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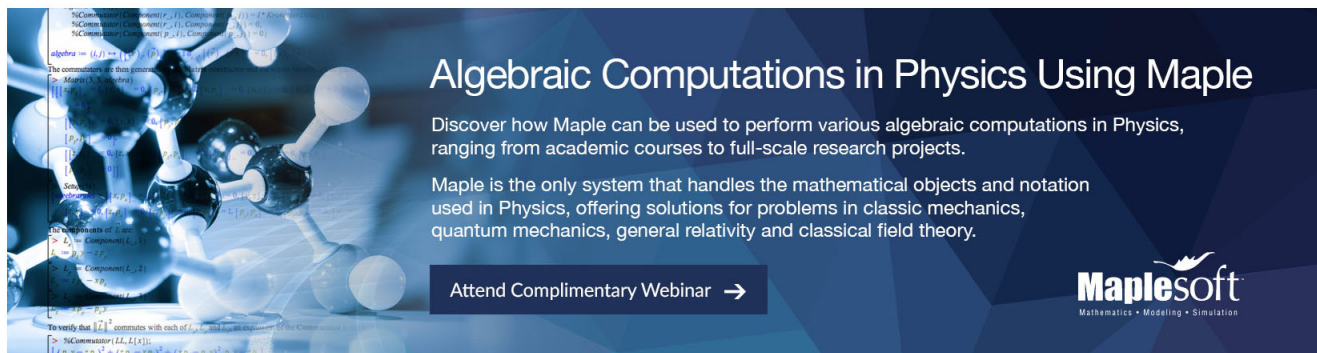Supersymmetric quantum mechanics: Engineered hierarchies of integrable potentials and related orthogonal polynomials
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Superintegrability with third order integrals of motion, cubic algebras, and supersymmetric quantum mechanics. II. Painlevé transcendent potentials

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We consider a superintegrable quantum potential in two-dimensional Euclidean space with a second and a third order integral of motion. The potential is written in terms of the fourth Painlevé transcendent. We construct for this system a cubic algebra of integrals of motion. The algebra is realized in terms of parafermionic operators and we present Fock-type representations which yield the corresponding energy spectra. We also discuss this potential from the point of view of higher order supersymmetric quantum mechanics and obtain ground state wave functions. © 2009 American Institute of Physics. [DOI: 10.1063/1.3096708]

I. INTRODUCTION

Over the years many articles have been devoted to superintegrable systems with second order integrals of motion.[1–12] Integrable and superintegrable systems with third order integrals have also been studied, albeit to a lesser degree.[13–19] This article is the second in a series[18] devoted to superintegrable systems in quantum mechanics in two-dimensional Euclidean space E_2. All classical and quantum potentials with one second and one third order integral of motion that separate in Cartesian coordinates in the two-dimensional Euclidean space were found by Gravel.[16] There are 21 quantum potentials and 8 classical ones. The systems investigated are of the form

$$H = \frac{P_x^2}{2} + \frac{P_y^2}{2} + g_1(x) + g_2(y), \tag{1.1}$$

$$A = \frac{P_x^2}{2} - \frac{P_y^2}{2} + g_1(x) - g_2(y), \tag{1.2}$$

$$B = \sum_{i+j+k=3} A_{ijk}\{L_3^i, p_1^j p_2^k\} + \{l_1(x,y), p_1\} + \{l_2(x,y), p_2\}, \tag{1.3}$$

where $\{,\}$ is an anticommutator and $L_3 = xP_2 - yP_1$ is the angular momentum. The constants A_{ijk} and functions V, l_1, and l_2 are known.[16]

The quantum case contains very interesting potentials written in term of higher transcendental functions. The irreducible potentials with rational functions were studied.[18] Polynomial algebras[18–31] and the parafermionic realizations of these algebras were found. The parafermionic realizations made it possible to construct Fock-type representations and to obtain the energy spectra. We also studied these potentials from the point of view of the supersymmetric quantum mechanics.[32–41]

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Among the 21 types of superintegrable quantum potentials, five of the irreducible ones are expressed in terms of Painlevé transcendents.[42] Let us present one of the superintegrable potentials of Ref. 16 written in terms of the fourth Painlevé transcendent, $P_4(z, \alpha, \beta)$,

$$g_1(x) = \frac{\omega^2}{2}x^2 + \epsilon\frac{\hbar\omega}{2}f'\left(\sqrt{\frac{\omega}{\hbar}}x\right) + \frac{\omega\hbar}{2}f^2\left(\sqrt{\frac{\omega}{\hbar}}x\right) + \omega\sqrt{\hbar\omega}xf\left(\sqrt{\frac{\omega}{\hbar}}x\right) + \frac{\hbar\omega}{3}(-\alpha + \epsilon),$$

