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INFINITE WORDS CONTAINING SQUARES AT EVERY POSITION *, **

James Currie[1] and Narad Rampersad[1]

Abstract. Richomme asked the following question: what is the infimum of the real numbers $\alpha > 2$ such that there exists an infinite word that avoids α-powers but contains arbitrarily large squares beginning at every position? We resolve this question in the case of a binary alphabet by showing that the answer is $\alpha = 7/3$.

Mathematics Subject Classification. 68R15.

1. Introduction

We consider the following question of Richomme [17]: what is the infimum of the real numbers $\alpha > 2$ such that there exists an infinite word that avoids α-powers but contains arbitrarily large squares beginning at every position? As we shall see, over the binary alphabet, the answer to Richomme's question is $\alpha = 7/3$.

First we recall some basic definitions. If α is a rational number, a word w is an *α-power* if there exist words x and x', with x' a prefix of x, such that $w = x^n x'$ and $\alpha = n + |x'|/|x|$. We refer to $|x|$ as a *period* of w, where $|x|$ denotes the length of the word x. An *α^+-power* is a word that is a β-power for some $\beta > \alpha$. A word is *α-power-free* if none of its factors is a β-power for any $\beta \geq \alpha$. A word is *α^+-power-free* if none of its factors is an α^+-power. A 2-power is called a *square*; a 2^+-power is called an *overlap*.

The motivation for Richomme's question comes from the observation that there exist aperiodic infinite binary words that contain arbitrarily large squares starting at every position. For instance, all Sturmian words have this property ([1], Prop. 2).

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Certain Sturmian words are not only aperiodic, but also avoid α-powers for some real number α. For instance, it is well-known that the Fibonacci word

$$\mathbf{f} = 0100101001001010010100\cdots$$

contains no $(2+\varphi)$-powers [14] (see also [13]), where $\varphi = (1+\sqrt{5})/2$ is the golden ratio. By contrast, the Thue-Morse word

$$\mathbf{t} = 0110100110010110100101110\cdots$$

is overlap-free [20] but does not contain squares beginning at every position (It is an easy exercise to show that $\mathbf{t}$ does not begin with a square). The squares occurring in the Thue-Morse word have been characterized by Pansiot [15] and Brlek [4], and the positions at which they occur were studied by Brown *et al.* [5].

Saari [18] also studied infinite words containing squares (not necessarily arbitrarily large) beginning at every position. He calls such words *squareful* (though he imposes the additional condition that the word contain only finitely many distinct minimal squares).

2. Overlap-free and 7/3-power-free squares

We begin by reviewing what is known concerning the overlap-free binary squares. Subsequently, we shall generalize this characterization to the 7/3-power-free binary squares.

Let μ denote the Thue–Morse morphism: *i.e.*, the morphism that maps $0 \to 01$ and $1 \to 10$. Define sets

$$A = \{00, 11, 010010, 101101\}$$

and

$$\mathcal{A} = \bigcup_{k \geq 0} \mu^k(A).$$

The set $\mathcal{A}$ is the set of squares appearing in the Thue-Morse word. Shelton and Soni [19] characterized the overlap-free squares (the result is also attributed to Thue by Berstel [2]), as being the conjugates of the words in $\mathcal{A}$ (a conjugate of x is a word y such that $x = uv$ and $y = vu$ for some u, v). An immediate application is the following theorem:

Theorem 2.1. *If* $\mathbf{w}$ *is an infinite overlap-free binary word, then there is a position* i *such that* $\mathbf{w}$ *does not contain a square beginning at position* i*.*

Proof. An easy computer search suffices to verify that any overlap-free word over $\{0, 1\}$ of length greater than 36 must contain the factor 010011. Let i denote any position at which 010011 occurs in $\mathbf{w}$. We claim that no square begins at position i. Suppose to the contrary that xx is such a square. By Shelton and Soni's result, except for the squares that are conjugates of A, every overlap-free square xx has

$|x|$ even. Since xx begins with 010011, the word xx is not in $\{01, 10\}^*$, and consequently xx is of the form $0y10y1$, where $y \in \{01, 10\}^\ell$ for some ℓ. However, this forces xx to be followed by 0 in $\mathbf{w}$, so that we have the overlap $xx0$ as a factor of $\mathbf{w}$, a contradiction. □

The next theorem generalizes the characterization of Shelton and Soni.

