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Citation: Journal of Mathematical Physics **50**, 122102 (2009); doi: 10.1063/1.3272003
View online: http://dx.doi.org/10.1063/1.3272003
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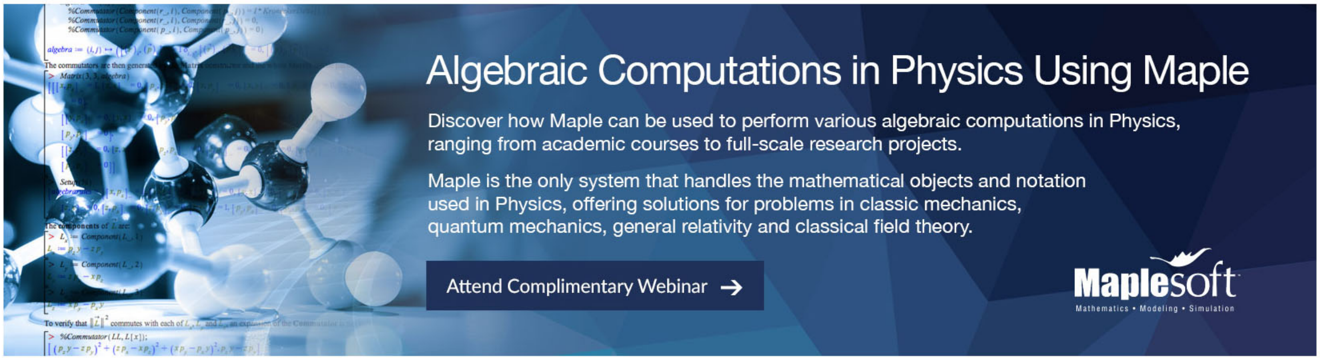

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JOURNAL OF MATHEMATICAL PHYSICS **50**, 122102 (2009)

Supersymmetry as a method of obtaining new superintegrable systems with higher order integrals of motion

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(Received 10 August 2009; accepted 12 November 2009; published online 18 December 2009)

The main result of this article is that we show that from supersymmetry we can generate new superintegrable Hamiltonians. We consider a particular case with a third order integral and apply Mielnik's construction in supersymmetric quantum mechanics. We obtain a new superintegrable potential separable in Cartesian coordinates with a quadratic and quintic integrals and also one with a quadratic integral and an integral of order of 7. We also construct a superintegrable system written in terms of the fourth Painlevé transcendent with a quadratic integral and an integral of order of 7. © *2009 American Institute of Physics.* [doi:10.1063/1.3272003]

I. INTRODUCTION

Superintegrability[1–14] and supersymmetric quantum mechanics (SUSYQMs)[15–21] have attracted a lot of attention in recent years. Both of these fields have important applications in quantum chemistry, atomic physics, molecular physics, nuclear physics, and condensed matter physics. Although they are two separate issues, many quantum systems such as the harmonic oscillator, the hydrogen atom, and the Smorodinsky–Winternitz potential are both superintegrable and supersymmetric.[21] Superintegrability with third order integrals was the object of a series of articles.[22–26] The systems studied have a second and a third order integrals. They were studied by means of cubic and deformed oscillator algebras. The supersymmetric quantum mechanics approach was used[25] and also higher order supersymmetric quantum mechanics[26] in order to calculate energies and wave functions. These articles indicate that superintegrability is closely connected to supersymmetry. We will show in this article that supersymmetry can provide a method of generating new superintegrable systems. We will consider two-dimensional systems separable in Cartesian coordinates. The separability implies the existence of a second order integral of motion.

Let us recall some definitions concerning superintegrability and supersymmetry. In classical mechanics a Hamiltonian system with Hamiltonian H and integrals of motion X_a,

$$H = \tfrac{1}{2} g_{ik} p_i p_k + V(\vec{x},\vec{p}), \quad X_a = f_a(\vec{x},\vec{p}), \quad a = 1, \ldots, n-1, \tag{1.1}$$

is called completely integrable (or Liouville integrable) if it allows n integrals of motion (including the Hamiltonian) that are well defined functions on phase space, are in involution $\{H,X_a\}_p = 0$, $\{X_a,X_b\}_p = 0$, $a,b = 1,\ldots,n-1$, and are functionally independent ($\{,\}_p$ is a Poisson bracket). A system is superintegrable if it is integrable and allows further integrals of motion $Y_b(\vec{x},\vec{p})$, $\{H,Y_b\}_p = 0$, $b = n, n+1, \ldots, n+k$ that are also well defined functions on phase space and the integrals $\{H, X_1, \ldots, X_{n-1}, Y_n, \ldots, Y_{n+k}\}$ are functionally independent. A system is maximally superintegrable if the set contains $2n-1$ such integrals. The integrals Y_b are not required to be in