(1.4)

$$g_2(x) = \frac{\omega^2}{2}y^2,$$

(1.5)

where $\epsilon = \pm 1$, $f' = df/dz$, and $z = \sqrt{\omega/\hbar}x$,

$$f''(z) = \frac{f'^2(z)}{2f(z)} + \frac{3}{2}f^3(z) + 4zf^2(z) + 2(z^2 - \alpha)f(z) + \frac{\beta}{f(z)},$$

(1.6)

$$f(z) = P_4(z, \alpha, \beta).$$

(1.7)

The six Painlevé transcendent functions appear in the theory of nonlinear differential equations. The occurence of Painlevé transcendents as superintegrable potentials seems somewhat surprising. It is less so once we remember the relation between the Schrödinger equation and the Korteweg–de Vries equation.[43] Solutions of the KdV include Painlevé transcendents. Unidimensional potentials expressed in terms of Painlevé transcendents were also obtained in the context of the dressing chain method[44–46] and conditionals and higher symmetries.[47] An important aspect of the fourth Painlevé transcendent is the existence of particular solutions in terms of rational functions and classical special functions for very specific values of the two parameters α and β.[48]

All Hamiltonians of Ref. 16 are, by construction, the sum of two unidimensional Hamiltonians ($H = H^x + H^y$). All the quantum potentials with rational function were related to supersymmetric quantum mechanics (SUSYQM).[19] Higher order SUSYQM and shape invariance have been investigated.[49–55] In the case of the potential given by Eqs. (1.4) and (1.5), the Hamiltonian H^y is the well known harmonic oscillator. The Hamiltonian H^x, the corresponding Schrödinger equation, has been obtained as a special case of third order shape invariance and solved.[51]

This article is organized in the following way. In Sec. II we construct the Fock-type representations for the superintegrable potential given by Eq. (1.1) by means of realizations of cubic algebras in terms of a parafermionic algebra. In Sec. III we will recall some aspects of third order shape invariance that are related to the potential given by Eqs. (1.4) and (1.5) with $\epsilon = -1$. We will also treat the case with $\epsilon = 1$. We will relate these results to those obtained using the approach involving the cubic algebra. In Sec. IV we will consider special cases and apply results of Secs. II and III.

II. CUBIC AND PARAFERMIONIC ALGEBRAS

We consider a quantum superintegrable Hamiltonian in E2 involving the fourth Painlevé trascendent. We have two cases $\epsilon = 1$ and $\epsilon = -1$ (with $\omega > 0$),

$$H = \frac{P_x^2}{2} + \frac{P_y^2}{2} + \frac{\omega^2}{2}y^2 + g_1(x),$$

(2.1)

with $g_1(x)$ given in (1.4). This Hamiltonian has two integrals of motion. The one of the second order is given by Eqs. (1.2) and (1.4). The third order one is given by the following equation:

$$B = \frac{1}{2}\{L, P_x^2\} + \frac{1}{2}\left\{\frac{\omega^2}{2}x^2 y - 3y g_1(x), P_x\right\} - \frac{1}{w^2}\left\{\frac{\hbar^2}{4}g_{1xxx}(x) + \left(\frac{\omega^2}{2}x^2 - 3g_1(x)\right)g_{1x}(x), P_y\right\},$$

(2.2)

where $L = xP_y - yPx$.

The operators A and B generate the following cubic algebra:

$$[A,B] \equiv C, \quad [A,C] = 16\omega^2\hbar^2 B,$$

$$[B,C] = -2\hbar^2 A^3 - 6\hbar^2 H A^2 + 8\hbar^2 H^3 + \frac{\omega^2\hbar^4}{3}(4\alpha^2 - 20 - 6\beta - 8\epsilon\alpha)A - 8\omega^2\hbar^4 H$$

$$+ \frac{\hbar^5\omega^3}{27}(-8\alpha^3 - 24\alpha - 36\alpha\beta + 24\epsilon\alpha^2 + 8\epsilon + 36\epsilon\beta).$$

(2.3)