Theorem 2.2. *The* $7/3$*-power-free binary squares are the conjugates of the words in* $\mathcal{A}$*.*

We defer the proof of Theorem 2.2 to Section 4. However, we end this section by proving an analogue, for 7/3-power-free words, of a well-known "progression lemma" for overlap-free words [3,7,10,11,16,19] that we shall need later.

The words $\mu^n(0)$ and $\mu^n(1)$, $n \geq 0$, are known as *Morse blocks*. Note that the reverse of a Morse block is a Morse block. We also have the following properties:

(a) if u is a Morse block of length 2^n, $n > 0$, then $u = u'u''$, where u' and u'' are distinct Morse blocks of length 2^{n-1};

(b) if u' and u'' are distinct Morse blocks of the same length, then $u'u''$ is a Morse block.

Lemma 2.3. *Let* $w = uvxy$ *be a binary* $7/3$*-power-free word with* $|u| = |v| = |x| = |y| = 2^n$*. If* u *and* v *are Morse blocks, then* x *is a Morse block.*

Proof. The proof is by induction on n. Clearly, the result holds for $n = 0$. By property (a) above, we have $u = u'u''$, where u' and u'' are distinct Morse blocks of length 2^{n-1}. If $u = v$, we have (1) $w = u'u''u'u''pqrs$, and if $u \neq v$, we have (2) $w = u'u''u''u'pqrs$, where $|p| = |q| = |r| = |s| = 2^{n-1}$. By induction, p, q, and r are also Morse blocks. By property (b) above, to show that pq is a Morse block it suffices to show that $p \neq q$. In case (1), $pq = u'u'$ creates the 5/2-power $u'u''u'u''u'$, and $pq = u''u''$ creates the cube $u''u''u''$. In case (2), $pq = u'u'$ creates the cube $u'u'u'$; and if $pq = u''u''$, then $r = u'$ creates the 7/3-power $u'u''u''u'u''u''u'$, and $r = u''$ creates the cube $u''u''u''$. Thus, $p \neq q$, as required. □

3. Infinite words containing squares at every position

We begin by showing that the answer to Richomme's question is at most 7/3.

Theorem 3.1. *There exists an infinite* $(7/3)^+$*-power-free binary word that contains arbitrarily large squares beginning at every position.*

Proof. We rely on the existence of a $(7/3)^+$-power-free morphism; that is, a morphism f such that $f(w)$ is $(7/3)^+$-power-free whenever w is $(7/3)^+$-power-free. Kolpakov, Kucherov, and Tarannikov [12] have given an example of such a morphism f:

$$\begin{array}{lcl} 0 & \rightarrow & 011010011001001101001 \\ 1 & \rightarrow & 100101100100110010110. \end{array}$$

For convenience we work instead with the following morphism g:

$$\begin{array}{rcl} 0 & \rightarrow & 011010011011001101001 \\ 1 & \rightarrow & 100101100110110010110, \end{array}$$

which we obtain by setting $g(0) = \overline{f(1)}$ and $g(1) = \overline{f(0)}$, where the overline denotes binary complementation.

We now define a sequence of words $(A_n)_{n\geq 0}$ as follows. Let $A_0 = 0110110$. For $n \geq 0$ define $A_{n+1} = (011010)^{-1}g(A_n)$, where the notation $(011010)^{-1}x$ denotes the word obtained by removing the prefix 011010 from the word x. Since $g(A_n)$ always begins with $g(0)$, it always has the prefix 011010, so the deletion operation is well-defined. The sequence $(A_n)_{n\geq 0}$ thus begins

$$\begin{array}{rcl} A_0 & = & 0110110 \\ A_1 & = & 011011001101001100101100110110010110100101100110110010110\cdots \\ & \vdots & \end{array}$$

Observe that since g and A_0 are $(7/3)^+$-power-free, A_n is also $(7/3)^+$-power-free. We first show that as $n \to \infty$, A_n converges to an infinite limit word $\mathbf{w}$; second, we show that $\mathbf{w}$ contains arbitrarily long squares beginning at every position.

We show by induction on n that A_n is a prefix of A_{n+1}. Clearly this is true for $n = 0$. Recall that $A_{n+1} = (011010)^{-1}g(A_n)$. Inductively, A_{n-1} is a prefix of A_n, so $g(A_{n-1})$ is a prefix of $g(A_n)$. Thus, $(011010)^{-1}g(A_{n-1}) = A_n$ is a prefix of $(011010)^{-1}g(A_n) = A_{n+1}$, as required. We conclude that A_n tends, in the limit, to a $(7/3)^+$-power-free word $\mathbf{w}$.