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evolution with $X_1, \dots, X_{n-1}$ nor with each other. The same definitions apply in quantum mechanics but $\{H, X_a, Y_b\}$ are well defined quantum mechanical operators, assumed to form an algebraically independent set.

In Sec. II, we recall definitions and results of supersymmetric quantum mechanics. We also discuss some results obtained by Mielnik.[27] Mielnik showed that the factorization of second order operators is not necessarily unique. Supersymmetric quantum mechanics allows to find the eigenfunctions, the energy spectrum, and creation and annihilation operators. In Sec. III, we will consider a two-dimensional Hamiltonian consisting of two one-dimensional Hamiltonians that are superpartners. Such systems are by construction separable in Cartesian coordinates so a second order integral exists. From the creation and annihilation operators of the one-dimensional part we can generate a higher order integral of motion. The system is thus superintegrable. We show how these results allow us to recover known superintegrable systems with a third order integral that are special cases of a Hamiltonian written in terms of the fourth Painlevé transcendent. In Sec. IV, we consider a particular case with a third order integral, apply the Mielnik's method, and obtain a new superintegrable potential separable in Cartesian coordinates with a quadratic and quintic integrals and also one with a quadratic and seventh order integrals. We also construct a superintegrable system written in terms of the fourth Painlevé transcendent with a quadratic and seventh order integrals.

II. SUPERSYMMETRY AND MIELNIK'S FACTORIZATION METHOD

We begin this section by recalling definitions and results of supersymmetric quantum mechanics. We define two first order operators,

$$A = \frac{\hbar}{\sqrt{2}}\frac{d}{dx} + W(x), \quad A^\dagger = -\frac{\hbar}{\sqrt{2}}\frac{d}{dx} + W(x). \tag{2.1}$$

We consider the following two Hamiltonians which are called "superpartners,"

$$H_1 = A^\dagger A = -\frac{\hbar^2}{2}\frac{d^2}{dx^2} + W^2 - \frac{\hbar}{\sqrt{2}}W', \quad H_2 = AA^\dagger = -\frac{\hbar^2}{2}\frac{d^2}{dx^2} + W^2 + \frac{\hbar}{\sqrt{2}}W'. \tag{2.2}$$

There are two cases. The first is $A\psi_0^{(1)} \neq 0$, $E_0^{(1)} \neq 0$, $A^\dagger\psi_0^{(2)} \neq 0$, and $E_0^{(2)} \neq 0$. We have

$$E_n^{(2)} = E_n^{(1)} > 0, \quad \psi_n^{(2)} = \frac{1}{\sqrt{E_n^{(1)}}}A\psi_n^{(1)}, \quad \psi_n^{(1)} = \frac{1}{\sqrt{E_n^{(2)}}}A^\dagger\psi_n^{(2)}, \tag{2.3}$$

and the two Hamiltonians are isospectral. This case corresponds to broken supersymmetry.

For the second case the supersymmetry is unbroken and we have $A\psi_0^{(1)}=0$, $E_0^{(1)}=0$, $A^\dagger\psi_0^{(2)} \neq 0$, and $E_0^{(2)} \neq 0$. Without lost of generality we take H_1 as having a zero energy ground state. We have

$$E_n^{(2)} = E_{n+1}^{(1)}, \quad E_0^{(1)} = 0, \quad \psi_n^{(2)} = \frac{1}{\sqrt{E_{n+1}^{(1)}}}A\psi_{n+1}^{(1)}, \quad \psi_{n+1}^{(1)} = \frac{1}{\sqrt{E_n^{(2)}}}A^\dagger\psi_n^{(2)}. \tag{2.4}$$

We can define the matrices

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \quad Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \quad Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}. \tag{2.5}$$

They satisfy the relations

$$[H,Q] = [H,Q^\dagger] = 0, \quad \{Q,Q\} = \{Q^\dagger,Q^\dagger\} = 0, \quad \{Q,Q^\dagger\} = H. \tag{2.6}$$

The operators Q, $Q^\dagger$ are called "supercharges." We have a sl(1|1) superalgebra and H_1 and H_2 are superpartners. Supersymmetric quantum mechanics allow us to obtain the creation and annihilation operators. The operators $M^\dagger$ and M with $b^\dagger$ and b, respectively, the creation and annihilation operators for the Hamiltonian H_1,

$$M = A^\dagger b A, \quad M^\dagger = A^\dagger b^\dagger A, \tag{2.7}$$

are thus the creation and annihilation operators for the Hamiltonian H_2.