The Casimir operator can be written as a polynomial in the Hamiltonian,

$$K = -16\hbar^2 H^4 + \frac{4\hbar^4\omega^2}{3}(4\alpha^2 - 8\alpha + 4 - \alpha\beta)H^2 - \frac{4\hbar^5\omega^3}{27}(8\alpha^3 - 24\epsilon\alpha^2 + 24\alpha + 36\alpha\beta - 8\epsilon - 36\epsilon\beta)H$$

$$- \frac{4\hbar^6\omega^4}{3}(4\alpha - 8\epsilon\alpha - 8 - 6\beta).$$

(2.4)

Realizations of cubic algebras in terms of parafermionic algebras have been discussed in our previous article.[19] Our potential belongs to the Case 2 of Ref. [19]. The cubic algebra has the form

$$[A,B] = C, \quad [A,C] = \delta B, \quad [B,C] = \mu A^3 + \nu A^2 + \xi A + \zeta,$$

(2.5)

where

$$\mu = \mu_0, \quad \nu = \nu_0 + \nu_1 H, \quad \xi = \xi_0 + \xi_1 H + \xi_2 H^2,$$

$$\zeta = \zeta_0 + \zeta_1 H + \zeta_2 H^2 + \zeta_3 H^3, \quad \delta = \delta_0 + \delta_1 H.$$

(2.6)

This algebra has been realized in terms of a deformed oscillator algebra of the form

$$[N, b^\dagger] = b^\dagger, \quad [N, b] = -b, \quad b^\dagger b = \Phi(N), \quad bb^\dagger = \Phi(N+1).$$

(2.7)

The structure function $\Phi(N)$ is given by

$$\Phi(N) = \left(\frac{K}{-4\delta} - \frac{\zeta}{4\sqrt{\delta}}\right) + \left(-\frac{\xi}{4} + \frac{\zeta}{2\sqrt{\delta}} + \frac{\nu\sqrt{\delta}}{12}\right)(N+u) + \left(\frac{-\nu\sqrt{\delta}}{4} + \frac{\xi}{4} + \frac{\mu\delta}{8}\right)(N+u)^2$$

$$+ \left(\frac{\nu\sqrt{\delta}}{6} - \frac{\mu\delta}{4}\right)(N+u)^3 + \left(\frac{\mu\delta}{8}\right)(N+u)^4.$$

(2.8)

We can use Eq. (2.4) to rewrite the structure function in terms of the Hamiltonian.

A. Case $\epsilon = 1$

From the general formula we obtain for our particular case the following structure function for $\epsilon = 1$:

$$\Phi(x) = -4\omega^2\hbar^4\left(x + u - \left(\frac{E}{2\hbar\omega} + \frac{1}{2}\right)\right)\left(x + u - \left(\frac{-E}{2\hbar\omega} + \frac{5}{6} - \frac{\alpha}{3}\right)\right),$$

$$\left(x+u-\left(\frac{-E}{2\hbar\omega}+\frac{1}{6}\left(\alpha+2-3i\sqrt{\frac{\beta}{2}}\right)\right)\right)\left(x+u-\left(\frac{-E}{2\hbar\omega}+\frac{1}{6}\left(\alpha+2+3i\sqrt{\frac{\beta}{2}}\right)\right)\right). \quad (2.9)$$

To obtain unitary representations,[25,26] we should impose three constraints given by

$$\Phi(p+1,u_i,k)=0, \quad \Phi(0,u,k)=0, \quad \phi(x)>0, \quad \forall \ x>0. \quad (2.10)$$

We have to distinguish the two cases, $\beta<0$ and $\beta>0$. For $\beta<0$ we get four possible values for u with $\Phi(0,u,k)=0$

$$u_1=\frac{-E}{2\hbar\omega}+\frac{5}{6}-\frac{\alpha}{3}, \quad u_2=\frac{-E}{2\hbar\omega}+\frac{1}{6}\left(\alpha+2+3\sqrt{\frac{-\beta}{2}}\right),$$

$$u_3=\frac{-E}{2\hbar\omega}+\frac{1}{6}\left(\alpha+2-3\sqrt{\frac{-\beta}{2}}\right), \quad u_4=\frac{E}{2\hbar\omega}+\frac{1}{2}. \quad (2.11)$$

We insert all these solutions for u and apply the constraint $\Phi(p+1,u_i,k)=0$, with $i=1,2,3,4$ to find the energy spectrum.