To see that $\mathbf{w}$ contains arbitrarily long squares beginning at every position, first let $u = 011010$ and observe that for every $n \geq 1$ we have

$$A_n = u^{-1}g(u)^{-1}[g^2(u)]^{-1}\cdots[g^{n-1}(u)]^{-1}g^n(0110110).$$

Let

$$v_n = g^{n-1}(u)g^{n-2}(u)\ldots g(u)u.$$

Since for words x, y we have $(xy)^{-1} = y^{-1}x^{-1}$, we have $A_n = v_n^{-1}g^n(0110110)$. Observe that since $|g(0)| = |g(1)| = 21$, we have

$$|v_n| = 6\sum_{j=0}^{n-1} 21^j = 6\left(\frac{21^n - 1}{21 - 1}\right) < 21^n = |g^n(0)|.$$

We see then that v_n is a prefix of $g^n(0)$. Write $g^n(0) = v_n x$, so that $A_n = xg^n(11)v_n xg^n(11)v_n x$. It follows that for $0 \leq j < |x| = 21^n - (3/10)(21^n - 1)$, A_n, and hence $\mathbf{w}$, contains a square of length 6×21^n beginning at position j. Since n may be taken to be arbitrarily large, the result follows. $\square$

The result of the previous theorem is optimal, as we now demonstrate.

Theorem 3.2. *If* $\mathbf{w}$ *is an infinite* $7/3$*-power-free binary word, then there is a position* i *such that* $\mathbf{w}$ *does not contain arbitrarily large squares beginning at position* i.

Proof. As in the proof of Theorem 2.1, an easy computer search suffices to verify that any 7/3-power-free word over $\{0, 1\}$ of length greater than 39 must contain the factor 010011. Let i be a position at which there is an occurrence of 010011 in $\mathbf{w}$. Suppose that there is a square xx beginning at position i. By Theorem 2.2, xx is a conjugate of a word in $\mathcal{A}$. In particular, since xx begins with 010011, we have $xx \notin \{01, 10\}^*$, and consequently $xx \notin \mathcal{A}$; that is, xx is a *non-identity conjugate* of a word in $\mathcal{A}$.

Case 1: xx is a conjugate of either $\mu^k(00)$ or $\mu^k(11)$ for some k. Then xx is a conjugate of a word of the form $uvuv$, where u and v are Morse blocks of the same length. Without loss of generality, we write $xx = u''vuvu'$, where $u'u'' = u$ and $u' \neq \epsilon \neq u''$.

Suppose that yy is another square beginning at position i. Suppose further that there are arbitrarily large squares beginning at position i, so that we may choose $|y| > |xx|$. We see then that there is an occurrence of $u''vuvu'$ at position $i + |y|$. Considering this later occurrence of $u''vuvu'$, and observing that Morse blocks of a given length are uniquely identified by their first letter (as well as by their last letter), we may apply Lemma 2.3 to this later occurrence of $u''vuvu'$ to conclude that the vuv of this occurrence is both preceded and followed by the Morse block u. Thus, $\mathbf{w}$ contains the 5/2-power $uvuvu$, a contradiction.

Case 2: xx is a conjugate of either $\mu^k(010010)$ or $\mu^k(101101)$ for some k. By a similar argument as in Case 1, we may suppose that xx has one of the forms $u''vuuvuu'$, $u''vvuvvu'$, or $u''uvuuvu'$, where u and v are Morse blocks of the same length, $u'u'' = u$, and $u' \neq \epsilon \neq u''$.

As before, we suppose the existence of a square yy beginning at position i, where $|y| > |xx|$. Then there is a later occurrence of xx at position $i + |y|$. Applying Lemma 2.3 to this later occurrence of xx, we deduce the existence of one of the 7/3-powers $uvuuvuu$, $uvvuvvu$, or $uuvuuvu$, a contradiction.

All cases yield a contradiction; we conclude that there does not exist arbitrarily large squares beginning at position i, as required. □

It is possible, however, to have an infinite 7/3-power-free binary word with squares beginning at every position; we are only prevented from having arbitrarily large squares beginning at every position.