Supersymmetric quantum mechanics with higher order supercharges has been studied.[28–32] The case with second order operators of the form

$$M^\dagger = \partial^2 - 2h(x)\,\partial + c(x), \quad M = \partial^2 + 2h(x)\,\partial + c(x) \tag{2.8}$$

was investigated. The case with a first and second order supersymmetry was also treated.

As far as we could find, the generalized ladder operators appeared first Ref. 33, but we shall follow a somewhat different approach. We present the further results in supersymmetric quantum mechanics by recalling results obtained by Mielnik[27] concerning the search of superpartners for the harmonic oscillator. He pointed out that the factorization is not unique. He presented a new derivation of an important class of potentials previously obtained by Abraham and Moses with the Gelfand–Levitan formalism.[34] Their energy and eigenfunctions can be directly obtained from the harmonic oscillator up to a zero mode state. In Sec. III, we will show how this family of Hamiltonians is related to superintegrable systems with third order integrals.

We consider the following Hamiltonian:

$$H_{\mathrm{osc}} = \frac{1}{2}\frac{d^2}{dx^2} + \frac{x^2}{2}. \tag{2.9}$$

We introduce the following first order operators:

$$a = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right), \quad a^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right). \tag{2.10}$$

The Hamiltonians H_1 and H_2 are superpartners and have, in fact, the shape invariance properties,

$$a^\dagger a = H_{\mathrm{osc}} - \tfrac{1}{2} = H_1, \quad a a^\dagger = H_{\mathrm{osc}} + \tfrac{1}{2} = H_2. \tag{2.11}$$

This construction allows us to find the energy spectrum and the eigenfunction algebraically. Mielnik[27] considered the Hamiltonian H_2 and showed that the operators a and $a^\dagger$ are not unique. He defined the following new operators:

$$b = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + \beta(x)\right), \quad b^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + \beta(x)\right), \tag{2.12}$$

and required

$$H_2 = H_{\mathrm{osc}} + \tfrac{1}{2} = b b^\dagger. \tag{2.13}$$

He obtained the following Riccati equation:[35]

$$\beta'(x) + \beta^2(x) = 1 + x^2. \tag{2.14}$$

The fact of knowing a particular solution ($\beta(x)=x$) allows to find the general solution.[35] He defined

$$\beta(x) = x + \phi(x) \tag{2.15}$$

and found

$$\phi(x) = \frac{e^{-x^2}}{\gamma + \int_0^x e^{-x'^2} dx'}, \tag{2.16}$$

where γ is a constant. There are two cases: with a singularity and without singularity. The inverted product $b^\dagger b$ was not H_2+const and was a new Hamiltonian,

$$H' = b^\dagger b = H_2 - \phi'(x) = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{x^2}{2} - \frac{d}{dx}\left(\frac{e^{-x^2}}{\gamma + \int_0^x e^{-x'^2} dx'}\right). \tag{2.17}$$

We can obtain from H_2 the creation and annihilation operators for H'. These operators are given by the following expression:

$$s^\dagger = b^\dagger a^\dagger b, \quad s = b^\dagger a b, \tag{2.18}$$

with a and $a^\dagger$ the annihilation and creation operators for H_2. The eigenfunctions and energy spectrum of the Hamiltonian H' can be obtained from Eq. (2.4). The coherent states have also been studied extensively.[36] This system is a special case of a one-dimensional part of a Hamiltonian separable in Cartesian coordinates written in terms of the fourth Painlevé transcendent.