Case 1:

$$\Phi(x)=4\hbar^4\omega^2x(p+1-x)\left(x+\frac{1}{2}-\frac{\alpha}{2}-\sqrt{\frac{-\beta}{8}}\right)\left(x+\frac{1}{2}-\frac{\alpha}{2}+\sqrt{\frac{-\beta}{8}}\right), \quad (2.12)$$

$$E=\hbar\omega\left(p+\frac{4}{3}-\frac{\alpha}{3}\right). \quad (2.13)$$

Case 2:

$$\Phi(x)=4\hbar^4\omega^2x(p+1-x)\left(x+\sqrt{\frac{-\beta}{2}}\right)\left(x-\frac{1}{2}+\frac{\alpha}{2}+\sqrt{\frac{-\beta}{8}}\right), \quad (2.14)$$

$$E=\hbar\omega\left(p+\frac{5}{6}+\frac{\alpha}{6}+\sqrt{\frac{-\beta}{8}}\right). \quad (2.15)$$

Case 3:

$$\Phi(x)=4\hbar^4\omega^2x(p+1-x)\left(x-\sqrt{\frac{-\beta}{2}}\right)\left(x-\frac{1}{2}+\frac{\alpha}{2}-\sqrt{\frac{-\beta}{8}}\right), \quad (2.16)$$

$$E=\hbar\omega\left(p+\frac{5}{6}+\frac{\alpha}{6}-\sqrt{\frac{-\beta}{8}}\right). \quad (2.17)$$

Case 4:

We get three solutions for this case with negative energy,

$$\Phi(x)=4\hbar^4\omega^2x(p+1-x)\left(\left(p+\frac{1}{2}\right)+\frac{\alpha}{2}+\sqrt{\frac{-\beta}{8}}-x\right)\left(\left(p+\frac{1}{2}\right)+\frac{\alpha}{2}-\sqrt{\frac{-\beta}{8}}-x\right), \quad (2.18)$$

$$E=-\hbar\omega\left(p+\frac{2}{3}+\frac{\alpha}{3}\right), \quad (2.19)$$

$$\Phi(x) = 4\hbar^4\omega^2 x(p+1-x)\left(\left(p+\frac{3}{2}\right)-\frac{\alpha}{2}-\sqrt{\frac{-\beta}{8}}-x\right)\left((p+1)-\sqrt{\frac{-\beta}{2}}-x\right), \quad (2.20)$$

$$E = -\hbar\omega\left(p+\frac{7}{6}-\frac{\alpha}{6}-\sqrt{\frac{-\beta}{8}}\right), \quad (2.21)$$

$$\Phi(x) = 4\hbar^4\omega^2 x(p+1-x)\left(\left(p+\frac{3}{2}\right)-\frac{\alpha}{2}+\sqrt{\frac{-\beta}{8}}-x\right)\left((p+1)+\sqrt{\frac{-\beta}{2}}-x\right), \quad (2.22)$$

$$E = -\hbar\omega\left(p+\frac{7}{6}-\frac{\alpha}{6}+\sqrt{\frac{-\beta}{8}}\right). \quad (2.23)$$

To obtain unitary representations we should also impose $\phi(x)$ to be a real function and $\phi(x)>0$ for $x>0$. The constraints do not allow all values for α and β so it may happen that only some of the states are physically meaningful. We can have one, two, or three infinite sequences of energies that correspond to each unitary representation.

For $\beta>0$ we have two solutions,

$$\Phi(x) = 4\hbar^4\omega^2 x(p+1-x)\left(x^2+(1-\alpha)x-\frac{\beta}{8}+\frac{\alpha^2}{4}-\frac{\alpha}{2}+\frac{1}{4}\right), \quad (2.24)$$

$$E = \hbar\omega\left(p+\frac{4}{3}-\frac{\alpha}{3}\right), \quad (2.25)$$

$$\Phi(x) = 4\hbar^4\omega^2 x(p+1-x)\left(x^2-(1+\alpha+2p)x+p^2+\alpha p+p-\frac{\beta}{8}+\frac{\alpha^2}{4}+\frac{\alpha}{2}+\frac{1}{4}\right), \quad (2.26)$$

$$E = -\hbar\omega\left(p+\frac{2}{3}+\frac{\alpha}{3}\right). \quad (2.27)$$

B. Case $\epsilon=-1$

For the case $\epsilon=-1$ we obtain the following expression for the structure function:

$$\Phi(x) = -4\omega^2\hbar^4\left(x+u-\left(\frac{E}{2\hbar\omega}+\frac{1}{2}\right)\right)\left(x+u-\left(\frac{-E}{2\hbar\omega}+\frac{1}{6}-\frac{\alpha}{3}\right)\right)\left(x+u-\left(\frac{-E}{2\hbar\omega}+\frac{1}{6}\left(\alpha+4\right.\right.\right.$$
$$\left.\left.\left.-3i\sqrt{\frac{\beta}{2}}\right)\right)\right)\left(x+u-\left(\frac{-E}{2\hbar\omega}+\frac{1}{6}\left(\alpha+4+3i\sqrt{\frac{\beta}{2}}\right)\right)\right). \quad (2.28)$$