Theorem 3.3. *There exists an infinite* $7/3$*-power-free binary word that contains squares beginning at every position.*

Proof. We show that a word constructed by Currie *et al.* [6] has the desired property. The construction is as follows. We define the following sequence of words:

$A_0 = 00$ and $A_{n+1} = 0\mu^2(A_n)$, $n \geq 0$. The first few terms in this sequence are

$$\begin{aligned} A_0 &= 00 \\ A_1 &= 001100110 \\ A_2 &= 001100110100110010110011010011001011 0 \\ &\vdots \end{aligned}$$

Currie *et al.* showed that as $n \to \infty$, this sequence converges to an infinite word $\mathbf{a}$, and further, $\mathbf{a}$ is 7/3-power-free. We show that $\mathbf{a}$ contains squares beginning at every position. We claim that for $n \geq 0$, $\mathbf{a}$ contains a word of the form xxx' at position $(4^n - 1)/3$, where $|x| = 4^{n+1}$ and x' is a prefix of x of length 4^n. Observe that for $n \geq 1$, by the construction of A_{n+1}, we have

$$A_{n+1} = 0\mu^2(0)\mu^4(0)\cdots\mu^{2(n-1)}(0)\mu^{2n}(A_1).$$

However, $\mu^{2n}(A_1) = xxx'$, where $x = \mu^{2n}(0011)$ has length 4^{n+1}; $x' = \mu^{2n}(0)$ is a prefix of x of length 4^n; and using the previous expression for A_{n+1}, $\mu^{2n}(A_1)$ occurs at position

$$\sum_{i=0}^{n-1} 4^i = \frac{4^n - 1}{3},$$

as claimed. It follows that for $i \in [(4^n-1)/3, (4^{n+1}-1)/3-1]$, $\mathbf{a}$ contains a square of length $2 \times 4^{n+1}$ at position i. This completes the proof. $\square$

Although the word constructed in the proof of Theorem 3.3 contains squares beginning at every position, it is not *squareful* in the sense of Saari [18], since it does not contain only finitely many minimal squares. For a squareful word, there exists a constant C such that at every position there is a square of length at most C. To see that this does not hold for the word $\mathbf{a}$ constructed above, we note that by repeated application of the factorization theorem of Karhumäki and Shallit [9] (Thm. 4.1 below), any infinite 7/3-power-free binary word contains occurrences of $\mu^n(0)$ for arbitrarily large n. However, $\mu^n(0)$ is a prefix of the Thue-Morse word, and we have already noted in the introduction that the Thue-Morse word does not begin with a square. Thus there cannot exist a constant C bounding the length of a minimal square in $\mathbf{a}$, so $\mathbf{a}$ is not squareful. In general, no infinite 7/3-power-free binary word can be squareful.

The result of Theorem 3.3 can be strengthened by applying a more general construction of Currie *et al.* [6].

Theorem 3.4. *For every real number $\alpha > 2$, there exists an infinite α-power-free binary word that contains squares beginning at every position.*

Proof. Since Theorem 3.3 establishes the result for $\alpha \geq 7/3$, we only consider $\alpha < 7/3$. We recall the following construction of Currie *et al.* ([6], Thm. 14). Let $s \geq 3$ and $t \geq 5$ be integers such that $2 < 3 - t/2^s < \alpha$, and such that the word obtained by removing the prefix of length t from $\mu^s(0)$ begins with 00. Let $\beta = 3 - t/2^s$.

We construct sequences of words A_n, B_n and C_n. Define $C_0 = 00$ and let u be the prefix of length t of $\mu^s(0)$. For each $n \geq 0$:

(1) Let $A_n = 0C_n$.
(2) Let $B_n = \mu^s(A_n)$.
(3) Let $C_{n+1} = u^{-1}B_n$.

Currie *et al.* showed that the C_n's converge to an infinite word $\mathbf{w}$ that is β^+-power-free, and hence, α-power-free. For $n \geq 1$, $\mathbf{w}$ begins with a prefix of the form

$$u^{-1}\mu^s(0)[\mu^s(u)]^{-1}\mu^{2s}(0)\dots[\mu^{ns}(u)]^{-1}\mu^{(n+1)s}(0)\mu^{(n+1)s}(00).$$

