &\min c^\top x \\ &\text{s.t. } Ax \geq b. \end{aligned} \tag{$\mathcal{P}$}$$

Its dual problem will be

$$\begin{aligned} &\max b^\top y \\ &\text{s.t. } A^\top y = c \\ &y \in \mathbb{R}^n_+. \end{aligned} \tag{$\mathcal{D}$}$$

where $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ are two vector variables and $c \in \mathbb{R}^m$, $b \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times m}$ are given. Then the following theorems are well-known.

Theorem 4.1.1 (Weak Duality Theorem [7] Page 21)**.** *If $x \in \mathbb{R}^m$ is feasible for $(\mathcal{P})$ and $y \in \mathbb{R}^n$ is feasible for $(\mathcal{D})$, then*

$$b^\top y \leq y^\top Ax \leq c^\top x.$$

Hence if $(\mathcal{P})$ is unbounded, then the $(\mathcal{D})$ is necessarily infeasible, and if $(\mathcal{D})$ is unbounded, then the $(\mathcal{P})$ is necessarily infeasible. Moreover, if $b^\top y^* = c^\top x^*$ with x^* and y^* feasible for $(\mathcal{P})$ and $(\mathcal{D})$, respectively, x^* and y^* must be the solutions of $(\mathcal{P})$ and $(\mathcal{D})$, respectively.

Theorem 4.1.2 (Strong Duality Theorem [7] Page 23)**.** *If either $(\mathcal{P})$ or $(\mathcal{D})$ has a finite optimal value, then so does the other, the optimal value exists. Suppose the optimal points are x^* and y^*, respectively, then the optimal value $c^\top x^* = b^\top y^*$.*

Then $(\mathcal{P})$ is solvable if and only if $(\mathcal{D})$ is solvable. The duality gap

$$c^\top x - b^\top y = y^\top Ax - y^\top b = y^\top (Ax - b) \geq 0.$$

will be nonnegative for any feasible pair (x, y) . If (x^*, y^*) are solutions of $(\mathcal{P})$ and $(\mathcal{D})$

respectively, then the optimality will indicate the complementary slackness, that is,

$$y^{*\top}(Ax^* - b) = 0.$$

Moreover, by changing the constraint condition from ≥ 0 to $\succeq_K 0$ with respect to some cone K, we get linear conic programming problem

$$\begin{aligned} &\min c^\top x \\ &\text{s.t. } Ax - b \succeq_K 0. \end{aligned} \tag{$\mathcal{CP}$}$$

Its dual problem will be

$$\begin{aligned} &\max b^\top \lambda \\ &\text{s.t. } A^\top \lambda = c \\ &\lambda \succeq_{K^*} 0. \end{aligned} \tag{$\mathcal{CD}$}$$

Then the $(\mathcal{CP})$ is solvable if and only if the $(\mathcal{CD})$ is solvable. The complementary condition in this case becomes

$$\lambda^{*\top}(Ax^* - b) = 0$$

where x^* and λ^* are optimal solutions for $(\mathcal{CP})$ and $(\mathcal{CD})$ respectively. When K is $\mathbb{R}^n_+$, the $(\mathcal{CP})$ is just $(\mathcal{P})$. Note that K may be a matrix cone. A classical example is *semidefinite programming (SDP)*. Here, the cone K is the set of $n \times n$ positive semidefinite matrices $\mathbb{S}^n_+$. It can be easily proved that $\mathbb{S}^n_+$ is self-dual with respect to Frobenius inner product. Then the primal semidefinite problem is

$$\min_{x \in \mathbb{R}^n} \{c^\top x : \mathcal{A}x - B \succeq 0\} \tag{4.5}$$

where $\mathcal{A}x - B = x_1 A_1 + \cdots + x_n A_n - B$ for some given $n \times n$ matrices $A_1, \ldots, A_n$. Its dual problem is

$$\max_{\Lambda \in \mathbb{S}^n_+} \{tr(B\Lambda) : \mathcal{A}^*\Lambda = c, \, \Lambda \succeq 0\}. \tag{4.6}$$

Then the complementarity slackness condition is $\Lambda(\mathcal{A}x - B) = \mathcal{A}x - B = 0$ where 0 denotes the n-dimensional zero matrix.

Although the complementarity problem idea occurs here, the connection between optimization and complementarity is implicit. A more explicit connection will be shown in the next section.

4.2 Practical examples

In the following, we will give some practical examples to show that many optimization problems can be interpreted as complementarity problems for not only $\mathbb{R}^n$ but some other cones such as second-order cone and positive semidefinite cones.

Example 4.2.1 (Cassel-Wald model [28] Section 5.1 Page 51)**.** *Suppose there are n commodities and m pure factors of production. Let c_k denotes the price of the k-th commodity, b_i denotes total inventory of the i-th factor, and a_{ij} denotes the consumption rate of the i-th factor which is required for producing one unit of the j-th commodity. Let $c = (c_1, \dots, c_n)^\top$, $b = (b_1, \dots, b_m)^\top$ and $A = (a_{ij})_{m \times n}$. Next, x_j denotes the output of the j-th commodity and p_i denotes the (shadow) price of the i-th factor so that $x = (x_1, \dots, x_n)^\top$ and $p = (p_1, \dots, p_m)^\top$. Let the vector b be fixed and let $c(x) : \mathbb{R}^n_+ :\to \mathbb{R}^n_+$. Denote $F(x, p) = ((A^\top p - c(x))^\top, (b - Ax)^\top)^\top$. Then the pair (x^*, p^*) is said to be in equilibrium if it is a solution to $CP(F, \mathbb{R}^{n+m}_+)$:*

$$\begin{aligned}&x^* \geq 0,\ p^* \geq 0\\&A^\top p^* - c(x^*) \geq 0,\ b - Ax^* \geq 0\\&(x^*)^\top[A^\top p^* - c(x^*)],\ (p^*)^\top[b - Ax^*] = 0.\end{aligned} \tag{4.7}$$

Example 4.2.2 (Robust linear programming [7] Section 2.4.1 Page 101). *Consider a linear programming*

$$\min_x\{c^\top x : Ax - b \geq 0\}$$

In many pratical circumstances, the data c, A and b are uncertain. but we know that they belong to a given set $\mathcal{U}$. Then the problem can be formulate as:

$$\{\min_x\{c^\top x : a_i^\top x - b_i \geq 0,\ i = 1, \ldots, m\} | (c, A, b) \in \mathcal{U}\}.$$

where $a_i^\top$ is the i-th row of A, b_i is the i-th entry of b and the set $\mathcal{U}$ is given as:

$$\mathcal{U} = \{(c, A, b) : \exists(\{u_i, u_i^\top u_i \leq 1\}_{i=0}^m) : c = c_* + P_0 u_0, (a_i^\top, b_i) = (a_i^{*\top}, b_i^*) + P_i u_i, i = 1, \ldots, m\}$$

where c_*, a_i^* and b_i^* are the "nominal data " and $P_i u_i$, $i = 0, 1, \ldots, m$ represent the data perturbations; the restrictions $u_i^\top u_i \leq 1$ enforce these perturbations to vary in the ellipsoids. The robust counterpart can be formulated as a conic quadratic programming (details can be seen in [7]):

$$\min_{x,t}\{t : \|P_0^\top x\|_2 \leq -c_*^\top x + t; \|P_i^\top (x^\top, -1)^\top\|_2 \leq a_i^{*\top} x - b_i^*,\ i = 1, \ldots, m\}$$

Let

$$A_0 = \begin{pmatrix} P_0^\top; & 0 \\ -c_*^\top; & 1 \end{pmatrix}$$

and

$$A_i = \begin{pmatrix} Q_i^\top; & 0 \\ a_i^{*\top}; & 0 \end{pmatrix}, \; i = 1, \ldots, m,$$

where Q_i *is obtained by deleting the last row of* P_i. *Let* q_i *denote the last row of* P_i. *So its dual problem is:*

$$\begin{aligned} &\max_{\mu,\nu} \sum_{i=1}^{m} (q_i \mu_i + \nu_i b_i^*) \\ s.\ t.\ &\sum_{i=1}^{m} A_i^\top \lambda_i = (0, \ldots, 0, 1)^\top \quad , \\ &\lambda_i = \begin{pmatrix} \mu_i \\ \nu_i \end{pmatrix} \in L^m \end{aligned}$$

where L^m *denotes m-dimensional Lorentz (second-order) cone. So if the conditions of strong duality theorem are satisfied, we will have the following complementarity relation:*

$$L^m \ni \lambda_i \perp A_i \begin{pmatrix} x \\ t \end{pmatrix} - \begin{pmatrix} q_i^\top \\ b_i^* \end{pmatrix} \in L^m.$$

Furthermore, suppose $f_r(\lambda) = \sum_{i=1}^{m} (q_i \mu_i + \nu_i b_i^*)$ *which is the objective function of the maximum problem and let*

$$\begin{aligned} G(x, t; \lambda) &= \sum_{i=1}^{m} A_i^\top \lambda_i - (0, \ldots, 0, 1)^\top \\ H_i(x, t; \lambda) &= A_i \begin{pmatrix} x \\ t \end{pmatrix} - \nabla f_r(\lambda_i). \end{aligned}$$

Then $(x^*, t^*; \lambda^*)$ is a solution of the primal and dual problems respectively if and only if it is a solution to $MiCP(G, H_i, L^m, \dim x + 1, 1)$, $i = 1, \dots, m$ when strong duality theorem is satisfied.

Example 4.2.3 (MAXCUT [7] Section 3.4.1 Page 175)**.** *Consider the maximum cut problem: Let $\mathscr{G}$ be an n-node graph, and let the arcs (i, j) of the graph be associated with nonnegative "weights" a_{ij}. The problem is to find a cut of the largest possible weight, i. e. to partition the set of nodes in two parts S, S' in such a way that the total weight of all arcs "linking S and S'" (i.e. with one incident node in S and the other one in S') is as large as possible. It can be formulated in the following way: let $x \in \mathbb{R}^n$ and the i-th entry of x, $x_i = 1$ for $i \in S$, $x_i = -1$ for $i \in S'$. The quantity $\frac{1}{2}\sum_{i,j=1}^n a_{ij}x_ix_j$ is the total weight of arcs with both ends either in S or S' minus the weight of the cut (S, S'); consequently, the quantity*

$$\frac{1}{2}\left[\frac{1}{2}\sum_{i,j=1}^n a_{ij} - \frac{1}{2}\sum_{i,j=1}^n a_{ij}x_ix_j\right] = \frac{1}{4}\sum_{i,j=1}^n a_{ij}(1 - x_ix_j) \tag{4.8}$$

is the weight of the cut (S, S'). Then the optimization problem is

$$\max_x \left\{\frac{1}{4}\sum_{i,j=1}^n a_{ij}(1 - x_ix_j) : x_i^2 = 1, i = 1, \dots n\right\}$$

For this problem, the semidefinite relaxation is:

$$\begin{aligned} &\max \frac{1}{4}\sum_{i,j=1}^n a_{ij}(1 - x_ix_j) \\ s.\ t.\ &X = [X_{ij}]_{i,j=1}^n = X^\top \succeq 0, \\ &X_{ii} = 1,\ i = 1, \dots, n, \end{aligned} \tag{4.9}$$

where $X_{ij} = x_i x_j$. *The optimial value is an upper bound for the optimal value of the maximum cut problem. Note that* a_{ij} *is given. The objective function can be regarded as* $\frac{1}{4}\sum_{i,j=1}^{n} a_{ij}x_i x_j$. *Denote* $A = (a_{ij})$. *Since the positive semidefinite cone is self-dual, the dual problem of above is:*

$$\begin{aligned} &\min_{y\in\mathbb{R}^n}(1,...,1)y \\ &s.\ t.\ diag(y_1,\dots,y_n) + \frac{1}{4}A \succeq 0. \end{aligned} \tag{4.10}$$

If both (4.9) *and* (4.10) *are strictly feasible, then we have the following complementarity problems:*

$$X\left[diag(y_1,\dots,y_n) + \frac{1}{4}A\right] = 0 \tag{4.11}$$

Similar to the above example, suppose that $f_m(X) = -\frac{1}{4}\sum_{i,j=1}^{n} a_{ij}x_i x_j$, *then by the calculus of matrix-value functions (for example, see [29] Theorem 2 Page 124),* $\nabla f_m(X) = -\frac{1}{4}A$. *Then let*

$$\begin{aligned} G(X;y) &= tr(A-I) - (1,...,1)^\top \\ H(X;y) &= diag(y) - \nabla f_m(\lambda_i). \end{aligned}$$

Then $(X^*; y^*)$ *is a solution of the primal and dual problem respectively if and only if it is a solution of* $MiCP(G, H, n, n\times n, \mathbb{S}^n_+)$ *when the conditions of strong duality theorem are satisfied.*

Consider the nonlinear optimization problem

$$\begin{aligned} &\min f(x) \\ &\text{subject to} \quad g_i(x) \le 0,\ i = 1,\dots,m, \\ &\qquad\qquad\quad h_i(x) = 0,\ i = 1,\dots,p, \\ &\qquad\qquad\quad x \in X_0, \end{aligned} \tag{4.12}$$

where the function $f : \mathbb{R}^n \mapsto \mathbb{R}$, $g_i : \mathbb{R}^n \mapsto \mathbb{R}$, $i = 1, \ldots, m$, and $h_i : \mathbb{R}^n \mapsto \mathbb{R}$, $i = 1, \ldots, p$ are continuously differentiable, and that the set $X_0 \subseteq \mathbb{R}^n$ is convex and closed. *Slater's condition* is as follows: there exists a point $x_s \in X_0$ such that $g_i(x_s) < 0$, $i = 1, \ldots, m$, $h_i(x_s) = 0$, $i = 1, \ldots, p$ and $x_x \in \operatorname{int} X_0$, if $p > 0$.

Theorem 4.2.1. *([60] Theorem 3.34, Page 127) Assume that $\hat{x}$ is the minimum of problem* (4.12)*, the function f is continuous at some feasible point x_0, and Slater's condition is satisfied. Then there exist $\hat{\lambda} \in \mathbb{R}^n_+$ and $\hat{\mu} \in \mathbb{R}^p$ such that*

$$0 \in \nabla f(\hat{x}) + \sum_{i=1}^{m} \hat{\lambda}_i \nabla g_i(\hat{x}) + \sum_{i=1}^{p} \hat{\mu}_i \nabla h_i(\hat{x}) + N_{X_0}(\hat{x}) \tag{4.13}$$

and

$$\hat{\lambda}_i g_i(\hat{x}) = 0, i = 1, \ldots, m. \tag{4.14}$$

Conversely, if for some feasible point $\hat{x}$ of (4.12) *and some $\hat{\lambda} \in \mathbb{R}^m_+$ and $\hat{\mu} \in \mathbb{R}^p$ conditions* (4.13) *and* (4.14) *are satisfied, then $\hat{x}$ is the global minimum of the problem.*

Definition 4.2.1. *Let $f : \mathbb{R}^q \mapsto \mathbb{R}$ be a function, $K \subset \mathbb{R}^m$ be a cone, A be a $p \times q$ matrix and $b \in \mathbb{R}^p$. Then, the problem*

$$CO(f, A, b, K, p, q) : \begin{cases} \min f(x) & \\ \textit{subject to} & Ax = b, \\ & x \in K \end{cases} \tag{4.15}$$

is called conic optimization problem.

4.3 The main result

In the previous sections, we stated the complementarity problems and the complementarity relation in linear (conic) programming problems. We also presented the Karush-Kuhn-Tucker condition which illustrated the properties of optimal solutions. Based on these results, we will prove the equivalence of a conic optimization problem and a mixed complementarity problem.

Theorem 4.3.1. *Let $f : \mathbb{R}^q \mapsto \mathbb{R}$ be a differentiable convex function at $v \in \mathbb{R}^q \setminus \{0\}$, $K \subseteq \mathbb{R}^q$ be a cone with a smooth boundary, A be a $p \times q$ matrix of full rank ($rank(A) = \min\{p, q\}$) and $b \in \mathbb{R}^p$. Suppose that the intersection of the interior of K and the linear subspace $\{v \in \mathbb{R}^q : Av = b\}$ is nonempty. Then, $\hat{v}$ is a solution of $CO(f, A, b, K)$ if and only if $(\hat{y}, \hat{v})$ is a solution of $MiCP(G, H, K, p, q)$, where $G(y, v) = b - Av$, $H(y, v) = \nabla f(v) - A^\top y$, which can be written explicitly as*

$$A\hat{v} = b, K \ni \hat{v} \perp \nabla f(\hat{v}) - A^\top \hat{y} \in K^*.$$

Proof. If $\hat{v}$ is a solution of $CO(f, A, b, K)$, by the preceding theorem, let $X_0 = K$, $h(v) = b - Av$ and $\hat{\lambda} = \hat{y}$. The equation (4.13) can be transformed to

$$0 \in \nabla f(\hat{v}) - A^\top y + N_K(\hat{v}).$$

By (2.8), we have that

$$\nabla f(\hat{v}) - A^\top \hat{y} \in K^* \text{ and } \nabla f(\hat{v}) - A^\top \hat{y} \perp \hat{v}. \tag{4.16}$$

The conditions $\hat{v} \in K$ and $A\hat{v} = b$ are obvious. Conversely, suppose that $(\hat{y}, \hat{v})$ is a solution of $MiCP(G, H, K)$. For any feasible solution v in $CO(f, A, b, K)$, we have

$$0 \leq \langle \nabla f(\hat{v}) - A^\top \hat{y}, v \rangle = \langle \nabla f(\hat{v}) - A^\top \hat{y}, v - \hat{v} \rangle = \langle \nabla f(\hat{v}), v - \hat{v} \rangle - \langle A^\top \hat{y}, v - \hat{v} \rangle. \quad (4.17)$$

Because $Av = A\hat{v} = b$ and $\langle A^\top \hat{y}, v - \hat{v} \rangle = \langle \hat{y}, Av - A\hat{v} \rangle = 0$, by the convexity of f, the inequality (4.17) and Proposition 4.1.1, we have

$$0 \leq \langle \nabla f(\hat{v}), v - \hat{v} \rangle \leq f(v) - f(\hat{v}). \quad (4.18)$$

Hence $f(\hat{v}) \leq f(v)$ for any v feasible. Therefore, $\hat{v}$ is a solution of $CO(f, A, b, K)$. □

Example 4.3.1. *Consider the KKT system (Proposition 1.2.1 in [12])*

Let Δ be defined as

$$\Delta \equiv \{v \in \mathbb{R}^n : Av = b, \ Cv \leq d\}, \quad (4.19)$$

where the matrix $A \in \mathbb{R}^{p \times q}$ have full rank, $C \in \mathbb{R}^{l \times q}$ and vectors $b \in \mathbb{R}^p$ and $d \in \mathbb{R}^l$ are given. A vector x is the solution of $VI(\Delta, F)$ if and only if there exist two vectors $\lambda \in \mathbb{R}^p$ and $\mu \in \mathbb{R}^l$ such that

$$\begin{cases} 0 = F(x) + C^\top \mu + A^\top \lambda, \\ 0 = b - Ax, \\ 0 \leq \mu \perp d - Cx \geq 0. \end{cases} \quad (4.20)$$

The MiCP resulted by the $VI(K, F)$ where K is defined by (4.19) *is called the* Karush-Kuhn-Tucker (KKT) *system of the VI. In this system, let $C = 0$, $d = 0$, $\lambda = -y$ and $F(x) = \nabla f(x)$. It is obvious that the dual cone of $\mathbb{R}^p$ is $\{0\}$. Then it can be transformed to*

$$\begin{cases} 0 = b - Ax, \\ \mathbb{R}^p \ni x \perp F(x) - A^\top y \in \{0\}, \end{cases}$$

which coincides with the mixed complementarity problem in Theorem 4.3.1.