III. HIGHER ORDER INTEGRALS OF MOTION AND SUSYQM

Let us consider a two-dimensional Hamiltonian separable in Cartesian coordinates $H_t(x,y,P_x,P_y)=H_x(x,P_x)+H_y(y,P_y)$ with creation and annihilation operators (polynomial in momenta) A_x, $A_x^\dagger$, A_y, and $A_y^\dagger$. These operators satisfy

$$[H_x, A_x^\dagger] = \lambda_x A_x^\dagger, \quad [H_y, A_y^\dagger] = \lambda_y A_y^\dagger. \tag{3.1}$$

The following operators,

$$f_1 = A_x^{\dagger m} A_y^n, \quad f_2 = A_x^m A_y^{\dagger n}, \tag{3.2}$$

commute with the Hamiltonian H,

$$[H_t, f_1] = [H_t, f_2] = 0 \tag{3.3}$$

if

$$m\lambda_x - n\lambda_y = 0, \quad m,n \in \mathbb{Z}^+. \tag{3.4}$$

Creation and annihilation operators allow us to construct polynomial integrals of motion.

The following sums are also polynomial integrals that commute with the Hamiltonian H:

$$I_1 = A_x^{\dagger m} A_y^n - A_x^m A_y^{\dagger n}, \quad I_2 = A_x^{\dagger m} A_y^n + A_x^m A_y^{\dagger n}. \tag{3.5}$$

There are the integrals I_1 and I_2. The system H_t is thus superintegrable. By construction, the Hamiltonian H_t possesses a second order integral ($K=H_x-H_y$). The integral I_2 is the commutator of I_1 and K. The Hamiltonian H_t is thus superintegrable. We will show how supersymmetry makes it possible to construct superintegrable systems from one-dimensional Hamiltonian H_x with creation and annihilation operators $A_x^\dagger$ and A_x. We choose in the y-axis a superpartner (or a family of superpartners). This Hamiltonian H_y possess creation and annihilation operators that can be obtain from Eq. (2.7). A direct consequence of supersymmetry is the relation $\lambda_x=\lambda_y$. We have thus the following integrals:

$$K = H_x - H_y, \quad I_1 = A_x^\dagger A_y - A_x A_y^\dagger, \quad I_2 = A_x^\dagger A_y + A_x A_y^\dagger. \tag{3.6}$$

Let us apply this construction to the interesting systems found by Mielnik. We take in the x axis the Hamiltonian H_2 given by Eq. (2.9) and in the y axis its superpartner H' given by Eq. (2.17). We obtain a superintegrable system with integrals given by Eq. (3.6) with Eqs. (2.10) and (2.18),

$$K=H_x-H_y, \quad I_1=a_x^\dagger s_y-a_x s_y^\dagger, \quad I_2=a_x s_y^\dagger+a_x^\dagger s_y, \tag{3.7}$$

where $a_x^\dagger$, a_x, $s_y^\dagger$, and s_y are, respectively, the creation and annihilation operators of H_2 and H'.

These integrals are of orders 2, 3, and 4. This superintegrable system appears in the investigation of superintegrable systems with a second and a third order integrals separable in Cartesian coordinates. This is a particular case of a Hamiltonian written in terms of the fourth Painlevé transcendent found by Gravel[23] and studied in Ref. 26.

IV. CONSTRUCTION OF NEW SUPERINTEGRABLE SYSTEMS

A. Hamiltonians involving the error function

We consider the following superintegrable systems obtained in Ref. 23 and studied in Ref. 25 from the point of view of cubic algebras and SUSYQM,

$$H_g=-\frac{\hbar^2}{2}\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)+\hbar^2\left[\frac{x^2+y^2}{8a^4}+\frac{1}{(x-a)^2}+\frac{1}{(x+a)^2}\right]. \tag{4.1}$$

We consider the case $a=ia_0$, $a_0\in\mathbb{R}$. Let us define the two operators,

$$c^\dagger=\frac{1}{\sqrt{2}}\left(-\hbar\frac{d}{dx}+\frac{\hbar}{2a_0^2}x+\hbar\left(\frac{1}{x-ia_0}+\frac{1}{x+ia_0}\right)\right), \tag{4.2}$$

$$c=\frac{1}{\sqrt{2}}\left(\hbar\frac{d}{dx}+\frac{\hbar}{2a_0^2}x+\hbar\left(\frac{1}{x-ia_0}+\frac{1}{x+ia_0}\right)\right). \tag{4.3}$$