Let $u' = u^{-1}\mu^s(0)$. Thus, for $n \geq 1$, $\mathbf{w}$ contains the word $\mu^{ns}(u')\mu^{(n+1)s}(00)$ at position

$$F_n = 2^s \sum_{i=0}^{n-1} 2^{is} - t \sum_{i=0}^{n-1} 2^{is} = (2^s - t)\left[\frac{2^{ns}-1}{2^s-1}\right],$$

where $\mu^{ns}(u')$ is a suffix of $\mu^{(n+1)s}(0)$, since

$$\mu^{ns}(u') = \mu^{ns}(u^{-1}\mu^s(0)) = [\mu^{ns}(u)]^{-1}\mu^{(n+1)s}(0).$$

Letting $|u'| = t'$ and defining $G_n = |\mu^{ns}(u')| = t' \times 2^{ns}$, we have

$$F_n = t'\left[\frac{2^{ns}-1}{2^s-1}\right] < G_n.$$

Since u' is a suffix of $\mu^s(0)$, and since $\mathbf{w}$ begins with $u'\mu^s(00)$, we see that for $j \in [0, t'-1]$, every factor of $\mathbf{w}$ of length 2^{s+1} starting at position j is a square. Similarly, for $n \geq 1$, we see that there is a square of length $2^{(n+1)s+1}$ starting at position j for every $j \in [F_n, F_n + G_n - 1]$. Since $F_{n+1} = F_n + G_n$, we have $\cup_{n\geq 0}[F_n, F_n + G_n - 1] = \mathbb{N}$; consequently, there is is a square at every position of $\mathbf{w}$. $\square$

4. Proof of Theorem 2.2

In this section we give the proof of Theorem 2.2. We begin with some lemmas, but first we recall the factorization theorem of Karhumäki and Shallit [9].

Theorem 4.1 (Karhumäki and Shallit)**.** *Let $x \in \{0,1\}^*$ be α-power-free, $2 < \alpha \leq 7/3$. Then there exist $u, v \in \{\epsilon, 0, 1, 00, 11\}$ and an α-power-free $y \in \{0,1\}^*$ such that $x = u\mu(y)v$.*

Lemma 4.2. *Let $xx \in \{0,1\}^*$ be $7/3$-power-free. If $xx = \mu(y)$, then $|y|$ is even. Consequently, y is a square.*

Proof. Suppose to the contrary that $|y| = |x|$ is odd. By an exhaustive enumeration one verifies that $|x| \geq 5$. Since $xx = \mu(y)$, we have $xx \in \{01, 10\}^*$. The first occurrence of x is therefore of the form $x \in 01\{01, 10\}^*1$ and the second occurrence is of the form $x \in 0\{01, 10\}^*01$ (or symmetrically, exchanging 0 and 1), which is impossible. $\square$

The next lemma is a version of Theorem 4.1 specifically applicable to squares.

Lemma 4.3. *Let $xx \in \{0, 1\}^*$ be 7/3-power-free. If $|xx| > 8$, then either*

(a) *$xx = \mu(y)$, where $y \in \{0, 1\}^*$; or*
(b) *$xx = \overline{a}\mu(y)a$, where $a \in \{0, 1\}$ and $y \in \{0, 1\}^*$.*

Proof. Applying Theorem 4.1, we write $xx = u\mu(y)v$. We first show that $|u| = |v| \leq 1$. Suppose that $u = 00$. Then xx begins with one of the words 000, 00100, or 001010. The first and third words are respectively a cube and a 5/2-power, a contradiction. Suppose that xx begins with 00100. The word x cannot have length 5, since in this case xx would contain the cube 000. We may assume then that $|x| \geq 6$. Consider the occurrence of 00100 as a prefix of the second x in xx. This occurrence cannot be preceded or followed by 0, as that would result in the cube 000. However, if 00100 is preceded and followed by 1, this results in the 7/3-power 1001001. We conclude that $u \neq 00$, and similarly, $u \neq 11$. A similar argument also holds for v. Since $|xx|$ is even, we must therefore have $|u| = |v|$, as required.