4.4 Notes and comments

In this chapter, we showed connections between complementarity problems and constrained optimization problems. The nonnegativity of variables is defined by a cone ranging from the nonnegative orthant (corresponding to KKT conditions) to some special cones such as positive semidefinite cones and Lorentz cones. Since the restriction of C in Theorem 5.2.1 is just to be a closed convex cone, this $MiCP$ formulation of constrained optimization problems will be a bridge for us to apply iterative methods to solve constrained optimization problems by using Theorem 5.2.1. Details will be given explicitly in Section 7.2.

Chapter 5

Complementarity problems and extended Lorentz cones

In this chapter, we will call a closed convex set isotone projection set with respect to a pointed closed convex cone if the projection onto the set is isotone (i.e., monotone) with respect to the order defined by the cone. We showed in Theorem 2.3.2 that a Cartesian product between an Euclidean space and any closed convex set in another Euclidean space is an isotone projection cone with respect to an extended Lorentz cone. We will use this property to find solutions of general mixed complementarity problems in an iterative way. This chapter and the next one are mainly from my joint work with S. Z. Németh [51, 53].

5.1 Complementarity problems

Recall the notion of a complementarity problem and the corresponding Picard iteration (3.12). It is natural to seek convergence conditions for x^n. This will be done by finding pointed closed convex cones L and conditions to be imposed on F such that the sequence $\{x^n\}_{n\in\mathbb{N}}$ to be L-increasing and L-bounded from above. These conditions will imply that

$\{x^n\}_{n\in\mathbb{N}}$ is convergent and its limit is a solution of $CP(F,K)$.

Denote by I the identity mapping.

Lemma 5.1.1. *Let $K \subset \mathbb{R}^m$ be a closed convex cone, K^* be its dual, $F : \mathbb{R}^m \to \mathbb{R}^m$ be a continuous mapping and L be a pointed closed convex cone. Consider the sequence $\{x^n\}_{n\in\mathbb{N}}$ defined by* (3.12)*. Suppose that the mappings P_K and $I - F$ are L-isotone, $x^0 \leq_L x^1$, and there exists a $y \in \mathbb{R}^m$ such that $x^n \leq_L y$, for all $n \in \mathbb{N}$ sufficiently large. Then, $\{x^n\}_{n\in\mathbb{N}}$ is convergent and its limit x^* is a solution of $CP(F,K)$.*

Proof. Since the mappings P_K and $I - F$ are L-isotone, the mapping $x \mapsto P_K \circ (I - F)$ is also L-isotone. Then, by using (3.12) and a simple inductive argument, it follows that $\{x^n\}_{n\in\mathbb{N}}$ is L-increasing. Since any pointed closed convex cone in $\mathbb{R}^m$ is regular, $\{x^n\}_{n\in\mathbb{N}}$ is convergent and hence its limit x^* is a solution of $CP(F,K)$. □

Remark 5.1.1.

1. *The condition $x^0 \leq_L x^1$ in Lemma 5.1.1 is satisfied when $x^0 \in K \cap F^{-1}(-L)$. Indeed, if $x^0 \in K \cap F^{-1}(-L)$, then $-F(x^0) \in L$ and $x^0 \in K$. Thus $x^0 \leq_L x^0 - F(x^0)$, and hence by the L-isotonicity of P_K we obtain $x^0 = P_K(x^0) \leq_L P_K(x^0 - F(x^0)) = x^1$.*

2. *The condition $x^0 \leq_L x^1$ in Lemma 5.1.1 is satisfied when $x^0 = 0$ and $-F(0) \in L$. Indeed, this is a particular case of the item above.*

Proposition 5.1.1. *Let L be a pointed closed convex cone, $K \subset \mathbb{R}^m$ be a closed convex cone such that $K \cap L \neq \varnothing$ and K^* be its dual, and $F : \mathbb{R}^m \to \mathbb{R}^m$ be a continuous mapping. Consider the sequence $\{x^n\}_{n\in\mathbb{N}}$ defined by* (3.12)*. Suppose that the mappings P_K and $I - F$ are L-isotone and $x^0 = 0 \leq_L x^1$. Let*

$$\Omega = K \cap L \cap F^{-1}(L) = \{x \in K \cap L : F(x) \in L\}$$

and

$$\Gamma = \{x \in K \cap L : P_K(x - F(x)) \leq_L x\}.$$

Consider the following assertions:

(i) $\Omega \neq \varnothing$*,*

(ii) $\Gamma \neq \varnothing$*,*

(iii) The sequence $\{x^n\}_{n\in\mathbb{N}}$ *is convergent and its limit* x^* *is a solution of* $CP(F, K)$*. Moreover,* x^* *is the* L*-least element of* Γ *and a lower* L*-bound of* Ω*.*

Then, $\Omega \subset \Gamma$ *and (i)* $\Longrightarrow$ *(ii)* $\Longrightarrow$ *(iii).*

Proof. Let us first prove that $\Omega \subset \Gamma$. Indeed, let $y \in \Omega$. Since the mappings P_K and $I - F$ are L-isotone, the mapping $P_K \circ (I - F)$ is also L-isotone. Hence, $y - F(y) \leq_L y$ implies $P_K(y - F(y)) \leq_L P_K(y) = y$, which shows that $y \in \Gamma$. Hence, $\Omega \subset \Gamma$. Thus, (i) $\Longrightarrow$ (ii) is trivial now.

we now prove (ii) $\Longrightarrow$ (iii):

Suppose that $\Gamma \neq \varnothing$. Since the mapping P_K and $I - F$ are L-isotione, the mapping, $P_K \circ (I - F)$ is also L- isotone. Similarly to the proof of Lemma 5.1.1, it can be shown that $\{x^n\}_{n\in\mathbb{N}}$ is L-increasing. Let $y \in \Gamma$ be arbitrary but fixed. We have $x^0 = 0 \leq_L y$. Now, suppose that $x^n \leq_L y$. Since the mapping $P_K \circ (I - F)$ is L-isotone, $x^n \leq_L y$ implies that $x^{n+1} = P_K(x^n - F(x^n)) \leq_L P_K(y - F(y)) \leq_L y$. Thus, we have by induction that $x^n \leq_L y$ for all $n \in \mathbb{N}$. Then, Lemma 5.1.1 implies that $\{x^n\}_{n\in\mathbb{N}}$ is convergent and its limit x^* is a solution of $CP(F, K)$. Since x^* is a solution of $CP(F, K)$, we have that $P_K(x^* - F(x^*)) = x^*$ and hence $x^* \in \Gamma$. Moreover, taking the limit in $x^n \leq_L y$, we get $x^* \leq_L y$ for any $y \in \Gamma$. Therefore, x^* is the L-least element of Γ. Since $\Omega \subset \Gamma$, x^* is the L-bound of Ω. □

We note that from the second item of Remark 5.1.1, it follows that condition $x^0 = 0 \leq_L x^1$ of Proposition 5.1.1 holds if $x^0 = 0$ and $-F(0) \in L$. We also remark that since the definition of Ω does not contain the projection onto K, (for a given F and K) it is easier to show that $\Gamma \neq \varnothing$ by first showing that $\Omega \neq \varnothing$.

5.2 Mixed complementarity problems

The following lemma extends the mixed complementarity problem in [12] by replacing $\mathbb{R}^q_+$ with an arbitrary nonempty closed convex cone in $\mathbb{R}^q$.

Lemma 5.2.1. *Let $K = \mathbb{R}^p \times C$, where C is an arbitrary nonempty closed convex cone in $\mathbb{R}^q$. Let $G : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p$, $H : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^q$ and $F = (G, H) : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p \times \mathbb{R}^q$. Then, the nonlinear complementarity problem $CP(F, K)$ is equivalent to the mixed complementarity problem $MiCP(G, H, C, p, q)$ defined by* (3.8)

Proof. It follows easily from the definition of the nonlinear complementarity problem $CP(F, K)$, by noting that $K^* = \{0\} \times C^*$. □

By using the notations of Lemma 5.2.1, the Picard iteration (3.12) can be rewritten as:

$$\begin{cases} x^{n+1} = x^n - G(x^n, u^n) \\ u^{n+1} = P_C(u^n - H(x^n, u^n)). \end{cases} \tag{5.1}$$

Consider the order defined by the extended Lorentz cone (2.9). Then, we obtain the following theorem.

Theorem 5.2.1. *Let $K = \mathbb{R}^p \times C$, where C is a closed convex cone, K^* be the dual of K, $G : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p$ and $H : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^q$ be continuous mappings, $F = (G, H)$:*

$\mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p \times \mathbb{R}^q$, and $L = L(p,q)$ be the extended Lorentz cone defined by (2.9). Let $x^0 = 0 \in \mathbb{R}^p$, $u^0 = 0 \in \mathbb{R}^q$ and consider the sequence $\{(x^n, u^n)\}_{n\in\mathbb{N}}$ defined by (5.1). Let $x, y \in \mathbb{R}^p$ and $u, v \in \mathbb{R}^q$. Suppose that $y - x \geq \|v-u\|e$ implies

$$y - x - G(y,v) + G(x,u) \geq \|v - u - H(y,v) + H(x,u)\|e,$$

and $x^1 \geq \|u^1\|e$ (in particular this holds when $-G(0,0) \geq \|H(0,0)\|e$).

Let

$$\Omega = \{(x,u) \in \mathbb{R}^p \times C : x \geq \|u\|e,\ G(x,u) \geq \|H(x,u)\|e\}$$

and

$$\Gamma = \{(x,u) \in \mathbb{R}^p \times C : x \geq \|u\|e,\ G(x,u) \geq \|u - P_C(u - H(x,u))\|e\}.$$

Consider the following assertions:

(i) $\Omega \neq \varnothing$,

(ii) $\Gamma \neq \varnothing$,

(iii) The sequence $\{(x^n, u^n)\}_{n\in\mathbb{N}}$ is convergent and its limit (x^*, u^*) is a solution of $MiCP(G,H,C,p,q)$. Moreover, (x^*, u^*) is a lower $L(p,q)$-bound of Ω and the $L(p,q)$-least element of Γ.

Then, $\Omega \subset \Gamma$ and (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

Proof. First observe that $K \cap L(p,q) \neq \varnothing$. By using the definition (2.9) of the extended Lorentz cone, it is easy to verify that

$$\Omega = K \cap L(p,q) \cap F^{-1}(L(p,q)) = \{z \in K \cap L(p,q) : F(z) \in L(p,q)\}$$

and

$$\Gamma = \{z \in K \cap L(p,q) : P_K(z - F(z)) \leq_L z\}.$$

Let $x, y \in \mathbb{R}^p$ and $u, v \in C$. Since $y - x \geq \|v - u\|e$ implies

$$y - x - G(y,v) + G(x,u) \geq \|v - u - H(y,v) + H(x,u)\|e,$$

it follows that $I - F$ is $L(p,q)$-isotone. Also, $x^1 \geq \|u^1\|e$ means that $(x^0, u^0) = (0,0) \leq_L (x^1, u^1)$ (in particular if $-G(0,0) \geq \|H(0,0)\|e$, or equivalently $-F(0,0) \in L(p,q)$, then by the second item of Remark 5.1.1, it follows that $(x^0, u^0) = (0,0) \leq_L (x^1, u^1)$). Hence, by Theorem 2.3.2, Proposition 5.1.1 (with $m = p + q$) and Lemma 5.2.1, it follows that $\Omega \subset \Gamma$ and (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii). $\square$

By Remark 3.1.1 and Theorem 5.2.1, we can get the following corollary:

Corollary 5.2.1. *Let $L = L(p,q)$ be the extended Lorentz cone defined by (2.9). Let C is a closed convex cone in $\mathbb{R}^p$ and $F : \mathbb{R}^q \to \mathbb{R}^q$ is a continuous mapping.Suppose $E : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p$ is a zero function, that is, $E(x,u) = 0$ for any $(x,u) \in \mathbb{R}^p \times \mathbb{R}^q$ and $\hat{F} : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^q$ is defined by $\hat{F}(x,u) = F(u)$. Let $x^0 = 0 \in \mathbb{R}^p$, $u^0 = 0 \in \mathbb{R}^q$ and consider the sequence $\{(x^n, u^n)\}_{n \in \mathbb{N}}$ defined by (5.1). Let $x, y \in \mathbb{R}^p$ and $u, v \in \mathbb{R}^q$. Suppose that $y - x \geq \|v - u\|e$ implies*

$$y - x \geq \|v - u - F(v) + F(u)\|e,$$

and $x^1 \geq \|u^1\|e$ (in particular this holds when $\hat{F}(0,0) = F(0) = 0$).

Let

$$\Omega = \{(x,u) \in \mathbb{R}^p \times C : x \geq \|u\|e,\ F(u) = 0\}$$

and

$$\Gamma = \{(x,u) \in \mathbb{R}^m \times C : x \geq \|u\|e, \ u = P_C(u - \hat{F}(x,u))\}.$$

Consider the following assertions:

(i) $\Omega \neq \varnothing$,

(ii) $\Gamma \neq \varnothing$,

(iii) The sequence $\{(x^n, u^n)\}_{n\in\mathbb{N}}$ *is convergent and its limit* (x^*, u^*) *is a solution of* $MiCP(E, \hat{F}, C, p, q)$. *Then* u^* *is the solution of* $CP(F, C)$

Then by Remark 3.1.1, $\Omega \subset \Gamma$ *and (i)* $\Longrightarrow$ *(ii)* $\Longrightarrow$ *(iii).*

5.3 An example

Let $L = L(2,2)$ be the extended Lorentz cone defined by (2.9). By using the notations of Theorem 5.2.1, suppose that $C = \{(u_1, u_2) : u_2 \geq u_1, u_1 \geq 0\}$ and $K = \mathbb{R}^2 \times C$. Let $f_1(x,u) = 1/12(x_1 + \|u\| + 12)$ and $f_2(x,u) = 1/12(x_2 + \|u\| - 7.2)$. Then it is easy to show that these two functions are $L(2,2)$-monotone. Let $w^1 = (1, 1, 1/6, 1/3)$ and $w^2 = (1, 1, 1/3, 1/6)$ so w^1 and w^2 are in $L(2,2)$. For any two vectors (x,u) and (y,v) in K, suppose $(x,u) \leq_L (y,v)$, we have $y_1 - x_1 \geq \|v - u\| \geq \|u\| - \|v\|$ by the triangle inequality. Hence,

$$f_1(y,v) - f_1(x,u) = \frac{1}{12}(y_1 - x_1 - (\|u\| - \|v\|)) \geq 0.$$

Similarly, we can prove that if $(x,u) \leq_L (y,v)$, then $f_2(y,v) - f_2(x,u) \geq 0$. Provided that K is convex, and $w^1, w^2 \in L(2,2)$, if $(x,u) \leq_L (y,v)$ holds, then

$$(f_1(y,v) - f_1(x,u))w^1 + (f_2(y,v) - f_2(x,u))w^2 \in L(2,2).$$

Thus, $f_1(x,u)w^1 + f_2(x,u)w^2 \leq_L f_1(y,v)w^1 + f_2(y,v)w^2$. Therefore, the mapping $f_1w^1 + f_2w^2$ is $L(2,2)$-isotone. Hence, choose the function

$$G(x,u) = \left(\frac{11}{12}x_1 - \frac{1}{12}x_2 - \frac{1}{6}\|u\| - \frac{2}{5}, -\frac{1}{12}x_1 + \frac{11}{12}x_2 - \frac{1}{6}\|u\| - \frac{2}{5}\right),$$

$$H(x,u) = \left(u_1 - \frac{1}{72}x_1 - \frac{1}{36}x_2 - \frac{1}{24}\|u\| + \frac{1}{30}, u_2 - \frac{1}{36}x_1 - \frac{1}{72}x_2 - \frac{1}{24}\|u\| - \frac{7}{30}\right),$$

so that to have

$$(x - G, u - H) = f_1w^1 + f_2w^2 = \left(f_1 + f_2, f_1 + f_2, \frac{1}{6}f_1 + \frac{1}{3}f_2, \frac{1}{3}f_1 + \frac{1}{6}f_2\right), \tag{5.2}$$

$L(2,2)$-isotone, where G, H, f_1 and f_2 are considered in the point (x,u). It is necessary to check that all the conditions in Theorem 5.2.1 are satisfied. First, since

$$-G(0,0;0,0) = (f_1(0,0;0,0) + f_2(0,0;0,0), f_1(0,0;0,0) + f_2(0,0;0,0)) = (0.4, 0.4)$$

and $\|H(0,0;0,0)\| = \sqrt{2}/6$, it is clear that $-G(0,0;0,0) \geq \|H(0,0;0,0)\|e$. Next, we will show that Ω is not empty. Consider the vector $(\bar{x},\bar{u}) = (31,31,3,4) \in K$. Obviously, $\bar{x} = (31,31) \geq \sqrt{3^2+4^2}e$, and since

$$G(31,31,3,4) = (31,31) - (f_1 + f_2, f_1 + f_2) = (24.6, 24.6)$$

and

$$H(31,31,3,4) = (3,4) - \left(\frac{1}{6}f_1 + \frac{1}{3}f_2, \frac{1}{3}f_1 + \frac{1}{6}f_2\right) = \left(\frac{23}{15}, \frac{34}{15}\right),$$

where the functions f_1 and f_2 are considered in the point $(\bar{x},\bar{u}) = (31,31,3,4)$, it is straightforward to check that $G(31,31,3,4) \geq \|H(31,31,3,4)\|e$. Thus, $(\bar{x},\bar{u}) \in \Omega$, which shows that $\Omega \neq \varnothing$.