We have

$$H_{s1}=b^\dagger b=\frac{P_x^2}{2}+\frac{\hbar^2x^2}{8a_0^4}+\frac{\hbar^2}{(x-ia_0)^2}+\frac{\hbar^2}{(x+ia_0)^2}+\frac{3\hbar^2}{4a_0^2}, \tag{4.4}$$

$$H_{s2}=bb^\dagger=\frac{P_x^2}{2}+\frac{\hbar^2x^2}{8a_0^4}+\frac{5\hbar^2}{4a_0^2}. \tag{4.5}$$

The Hamiltonian H_g is the sum up to a constant of H_{s1} and H_{s2}. We apply Mielnik's procedure to the Hamiltonian H_{s1} to find all the superpartners. We define the following operator:

$$d=\frac{\hbar}{2}\left(\frac{d}{dx}+\beta(x)\right), \quad d^\dagger=\frac{\hbar}{2}\left(-\frac{d}{dx}+\beta(x)\right), \tag{4.6}$$

and demand $H_{s1}=d^\dagger d$. We obtain the following Riccati equation:

$$\beta'(x)+\beta^2(x)=\frac{\hbar^2x^2}{8a_0^4}+\frac{\hbar^2}{(x-ia_0)^2}+\frac{\hbar^2}{(x+ia_0)^2}+\frac{3\hbar^2}{4a_0^2}. \tag{4.7}$$

We know a particular solution,

$$\beta_0=\frac{1}{2a_0^2}x+\left(\frac{1}{x-ia_0}+\frac{1}{x+ia_0}\right). \tag{4.8}$$

Because we know a particular solution we can found the general solution. We consider

$$\beta=\beta_0(x)+\phi(x) \tag{4.9}$$

and obtain the following equation:

$$\phi'(x) + \phi^2(x) + 2\beta_0(x)\phi(x) = 0. \tag{4.10}$$

We consider the transformation $z(x)=1/\phi(x)$ and obtain a first order linear inhomogeneous equation,

$$-z'(x) + 2\beta_0(x)z(x) + 1 = 0. \tag{4.11}$$

We obtain

$$z(x) = e^{x^2/2a_0^2}(a_0^2+x^2)^2\gamma + \frac{1}{4a_0^3}(a_0^2+x^2)\left(2a_0x + e^{x^2/2a_0^2}\sqrt{2\pi}(a_0^2+x^2)\mathrm{Erf}\left(\frac{x}{\sqrt{2}a_0}\right)\right), \tag{4.12}$$

$$\beta(x) = \frac{1}{2a_0^2}x + \left(\frac{1}{x-ia_0} + \frac{1}{x+ia_0}\right) + \frac{1}{e^{x^2/2a_0^2}(a_0^2+x^2)^2\gamma + \frac{1}{4a_0^3}(a_0^2+x^2)\left(2a_0x + e^{x^2/2a_0^2}\sqrt{2\pi}(a_0^2+x^2)\mathrm{Erf}\left(\frac{x}{\sqrt{2}a_0}\right)\right)}. \tag{4.13}$$

Using the function $z(x)$ given by Eq. (4.12) the family of superpartner is thus given by

$$H_\gamma = H_{s1} - \phi'(x) = \frac{P_x^2}{2} + \frac{\hbar^2x^2}{8a_0^4} + \frac{\hbar^2}{(x-ia_0)^2} + \frac{\hbar^2}{(x+ia_0)^2} + \frac{3\hbar^2}{4a_0^2} - \frac{d}{dx}\left[\frac{1}{e^{x^2/2a_0^2}(a_0^2+x^2)^2\gamma + \frac{1}{4a_0^3}(a_0^2+x^2)\left(2a_0x + e^{x^2/2a_0^2}\sqrt{2\pi}(a_0^2+x^2)\mathrm{Erf}\left(\frac{x}{\sqrt{2}a_0}\right)\right)}\right]. \tag{4.14}$$