If $|u| = |v| = 0$, then we have established (a). If $|u| = |v| = 1$, it remains to show that $u = \overline{v}$. If $u = v$, then we have $xx = u\mu(y)u$. Since x begins and ends with u, we have $x = u\mu(y') = \mu(y'')u$, where $y = y'y''$. Let $z' = \mu(y')$ and $z'' = \mu(y'')$. Then z'' begins with u and hence with $u\overline{u}$. This implies that z' begins with $\overline{u}$ and hence with $\overline{u}u$. This in turn implies (since $|xx| > 8$) that z'' begins with $u\overline{u}u\overline{u}$, and hence that z' begins with $\overline{u}u\overline{u}u$. Now we see that x begins with the 5/2-power $u\overline{u}u\overline{u}u$, which is a contradiction. We conclude that $u \neq v$, as required. $\square$

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. Let xx be a shortest 7/3-power-free square that is not a conjugate of a word in $\mathcal{A}$. That $|xx| > 8$ is easily verified computationally. Applying Lemma 4.3 leads to two cases.

Case 1: $xx = \mu(y)$. By Lemma 4.2, y is a square. Since $xx = \mu(y)$ is not a conjugate of a word in $\mathcal{A}$, the word y is also not a conjugate of a word in $\mathcal{A}$, contradicting the minimality of xx.

Case 2: $xx = \overline{a}\mu(y)a$. Then $a\overline{a}\mu(y) = \mu(ay)$ is also a square zz. We show that zz is 7/3-power-free, and consequently, by Lemma 4.2, that ay is a 7/3-power-free square, contradicting the minimality of xx.

Suppose to the contrary that zz contains a 7/3-power $s = rrr'$, where r' is a prefix of r and $|r'|/|r| \geq 1/3$. The word s must occur at the beginning of zz and we must have $|s| > |x|$; otherwise, xx would contain an occurrence of s, contradicting the assumption that xx is 7/3-power-free. We have four cases, depending on the relative sizes of $|r|$ and $|z|$, as illustrated in Figures 1–4. Note that $|z| \neq 2|r|$, since otherwise zz, and hence xx, would be a 4-power. By analyzing the overlaps

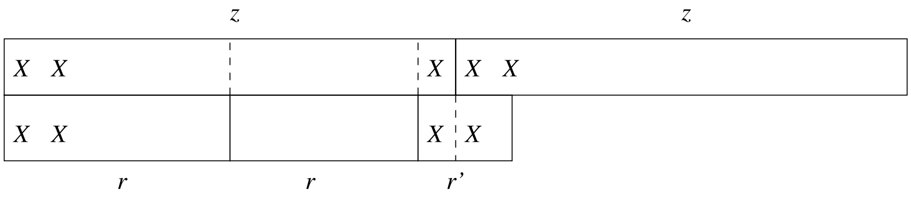

Figure 1. The case where $2|r| < |z| \leq 2|r| + |r'|/2$.

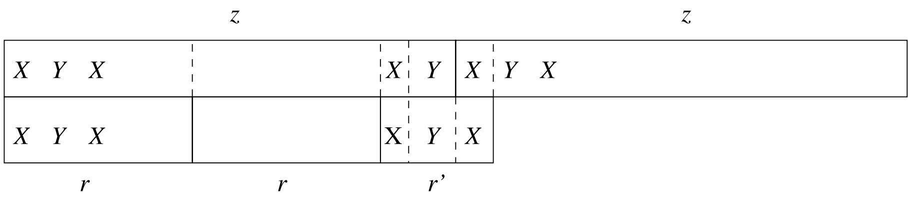

Figure 2. The case where $|z| > 2|r| + |r'|/2$.

between zz and rrr', denoted X in the figures, we derive a contradiction in each case.

Case 2a: $2|r| < |z| \leq 2|r| + |r'|/2$ (Fig. 1). In this case, r' has a non-empty prefix X that is also a suffix of z. Then X is also a prefix of z and XX is a prefix of r'. Consequently, XX is a prefix of z and xx contains the cube XXX. This is a contradiction.

Case 2b: $|z| > 2|r| + |r'|/2$ (Fig. 2). In this case, r' is of the form XYX, where XY is a suffix of z and X is a non-empty prefix of z. Since $r' = XYX$, then z has XYX as a prefix. In particular, xx contains a 7/3-power-free square $XYXY$. By the assumed minimality of xx, $XYXY$ is a conjugate of a word in $\mathcal{A}$. If $XYXY = \mu^k(w)$, where w is a conjugate of a word in A (not $\mathcal{A}$!), then $XYXY$ can be written either as A_1A_1 or as $A_1A_2A_3A_1A_2A_3$, where the A_i's are all Morse blocks of the same length. Since $XYXY$ is followed by X in zz, and since the Morse blocks of a given length are uniquely identified by their first letter, by Lemma 2.3 $XYXY$ is followed by the Morse block A_1, creating either a cube or a 7/3-power in xx, contrary to our assumption.