Now, we begin to solve the $MiCP(G, H, C, p, q)$. Suppose that (x, u) is its solution. Since $G(x, u) = 0$, and

$$x - G(x, u) = (f_1 + f_2, f_1 + f_2),$$

where $f_i = f_i(x, u), i = 1, 2$, we have $x_1 = x_2 = f_1 + f_2$. Moreover, since

$$x_1 = \frac{1}{12}(x_1 + x_2) + \frac{1}{6}\|u\| + 0.4,$$

we get

$$x_1 = x_2 = \frac{1}{5}\|u\| + \frac{12}{25}. \tag{5.3}$$

The perpendicularity $u \perp H(x, u)$ implies

$$\langle u, H(x, u)\rangle = u_1(u_1 - \frac{1}{6}f_1 - \frac{1}{3}f_2) + u_2(u_2 - \frac{1}{3}f_1 - \frac{1}{6}f_2) = 0.$$

Thus,

$$u_1^2 + u_2^2 = \|u\|^2 = f_1\left(\frac{1}{6}u_1 + \frac{1}{3}u_2\right) + f_2\left(\frac{1}{3}u_1 + \frac{1}{6}u_2\right). \tag{5.4}$$

We will find all nonzero solutions on the boundary of C.

Case1: $u_1 = u_2, u_1 > 0$. Then, $\|u\| = \sqrt{2}u_1 = \sqrt{2}u_2$.

Hence, from (5.4), we get

$$2u_1 = \frac{1}{2}(f_1 + f_2).$$

By (5.3), we can conclude

$$u_1 = u_2 = \frac{120 + 6\sqrt{2}}{995}, \tag{5.5}$$

which implies that

$$(x,u) = \left(\frac{480+24\sqrt{2}}{995}, \frac{480+24\sqrt{2}}{995}, \frac{120+6\sqrt{2}}{995}, \frac{120+6\sqrt{2}}{995}\right). \tag{5.6}$$

Case 2: $u_1 = 0$, i.e., $\|u\| = u_2$. Equation (5.4) can be transformed into

$$u_2\left(u_2 - \frac{1}{3}f_1 - \frac{1}{6}f_2\right) = 0. \tag{5.7}$$

By using (5.4) again, we get $u_2 = 4/15$, so $u = (0, 4/15)$ and

$$(x,u) = \left(\frac{8}{15}, \frac{8}{15}, 0, \frac{4}{15}\right). \tag{5.8}$$

If the Picard iteration shown in (5.1) is applied and $(0,0,0,0)$ is the starting point, then we obtain

$$\begin{cases} x^{n+1} = x^n - G(x^n,u^n) = (f_1^n + f_2^n)e, \\ \\ u^{n+1} = P_C(u^n - H(x^n,u^n)) = P_C\left(\frac{1}{6}f_1^n + \frac{1}{3}f_2^n, \frac{1}{3}f_1^n + \frac{1}{6}f_2^n\right), \end{cases} \tag{5.9}$$

where $f_i^n = f_i(x^n,u^n), i = 1,2$. So, we have $x_1^{n+1} = x_2^{n+1}$. As we start from $(0,0,0,0)$, $x_1^j = x_2^j \geq 0$ for all $j \in \mathbb{N}$. Furthermore, define the set S by

$$S = \left\{(x,u) \in \mathbb{R}^2 \times \mathbb{R}^2 : 0 \leq x_1 = x_2 < \frac{8}{15},\ u_1 = 0,\ 0 \leq u_2 < \frac{4}{15}\right\}. \tag{5.10}$$

We will prove by induction that $(x^n,u^n) \in S$, for all $n \in \mathbb{N}$. We have $(x^0,u^0) = (0,0,0,0) \in S$, and we need to show that as long as $(x^n,u^n) \in S$, (x^{n+1},u^{n+1}) defined by (5.9) is in S.

Indeed, by using the above analysis, $x_1^n = x_2^n$. By $u_1^n = 0$, $\|u^n\| = u_2^n$. If $0 \le x_1^n = x_2^n < 8/15$ and $0 \le u_2^n < 4/15$, we have

$$0 < x_1^{n+1} = x_2^{n+1} = f_1^n + f_2^n = \frac{1}{6}(x_1^n + u_2^n) + \frac{2}{5} < \frac{1}{6}\left(\frac{4}{15} + \frac{8}{15}\right) + \frac{2}{5} = \frac{8}{15}.$$

On the other hand, it can be deduced that,

$$u^n - H(x^n, u^n) = \left(\frac{1}{24}(x_1^n + u_2^n) - \frac{1}{30}, \frac{1}{24}(x_1^n + u_2^n) + \frac{7}{30}\right).$$

Then, the first entry of $u^n - H(x^n, u^n)$ is smaller than $(1/24)(8/15+4/15) - 1/30 = 0$ and the second entry is positive and smaller than $(1/24)(8/15 + 4/15) + 7/30 = 4/15$. Thus, the projection of it onto C must be on the line $\{(u_1, u_2) : u_1 = 0,\ u_2 \ge 0\}$. Moreover, $u_2^{n+1} = (u^n - H(x^n, u^n))_2 < \frac{4}{15}$. Hence, by the equation (5.9),

$$u^{n+1} = (u_1^{n+1}, u_2^{n+1}) = P_C(u^n - H(x^n, u^n)) = \left(0, \frac{1}{3}f_1^n + \frac{1}{6}f_2^n\right).$$

Therefore, equation (5.9) can be transformed into

$$\begin{cases} x_1^{n+1} = x_2^{n+1} = \dfrac{1}{6}\left(x_1^n + u_2^n + \dfrac{12}{5}\right) \\ \\ u_2^{n+1} = \dfrac{1}{24}\left(x_1^n + u_2^n + \dfrac{28}{5}\right) \end{cases} \tag{5.11}$$

Observing that

$$x_1^{n+1} = 4u_2^{n+1} - \frac{8}{15}, \tag{5.12}$$

and by substituting (5.12) (with $n+1$ replaced by n) into $(5.11)_1$, we get $u_2^{n+1} =$

$(5/24)u_2^n + 19/90$ and $x_1^{n+1} = (5/24)x_1^n + 19/45$. Hence,

$$\begin{cases} x_1^{n+1} - \dfrac{8}{15} = \dfrac{5}{24}\left(x_1^n - \dfrac{8}{15}\right) = \left(\dfrac{5}{24}\right)^n \left(x_1^1 - \dfrac{8}{15}\right), \\ \\ u_2^{n+1} - \dfrac{4}{15} = \dfrac{5}{24}\left(u_2^n - \dfrac{4}{15}\right) = \left(\dfrac{5}{24}\right)^n \left(u_2^1 - \dfrac{4}{15}\right). \end{cases} \tag{5.13}$$

Therefore, when n goes to infinity, the sequence (x^n, u^n) converges to $(8/15, 8/15, 0, 4/15)$ which is a solution shown in Case 2.

5.4 How wide is the family of K-isotone mappings?

The section is entirely for the purpose of convincing the reader that the family of K-isotone mappings which occur in the condition "$I - F$ is K-isotone" of Proposition 5.1.1 and the corresponding condition in Theorem 5.2.1 is very wide.

Let $K, S \subset \mathbb{R}^m$ be pointed closed convex cones such that $K \subset S$. The function $f : \mathbb{R}^m \mapsto \mathbb{R}$ is called *K-monotone* if $x \leq_K y$ implies $f(x) \leq f(y)$. Both the K-monotone functions and the K-isotone mappings form a cone. If $f_1, \dots, f_\ell : \mathbb{R}^m \mapsto \mathbb{R}$ are K-monotone and $w^1, \dots, w^\ell \in K$, then it is easy to see that the mapping $F : \mathbb{R}^m \mapsto \mathbb{R}^m$ defined by

$$F(x) = f_1(x)w^1 + \cdots + f_\ell(x)w^\ell \tag{5.14}$$

is K-isotone. It is obvious that any S-monotone function is also K-monotone. Hence, if $f_1, \dots, f_\ell : \mathbb{R}^m \mapsto \mathbb{R}$ are S-monotone, then the mapping F defined by (5.14) is K-isotone. The pointed closed convex cone S is called simplicial if there exists linearly independent vectors $u^1, \dots, u^m \in \mathbb{R}^m$ such that

$$S = \operatorname{cone}\{u^1, \dots, u^m\} := \{\lambda_1 u^1 + \dots + \lambda_m u^m : \lambda_1, \dots, \lambda_m \geq 0\}. \tag{5.15}$$

The vectors $u^1, \dots, u^m$ are called the *generators* of S and we say that S is *generated* by $u^1, \dots, u^m$. It can be shown that the dual S^* of a simplicial cone S is simplicial. Moreover, if $U := (u^1, \dots, u^m)$ (that is an $m \times m$ matrix with columns $u^1, \dots, u^m$) and $U^{-\top} = (v^1, \dots, v^m)$ where $U^{-\top} = (U^\top)^{-1}$, then $S^* = \operatorname{cone}\{v^1, \dots, v^m\}$ [9]. Let $\{e^1, \dots, e^m\}$ be the set of standard unit vectors in $\mathbb{R}^m$. Obviously, we have $\mathbb{R}^m_+ = \{\lambda_1 e^1 + \dots + \lambda_m e^m : \lambda_1, \dots, \lambda_m \geq 0\}$ where $\mathbb{R}^m_+$ is the nonnegative orthant. Let S be the simplicial cone defined by (5.15). If $f : \mathbb{R}^m \to \mathbb{R}$ is $\mathbb{R}^m_+$-monotone, then $\hat{f} : \mathbb{R}^m \mapsto \mathbb{R}$ defined by $\hat{f}(x_1 u^1 + \dots + x_m u^m) = f(x_1 e^1 + \dots + x_m e^m)$ is S-monotone. Let $U^{-1} = (w^1, \dots, w^m)$, $\hat{f}(x_1 e^1 + \dots + x_m e^m) = f(x_1 w^1 + \dots + x_m w^m)$. $g_1, \dots, g_m : \mathbb{R} \mapsto \mathbb{R}$ are monotone increasing, then obviously $g : \mathbb{R}^m \to \mathbb{R}$ defined by

$$g(x_1 u^1 + \dots + x_m u^m) = g_1(x_1) + \dots + g_m(x_m) \tag{5.16}$$

is S-monotone. So $g(x_1 e^1 + \dots + x_m e^m) = g_1((e^1)^\top((x_1 w^1 + \dots + x_m w^m)) + \dots + g_m((e^m)^\top((x_1 w^1 + \dots + x_m w^m))$. Moreover, if $f : \mathbb{R}^m \mapsto \mathbb{R}$ is S-monotone and $\psi : \mathbb{R} \mapsto \mathbb{R}$ is monotone increasing, then it is straightforward to see that $\psi \circ f$ is also S-monotone. Hence, if all mappings f_i in (5.14) are formed by using a (not necessarily linear) combination of (5.16), the previous property and the conicity of the S-monotone functions, then the mapping F defined by (5.14) is K-isotone for any pointed closed convex cone contained in S. For any such cone K it is easy to construct a simplicial cone S which contains K. From the definition of the dual of a cone it follows that $\mathbb{R}^m = \{0\}^* = (K \cap (-K))^* = K^* + (-K)^* = K^* - K^*$. Thus, the smallest linear subspace of $\mathbb{R}^m$ containing K^* is $\mathbb{R}^m$

and hence the interior of K^* is nonempty (see [58] Theorem 1.1 Page 4). Therefore, there exist m linearly independent vectors in K^*, that is, K^* contains a simplicial cone T. Let S be the dual of T. Then, obviously $K \subset S$.

The above constructions show that for any pointed closed convex cone the family of K-isotone mappings, used in Proposition 5.1.1 and Theorem 5.2.1 is very wide. Moreover, there may be many K-isotone mappings which are not of the above type. This topic is worth to be investigated in the future.

5.5 Notes and comments

A main result of this thesis is provided in this chapter. Although we still considered the L-isotonicity, we solved a mixed complementarity problem with respect to a general closed convex cone C by an order defined by the extended Lorentz cones rather than using simplicial cones particularly restricted by isotonicity properties of the projection onto them. This is the main difference between this chapter and the previous papers solving complementarity problems by using the isotonicity of the projection. As shown in the previous chapters, a variety of optimization problems can be formulated as complementarity problems. In Chapter 7, we will show these complementarity problems in detail. Moreover, we will show an iterative method to solve some conic optimization problems using Theorem 5.2.1.

Chapter 6

Extended Lorentz Cones and Variational Inequalities on Cylinders

6.1 Preliminaries

Solutions of a variational inequality problem defined by a closed and convex set and a mapping are found by imposing conditions for the monotone convergence with respect to a cone of the Picard iteration corresponding to the composition of the projection onto the defining closed and convex set and the difference of the identity mapping and the defining mapping. One of these conditions is the isotonicity of the projection onto the defining closed and convex set. If the closed and convex set is a cylinder and the cone is an extented Lorentz cone, then this condition can be dropped because it is automatically satisfied. In this case a large class of affine mappings and cylinders which satisfy the conditions of monotone convergence above is presented. The obtained results are further specialized for unbounded box constrained variational inequalities. In a particular case of a cylinder with a base being a cone, the variational inequality is reduced to a generalized mixed complementarity problem which has been already considered in "Németh, S.Z., Zhang, G. Extended Lorentz cones and mixed complementarity problems. J. Global Optim., 62(3): 443-457 (2015)" and previous chapters. This chapter is mainly based on my joint paper with S.Z. Németh [53].

6.2 Variational inequalities

Let $K \subset \mathbb{R}^m$ be a closed convex set and $F : \mathbb{R}^m \to \mathbb{R}^m$ be a mapping. It is known that x^* is a solution of the variational inequality $VI(F, K)$ defined by F and K(see Definition 3.1.1) if and only if it is a fixed point of the mapping $I - F_K^{\mathrm{nat}} = P_K \circ (I - F)$, where I is the identity mapping of $\mathbb{R}^m$ and F_K^{nat} is the natural mapping associated to $VI(F, K)$ defined by $F_K^{\mathrm{nat}} = I - P_K \circ (I - F)$ [12]. Recall the Picard iteration defined by

$$x^{n+1} = P_K(x^n - F(x^n)). \tag{6.1}$$

If F is continuous and $\{x^n\}_{n\in\mathbb{N}}$ is convergent to x^*, then it follows that x^* is a fixed point of the mapping $P_K \circ (I - F)$ and hence a solution of $VI(F, K)$. Therefore, it is natural to seek convergence conditions for x^n. Let us first state the following simple lemma:

Lemma 6.2.1. *Let $K \subset \mathbb{R}^m$ be a closed convex set, $F : \mathbb{R}^m \to \mathbb{R}^m$ be a continuous mapping and L be a pointed closed convex cone. Consider the sequence $\{x^n\}_{n\in\mathbb{N}}$ defined by* (6.1)*. Suppose that the mappings P_K and $I - F$ are L-isotone, $x^0 \leq_L x^1$, and there exists a $y \in \mathbb{R}^m$ such that $x^n \leq_L y$, for all $n \in \mathbb{N}$ sufficiently large. Then, $\{x^n\}_{n\in\mathbb{N}}$ is convergent and its limit x^* is a solution of $VI(F, K)$.*

Proof. Since the mappings P_K and $I - F$ are L-isotone, the mapping $x \mapsto P_K \circ (I - F)$ is also L-isotone. Then, by using (6.1) and a simple inductive argument, it follows that $\{x^n\}_{n\in\mathbb{N}}$ is L-increasing. Since any cone in $\mathbb{R}^m$ is regular, $\{x^n\}_{n\in\mathbb{N}}$ is convergent and hence its limit x^* is a fixed point of $P_K \circ (I - F)$ and therefore a solution of $VI(F, K)$. $\square$

Remark 6.2.1. *Consider the assumptions of Lemma 5.1.1. If we further suppose that $I - F$ is nonexpansive, then $P_K \circ (I - F)$ is also nonexpansive. Hence the limit in Lemma 5.1.1 is robust in the sense that if the starting points x^0 and y^0 are close to each other, then the corresponding limits x^* and y^* are also closed to each other.*

Remark 6.2.2. *The condition $x^0 \leq_L x^1$ in Lemma 6.2.1 is satisfied when $x^0 \in K \cap F^{-1}(-L)$. Indeed, if $x^0 \in K \cap F^{-1}(-L)$, then $-F(x^0) \in L$ and $x^0 \in K$. Thus $x^0 \leq_L x^0 - F(x^0)$, and hence by the isotonicity of P_K we obtain $x^0 = P_K(x^0) \leq_L P_K(x^0 - F(x^0)) = x^1$.*

Proposition 6.2.1. *Let $K \subset \mathbb{R}^m$ be a closed convex set, $F : \mathbb{R}^m \to \mathbb{R}^m$ be a continuous mapping and L be a pointed closed convex cone. Consider the sequence $\{x^n\}_{n\in\mathbb{N}}$ defined by* (6.1). *Suppose that the mappings P_K and $I - F$ are L-isotone and $x^0 \leq_L x^1$. Denote by I the identity mapping. Let*

$$\Omega = \{x \in K \cap (x^0 + L) : F(x) \in L\},$$

$$\Gamma = \{x \in K \cap (x^0 + L) : P_K(x - F(x)) \leq_L x\}.$$

Consider the following assertions:

(i) $\Omega \neq \varnothing$,

(ii) $\Gamma \neq \varnothing$,

(iii) The sequence $\{x^n\}_{n\in\mathbb{N}}$ is convergent and its limit x^* is a solution of $VI(F, K)$. Moreover, x^* is the L-least element of Γ.