The eigenfunctions and energy spectrum of Hamiltonian H_{s_1} have been obtained in Ref. 25 from supersymmetry. The eigenfunctions and energy spectrum of H_γ can be obtained directly from H_{s_1} and Eq. (2.4). We can also obtain the creation and annihilation operators from those of H_{s1}. If we take $H_x=H_{s1}$ and $H_y=H_\gamma$ (the Hamiltonian H_γ is thus now given in term of the variable y), we obtain a new superintegrable Hamiltonian,

$$H_e = H_x + H_y = \frac{P_x^2}{2} + \frac{P_y^2}{2} + \frac{\hbar^2y^2}{8a_0^4} + \frac{\hbar^2}{(y-ia_0)^2} + \frac{\hbar^2}{(y+ia_0)^2} + \frac{3\hbar^2}{4a_0^2} - \frac{d}{dy}\left[\frac{1}{e^{y^2/2a_0^2}(a_0^2+y^2)^2\gamma + \frac{1}{4a_0^3}(a_0^2+y^2)\left(2a_0y + e^{y^2/2a_0^2}\sqrt{2\pi}(a_0^2+y^2)\mathrm{Erf}\left(\frac{y}{\sqrt{2}a_0}\right)\right)}\right] + \frac{\hbar^2x^2}{8a_0^4} + \frac{\hbar^2}{(x-ia_0)^2} + \frac{\hbar^2}{(x+ia_0)^2} + \frac{3\hbar^2}{4a_0^2}. \tag{4.15}$$

The creation and annihilation operators for the Hamiltonian H_{s1} are

$$m_x^\dagger = c_x^\dagger a_x^\dagger c_x, \quad m_x = c_x^\dagger a_x c_x, \tag{4.16}$$

with

$$a_x = \frac{\hbar}{2a_0^2}\left(x + 2a_0^2\frac{d}{dx}\right), \quad a_x^\dagger = \frac{\hbar}{2a_0^2}\left(x - 2a_0^2\frac{d}{dx}\right). \tag{4.17}$$

The creation and annihilation operators of the Hamiltonian H_γ are

$$r_y^\dagger = d_y^\dagger m_y^\dagger d_y, \quad r_y = d_y^\dagger m_y d_y. \tag{4.18}$$

We can find from Eq. (3.6) the integrals of motion of the Hamiltonian H_e of the orders 2, 7, and 8,

$$K = H_x - H_y, \quad I_1 = m_x^\dagger r_y - m_x r_y^\dagger, \quad I_2 = m_x^\dagger r_y + m_x r_y^\dagger. \tag{4.19}$$

The integral I_2 is given by the commutator of the integrals K and I_1.

Because the harmonic oscillator is also isospectral to H_γ we have also the following superintegrable systems where we take $H_x{=}H_{s2}$ and $H_y{=}H_\gamma$:

$$\begin{aligned} H_f = H_x + H_y = {} & \frac{P_x^2}{2} + \frac{P_y^2}{2} + \frac{\hbar^2 x^2}{8a_0^4} + \frac{\hbar^2 y^2}{8a_0^4} + \frac{\hbar^2}{(y - ia_0)^2} + \frac{\hbar^2}{(y + ia_0)^2} + \frac{9\hbar^2}{4a_0^2} \\ & - \frac{d}{dy}\left[\frac{1}{e^{y^2/2a_0^2}(a_0^2 + y^2)^2\gamma + \frac{1}{4a_0^3}(a_0^2 + y^2)\left(2a_0 y + e^{y^2/2a_0^2}\sqrt{2\pi}(a_0^2 + y^2)\mathrm{Erf}\left(\frac{y}{\sqrt{2}a_0}\right)\right)}\right]. \end{aligned} \tag{4.20}$$

We have from Eq. (3.6) the following integrals of orders 2, 5, and 6:

$$K = H_x - H_y, \quad I_1 = a_x^\dagger r_y - a_x r_y^\dagger, \quad I_2 = a_x^\dagger r_y + a_x r_y^\dagger. \tag{4.21}$$

The integral I_2 is given by the commutator of the integrals K and I_1.

B. Hamiltonians with fourth Painlevé transcendent

The following superintegrable system written in terms of the fourth Painlevé transcendent can also be related to supersymmetric quantum mechanics:[26]

$$H_{p1} = \frac{P_x^2}{2} + \frac{P_y^2}{2} + g_1(x) + g_2(y), \tag{4.22}$$

$$g_1(x) = \frac{\omega^2}{2}x^2 + \epsilon\frac{\hbar\omega}{2}f'\left(\sqrt{\frac{\omega}{\hbar}}x\right) + \frac{\omega\hbar}{2}f^2\left(\sqrt{\frac{\omega}{\hbar}}x\right) + \omega\sqrt{\hbar\omega}xf\left(\sqrt{\frac{\omega}{\hbar}}x\right) + \frac{\hbar\omega}{3}(-\alpha + \epsilon), \tag{4.23}$$

$$g_2(y) = \frac{\omega^2}{2}y^2. \tag{4.24}$$

This Hamiltonian has a second and third order integrals.