If $XYXY \neq \mu^k(w)$, where w is a conjugate of a word in A, then we may write $XYXY$ as one of $uBABv$, $uBAABAv$, $uBBABBv$, or $uABAABv$, where A and B are Morse blocks, $B = \overline{A}$, and $vu = A$.

If $XYXY = uBABv$, then, since u is a non-empty suffix of A and v is a non-empty prefix of A, and since the Morse blocks of a given length are uniquely identified by their first letter (as well as by their last letter), we can apply Lemma 2.3 to conclude that BAB is preceded and followed by A. Thus xx contains the 5/2-power $ABABA$, contrary to our assumption. Similarly, if $XYXY = uBAABAv$, then $BAABA$ is preceded and followed by A, creating the 7/3-power $ABAABAA$.

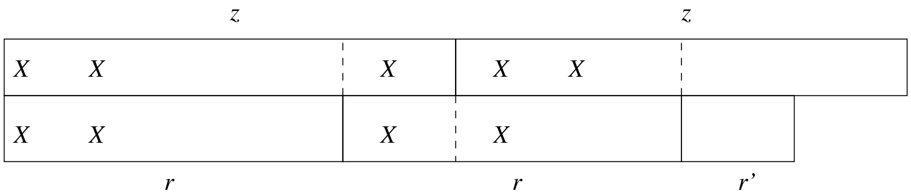

FIGURE 3. The case where $|z| \leq 3/2 \times |r|$.

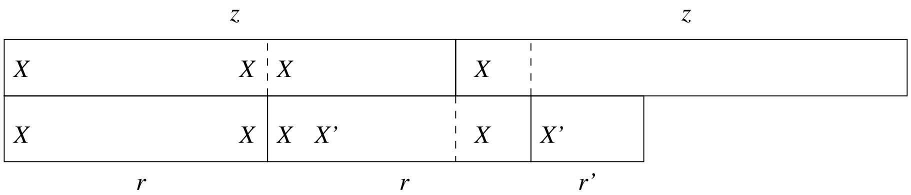

FIGURE 4. The case where $3/2 \times |r| < |z| < 2|r|$.

The other possibilities for $XYXY$ lead to the existence of a 7/3-power in xx by a similar argument.

Case 2c: $|z| \leq 3/2 \times |r|$ (Fig. 3). In this case, r has a non-empty prefix X that is also a suffix of z. Then X is also a prefix of z and XX is a prefix of r. Consequently, XX is a prefix of z and xx contains the cube XXX. This is a contradiction.

Case 2d: $3/2 \times |r| < |z| < 2|r|$ (Fig. 4). In this case, r has a non-empty suffix X that is also a prefix of z. Then X is also a prefix of r and r' begins with some prefix X' of X. Consequently, rr, and indeed xx, contains a factor XXX' that is at least a 7/3-power (since $|r'|/|r| \geq 1/3$). This is a contradiction.

Since in all cases we have derived a contradiction by showing that xx contains a 7/3-power, we conclude that our assumption that zz contains a 7/3-power is false. Recalling that $zz = \mu(ay)$ and that ay is necessarily a square, we conclude that ay is a 7/3-power-free square, contradicting the minimality of xx. We conclude that there exists no 7/3-power-free square xx that is not a conjugate of a word in $\mathcal{A}$. □

We claim that the constant 7/3 in Theorem 2.2 is best possible. To see this, note that the word

$$0110100110110010110011010011011001011 0$$

is a $(7/3)^+$-power-free square, but is not a conjugate of a word in $\mathcal{A}$.

5. Conclusion

We have only considered words over a binary alphabet. It remains to consider whether similar results hold over a larger alphabet. For instance, does there exist an infinite overlap-free ternary word that contains squares beginning at every position? Richomme observes that over any alphabet there cannot exist an infinite overlap-free word containing infinitely many squares at every position. He points out that this follows from the following result of Ilie ([8], Lem. 2): in any word, if vv and uu are two squares at position i and ww is a square at position $i+1$, then either $|w| = |u|$ or $|w| = |v|$ or $|w| \geq 2|v|$. An easy consequence of this result is that in any infinite word, if infinitely many distinct squares begin at position i and ww is a square beginning at position $i+1$, then $|w| = |u|$ for some square uu occurring at position i, and hence there is an overlap at position i.

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