Then, $\Omega \subset \Gamma$ and (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

Proof. Let us first prove that $\Omega \subset \Gamma$. Indeed, let $y \in \Omega$. Since P_K is L-isotone, $y - F(y) \leq_L y$ implies $P_K(y - F(y)) \leq_L P_K(y) = y$, which shows that $y \in \Gamma$. Hence, $\Omega \subset \Gamma$. Thus, (i) $\Longrightarrow$ (ii) is trivial now.

(ii) $\Longrightarrow$ (iii):

Suppose that $\Gamma \neq \varnothing$. Since the mappings P_K and $I - F$ are L-isotone, the mapping $P_K \circ (I - F)$ is also L-isotone. Similarly to the proof of Lemma 6.2.1, it can be shown that $\{x^n\}_{n\in\mathbb{N}}$ is L-increasing. Let $y \in \Gamma$ be arbitrary but fixed. We have $y - x^0 \in L$, that is $x^0 \leq_L y$. Suppose that $x^n \leq_L y$. We show, by induction, that $x^n \leq_L y$ for all $n \geq 0$. Since the mapping $P_K \circ (I - F)$ is L-isotone, $x^n \leq_L y$ implies that

$$x^{n+1} = P_K(x^n - F(x^n)) \leq_L P_K(y - F(y)) \leq_L y.$$

Thus, $x^n \leq_L y$ for all $n \in \mathbb{N}$. Then, Lemma 6.2.1 implies that $\{x^n\}_{n\in\mathbb{N}}$ is convergent and its limit $x^* \in K \cap (x^0 + L)$ is a solution of $VI(F, K)$. Since x^* is a solution of $VI(F, K)$, we have that $P_K(x^* - F(x^*)) = x^*$ and hence $x^* \in \Gamma$. Moreover, the relation $x^n \leq_L y$ in limit gives $x^* \leq y$. Therefore, x^* is the smallest element of Γ with respect to the partial order defined by L. □

6.3 Variational Inequalities on cylinders

Let p, q be positive integers and $m = p + q$. By a *cylinder* we mean a set $K = \mathbb{R}^p \times C \subset \mathbb{R}^p \times \mathbb{R}^q \equiv \mathbb{R}^m$, where C is a nonempty, closed and convex subset of $\mathbb{R}^q$. In this section we will specialize the results of the previous section for variational inequalities on cylinders.

Lemma 6.3.1. *Let $K = \mathbb{R}^p \times C$, where C is an arbitrary nonempty closed convex set in*

$\mathbb{R}^q$. *Let* $G : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p$, $H : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^q$ *and*

$$F = (G, H) : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p \times \mathbb{R}^q.$$

Then, the variational inequality $VI(F, K)$ *is equivalent to the problem of finding a vector* $(x, u) \in \mathbb{R}^p \times C$ *such that*

$$G(x, u) = 0, \ \ \langle v - u, H(x, u)\rangle \geq 0, \ \ \forall v \in C \tag{6.2}$$

for any $v \in C$.

Proof. The variational inequality $VI(F, K)$ is equivalent to finding an $(x, u) \in \mathbb{R}^p \times C$ such that

$$\langle y - x, G(x, u)\rangle + \langle v - u, H(x, u)\rangle \geq 0, \ \forall (y, v) \in \mathbb{R}^p \times C. \tag{6.3}$$

Let $(x, u) \in \mathbb{R}^p \times C$ be a solution of (6.3). If we choose $v = u \in C$ in (6.3), then we get $\langle y - x, G(x, u)\rangle \geq 0$ for any $y \in \mathbb{R}^p$. Hence, $G(x, u) = 0$ and $\langle (v - u), H(x, u)\rangle \geq 0$. Conversely, if $(x, u) \in \mathbb{R}^p \times C$ is a solution of (6.2), then it is easy to see that it is a solution of (6.3). □

By using the notation of Lemma (6.3.1), the Picard iteration (3.12) can be rewritten as

$$\begin{cases} x^{n+1} = x^n - G(x^n, u^n) \\ u^{n+1} = P_C(u^n - H(x^n, u^n)). \end{cases} \tag{6.4}$$

Consider the partial order defined by the extended Lorentz cone (2.9). Then, we obtain the following proposition.

Theorem 6.3.1. *Let* $K = \mathbb{R}^p \times C$, *where* C *is a nonempty, closed and convex subset* *of $\mathbb{R}^q$. Let $G : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p$, and $H : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^q$ be continuous mappings, and $F = (G, H) : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p \times \mathbb{R}^q$. Let $(x^0, u^0) \in \mathbb{R}^p \times C$ and consider the sequence $(x^n, u^n)_{n\in\mathbb{N}}$ defined by (6.4). Let $x, y \in \mathbb{R}^p$ and $u, v \in \mathbb{R}^q$. Suppose that $x^1 - x^0 \geq \|u^1 - u^0\|e$ (in particular, by Remark 6.2.2, this holds if $u^0 \in C$ and $-G(x^0, u^0) \geq \|H(x^0, u^0)\|e$) and that $y - x \geq \|v - u\|e$ implies*

$$y - x - G(y, v) + G(x, u) \geq \|v - u - H(y, v) + H(x, u)\|e.$$

Let

$$\Omega = \{(x, u) \in \mathbb{R}^p \times C : x - x^0 \geq \|u - u^0\|e \ , \ G(x, u) - x^0 \geq \|H(x, u) - u^0\|e\}$$

and

$$\begin{aligned}\Gamma = \{(x, u) \in \mathbb{R}^p \times C : x - x^0 \geq \|u - u^0\|e,& \\ G(x, u) - x^0 \geq \|u - u^0 - P_C(u - H(x, u))\|e\}&\end{aligned}$$

Consider the following assertions

(I) $\Omega \neq \varnothing$,

(II) $\Gamma \neq \varnothing$,

(III) The sequence $\{(x^n, u^n)\}_{n\in\mathbb{N}}$ is convergent and its limit (x^*, u^*) is a solution of $VI(F, K)$. Moreover, (x^*, u^*) is the smallest element of Γ with respect to the partial order defined by the extended Lorentz cone $L(p, q)$ defined by (2.9).

Then, $\Omega \subset \Gamma$ and (I) $\Longrightarrow$ (II) $\Longrightarrow$ (III).

Proof. Let L be the extended Lorentz cone defined by (2.9). First observe that $K \cap (x^0 +$

$L(p,q)) \neq \varnothing$. By using the definition of the extended Lorentz cone, it is easy to verify that

$$\Omega = K \cap ((x^0,u^0) + L(p,q)) \cap F^{-1}(L) = \{z \in K \cap ((x^0,u^0) + L(p,q)) : F(z) \in L\}$$

and

$$\Gamma = \{z \in K \cap ((x^0,u^0) + L) : P_K(z - F(z)) \leq_{L(p,q)} z\}.$$

Let $x, y \in \mathbb{R}^p$ and $u, v \in C$. Since $y - x \geq \|v-u\|e$ implies

$$y - x - G(y,v) + G(x,u) \geq \|v - u - H(y,v) + H(x,u)\|e,$$

it follows that $I - F$ is $L(p,q)$-isotone. Hence, by Proposition 6.2.1 (with $m = p+q$), Lemma 6.3.1 and Theorem 2.3.2, it follows that $\Omega \subseteq \Gamma$ and (I) $\Longrightarrow$ (II) $\Longrightarrow$ (III). $\square$

6.4 Affine Variational Inequalities on Cylinders

Throughout this section, we will use the notation of Proposition 5.1.1, and we will suppose that $\mathrm{int}(L)$ is nonempty and P_K is L-isotone, which is true for the extended Lorentz cone $L = L(p,q)$. We will present a large class of monotone solvable affine variational inequalities for which (5.1) is monotone and convergent.

Lemma 6.4.1. *If*

1. *the mapping $I - F$ is L-isotone, $x^0 \in K$ and $F(x^0) \in -L$*

and if there exists an $x \in \mathbb{R}^m$ such that

2. *we have the inclusions $x \in K$, $F(x) \in L$ and $x - x^0 \in L$,*

then (6.1) *is convergent (*(6.4) *is convergent if $L = L(p,q)$).*

Proof. By Remarks 6.2.2 and Condition 1 of the lemma we have $x^0 \leq_L x^1$, and Condition 2 of the lemma means that $x \in \Omega$, that is, Ω is nonempty. Hence, the result follows from Proposition 5.1.1. □

For any $m \times m$ matrix M and set $\Lambda \subseteq \mathbb{R}^m$ denote by $\operatorname{int} \Lambda$ the interior of Λ and by $\|M\|$ the operator norm of M, i.e.,

$$\|M\| = \min\{c \geq 0 : \|Mx\| \leq c\|x\| \text{ for all } x \in \mathbb{R}^m\},$$

and let $M\Lambda := \{Mx : x \in \Lambda\}$.

Lemma 6.4.2. *Suppose that F is an affine mapping, that is, $F(z) = Az + b$, $\forall z \in \mathbb{R}^m$, where A is a constant $m \times m$ nonsingular matrix and $b \in \mathbb{R}^m$ is a constant vector. Let $x^0 \in A^{-1}(-b-L)$. If $(I-A)L \subseteq L$ and $AL \cap \operatorname{int}(L) \neq \varnothing$, then there exists an $x \in \mathbb{R}^m$ such that $F(x) \in L$ and $x - x^0 \in L$ and if K is a closed and convex set such that $x, x^0 \in K$, then* (6.1) *(*(6.4) *if $L = L(p,q)$) is convergent.*

Proof. Note that $x^0 \in A^{-1}(-b - L)$ means that $F(x^0) \in -L$ and $(I - A)L \subseteq L$ is equivalent to $I - F$ is L-isotone (as remarked by one of the reviewers of [53], in case of $L = \mathbb{R}^m_+$, this implies that A has the Z-property, that is, it has nonpositive off-diagonal entries). Let

$$Ay \in AL \cap \operatorname{int}(L).$$

Then, $y \in L$ and $Ay \in \operatorname{int}(L)$. Hence, there exists a sufficiently large positive real number λ such that $(1/\lambda)(Ax^0 + b) + Ay \in L$. Then, $Ax + b \in L$, where $x = x^0 + \lambda y$. Hence, $F(x) \in L$ and $x - x^0 \in L$. Choose K to be a closed and convex set such that K contains x^0 and x (for example in case of the box constrained variational inequalities of the next section choose the box large enough to contain x^0 and x). Then, Conditions 1-2 of Lemma

6.4.1 are satisfied and therefore (6.1) ((6.4) if $L = L(p,q)$) is convergent. $\square$

In conclusion, satisfying Conditions 1-2 of Lemma 6.4.1 reduces to finding nonsingular matrices A such that $(I-A)L \subseteq L$ and $AL \cap \mathrm{int}(L) \neq \varnothing$. Let us concentrate on the latter problem for $L = L(p,q)$, the extended Lorentz cone, when the conditions of Proposition 5.1.1 become the conditions of Theorem 5.2.1.

Usually I_r denotes the identity matrix in $\mathbb{R}^r$. However, in our case the notation I will always be unambigous and therefore we omit the index r.

Proposition 6.4.1. *Let $\alpha \in]0,1[$ be a real constant, S be a $p \times p$ positive matrix with all entries in the main diagonal from the interval $]\alpha, 1[$ and the sum of the elements in each of its row less than 1, T be a $q \times q$ matrix such that $\|T\| \leq \alpha$, and A be the block diagonal matrix given by $A := \left(\begin{smallmatrix} I-S & 0 \\ 0 & I-T \end{smallmatrix}\right)$. Then, $(I-A)L(p,q) \subseteq L(p,q)$ and $AL(p,q) \cap \mathrm{int}(L(p,q)) \neq \varnothing$.*

Proof. For any $(x,u) \in \mathbb{R}^p \times \mathbb{R}^q$ we have

$$A(x,u) = ((I-S)x, (I-T)u).$$

Hence, $(x,u) \in L(p,q)$ is equivalent to $x \geq \|u\|e$ and $A(x,u) \in \mathrm{int}(L(p,q))$ is equivalent to $(I-S)x > \|(I-T)u\|e$ where $x > y$ for $x, y \in \mathbb{R}^p$ means $x_i > y_i$ for each $i = 1, \ldots, p$. From the restrictions imposed on S, it follows that both inequalities will be satisfied if the components of x are equal and large enough. Hence, $(x,u) \in \mathrm{int}(L(p,q)) \neq \varnothing$ and therefore $AL(p,q) \cap \mathrm{int}(L(p,q)) \neq \varnothing$. We have $I - A = \left(\begin{smallmatrix} S & 0 \\ 0 & T \end{smallmatrix}\right)$ and thus

$$(I-A)(x,u) = (Sx, Tu),$$

for any $(x,u) \in L(p,q)$ (i.e., $x \geq \|u\|e$ which implies $x \geq 0$). Since $S - \alpha I$ is a positive matrix and $x \geq 0$ we get

$$Sx \geq \alpha x \geq \|T\|x \geq \|T\|\|u\|e \geq \|Tu\|e.$$

Therefore, $(I - A)(x, u) = (Sx, Tu) \in L(p, q)$, which shows that

$$(I - A)L(p, q) \subseteq L(p, q).$$

□

Remark 6.4.1. *Going back to a general cone L with nonempty interior, in the literature the matrices $I - A$ for which $(I - A)L \subseteq L$ are called L-positive and form the cone $P(L)$. It is known that $P(L)$ has a nonempty interior as well (see Lemma 5 of Schneider and Vidyasagar [61]). Hence, the inclusion $(I - A)L(p, q) \subseteq L(p, q)$ also holds for some open set in any neighbourhood of the matrices of the type $\left(\begin{smallmatrix} I-S & 0 \\ 0 & I-T \end{smallmatrix}\right)$ constructed above. By continuity reasons, this open set can be chosen so that to satisfy $AL(p, q) \cap \mathrm{int}(L(p, q)) \neq \varnothing$ as well. We conclude that the set of affine mappings F satisfying the condition of Theorem 6.3.1 is large.*

6.5 Unbounded box constrained variational inequalities

Let p, q be positive integers, $m = p + q$ and $K = \prod_{\ell=1}^{m} [a_\ell, b_\ell]$ be a box, where $a_\ell, b_\ell \in \mathbb{R} \cup \{-\infty, \infty\}$ and $a_\ell < b_\ell$, for all $\ell \in \{1, \ldots, m\}$. The i-th entry of the projection mapping is (see Example 1.5.10 in [12]):

$$(P_K(x))_i = P_{[a_i,b_i]}(x_i) = \mathrm{mid}(a_i, b_i, x_i) = \begin{cases} a_i \text{ if } x_i \leq a_i, \\ x_i \text{ if } a_i \leq x_i \leq b_i, \\ b_i \text{ if } b_i \leq x_i. \end{cases} \tag{6.5}$$

Let $B = \prod_{i=1}^{p}[a_i, b_i] \subseteq \mathbb{R}^p$ and $C = \prod_{j=1}^{q}[a_{p+j}, b_{p+j}] \subseteq \mathbb{R}^q$. So we have

$$P_K(y, v) = (P_B(y), P_C(v)) \tag{6.6}$$

and the Picard iteration (6.1) becomes

$$x_i^{n+1} = \mathrm{mid}(a_i, b_i, (x^n - F(x^n))_i). \tag{6.7}$$

Let $L(p, q)$ be the extended Lorentz cone defined by (2.9). The next proposition shows that the $L(p, q)$-isotonicity of a box is equivalent to the box being a cylinder.

Proposition 6.5.1. *Let $L(p, q)$ be the extended Lorentz cone. Then, the projection mapping P_K is $L(p, q)$-isotone if and only if $K = \mathbb{R}^p \times C$ where $C = \prod_{j=1}^{q}[a_{p+j}, b_{p+j}]$.*

Proof. The sufficiency follows easily from item 1 of Theorem 2.3.2. For the sake of completeness we provide a proof here. Suppose that $B = \mathbb{R}^p$. If $(x, u) \leq_{L(p,q)} (y, v)$, that is, $y - x \geq \|v - u\|e$, then by the nonexpansivity of P_C we get

$$P_B(y) - P_B(x) = y - x \geq \|v - u\|e \geq \|P_C(v) - P_C(u)\|e,$$

which is equivalent to

$$P_K(x,u) = (P_B(x), P_C(u)) \leq_{L(p,q)} (P_B(y), P_C(v)) = P_K(y,v).$$

Hence, P_K is $L(p,q)$-isotone. Although, the necessity could also be derived from item 3 of the same theorem, it is more clear to prove this directly as follows. Suppose that P_K is $L(p,q)$-isotone. We need to prove that $a_i = -\infty, b_i = \infty$, for any $i = 1, \dots p$. Assume to the contrary, that there exist at least one $k \in \{1, \dots, p\}$ such that either a_k or b_k is a finite real number.

Assume that b_k is a finite real number. The case a_k is a finite real number can be treated similarly. Let u and v be two different vectors in C. Then, $P_C(u) = u$ and $P_C(v) = v$. It is easy to choose $x, y \in \mathbb{R}^p$ such that $y_i - x_i \geq \|v - u\|$ and $b_i \leq x_i \leq y_i$ for all $i \in \{1, \dots, p\}$. For example, we may choose $x_i = \delta_{ik} b_k$ and $y_i = \delta_{ik} b_k + \|v - u\|$, for all $i = \{1, \dots, p\}$, where δ_{ik} is the Kronecker symbol, that is, $\delta_{ik} = 0$ when $i \neq k$ and $\delta_{ii} = 1$. Then, $(x,u) \leq_L (y,v)$ and by (6.5) we have $(P_K(y,v))_k = (P_K(x,u))_k$, or equivalently $(P_B(y))_k = (P_B(x))_k$. Hence, by (6.6) and the L-isotonicity of P_K we get

$$0 = (P_B(y) - P_B(x))_k \geq \|P_C(v) - P_C(u)\| = \|v - u\| > 0,$$

which is a contradiction.