The function f is the fourth Painlevé transcendent and $f' = df/dz$, $z = \sqrt{\frac{\omega}{\hbar}}x$,

$$f''(z) = \frac{f'^2(z)}{2f(z)} + \frac{3}{2}f^3(z) + 4zf^2(z) + 2(z^2 - \alpha)f(z) + \frac{\beta}{f(z)}, \tag{4.25}$$

$$f(z) = P_4(z, \alpha, \beta). \tag{4.26}$$

We will show that we can find new superintegrable systems from the superpartners of a one-dimensional Hamiltonian with potential g_1 given by Eq. (4.23). This system was discussed in Refs.

26 and 30 and has a first and second order supersymmetry that allow to get the eigenfunctions and the energy spectrum. This system can have three, two, or one infinite sequence of levels depending on parameters α and β. When a potential possesses only one infinite sequence of energies, this potential may also allow singlet or doublet states.

Let us consider

$$H_i = P_x^2 + V_i(x), \quad i = 1,2, \tag{4.27}$$

with a supersymmetry of orders 1 and 2 with the following operators:

$$q^\dagger = \frac{\hbar}{\sqrt{2}}(\partial + W(x)), \quad q = -\frac{\hbar}{\sqrt{2}}(\partial + W(x)), \tag{4.28}$$

$$M^\dagger = \partial^2 - 2h(x)\,\partial + b(x), \quad M = \partial^2 + 2h(x)\,\partial + b(x). \tag{4.29}$$

From first order supersymmetry we have

$$V_1 = W'(x) + W^2(x), \quad V_2 = -W'(x) + W^2(x) - \frac{2\omega}{\hbar} \tag{4.30}$$

(another relations can be obtained from the supersymmetry of second order). The compatibility condition leads to

$$W(x) = -h(x) - \sqrt{\frac{\omega}{\hbar}}x. \tag{4.31}$$

The potentials V_1 and V_2 are obtained from (4.23) putting, respectively, $\epsilon=-1$ and $\epsilon=1$ and adding $\hbar\omega(\alpha/3-\epsilon/3-1)$ [with $h(x)=\sqrt{\frac{\omega}{\hbar}}f\left(\sqrt{\frac{\omega}{\hbar}}x\right)$]. We can apply the method to the Hamiltonian $H_1(x)$ and find new operators $k^\dagger$ and k that factorize H_1,

$$k = \frac{\hbar}{2}\left(\frac{d}{dx} + \beta(x)\right), \quad k^\dagger = \frac{\hbar}{2}\left(-\frac{d}{dx} + \beta(x)\right). \tag{4.32}$$

This leads to a Riccati equation that we can solve because we know the particular solution $W(x)$, and we find

$$z(x) = \frac{1}{\phi(x)} = e^{\int^x 2W(x')dx'}\left(\gamma + \int^x e^{\int^{x'} 2W(x'')dx''}dx'\right), \tag{4.33}$$

$$\beta(x) = W(x) + \frac{1}{z(x)}. \tag{4.34}$$

We obtain

$$H_{\mathrm{SUSY}} = \frac{P_x^2}{2} - \frac{d}{dx}(\phi(x)) + \frac{\omega^2}{2}x^2 - \frac{\hbar\omega}{2}f'\left(\sqrt{\frac{\omega}{\hbar}}x\right) + \frac{\omega\hbar}{2}f^2\left(\sqrt{\frac{\omega}{\hbar}}x\right) + \omega\sqrt{\hbar\omega}xf\left(\sqrt{\frac{\omega}{\hbar}}x\right) - \hbar\omega. \tag{4.35}$$

The eigenfunctions and the corresponding energy spectrum of H_1, H_2, and thus H_{p_1} were discussed in Refs. 26 and 30. Thus we can obtain directly with Eq. (2.4) eigenfunctions and corresponding energy spectrum of Hamiltonian H_{SUSY} given by Eq. (4.35). We also know the creation and annihilation operators of the Hamiltonian H_1 (and H_2) and we can obtain from them the creation and annihilation operators for H_{SUSY} by the supersymmetry. From these operators, we can form two integrals of motion and we have from the separation of variables in Cartesian coordinates an integral of order of 2. This system is superintegrable.