$\square$

Hence, let $K = \mathbb{R}^p \times C$ where $C = \prod_{j=1}^q [a_{p+j}, b_{p+j}]$, so the Picard iteration (6.4) can be transformed to

$$\begin{cases} x^{n+1} = x^n - G(x^n, u^n), \\ u_i^{n+1} = \text{mid}(a_i, b_i, u_i^n - H_i(x^n, u^n));\ i = 1, \dots, q, \end{cases}$$

where $H_i(x^n, u^n)$ denotes the ith entry of $H(x^n, u^n)$. Then results of Theorem 6.3.1 will hold. In the next section we will present an example for this particularized result.

6.6 Numerical example

Let $K = \mathbb{R}^2 \times C$ where $C = [0, 10] \times [0, 10]$. Let L be the extended Lorentz cone defined by (2.9). Let $f_1(x, u) = 1/12(x_1 + \|u\| + 12)$ and $f_2(x, u) = 1/12(x_2 + \|u\| - 7.2)$. Then it is easy to show that these two functions are L-monotone. Let $w^1 = (1, 1, 1/6, 1/3)$ and $w^2 = (1, 1, 1/3, 1/6)$ so w^1 and w^2 are in L. For any two vectors (x, u) and (y, v) in K, suppose $(x, u) \leq_L (y, v)$, we have $y_1 - x_1 \geq \|v - u\| \geq \|u\| - \|v\|$ by the triangle inequality. Hence,

$$f_1(y, v) - f_1(x, u) = \frac{1}{12}(y_1 - x_1 - (\|u\| - \|v\|)) \geq 0.$$

Similarly we can prove that if $(x, u) \leq_L (y, v)$, then $f_2(y, v) - f_2(x, u) \geq 0$. It is obvious that K is convex, and $w^1, w^2 \in L$. If $(x, u) \leq_L (y, v)$ holds, then

$$(f_1(y, v) - f_1(x, u))w^1 + (f_2(y, v) - f_2(x, u))w^2 \in L.$$

Thus, $f_1(x, u)w^1 + f_2(x, u)w^2 \leq_L f_1(y, v)w^1 + f_2(y, v)w^2$. Therefore, the mapping $f_1 w^1 + f_2 w^2$ is L-isotone. Hence, choose the function

$$(x - G, u - H) = f_1 w^1 + f_2 w^2 = \left(f_1 + f_2, f_1 + f_2, \frac{1}{6} f_1 + \frac{1}{3} f_2, \frac{1}{3} f_1 + \frac{1}{6} f_2\right), \qquad (6.8)$$

where G, H, f_1 and f_2 are considered at the point (x, u). It is necessary to check that all the conditions in Theorem 6.3.1 are satisfied. First, since

$$-G(0,0;0,0) = (f_1(0,0;0,0) + f_2(0,0;0,0), f_1(0,0;0,0) + f_2(0,0;0,0)) = (0.4, 0.4)$$

and $\|H(0,0;0,0)\| = \sqrt{2}/6$, it is clear that $-G(0,0;0,0) \geq \|H(0,0;0,0)\|e$. Next, we will show that Ω is not empty, where

$$\Omega = \{(x,u) \in \mathbb{R}^p \times C : x - x^0 \geq \|u - u^0\|e \ , \ G(x,u) - x^0 \geq \|H(x,u) - u^0\|e\}$$

Consider the vector $(\bar{x}, \bar{u}) = (31, 31, 3, 4) \in K$. Obviously, $\bar{x} = (31, 31) \geq \sqrt{3^2 + 4^2}e$, and since

$$G(31, 31, 3, 4) = (31, 31) - (f_1 + f_2, f_1 + f_2) = (24.6, 24.6)$$

and

$$H(31, 31, 3, 4) = (3, 4) - \left(\frac{1}{6}f_1 + \frac{1}{3}f_2, \frac{1}{3}f_1 + \frac{1}{6}f_2\right) = \left(\frac{23}{15}, \frac{34}{15}\right),$$

where the functions f_1 and f_2 are considered at the point $(\bar{x}, \bar{u}) = (31, 31, 3, 4)$, it is straightforward to check that $G(31, 31, 3, 4) \geq \|H(31, 31, 3, 4)\|e$. Thus, $(\bar{x}, \bar{u}) \in \Omega$, which shows that $\Omega \neq \varnothing$.

Now, we begin to solve the VI. Suppose that (x, u) is a solution. First consider the case $F(x, u) = 0$. Since $G(x, u) = 0$, and

$$x - G(x, u) = (f_1 + f_2, f_1 + f_2),$$

where $f_i = f_i(x,u), i = 1,2$, we have $x_1 = x_2 = f_1 + f_2$. Moreover, since

$$x_1 = \frac{1}{12}(x_1 + x_2) + \frac{1}{6}\|u\| + 0.4,$$

we get

$$x_1 = x_2 = \frac{1}{5}\|u\| + \frac{12}{25}. \tag{6.9}$$

Since $H(x,u) = 0$, we have:

$$\begin{cases} u_1 &= \frac{1}{6}f_1 + \frac{1}{3}f_2 \\ u_2 &= \frac{1}{3}f_1 + \frac{1}{6}f_2. \end{cases} \tag{6.10}$$

Then we can simplify the equations (6.10) by using (6.9), to obtain:

$$\begin{cases} u_1 &= \frac{1}{20}\|u\| - \frac{1}{75} \\ u_2 &= \frac{1}{20}\|u\| + \frac{19}{75}. \end{cases} \tag{6.11}$$

So $u_2 = u_1 + \frac{4}{15}$. Substiting into (6.10), we have

$$u_1\left(u_1 - \frac{28}{995}\right) = 0. \tag{6.12}$$

But (6.11) can be transformed to

$$19u_1 + u_2 = \|u\|,$$

or equivalently, by squaring both sides

$$361u_1^2 + u_2^2 + 38u_1u_2 = u_1^2 + u_2^2.$$

Hence,

$$2u_1(180u_1 + 19u_2) = 0. \tag{6.13}$$

So, if $u_1 = \frac{28}{995}$ in (6.6), then $u_2 = u_1 + \frac{4}{15}$ and (6.13) will not hold. Therefore, in this case, the only solution is

$$(x, u) = \left(\frac{8}{15}, \frac{8}{15}, 0, \frac{4}{15}\right).$$

Now, consider the case $F(x, u) \neq 0$. By using the variational inequality in (6.3) and equation (6.9), we get

$$(v_1 - u_1)(300u_1 - 15\|u\| + 4) + (v_2 - u_2)(300u_2 - 15\|u\| - 76) \geq 0. \tag{6.14}$$

Note that the solution in this case should be on the boundary of K. If $u_1 = 10$, then $10 \leq \|u\| \leq 20$. So, the first term of (6.14) is always negative. For any fixed u_2 we can always find a v_2 close enough to u_2 such that (6.14) doesn't hold. If $u_2 = 0$, the second term of (6.14) is always negative. Similarly we could also find a v_1 close enough to u_1 such that (6.14) doesn't hold. Then the only possibility left is the case when $u_1 = 0$. Hence, $u_2 = \|u\|$, and (6.14) can be simplified to

$$(19v_2 - v_1 - 19u_2)(15u_2 - 4) \geq 0$$

for any $v = (v_1, v_2) \in C$. Then the only solution is $u = (0, \frac{4}{15})$ which is the same as the former case.

The Picard iteration can be completed by using Microsoft Excel. Note that since the variational inequality is box constrained, the iteration shown by (6.7) will be calculated by using the *median* function as shown in the following table. More precisely, we gave the original value in the first row for $n = 1$ and calculated the value in the following row where the column for u_1^n and u_2^n were obtained by taking the *median* of the upper bound, lower bound and $u^n - H(x^n, u^n)$. In the following four tables, we will iterate from four points in different directions of the set C.

n	x_1^n	x_2^n	u_1^n	u_2^n
1	2	10	11	6
2	$\frac{295}{98}$	$\frac{697}{719}$	$\frac{81}{133}$	$\frac{7}{8}$
3	$\frac{14}{55}$	$\frac{125}{344}$	$\frac{12}{89}$	$\frac{2}{5}$
4	$\frac{50}{77}$	$\frac{463}{713}$	$\frac{19}{655}$	$\frac{21}{71}$
5	$\frac{7}{13}$	$\frac{7}{13}$	$\frac{1}{785}$	$\frac{15}{56}$
6	$\frac{31}{58}$	$\frac{101}{189}$	0	$\frac{4}{15}$
7	$\frac{8}{15}$	$\frac{159}{298}$	0	$\frac{4}{15}$
8	$\frac{8}{15}$	$\frac{8}{15}$	0	$\frac{4}{15}$
9	$\frac{8}{15}$	$\frac{8}{15}$	0	$\frac{4}{15}$
10	$\frac{8}{15}$	$\frac{8}{15}$	0	$\frac{4}{15}$
11	$\frac{8}{15}$	$\frac{8}{15}$	0	$\frac{4}{15}$
12	$\frac{8}{15}$	$\frac{8}{15}$	0	$\frac{4}{15}$

n	x_1^n	x_2^n	u_1^n	u_2^n
1	-6	-10	6	11
2	$\frac{121}{43}$	$\frac{1251}{514}$	$\frac{122}{511}$	$\frac{47}{93}$
3	$\frac{63}{85}$	$\frac{232}{313}$	$\frac{29}{558}$	$\frac{29}{91}$
4	$\frac{56}{97}$	$\frac{474}{821}$	$\frac{9}{818}$	$\frac{5}{18}$
5	$\frac{51}{94}$	$\frac{389}{717}$	$\frac{2}{869}$	$\frac{7}{26}$
6	$\frac{38}{71}$	$\frac{372}{695}$	0	$\frac{4}{15}$
7	$\frac{8}{15}$	$\frac{356}{667}$	0	$\frac{4}{15}$
8	$\frac{8}{15}$	$\frac{423}{793}$	0	$\frac{4}{15}$
9	$\frac{8}{15}$	$\frac{8}{15}$	0	$\frac{4}{15}$
10	$\frac{8}{15}$	$\frac{8}{15}$	0	$\frac{4}{15}$
11	$\frac{8}{15}$	$\frac{8}{15}$	0	$\frac{4}{15}$
12	$\frac{8}{15}$	$\frac{8}{15}$	0	$\frac{4}{15}$

n	x_1^n	x_2^n	u_1^n	u_2^n
1	-5	4	-12	7
2	$\frac{115}{17}$	$\frac{1187}{212}$	$\frac{321}{952}$	$\frac{32}{53}$
3	$\frac{63}{76}$	$\frac{63}{76}$	$\frac{32}{433}$	$\frac{16}{47}$
4	$\frac{31}{52}$	$\frac{127}{213}$	$\frac{12}{763}$	$\frac{24}{85}$
5	$\frac{47}{86}$	$\frac{47}{86}$	$\frac{2}{607}$	$\frac{17}{63}$
6	$\frac{52}{97}$	$\frac{52}{97}$	0	$\frac{23}{86}$
7	$\frac{8}{15}$	$\frac{63}{118}$	0	$\frac{4}{15}$
8	$\frac{8}{15}$	$\frac{295}{553}$	0	$\frac{4}{15}$
9	$\frac{8}{15}$	$\frac{8}{15}$	0	$\frac{4}{15}$
10	$\frac{8}{15}$	$\frac{8}{15}$	0	$\frac{4}{15}$
11	$\frac{8}{15}$	$\frac{8}{15}$	0	$\frac{4}{15}$
12	$\frac{8}{15}$	$\frac{8}{15}$	0	$\frac{4}{15}$

n	x_1^n	x_2^n	u_1^n	u_2^n
1	8	-19	-9	-15
2	$\frac{424}{37}$	$\frac{4109}{168}$	$\frac{125}{866}$	$\frac{113}{44}$
3	$\frac{141}{91}$	$\frac{1296}{657}$	$\frac{36}{157}$	$\frac{1}{2}$
4	$\frac{11}{15}$	$\frac{463}{713}$	$\frac{39}{782}$	$\frac{25}{79}$
5	$\frac{19}{33}$	$\frac{96}{131}$	$\frac{4}{379}$	$\frac{23}{83}$
6	$\frac{45}{83}$	$\frac{80}{139}$	$\frac{1}{453}$	$\frac{25}{93}$
7	$\frac{38}{71}$	$\frac{45}{83}$	0	$\frac{4}{15}$
8	$\frac{8}{15}$	$\frac{213}{398}$	0	$\frac{4}{15}$
9	$\frac{8}{15}$	$\frac{372}{697}$	0	$\frac{4}{15}$
10	$\frac{8}{15}$	$\frac{447}{838}$	0	$\frac{4}{15}$
11	$\frac{8}{15}$	$\frac{8}{15}$	0	$\frac{4}{15}$
12	$\frac{8}{15}$	$\frac{8}{15}$	0	$\frac{4}{15}$

6.7 Notes and comments

In this chapter, we extended the results of Chapter 5. Note that $VI(F,K)$ is defined on a set K rather than a cone K. We also considered the affine variational inequalities on cylinders. In the corresponding section, we used properties involving such positive operators of extended Lorentz cones. In Chapter 8, we will show more detailed results about positive operators. In [14], Gabay and Moulin proved the following lemma:

Lemma 6.7.1. *Let $\mathcal{G} = [P, \{S_i, u_i\}_{i\in P}]$ be a strategic game such that S_i is a closed and convex subset of a Hilbert space X_i and u_i is Fréchet differentiable and pseudo-concave with respect to its own actions, for each i. If $S = \prod_{i\in P} S_i$, $X = \prod_{i\in P} X_i$ and $F : S \to X$*

is defined by

$$F(x) = (-\nabla_1 u_1(x), \ldots, -\nabla_{|P|} u_I(x)),$$

then x^* *is a Nash equilibrium of* $\mathcal{G}$ *if and only if it is a solution of* $VI(F, S)$.

This results built a bridge between variational inequalities and Nash equilibrium. In Chapter 7, we will adapt the results of this chapter to Nash equilibrium and formulate a theorem corresponding to Theorem 6.3.1.

Chapter 7

Applications of Extended Lorentz cones

In this chapter, we will introduce some application of previous chapters.

7.1 Applications in Game theory

A well-known application of the saddle point is in Game theory. In this chapter, we will just discuss noncooperative games. Before introducing that more deeply, let's first introduce the seminal concept, *Nash equilibrium* (see [41, 42]) which is the corner stone of modern economics. The Nash equilibrium point is a choice of strategies of two players such that no player can be better off by a unilateral change of his strategy. More explicitly, suppose that P is the set of players. Let the strategy set of player i be $K_i \subseteq \mathbb{R}^{n_i}$. Note that K_i can be a countably or uncountably infinite set and K_i is independent of the other player's. The player i's cost function is $c_i(x)$ where $x = (x_i, i = 1, 2, \dots, |P|)$ describes the players' strategies and $x_i \in \mathbb{R}^{n_i}$. Then the player i's task is to determine his strategy under the condition that the other players' strategies are fixed but arbitrary. In the following, we will see that variational inequality and complementarity problems have a very wide range of application in game theory. Let us first introduce some basic definitions. Here for economists and mathematical economists, the payoff was measured mostly by a class of utility functions [35]:

Definition 7.1.1. *Let O be a set of outcomes and $\succsim$ be a preference relation over O which satisfies the following axioms*

(A1) Completeness, that is, for any $x, y \in O$, either $x \succsim y$ or $y \succsim x$, or both

(A2) Reflexivity, that is, for any $x \in O$, $x \succsim x$

(A3) Transitivity, that is, if $x \succsim y$ and $y \succsim z$ then $x \succsim z$

A function $u : O \to \mathbb{R}$ is called a utility function representing $\succsim$ if for all $x, y \in O$

$$x \succsim y \Leftrightarrow u(x) \geq u(y)$$

Then we can define the strategy set [34]:

Definition 7.1.2. *Let $\mathscr{H}_i$ denote the collection of the information set of player i, $\mathscr{A}$ the possible actions in the game, and $C(H) \subseteq \mathscr{A}$ the set of actions possible at the information set H. A strategy for player i is a function $s_i : \mathscr{H}_i \to \mathscr{A}$ such that $s_i \in C(H)$ for all $H \in \mathscr{H}_i$*

Then we can define a game in the normal form [34] and the Nash equilibrium [35]:

Definition 7.1.3. *For a game with players set P, the normal form representation $\mathcal{G}$ specifies for each player i a set of strategies S_i and a payoff function $u_i(s_1, \ldots s_{|P|})$ giving the (expected) utility function arising from strategies $(s_1, \ldots, s_{|P|})$. Formally, we write $\mathcal{G} = [P, \{S_i, u_i\}_{i \in P}]$.*

Definition 7.1.4. *A strategy vector $s^* = (s_1^*, \ldots, s_{|P|}^*)$ is a Nash equilibrium if for each player i and each strategy $s_i \in S_i$ the following inequality holds:*

$$u_i(s^*) \geq u_i(s_i, s_{-i}^*),$$

where $(s_i, s_{-i}^*) = (s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_{|P|}^*)$.

For example, let us consider the following profit matrix simplified from Prisoner's Dilemma [35, p. 88]:

	C	D
C	2, 2	0, 3
D	0, 3	1, 1

In this game, the first column and first row denote the strategy sets of player 1 and player 2, respectively. So the strategy set $S_1 = \{C, D\}$ (cooperation or defection). The numbers in each entry denote the utility for player 1 and 2, respectively. For example, $u_1(C, D) = 0$ and $u_2(C, D) = 3$. It is easy to verify that the Nash equilibrium point is $s^* = (D, D)$. Sometimes, the utility function is continuous and differentiable. The Nash equilibrium can be transformed to a solution of a variational inequality:

Proposition 7.1.1. *([12] Proposition 2.2.9Page 156) Let each strategy set of player i, $S_i \subseteq \mathbb{R}^{n_i}$, be compact and convex and each u_i be continuously differentiable. Suppose that for each fixed s_{-i}^*, the function $-u_i(s_i, s_{-i}^*)$ is convex in s_i. Then $s^* = (s_1^*, \ldots, s_{|P|}^*)$ is a Nash Equilibrium if and only if $x \in SOL(F, S)$ where*

$$S = \prod_{i \in P} S_i \ , \ F(s) = -(\nabla_i u_i(s))_{i \in P}.$$

Proof. By the definition of Nash equilibrium, the function $-u_i(x_i, x^*_{-i})$ will get its minimum point on x^*_i for each $i = 1, \ldots, |P|$. Then by the Propsition 4.1.1, we have

$$(x_i - x_i^*)^\top \nabla_{x_i} c_i(x^*) \leq 0, \; \forall x_i \in S_i. \tag{7.1}$$

If we concatenate the individual variational inequalities, it is very clear that x^* solves $VI(F, S)$ as long as it is the Nash equilibrium point.

Conversely, if x^* is the solution of $VI(F, S)$, we have

$$(y - x^*)^T F(x^*) \geq 0, \; \forall y \in S. \tag{7.2}$$

Let y_j, the j-th entry of y, be x_j^* when $j \neq i$ and keep the ith entry $y_i \in S$ arbitrary. Then (7.2) implies (7.1). □

By Corollary 3.2.1, we have the following proposition ([12]).

Proposition 7.1.2. *Let each strategy set of player i, $S_i \subseteq \mathbb{R}^{n_i}$, be compact and convex and each u_i be continuously differentiable. Suppose that for each fixed s^*_{-i}, the function $-u_i(s_i, s^*_{-i})$ is convex in s_i. Then the set of Nash equilibrium is nonempty and compact.*

Proof. By Proposition 7.1.1, under this conditions, the Nash equilbrium problem is equivalent to the $VI(F, S)$ where

$$S = \prod_{i \in P} S_i \text{ and } F(x^*) = (-\nabla_{x_i} u_i(x^*))_{i \in P}.$$

Because each K_i is compact and convex, it is easy to check that their Cartesian product K is compact and convex. Meanwhile, F is continuous since each u_i is continuously differentiable. By Corollary 3.2.1, we obtain the required conclusion. □

Recall the Picard iteration (5.1):

$$\begin{cases} x^{n+1} = x^n - G(x^n, u^n), \\ u^{n+1} = P_C(u^n - H(x^n, u^n)). \end{cases}$$

By applying Theorem 6.3.1, the reformulation of the Nash equilibrium as a variational inequality, we get:

Theorem 7.1.1. *Let $\mathcal{G} = [P, \{S_i, u_i\}_{i\in P}]$ be a strategic game such that each strategy set of player i, $S_i \subseteq \mathbb{R}$, is compact and convex, $P = \{1, \dots, |P|\}$ is the set of players, $|P| = p + q$, and u_i is differentiable. Denote $F(s) = -(\nabla_i u_i(s))_{i\in P} = (G, H)$ where $G : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p$, $H : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^q$ are continuous mappings. Let $K = \mathbb{R}^p \times C$, where C is a nonempty, closed and convex subset of $\mathbb{R}^q$ and $\prod_{i=p+1}^{p+q} S_i \subseteq C$. Let $(x^0, u^0) \in \mathbb{R}^p \times C$ and consider the sequence $(x^n, u^n)_{n\in\mathbb{N}}$ defined by (6.4). Let $x, y \in \mathbb{R}^p$ and $u, v \in \mathbb{R}^q$. Suppose that $x^1 - x^0 \geq \|u^1 - u^0\|e$ (in particular, by Remark 6.2.2, this holds if $u^0 \in C$ and $-G(x^0, u^0) \geq \|H(x^0, u^0)\|e$) and that $y - x \geq \|v - u\|e$ implies*

$$y - x - G(y, v) + G(x, u) \geq \|v - u - H(y, v) + H(x, u)\|e.$$

Let

$$\Omega = \{(x, u) \in \mathbb{R}^p \times C : x - x^0 \geq \|u - u^0\|e \ , \ G(x, u) - x^0 \geq \|H(x, u) - u^0\|e\}.$$

If Ω is nonempty, then the sequence $\{(x^n, u^n)\}_{n\in\mathbb{N}}$ is convergent and its limit $(x^*, u^*) \in SOL(F, K)$. Moreover, if its limit (x^*, u^*) is an element of $S = \prod_{i=1}^{p+q} S_i$, then (x^*, u^*) is a Nash equilibrium.

Proof. By Theorem 6.3.1, we know that (x^*, u^*) is the solution of $VI(F, K)$. Then, by Proposition 7.1.1 it is a Nash equilibrium point since $S \subseteq K$. $\square$

Remark 7.1.1. *Note that in some circumstances, the Nash equilibrium point is not unique. But in this theorem, we cannot find more than one Nash equilibrium point, since the limit of a sequence is unique.*

Example 7.1.1. *Let us consider a classical example in Game theory, the Cournot model [75] . Suppose there are only two firms, firm 1 and firm 2 in the markets. These two firms produce same product. Both of them want to maximize their profit by setting their quantities simutaneously. The price* $P(Q_1, Q_2)$ *of the market is given by*

$$P(Q_1, Q_2) = \frac{a - Q_1 - Q_2}{4},$$

where Q_1 *and* Q_2 *are quantities of firm 1 and 2, respectively. Obviously,* $[0, a]$*, which is compact and convex, is the strategy set for each of them, we suppose their cost is zero. Then their utility (profit) functions are*

$$\pi_i = \frac{Q_i(a - Q_1 - Q_2)}{4}\ ,\ i = 1, 2.$$

Hence $p = q = 1$*, then it is easy to show that* $F = (-\nabla_1 \pi_1, -\nabla_2 \pi_2)$ *is* $L(1,1)$*-isotone. Let* $(x_0, u_0) = (a/20, a/10)$ *and* $C = \mathbb{R}$*, we have* $(x_1, u_1) = (a/4, a/80)$*,* $x^1 - x^0 \geq |u^1 - u^0|$*. All conditions of Theorem 7.1.1 are satisfied. Then by the* (6.4)*, we have*

$$\begin{cases} x^{n+1} = \frac{2x^n - u^n + a}{4}, \\ u^{n+1} = \frac{2u^n - x^n + a}{4}. \end{cases}$$

Hence, we have:

$$x^n = u^n + \left(\frac{3}{4}\right)^n (x^0 - u^0).$$

Then we get:

$$u^{n+1} + \left(\frac{3}{4}\right)^{n+1} \left(\frac{1}{2}\right)(x_0 - u_0) - \frac{a}{3} = \frac{1}{4}\Big[u^n + \left(\frac{3}{4}\right)^n \left(\frac{1}{2}\right)(x_0 - u_0) - \frac{a}{3}\Big].$$

It implies that

$$u^{n+1} = -\left(\frac{3}{4}\right)^{n+1} \left(\frac{1}{2}\right)(x_0 - u_0) + \frac{a}{3} + \left(\frac{1}{4}\right)^{n+1} \Big[u^0 + (\frac{1}{2})(x_0 - u_0) - \frac{a}{3}\Big].$$

Therefore, when $n \to \infty$, we get $u^* = a/3$, similarly, $x^* = a/3$. Then $(a/3, a/3)$ is a Nash equilibrium point which is also shown in [75].

7.2 Applications to conic optimization problems

In Theorem 4.3.1, we showed that a class of constrained optimization problems can be reformulated as mixed complementarity problems. In Theorem 5.2.1, we showed that a mixed complementarity problem defined on the Cartesian product of a Euclidean space and a closed convex cone can be solved by the Picard's iteration. Based on these two theorems, we obtain the following:

Theorem 7.2.1. *Let $f : \mathbb{R}^q \mapsto \mathbb{R}$ be a continuously differentiable convex function at $v \in \mathbb{R}^q \setminus \{0\}$, $K \subseteq \mathbb{R}^q$ be a closed convex cone with smooth boundary, A be a $p \times q$ matrix of full rank and $b \in \mathbb{R}^p$. Suppose that the intersection of the interior of K and the linear subspace $\{v \in \mathbb{R}^q : Av = b\}$ is nonempty. Let $L = L(p,q)$ be the extended Lorentz cone defined by (2.9). Let $x^0 = 0 \in \mathbb{R}^p$, $u^0 = 0 \in \mathbb{R}^q$ and consider the sequence $\{(x^n, u^n)\}_{n\in\mathbb{N}}$ defined by (5.1) with $G(x,u) = Au - b$ and $H(x,u) = \nabla f(u) - A^\top x$. Let $x, y \in \mathbb{R}^p$ and*

$u, v \in \mathbb{R}^q$. Suppose that $y - x \geq \|v - u\|e$ implies

$$y - x + A(u - v) \geq \|v - u + \nabla f(u) - \nabla f(v) + A^\top (y - x)\|e,$$

and $x^1 \geq \|u^1\|e$ (in particular this holds when $b \geq \|\nabla f(0)\|e$).

Let

$$\Omega = \{(x, u) \in \mathbb{R}^p \times C : x \geq \|u\|e,\ Au - b \geq \|\nabla f(u) - A^\top x\|e\}$$

and

$$\Gamma = \{(x, u) \in \mathbb{R}^p \times C : x \geq \|u\|e,\ Au - b \geq \|u - P_C(u - \nabla f(u) + A^\top x)\|e\}.$$

Consider the following assertions:

(i) $\Omega \neq \varnothing$,

(ii) $\Gamma \neq \varnothing$,

(iii) The sequence $\{(x^n, u^n)\}_{n\in\mathbb{N}}$ is convergent and its limit (x^*, u^*) is a solution of $CO(f, A, b, K)$. Moreover, (x^*, u^*) is a lower $L(p, q)$-bound of Ω and the $L(p, q)$-least element of Γ.

Then, $\Omega \subset \Gamma$ and (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

Proof. By Theorem 4.3.1, u^* is a solution of $CO(f, A, b, K)$ if and only if there exists x^* such that (x^*, v^*) is a solution of $MiCP(G, H, K, p, q)$. Hence, Theorem 5.2.1 implies our result. $\square$

Example 7.2.1. *Consider the following nonlinear optimization problems where $p = 1$*

and $q = 2$

$$\begin{aligned} &\min \frac{1}{2}\|u\|^2 \\ &s.\ t.\ u_1 + u_2 = 2, \\ &u \in \mathbb{R}^2_+. \end{aligned} \tag{7.3}$$

It is easy to conclude that the optimal solution is $u = (1,1)^\top$. However, here $y - x \geq \|v - u\|$ doesn't imply $y - x + A(v-u) \geq \|v - u + \nabla f(u) - \nabla f(v) + A^\top(y-x)\|e$, where $A = (1,1)$ and $b = 2$. Let us transform the above problem to an equivalent form:

$$\begin{aligned} &\min \frac{1}{2}\|u\|^2 \\ &\text{s. t. } \frac{1}{5}(u_1 + u_2) = \frac{2}{5}, \\ &u \in \mathbb{R}^2_+. \end{aligned}$$

Here, $A = (1/5, 1/5)$, $b = 2/5$, $H(x,u) = \nabla f(u) - A^\top x = (u_1 - 1/5x, u_2 - 1/5x)^\top$. Therefore, if $y - x \geq \|v - u\|$, then

$$\begin{aligned} y - x + A(v-u) &= y - x + \frac{[(v_1 - u_1) + (v_2 - u_2)]}{5} \\ &\geq y - x - \frac{2}{5}(y-x) \\ &= \frac{3}{5}(y-x) \end{aligned}$$

and

$$\|v - u + \nabla f(u) - \nabla f(v) + A^\top(y-x)\| \leq \frac{2}{5}(y-x).$$

Hence, $y - x + A(v-u) \geq \|v - u + \nabla f(u) - \nabla f(v) + A^\top(y-x)\|e$ and

$$\Omega = \left\{(x,u) \in \mathbb{R}^p \times C : x \geq \|u\|e,\ \frac{1}{5}u_1 + \frac{1}{5}u_2 - \frac{2}{5} \geq \|u_1 - \frac{1}{5}x, u_2 - \frac{1}{5}x\|\right\}$$

Then $(x,u)=(5,1,1)\in\Omega$. It is obvious that $b\geq\|\nabla f(0)\|$ where $b=\frac{2}{5}$. Thus we have

$$\begin{cases} x^{n+1}=x^n+\frac{2}{5}-\frac{1}{5}u_1^n-\frac{1}{5}u_2^n, \\ u^{n+1}=P_{\mathbb{R}^2_+}((\frac{1}{5}x^n,\frac{1}{5}x^n)^\top). \end{cases}$$

If we start from $(x;u)^\top=(0;0,0)$, it is easy to see that x^n is always positive, then the iteration becomes

$$\begin{cases} x^{n+1}=x^n+\frac{2}{5}-\frac{1}{5}u_1^n-\frac{1}{5}u_2^n, \\ u^{n+1}=(\frac{1}{5}x^n,\frac{1}{5}x^n)^\top. \end{cases}$$

Then we have:

$$x^{n+1}=x^n+\frac{2}{5}-\frac{2}{25}x^{n-1}.$$

Let $a^n=x^n-5$, Hence,

$$a^{n+1}=a^n-\frac{2}{25}a^{n-1}.$$

The corresponding characteristic equation (see Appendix) will be

$$\lambda^2=\lambda-\frac{2}{25}.$$

The roots of the above equation are

$$\lambda_1=\frac{5+\sqrt{17}}{10}\text{ and }\lambda_2=\frac{5-\sqrt{17}}{10}.$$

Therefore, there exists two constants c_1,c_2, such that $a^n=c_1{\lambda_1}^n+c_2{\lambda_2}^n$. When $n\to\infty$, we have $a^n\to 0$. Then $x^*=5$ and $u^*=(1,1)^\top$ which is the optimal solution.

Remark 7.2.1. *Note that the iteration will not converge in some special cases. For example, if we revise $G(x,u)=b-Ax$, then the corresponding conditions will be changed:*

the inequality $y - x \geq \|v - u\|e$ *must imply*

$$y - x + A(v - u) \geq \|v - u + \nabla f(u) - \nabla f(v) + A^\top (x - y)\|e,$$

and $x^1 \geq \|u^1\|e$ *(in particular this holds when* $-b \geq \|\nabla f(0)\|e$*). So we need to transform the numerical example into*

$$\begin{aligned}
&\min \frac{1}{2}\|u\|^2 \\
&s.\ t.\ \ -\frac{1}{5}(u_1 + u_2) = -\frac{2}{5} \\
&u \in \mathbb{R}^2_+
\end{aligned}$$

to satisfy the above implications. Hence the iteration will be

$$\begin{cases} x^{n+1} = x^n + \frac{2}{5} - \frac{1}{5}u_1^n - \frac{1}{5}u_2^n, \\ u^{n+1} = P_{\mathbb{R}^2_+}(-(\frac{1}{5}x^n, \frac{1}{5}x^n)^\top). \end{cases}$$

Although $(x, u) = (-5, 1, 1)$ *is the fixed point of the above iteration, if we start from* $(x_0, u_0) = (0, 0, 0)$*, it is easy to check that when n increases,* u^n *will stay at* $(0, 0)$ *and* x^n *will increase by step length* 0.4*. Hence, the iteration will not converge. Indeed, we have*

$$\Omega = \left\{ (x, u) \in \mathbb{R}^p \times C : x \geq \|u\|e,\ \frac{1}{5}u_1 + \frac{1}{5}u_2 - \frac{2}{5} \geq \|u_1 + \frac{1}{5}x, u_2 + \frac{1}{5}x\| \right\},$$

which is easy to check that it is empty. So, we cannot apply Theorem 7.2.1. This example shows that the condition $\Omega \neq \varnothing$ *is necessary for the convergence of the iteration in Theorem 7.2.1.*

7.3 Notes and comments

Lemma 6.7.1 and Theorem 7.1.1 are based on the equivalence of Nash equilibrium and the corresponding variational inequality problem. Originally, in [12], the function is given by cost functions and the consumer i is aiming to minimize the cost function c_i. In the above lemma and Theorem 7.1.1, we used utility functions instead. Then, the consumer i aims to maximize the utility function u_i which is equivalent to minimize $-u_i$.

Unlike the KKT conditions, Section 7.2 showed an iterative scheme for solving constrained optimization problems rather than solving a system of nonlinear equations. We showed a simple example as well. In Theorem 7.2.1, the function f is defined on $\mathbb{R}^q \to \mathbb{R}$. In the future, we may consider the functions defined on some other cones such as Lorentz cone or $\mathbb{S}^n_+$.

Chapter 8

Positive operators of the Extended Lorentz cones

In the Sections 5.4 and 6.4 we encountered the isotonicity of mappings with respect to Extended Lorentz cones related to complementarity problems and variational inequalities, respectively. The mappings in Section 6.4 are linear. These motivates the study of positive operators (i. e, linear isotone mappings) of extended Lorentz cones.

8.1 Introduction

Recall the mutually dual (p, q)-type extended Lorentz cones:

$$L(p, q) = \{(x, u) \in \mathbb{R}^p \times \mathbb{R}^q : x \geq \|u\|e\}$$

and

$$M(p, q) = \{(x, u) \in \mathbb{R}^p \times \mathbb{R}^q : \langle x, e\rangle \geq \|u\|, x \geq 0\},$$

where $e = (1, \dots, 1) \in \mathbb{R}^p$. The extended Lorentz cones $L(p, q)$ and $M(p, q)$ become Lorentz cones exactly in the special case $p = 1$. This is the only case when $L(p, q)$ is self-dual.