The creation and annihilation operators of H_1 are given by the following third order operators:

$$a^{\dagger}=q^{\dagger}M^{\dagger},\quad a=M^{\dagger}q, \tag{4.36}$$

$$M^{\dagger}=\left(\frac{d}{dx}+W_1\right)\left(\frac{d}{dx}+W_2\right),\quad M=\left(-\frac{d}{dx}+W_1\right)\left(-\frac{d}{dx}+W_2\right), \tag{4.37}$$

with

$$W_{1,2}=-\frac{1}{2}\sqrt{\frac{\omega}{\hbar}}f\left(\sqrt{\frac{\omega}{\hbar}}x\right)\pm\left(\frac{\frac{1}{2}\sqrt{\frac{\omega}{\hbar}}f'\left(\sqrt{\frac{\omega}{\hbar}}x\right)-\sqrt{-\beta}\frac{\omega}{\sqrt{2}\hbar}}{\frac{1}{2}\sqrt{\frac{\omega}{\hbar}}f\left(\sqrt{\frac{\omega}{\hbar}}x\right)}\right). \tag{4.38}$$

The creation and annihilation operators of H_{SUSY} are given by

$$v^{\dagger}=k^{\dagger}a^{\dagger}k,\quad v=k^{\dagger}ak. \tag{4.39}$$

The operators $v^{\dagger}$ and v are quintic operators. If we take $H_x(x)=H_1$ and $H_y(y)=H_{\mathrm{SUSY}}$, we obtain the following Hamiltonian:

$$\begin{aligned}H_{ss}=&\frac{P_x^2}{2}+\frac{\omega^2}{2}x^2-\frac{\hbar\omega}{2}f'\left(\sqrt{\frac{\omega}{\hbar}}x\right)+\frac{\omega\hbar}{2}f^2\left(\sqrt{\frac{\omega}{\hbar}}x\right)+\omega\sqrt{\hbar\omega}xf\left(\sqrt{\frac{\omega}{\hbar}}x\right)-\hbar\omega-\frac{d}{dy}(\phi(y))+\frac{\omega^2}{2}y^2\\&-\frac{\hbar\omega}{2}f'\left(\sqrt{\frac{\omega}{\hbar}}y\right)+\frac{\omega\hbar}{2}f^2\left(\sqrt{\frac{\omega}{\hbar}}y\right)+\omega\sqrt{\hbar\omega}yf\left(\sqrt{\frac{\omega}{\hbar}}y\right)-\hbar\omega,\end{aligned} \tag{4.40}$$

with the integrals of motion

$$K=H_x-H_y,\quad I_1=a_x^{\dagger}v_y-a_xv_y^{\dagger},\quad I_2=a_x^{\dagger}v_y+a_xv_y^{\dagger}. \tag{4.41}$$

The integral I_2 is given by the commutator of the integrals K and I_1. The integral K is of order of 2, I_1 is of order of 7, and I_2 of order of 8.

V. CONCLUSION

In this article, we showed how supersymmetric quantum mechanics gives us a method of obtaining new superintegrable systems with higher order integrals of motion. Supersymmetry in quantum mechanics makes it possible to find eigenfunctions and energy spectra from a superpartner using Eqs. (2.3) and (2.4). From a one-dimensional Hamiltonian and its superpartner we have constructed a two-dimensional superintegrable system and its integrals. The integrals are given by the Eq. (3.6).

We discussed results obtained by Mielnik[27] in context of SUSYQM. We showed how we can generate a superintegrable system from the Hamiltonian he obtained and recover a particular case of a system with a third order integral from Ref. 23 and studied in Ref. 26.

From the method, we have explicitly constructed superintegrable systems written in terms of the error function and the fourth Painlevé transcendent. These systems have higher integrals of motion. They possess, respectively, a second and a quintic integrals and a second and seventh order one. The supersymmetry allows also to find the wave functions and the energy spectrum.

This method of generating superintegrable systems can be applied to other systems obtained in the context of supersymmetric quantum mechanics. The results can be generalized in higher dimensions.

ACKNOWLEDGMENTS

The research of I.M. was supported by a postdoctoral research fellowship from FQRNT of Quebec. The author thanks P. Winternitz for very helpful comments and discussions.

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