Let $m = p + q$. The set $\Gamma(C)$ of *positive operators* ([30]) of a cone $C \subseteq \mathbb{R}^p \times \mathbb{R}^q \equiv \mathbb{R}^{p+q}$ is defined by

$$\Gamma(C) = \{A \in \mathbb{R}^{(p+q)\times(p+q)} : AC \subseteq C\}.$$

The set of positive operators is a cone in $\mathbb{R}^{m\times m}$ [30]. It can be easily checked that A is a positive operator of C if and only if $A^\top$ is a positive operator of C^*. The authors of [30] introduced the characteristic matrix of the Lorentz cones as:

$$J_m = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{bmatrix}$$

and showed that the Lorentz cone can be represented as

$$L(1, q) = \{x \in \mathbb{R}^m : x^\top J_m x \geq 0 \text{ and } x_m \geq 0\}.$$

Moreover, they proved the following theorem which characterizes a positive operator by a positive semidefiniteness condtion [30]:

Theorem 8.1.1. *Let $A \in \mathbb{R}^{m\times m}$. If $A \in \Gamma(L(1, q) \cup \Gamma(-L(1, q))$, then there exists a $\mu \geq 0$*

$$A^\top J_{1+q} A \succeq \mu J_{1+q} \tag{8.1}$$

Conversely, if $rank(A) \neq 1$ *and there is a* $\mu \geq 0$ *such that* (8.1) *holds, then*

$$A \in \Gamma(L(1,q)) \cup \Gamma(-L(1,q)).$$

Since $L(p,q)$ and $M(p,q)$ are extensions of the second-order cone, the problem of finding the positive operators of $L(p,q)$ and $M(p,q)$ arises naturally. The aim of this chapter is to find both necessary conditions and sufficient conditions for a linear operator to be a positive operator of $L(p,q)$ or $M(p,q)$ and state the similarities and differences between the case $p = 1$ and $p > 1$. In [70] Sznajder determined all automorphism operators of $L(p,q)$. In particular, these operators are also positive operators of $L(p,q)$. This shows that the problem of finding all positive operators of $L(p,q)$ (or $M(p,q)$) is more difficult than the one solved by Sznajder. Although this problem is still open, the present note presents some partial results, by finding necessary conditions and sufficient conditions for a linear operator to be a positive operator of $L(p,q)$ (or $M(p,q)$).

The structure of the chapter is as follows. First we introduce some notations. Then, we will prove a lemma about the characterization of an extended Lorentz cone. Finally, based on this lemma, we present necessariy conditions and sufficient conditions for a linear operator to be a positive operator of an extended Lorentz cone.

8.2 Notations

The complementarity set of K is defined by

$$C(K) = \{(x,s) : x \in K, s \in K^*, \langle x,s \rangle = 0\} \text{ [70].}$$

A matrix $A \in \mathbb{R}^{m\times m}$ is said to be *Lyapunov-like* on K if $\langle Ax,s \rangle = 0$ for all $(x,s) \in C(K)$ (see [59]) . Such matrix (transformations) are also characterized by the condition (see

[13], [61])

$$e^{tA} \in \mathrm{Aut}(K) \text{ for all } t \in \mathbb{R},$$

where $\mathrm{Aut}(K)$ denotes the automorphism group of the cone K (an automorphism group of a cone K is a set of invertible operators A such that $A(K) = K$). Note that any automorphism is a positive operator.

8.3 Main results

First we need to present a lemma.

Lemma 8.3.1. *Let $M(p,q)$ be the extended Lorentz cone defined by* (2.10)*. Then,*

$$M(p,q) = \{z = (x,u) \in \mathbb{R}^p \times \mathbb{R}^q : z^\top Jz \geq 0,\ x \in \mathbb{R}^p_+\}, \tag{8.2}$$

where

$$J = \begin{bmatrix} ee^\top & 0 \\ 0 & -I \end{bmatrix}$$

and $e \in \mathbb{R}^p$ is the vector with all entries 1.

Proof. Suppose $\Omega = \{z = (x,u) \in \mathbb{R}^p \times \mathbb{R}^q : z^\top Jz \geq 0, x \in \mathbb{R}^p_+\}$. We have $z = (x,u) \in \Omega$ if and only if $z^\top Jz \geq 0$ and $x \geq 0$, or equivalently.

$$0 \leq z^\top Jz = x^\top ee^\top x - u^\top u = \langle x, e\rangle^2 - \|u\|^2$$

and $x \geq 0$. Thus, $z = (x,u) \in \Omega$ if and only if $\langle x, e\rangle \geq \|u\|$ and $x \geq 0$. Hence, $\Omega = M(p,q)$. □

The next theorem states necessary conditions for a linear operator to be a positive operator of the extended Lorentz cone $M(p,q) = (L(p,q))^*$, where $L(p,q)$ is defined by (2.9) and $M(p,q)$ is defined by (2.10).

Theorem 8.3.1 (Necessary conditions for positive operators of $\boldsymbol{M(p,q)}$). *Let $p > 1$ and $q > 0$ be integers. Let $A \in \mathbb{R}^{(p+q)\times(p+q)}$. If A is a positive operator of $M(p,q)$, then the following hold:*

(i) The transpose of the first p rows of A are in $L(p,q)$.

(ii) The first p columns of A are in $M(p,q)$.

(iii) by adding any i-th column $i = 1,\dots,p$ to the linear combination of the columns $p+1,\dots,p+q$ with coefficients $u_1,\dots,u_q$ such that the Euclidean norm of $u = (u_1,\dots,u_q)^\top$ is one, we obtain an element in $M(p,q)$.

(iv) The sum of any i-th column $i = 1,\dots,p$ with any $(p+j)$-th column $j = 1,\dots,q$ is in M(p, q).

(v) If A is $M(p,q)$-Lyapunov like, then $e^{tA} \in \mathrm{Aut}(M(p,q))$ and hence it is in particular a positive operator of $M(p,q)$, for any $t \in \mathbb{R}$.

Proof. (i) Since A is a positive operator of $M(p,q)$, the first p entries of Az are nonnegative for any $z \in M(p,q)$. Hence, the inner product of z and any row vector of the first p rows of A is nonnegative. Therefore these row vectors must be in the dual cone of $M(p,q)$, that is, in $L(p,q)$.

(ii) Since A^T is a positive operator of $L(p,q) = (M(p,q))^*$, (ii) follows similarly to (i).

(iii) Let

$$\beta_i = \alpha_i + \sum_{j=1}^{q} u_j \alpha_{p+j},$$

where α_t is the t-th column of A. Then, for any $z \in L(p,q)$

$$\langle z, \beta_i \rangle = \langle z, \alpha_i \rangle + \sum_{j=1}^{q} u_j \langle z, \alpha_{p+j} \rangle.$$

By the Cauchy-Schwarz inequality, we have:

$$\sqrt{\sum_{j=1}^{q} \langle z, \alpha_{p+j} \rangle^2} = \sqrt{\sum_{j=1}^{q} \langle z, \alpha_{p+j} \rangle^2} \sqrt{\sum_{j=1}^{q} u_i^2} \geq -\sum_{j=1}^{q} u_j \langle z, \alpha_{p+j} \rangle.$$

So

$$\langle z, \beta_i \rangle \geq \langle z, \alpha_i \rangle - \sqrt{\sum_{j=1}^{q} \langle z, \alpha_{p+j} \rangle^2}.$$

As the matrix A is a positive operator of $M(p,q)$, $A^\top$ is a positive operator of $L(p,q)$ and therefore $A^\top z \in L(p,q)$. Since $\langle z, \alpha_k \rangle$ is the k-th entry of $A^\top z$, by the definition of $L(p,q)$, we have that the right hand side of the above inequality is nonnegative. Thus, $\beta_i \in M(p,q)$.

(iv) Obviously, it is a special case of the above assertion.

(v) See [13], [61].

□

Theorem 8.3.2 (A sufficient condition for positive operators)**.** *If there exists a $\lambda \geq 0$ such that $A^\top JA - \lambda J$ is positive semidefinite and the transpose of the first p rows of A are in $L(p,q)$, then A is a positive operator of $M(p,q)$.*

Proof. Since the transpose of the first p rows of A are in $L(p,q)$, the first p entries of Az are nonnegative for any $z \in M(p,q)$. Since $A^\top JA - \lambda J$ is positive semidefinite, we have

$$0 \leq z^\top (A^\top JA - \lambda J) z = (Az)^\top J (Az) - \lambda z^\top J z. \tag{8.3}$$

By Lemma 8.3.1, $\lambda z^\top Jz \geq 0$. So, from (8.3), we get $(Az)^\top J(Az) \geq 0$, hence by lemma 8.3.1, $Az \in M(p,q)$. Thus, we have that A is a positive operator of $M(p,q)$. □

Similar necessary conditions and sufficient conditions can be given for the positive operators of $L(p,q)$.

Proposition 6.4.1 provides another sufficient condition for positive operators. However, one of the conditions of that proposition is not related to positive operators, it serves another purpose. Therefore, we state a more general result here:

Theorem 8.3.3. *Let:* $A \in \mathbb{R}^{(p+q)\times(p+q)}$

$$A = \begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix}$$

where $S \in \mathbb{R}^{p\times p}$, $T \in \mathbb{R}^{q\times q}$ *and* $\|T\| = \alpha$ *for some* $\alpha > 0$. *Then,*

(I) A is a positive operator of $L(p,q)$ if and only if each entry of S is nonnegative and the sums of each row of S are at least α.

(II) Moreover, there exist $\lambda \geq 0$ such that $A^\top JA - \lambda J$ is positive semidefinite if and only if the sums of each column of S are the same and at least α.

Proof. (I) "$\Leftarrow$" : Suppose that $(x,u) \in L(p,q)$, then $x_j \geq \|u\|$ where x_j is the jth entry of x. Let $(y,v) = A(x,u) \in \mathbb{R}^p \times \mathbb{R}^q$, then $y_i = \sum_{j=1}^p s_{ij}x_j \geq \|u\| \sum_{j=1}^p s_{ij} \geq \|T\|\|u\| \geq \|Tu\| = \|v\|$, for any ith entry of y. Then, $A(x,u) \in L(p,q)$, and hence A is a positive operator of $L(p,q)$.

"$\Rightarrow$": Suppose that A is a positive operator of $L(p,q)$. By Theorem 8.3.1, it is obvious that each entry of S is nonnegative. Let $u_0 \neq 0$ be the vector such that $\|Tu_0\| = \alpha\|u_0\|$ (this u_0 always exists). The vector $(\|u_0\|e, u_0) \in L(p,q)$. Suppose that the sum of the jth row of S, $s_j < \alpha$. Then, $A(\|u_0\|e, u_0) = (\|u_0\|Se, Tu_0)$ and the jth entry of this vector will be $s_j\|u_0\| < \alpha\|u_0\| = \|Tu_0\|$. That means, $A(\|u_0\|e, u_0) \notin L(p,q)$, which is a contradiction.

(II) "$\Leftarrow$": By the assumptions, if the sum of each column of S is s, we can conclude that $A^\top JA - \lambda J$ is :

$$\begin{bmatrix} S^\top & 0 \\ 0 & T^\top \end{bmatrix} \begin{bmatrix} ee^\top & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix} - \lambda \begin{bmatrix} ee^\top & 0 \\ 0 & -I \end{bmatrix} = \begin{bmatrix} (s^2-\lambda)ee^\top & 0 \\ 0 & \lambda I - T^\top T \end{bmatrix}. \tag{8.4}$$

Since $ee^\top$ is positive semidefinite, there exists a $\lambda \in (\alpha^2, s^2)$ such that the $A^\top JA - \lambda J$ is positive semidefinite.

$\Rightarrow$: Suppose the $s_i \neq s_j$ where s_i and s_j are the sums of the ith and jth column, respectively. Similar to the above equation, the right lower block of $A^\top JA - \lambda J$ will be $\lambda I - T^\top T$. So if $A^\top JA - \lambda J$ is positive semidefinite, then $\lambda \geq \alpha^2 > 0$. Then the upper left block of $A^\top JA - \lambda J$ will be in this form:

$$\begin{bmatrix} \ddots & & & & \\ & s_i^2 - \lambda & \dots & s_is_j - \lambda & \\ & \vdots & \ddots & \vdots & \\ & s_is_j - \lambda & \dots & s_j^2 - \lambda & \\ & & & & \ddots \end{bmatrix}$$

If $A^\top JA - \lambda J$ is positive semidefinite, then $(s_i^2 - \lambda)(s_j^2 - \lambda) - (s_is_j - \lambda)^2 \geq 0$. Thus $\lambda(s_i - s_j)^2 \leq 0$. Since $s_i \neq s_j$, $\lambda \leq 0$, which is contradictory to $\lambda \geq \alpha^2 > 0$. Now suppose the sums of each row of S are the same. Then, from equation (8.4), s must be at least α.

$\square$

8.4 Numerical example

In this section, we will show some numerical examples. Let us first consider the following matrix satisfying the condtions of both I and II. Let: $A \in \mathbb{R}^{4\times 4}$

$$A = \begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix} = \begin{bmatrix} 0.25 & 0.45 & 0 & 0 \\ 0.45 & 0.25 & 0 & 0 \\ 0 & 0 & 0.2 & 0.3 \\ 0 & 0 & 0.3 & 0.2 \end{bmatrix}.$$

Here S is a 2×2 constant matrix where the sums of each row is at least 0.55 and the sum of each column is 0.7. T is a 2×2 matrix with $\|T\| \leq 0.5 < 0.55$ since the eigenvalues of T are 0.5 and -0.1. So by the result of Theorem 8.3.3 I, $AL(p,q) \subseteq L(p,q)$. Since $A = A^\top$, then A is a positive operator of $M(p,q)$, too. On the other hand, since every entry of S is positive, the first 2 lines of A are in $L(p,q)$ and

$$A^\top JA - \lambda J = \begin{bmatrix} 0.49-\lambda & 0.49-\lambda & 0 & 0 \\ 0.49-\lambda & 0.49-\lambda & 0 & 0 \\ 0 & 0 & \lambda - 0.13 & -0.12 \\ 0 & 0 & -0.12 & \lambda - 0.13 \end{bmatrix}.$$

Then for every $\lambda \in [0.25, 0.49]$, $A^\top JA - \lambda J$ is positive semidefinite, and from Theorem 8.3.2 we can also conclude that A is a positive operator of $M(p,q)$.

Now, consider a slightly different example :

$$A_1 = \begin{bmatrix} 0.25 & 0.45 & 0 & 0 \\ 0.3 & 0.4 & 0 & 0 \\ 0 & 0 & 0.2 & 0.3 \\ 0 & 0 & 0.3 & 0.2 \end{bmatrix}.$$

By using Theorem 8.3.3 (I), it is easy to verify that A is a positive operator of $L(p,q)$. However,

$$A_1^\top J A_1 - \lambda J = \begin{bmatrix} 0.3025-\lambda & 0.4675-\lambda & 0 & 0 \\ 0.4675-\lambda & 0.7225-\lambda & 0 & 0 \\ 0 & 0 & \lambda-0.13 & -0.12 \\ 0 & 0 & -0.12 & \lambda-0.13 \end{bmatrix}.$$

If $A_1^\top J A_1 - \lambda J$ is positive semidefinite, then $0.25 \leq \lambda \leq 0.3025$ and $\det(S^\top ee^\top S - \lambda ee^\top) \geq 0$. So

$$(0.3025-\lambda)(0.7225-\lambda) - (0.4675-\lambda)(0.4675-\lambda) \geq 0.$$

Then $\lambda \leq 0$, which is a contradiction. That means there is no λ such that $A_1^\top J A_1 - \lambda J$ is positive semidefinite. Hence the conditions of Theorem 8.3.2 are not satisfied which shows that the theorem describes only a sufficient condition for positivity.

8.5 Notes and comments

Theorem 2.3 in [30] (Theorem 8.1.1 in this chapter) showed a necessary and sufficient condition for a linear operator to be a positive operator of Lorentz cone. But when $p > 1$, the study of positive operators of an extended Lorentz cone $L(p,q)$ (or $M(p,q)$) is much more difficult. In Theorem 8.3.3, we showed a necessary and sufficient condition for a block diagonal linear operator A to be a positive operator of $L(p, q)$. In general, if

$$A = \begin{bmatrix} S & R \\ W & T \end{bmatrix},$$

where $R \in \mathbb{R}^{p\times q}$ and $W \in \mathbb{R}^{q\times p}$, then $A(x^\top, u^\top)^\top = (Sx + Ru, Wx + Tu)^\top$. Hence, each entry is determined by both x and u. It is difficult to ensure the first p entries of $A(x^\top, u^\top)^\top$, that is, $Sx + Ru$, to be all positive or all negative simutaneously. It seems hard to find a unified feature of such a positive operator A. The example A_1 shows that even if $A^\top JA - \lambda J$ will not be positive semidefinite for $\lambda \geq 0$, A may still be a positive operator of the extended Lorentz cone.

To improve our results, we may consider some other direction to investigate the necessary conditions and sufficient conditions for a linear operator to be a positive operator. For example, we may consider that each entry of S is sufficiently larger than $\|R\|$. Then each entry of $Sx + Ru$ will be positive. But we would still need to consider that these entries are larger than $\|Wx + Tu\|$, which is a difficult task.

Chapter 9

Conclusion and future works

In this thesis, we extended the notion of Lorentz cones and we showed that the projection onto a set given as the Cartesian product between an Euclidean space and any closed convex set C in another Euclidean space (called cylinder) is isotone (i.e., monotone) with respect to the order defined by an extended Lorentz cone L. We called such sets L-isotone projection sets and generated all of them. When C is a closed convex cone we used the L-isotonicity of the above Cartesian product to show the convergence of a Picard type iteration to a solution of a general mixed complementarity problem, and we have given some examples. Moreover, we presented a Picard iteration for solving a variational inequality on a cylinder via a fixed point formulation. The iteration is monotonically convergent to the solution of the variational inequality with respect to the partial order defined by an extended Lorentz cone. The monotone convergence is based on the isotonicity of the projection onto a cylinder with respect to the partial order defined by the extended Lorentz cone. Our iterative idea also works when C is a general closed convex set which is not a closed convex cone.

A more ambitious plan would be to find all pairs of closed convex cones (K, L) (or more generally, pairs of closed convex sets (K, L) with L a pointed closed convex cone), in a Euclidean space such that K is L-isotone. An even more ambitious plan is to determine all the triples (K, L, M) such that K is a closed convex set, L and M are pointed closed convex cones and $y - x \in L$ implies $P_K y - P_K x \in M$. Although these plans are quite utopistic, any positive step in this direction could lead to interesting applications to complementarity problems (variational inequalities).

Related to these problems we state

1. Given a cone, determine all closed and convex sets onto which the projection is isotone with respect to the order defined by the cone.

2. Given a closed and convex set, determine all cones such that the projection onto the closed and convex set is isotone with to respect the partial order defined by the cone.

3. Determine the closed and convex sets for which there exists a cone, such that the projection onto the closed and convex set is isotone with respect to the partial order defined by the cone.

Although the above questions are difficult to answer in general, any particular result about them can be important for solving complementarity problems and/or variational inequalities by using a monotone convergence. Moreover, any such result could be important in statistics as well, where the isotonicity of the projection may occur in various algorithms (see for example the algorithms considered in Guyader, Jegou, Németh and Németh [17]). Some partial results related to Questions 1, 2 and 3 above can be found in Németh and Németh [47, 48] and in this thesis, but there is still much to be done.

Chapter 4 presented an explicit connection between conic optimization and complementarity problems, connection which comes from the complementary slackness relation of the Karush-Kuhn-Tucker conditions. Although the complementary slackness suggests that such a connection should exist, it is difficult to find it explicitly in the literature. Hopefully, this chapter will be a useful reference for some readers.

In Chapter 7, we combined results from previous chapters and applied them in game theory and conic optimization. In fact, Theorem 7.1.1 doesn't ensure that all the Nash equilibrium points are found. So, how can we get all the Nash equilibrium points under the circumstances of Theorem 7.1.1? Note that in Theorem 7.1.1 and Theorem 7.2.1, the strategy set must be compact and convex. We can raise the following questions: If we drop at least one of these two conditions (closed or convex), can we still find a class of variational inequalities whose solution is equivalent to the Nash equilibrium points (for example, where K_i is finite for each $i \in P$)? Algorithmic game theory is a hot area nowadays; how can Theorem 7.1.1 be applied in this area? In Theorem 7.1.1, we considered games with the differentiable utility functions. In the future we aim to generalize this results to continuous games defined in [55, 68]. Moreover, in classical microeconomic theory, all the assumptions of utility functions have corresponding economic interpretations. How can we interprete these conditions in Theorem 7.1.1? These open questions will be interesting in our further studies. In the future, in Section 7.2 (similar to Section 7.1), we can consider applying it to numerical optimization algorithms in terms of explicit functions G, H. In Chapters 4 and 7, the constraint function G must be linear, how will the results change if we drop this condition? We may first consider the case when G is quadratic. This is an open question, too.

The positive operators on the Lorentz cone has been completely classified in Loewy and Schneider [30]. This classification suggests that the class of affine mappings satisfying Theorem 5.2.1 is even much larger than the one presented in Section 6.4. However the complete classification of the affine mappings which satisfy the conditions of Theorem

5.2.1 is still an open question. In Chapter 8, we showed necessary conditions and sufficient conditions for a linear operator to be a positive operator of an extended Lorentz cone. Since the first p entries (rather than the first entry only of vectors in $L(1,q)$) must all be nonnegative, some extra conditions (such as the first p lines are in L) are needed to ensure that A is a positive operator when $A^\top JA - \lambda J$ is positive semidefinite. In Section 8.3, we further studied the Proposition 6.4.1 and showed sufficient and necessary conditions for a block diagonal operator to be a positive operator of an extended Lorentz cone. In the future, we will consider a more general form of linear operators and wish to find the necessary and sufficient conditions for them to be positive operators of extended Lorentz cones.

Appendix

To solve a constant-recursive sequence of the form:

$$ax_{n+2} + bx_{n+1} + cx_n = 0,$$

we can consider the following characteristic equation (polynomial):

$$a\lambda^2 + b\lambda + c = 0.$$

If two (not necessarily real) roots are λ_1 and λ_2. For each $n \in \mathbb{N}$, x_n will take the form of $d_1\lambda_1^n + d_2\lambda_2^n$ where d_1 and d_2 are determined by the values of x_1 and x_2. For example, consider the Fibonacci number:

$$x_{n+2} = x_{n+1} + x_n,$$

where $x_1 = x_2 = 1$. Then its characteristic equation is $\lambda^2 - \lambda - 1 = 0$. Then:

$$\lambda_1 = \frac{1+\sqrt{5}}{2}\ ,\ \lambda_1 = \frac{1-\sqrt{5}}{2}$$

and

$$x_n = d_1\lambda_1^n + d_2\lambda_2^n.$$

Let $n = 1$ and $n = 2$, we get:

$$d_1 = \frac{1}{\sqrt{5}} \text{ and } d_2 = \frac{-1}{\sqrt{5}}.$$

Then we have:

$$x_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

More details can be found in [16].

List of papers

List of published papers

S. Z. Németh and G. Zhang. Extended Lorentz cones and mixed complementarity problems. *Journal of Global Optimization*, 62(3):443–457, 2015

S. Z. Németh and G. Zhang. Extended Lorentz cones and variational inequalities on cylinders. *Journal of Optimization theory and applications*, 168(3):756-768, 2016

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S. Z. Németh and G. Zhang. Conic optimization and complementarity problems. *ArXiv e-prints* arXiv:1607.05161v1, July 2016

S. Z. Németh and G. Zhang. Positive operators of Extended Lorentz cones. *ArXiv e-prints* arXiv:1608.07455, August 2016

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[1] M. Abbas and S. Z. Németh. Solving nonlinear complementarity problems by isotonicity of the metric projection. *J. Math. Anal. Appl.*, 386(2):882–893, 2011.

[2] M. Abbas and S. Z. Németh. Finding solutions of implicit complementarity problems by isotonicity of the metric projection. *Nonlinear Anal.*, 75(4):2349–2361, 2012.

[3] M. Abbas and S. Z. Németh. Implicit complementarity problems on isotone projection cones. *Optimization*, 61(6):765–778, 2012.

[4] M. Aghassi and D. Bertsimas. Robust game theory. *Math. Program.*, 107(1-2, Ser. B):231–273, 2006.

[5] F. Alizadeh and D. Goldfarb. Second-order cone programming. *Math. Program.*, 95(1, Ser. B):3–51, 2003. ISMP 2000, Part 3 (Atlanta, GA).

[6] R. Andreani, A. Friedlander, M. P. Mello, and S. A. Santos. Box-constrained minimization reformulations of complementarity problems in second-order cones. *J. Global Optim.*, 40(4):505–527, 2008.

[7] A. Ben-Tal and A. Nemirovski. *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications.* MPS-SIAM Series on Optimization. Society for Industrial and Applied Mathematics, 2001.

[8] D.P. Bertsekas and J.N. Tsitsiklis. *Parallel and Distributed Computation: Numerical Methods*. Athena scientific optimization and computation series. Athena Scientific, 1997.

[9] S. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge University Press, Cambridge, 2004.

[10] F. E. Browder. Continuity properties of monotone non-linear operators in Banach spaces. *Bull. Amer. Math. Soc.*, 70:551–553, 1964.

[11] J.-S. Chen and P. Tseng. An unconstrained smooth minimization reformulation of the second-order cone complementarity problem. *Math. Program.*, 104(2-3, Ser. B):293–327, 2005.

[12] F. Facchinei and J-S. Pang. *Finite-dimensional variational inequalities and complementarity problems. Vol. II*. Springer-Verlag, New York, 2003.

[13] J. Faraut and A. Korányi. *Analysis on symmetric cones*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1994. Oxford Science Publications.

[14] D. Gabay and H. Moulin. On the uniqueness and stability of Nash-equilibria in noncooperative games. In *Applied stochastic control in econometrics and management science*, volume 130 of *Contrib. Econom. Anal.*, pages 271–293. North-Holland, Amsterdam-New York, 1980.

[15] M. Seetharama Gowda, R. Sznajder, and J. Tao. Some **P**-properties for linear transformations on Euclidean Jordan algebras. *Linear Algebra Appl.*, 393:203–232, 2004.

[16] D.H. Greene and D.E. Knuth. *Mathematics for the analysis of algorithms.* Progress in computer science. Birkhäuser, 1981.

[17] A. Guyader, N. Jégou, A. B. Németh, and S. Z. Németh. A geometrical approach to iterative isotone regression. *Appl. Math. Comput.*, 227:359–369, 2014.

[18] L. Hurwicz. The design of mechanisms for resource allocation. In *Frontiers of quantitative economics (Invited Papers, Econometric Soc. Winter Meetings, Toronto, Ont., 1972), Vol. II*, pages 3–42. Contributions to Economic Analysis, Vol. 87. North-Holland, Amsterdam, 1974.

[19] G. Isac and A. B. Németh. Every generating isotone projection cone is latticial and correct. *J. Math. Anal. Appl.*, 147(1):53–62, 1990.

[20] G. Isac and A. B. Németh. Isotone projection cones in Hilbert spaces and the complementarity problem. *Boll. Un. Mat. Ital. B.*, 7(4):773–802, 1990.

[21] G. Isac and A. B. Németh. Projection methods, isotone projection cones, and the complementarity problem. *J. Math. Anal. Appl.*, 153(1):258–275, 1990.

[22] G. Isac and A. B. Németh. Isotone projection cones in Euclidean spaces. *Ann. Sci. Math. Québec*, 16(1):35–52, 1992.

[23] A. N. Iusem and B. F. Svaiter. A variant of Korpelevich's method for variational inequalities with a new search strategy. *Optimization*, 42(4):309–321, 1997.

[24] R. Kachurovskii. On monotone operators and convex functionals. *Uspeкhi. Mat. Nauk*, 15(4):213–215, 1960.

[25] E. N. Khobotov. A modification of the extragradient method for solving variational inequalities and some optimization problems. *Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki*, 27(10):1462–1473, 1987.

[26] C.-H. Ko, J.-S. Chen, and C.-Y. Yang. Recurrent neural networks for solving second-order cone programs. *Neurocomputing*, 74:3464–3653, 2011.

[27] M. Kočvara and J. V. Outrata. On optimization of systems governed by implicit complementarity problems. *Numer. Funct. Anal. Optim.*, 15(7-8):869–887, 1994.

[28] I. Konnov. *Equilibrium Models and Variational Inequalities*. Mathematics in Science and Engineering. Elsevier Science, 2007.

[29] P.D. Lax. *Linear Algebra and Its Applications*. Number v. 10 in Linear algebra and its applications. Wiley, 2007.

[30] R. Loewy and H. Schneider. Positive operators on the n-dimensional ice cream cone. *Journal of Mathematical Analysis and Applications*, 49:375–392, 1975.

[31] G.-M. Luo, X. An, and J.-Y. Xia. Robust optimization with applications to game theory. *Appl. Anal.*, 88(8):1183–1195, 2009.

[32] N.G. Mankiw and M.P. Taylor. *Macroeconomics*. Cengage Learning, 2014.

[33] P. Marcotte. Application of Khobotov's algorithm to variational inequalities and network equilibrium problems. *Information Systems and Operational Research*, 29:258–270, 1991.

[34] A. Mas-Colell, M.D. Whinston, and J.R. Green. *Microeconomic Theory*. Oxford student edition. Oxford University Press, 1995.

[35] M.l Maschler, E. Solan, and S. Zamir. *Game theory.* Cambridge University Press, Cambridge, 2013. Translated from Hebrew by Ziv Hellman and edited by Mike Borns.

[36] C. W. McArthur. In what spaces is every closed normal cone regular? *Proc. Edinburgh Math. Soc. (2)*, 17:121–125, 1970/1971.

[37] N. McCarty and A. Meirowitz. *Political Game Theory: An Introduction.* Analytical Methods for Social Research. Cambridge University Press, 2007.

[38] G. Minty. Monotone operators in Hilbert spaces. *Duke Math. J.*, 29:341–346, 1962.

[39] G. Minty. On a "monotonicity" method for the solution of non-linear equations in Banach spaces. *Proc. Nat. Acad. Sci. USA*, 50:1038–1041, 1963.

[40] A. Nagurney. *Network Economics - A Variational Inequality Approach.* Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.

[41] J. F. Nash, Jr. The bargaining problem. *Econometrica*, 18(2):155âĂŞ162, 1950.

[42] J. F. Nash, Jr. Equilibrium points in n-person games. *Proc. Nat. Acad. Sci. U. S. A.*, pages 48–49, 1950.

[43] J. F. Nash, Jr. Non-cooperative games. *Ann. of Math. (2)*, pages 286–295, 1951.

[44] J. F. Nash, Jr. Two-person cooperative games. *Econometrica*, 21(1):128–140, 1953.

[45] A. B. Németh and S. Z. Németh. How to project onto an isotone projection cone. *Linear Algebra Appl.*, 433(1):41–51, 2010.

[46] A. B. Németh and S. Z. Németh. A duality between the metric projection onto a cone and the metric projection onto its dual. *J. Math. Anal. Appl.*, 392(2):172–178, 2012.

[47] A. B. Németh and S. Z. Németh. Lattice-like operations and isotone projection sets. *Linear Algebra Appl.*, 439(10):2815–2828, 2013.

[48] A. B. Németh and S. Z. Németh. Lattice-like subsets of Euclidean Jordan algebras. arXiv:1401.3581, 2014.

[49] S. Z. Németh. Iterative methods for nonlinear complementarity problems on isotone projection cones. *J. Math. Anal. Appl.*, 350(1):340–347, 2009.

[50] S. Z. Németh. A duality between the metric projection onto a convex cone and the metric projection onto its dual in Hilbert spaces. *Nonlinear Anal.*, 97:1–3, 2014.

[51] S. Z. Németh and G. Zhang. Extended Lorentz cones and mixed complementarity problems. *J. Global Optim.*, 62(3):443–457, 2015.

[52] S. Z. Németh and G. Zhang. Conic optimization and complementarity problems. *ArXiv e-prints*, July 2016.

[53] S. Z. Németh and G. Zhang. Extended Lorentz cones and variational inequalities on cylinders. *J. Optim. Theory Appl.*, 168(3):756–768, 2016.

[54] S. Z. Németh and G. Zhang. Positive operators of Extended Lorentz cones. *ArXiv e-prints*, August 2016.

[55] H. Nishimura and E. A. Ok. Solvability of variational inequalities on Hilbert lattices. *Math. Oper. Res.*, 37(4):608–625, 2012.

[56] R. Nishimura, S. Hayashi, and M. Fukushima. Robust Nash equilibria in N-person non-cooperative games: uniqueness and reformulation. *Pac. J. Optim.*, 5(2):237–259, 2009.

[57] R. Nishimura, S. Hayashi, and M. Fukushima. Semidefinite complementarity reformulation for robust Nash equilibrium problems with Euclidean uncertainty sets. *J. Global Optim.*, 53(1):107–120, 2012.

[58] R. Tyrrell Rockafellar. *Convex analysis*. Princeton University Press, Princeton, NJ, 1997.

[59] G. Rudolf, N. Noyan, D. Papp, and F. Alizadeh. Bilinear optimality constraints for the cone of positive polynomials. *Math. Program.*, 129(1, Ser. B):5–31, 2011.

[60] A. Ruszczyński. *Nonlinear optimization*. Princeton University Press, Princeton, NJ, 2006.

[61] H. Schneider and M. Vidyasagar. Cross-positive matrices. *SIAM J. Numer. Anal.*, 7:508–519, 1970.

[62] L. S. Shapley. Cores of convex games. *International Journal of Game Theory*, 1(1):11–26, 1971.

[63] L. S. Shapley. A value for n-person games [from *contributions to the theory of games, vol. 2*, 307–317, Princeton Univ. Press, Princeton, NJ, 1953; MR **14**, 779]. In *The Shapley value*, pages 31–40. Cambridge Univ. Press, Cambridge, 1988.

[64] M. Sibony. Méthodes itératives pour les équations et inéquations aux dérivées partielles non linéaires de type monotone. *Calcolo*, 7:65–183, 1970.

[65] J.M. Smith. *Evolution and the Theory of Games*. Cambridge University Press, 1982.

[66] M. V. Solodov and B. F. Svaiter. A new projection method for variational inequality problems. *SIAM J. Control Optim*, 37(3):765–776, 1999.

[67] M. V. Solodov and P. Tseng. Modified projection-type methods for monotone variational inequalities. *SIAM J. Control Optim*, 34(5):1814–1830, 1996.

[68] Noah D. Stein, Asuman Ozdaglar, and Pablo A. Parrilo. Separable and low-rank continuous games. *International Journal of Game Theory*, 37(4):475–504, 2008.

[69] D. Sun. A class of iterative methods for nonlinear projection equations. *J. Optim. Theory Appl.*, 91(1):123–140, 1996.

[70] R. Sznajder. The Lyapunov rank of extended second order cones. *J. Global Optim.*, 66(3):585–593, 2016.

[71] J. Tirole. *The Theory of Industrial Organization*. Cambridge, Mass., 1988.

[72] M. J. Todd. A note on computing equilibria in economies with activity analysis models of production. *J. Math. Econom.*, 6(2):135–144, 1979.

[73] P. Tseng. On linear convergence of iterative methods for the variational inequality problem. *J. Comput. Appl. Math.*, 60(1-2):237–252, 1995. Linear/nonlinear iterative methods and verification of solution (Matsuyama, 1993).

[74] L. Vandenberghe, B. L. De Moor, and J. Vandewalle. The generalized linear complementarity problem applied to the complete analysis of resistive piecewise-linear circuits. *IEEE Trans. Circuits and Systems*, 36(11):1382–1391, 1989.

[75] H.R. Varian. *Intermediate Microeconomics: A Modern Approach*. Norton, 2011.

[76] J. von Neumann and O. Morgenstern. *Theory of games and economic behavior*. Princeton University Press, Princeton, NJ, anniversary edition, 2007. With an introduction by Harold W. Kuhn and an afterword by Ariel Rubinstein.

[77] K. Yonekura and Y. Kanno. Second-order cone programming with warm start for elastoplastic analysis with von Mises yield criterion. *Optim. Eng.*, 13(2):181–218, 2012.

[78] A. Youdine. Solution des deux problÃÍmes de la theÃĺorie des espaces semi-ordonnÃĺs. *C.R.Acad.Sci. URSS*, 23:418âĂŞ422, 1939.

[79] E. H. Zarantonello. Projections on convex sets in Hilbert space and spectral theory. II. Spectral theory. In *Contributions to nonlinear functional analysis (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1971)*, pages 343–424. Academic Press, New York, 1971.

[80] L. L. Zhang, J. Y. Li, H. W. Zhang, and S. H. Pan. A second order cone complementarity approach for the numerical solution of elastoplasticity problems. *Comput. Mech.*, 51(1):1–18, 2013.

[81] Y. B. Zhao, J. Y. Han, and H. D. Qi. Exceptional families and existence theorems for variational inequality problems. *J. Optim. Theory Appl.*, 101(2):475–495, 1999.

[82] Y-B. Zhao and D. Li. A globally and locally superlinearly convergent non-interior-point algorithm for P_0 LCPs. *SIAM J. Optim.*, 13(4):1195–1221 (electronic), 2003.