Four cases occur for $\beta<0$,

$$u_1 = \frac{-E}{2\hbar\omega}+\frac{1}{6}-\frac{\alpha}{3}, \quad u_2 = \frac{-E}{2\hbar\omega}+\frac{1}{6}\left(\alpha+4+3\sqrt{\frac{-\beta}{2}}\right),$$

$$u_3 = \frac{-E}{2\hbar\omega}+\frac{1}{6}\left(\alpha+4-3\sqrt{\frac{-\beta}{2}}\right) \quad u_4 = \frac{E}{2\hbar\omega}+\frac{1}{2}. \quad (2.29)$$

Case 1:

$$\Phi(x) = 4\hbar^4\omega^2 x(p+1-x)\left(x - \frac{1}{2} - \frac{\alpha}{2} - \sqrt{\frac{-\beta}{8}}\right)\left(x - \frac{1}{2} - \frac{\alpha}{2} + \sqrt{\frac{-\beta}{8}}\right), \tag{2.30}$$

$$E = \hbar\omega\left(p + \frac{2}{3} - \frac{\alpha}{3}\right). \tag{2.31}$$

Case 2:

$$\Phi(x) = 4\hbar^4\omega^2 x(p+1-x)\left(x + \sqrt{\frac{-\beta}{2}}\right)\left(x + \frac{1}{2} + \frac{\alpha}{2} + \sqrt{\frac{-\beta}{8}}\right), \tag{2.32}$$

$$E = \hbar\omega\left(p + \frac{7}{6} + \frac{\alpha}{6} + \sqrt{\frac{-\beta}{8}}\right). \tag{2.33}$$

Case 3:

$$\Phi(x) = 4\hbar^4\omega^2 x(p+1-x)\left(x - \sqrt{\frac{-\beta}{2}}\right)\left(x + \frac{1}{2} + \frac{\alpha}{2} - \sqrt{\frac{-\beta}{8}}\right), \tag{2.34}$$

$$E = \hbar\omega\left(p + \frac{7}{6} + \frac{\alpha}{6} - \sqrt{\frac{-\beta}{8}}\right). \tag{2.35}$$

Case 4: We get three solutions for this case with negative energy,

$$\Phi(x) = 4\hbar^4\omega^2 x(p+1-x)\left(\left(p + \frac{3}{2}\right) + \frac{\alpha}{2} - \sqrt{\frac{-\beta}{8}} - x\right)\left(\left(p + \frac{3}{2}\right) + \frac{\alpha}{2} + \sqrt{\frac{-\beta}{8}} - x\right), \tag{2.36}$$

$$E = -\hbar\omega\left(p + \frac{4}{3} + \frac{\alpha}{3}\right), \tag{2.37}$$

$$\Phi(x) = 4\hbar^4\omega^2 x(p+1-x)\left(\left(p + \frac{1}{2}\right) - \frac{\alpha}{2} - \sqrt{\frac{-\beta}{8}} - x\right)\left((p+1) - \sqrt{\frac{-\beta}{2}} - x\right), \tag{2.38}$$

$$E = -\hbar\omega\left(p + \frac{5}{6} - \frac{\alpha}{6} - \sqrt{\frac{-\beta}{8}}\right), \tag{2.39}$$

$$\Phi(x) = 4\hbar^4\omega^2 x(p+1-x)\left(\left(p + \frac{1}{2}\right) - \frac{\alpha}{2} - \sqrt{\frac{-\beta}{8}} - x\right)\left((p+1) + \sqrt{\frac{-\beta}{2}} - x\right), \tag{2.40}$$

$$E = -\hbar\omega\left(p + \frac{5}{6} - \frac{\alpha}{6} + \sqrt{\frac{-\beta}{8}}\right). \tag{2.41}$$

One interesting aspect of this potential is that we can have three, two, or one series of equidistant energy levels.

For $\beta > 0$ we get the following solution:

$$\Phi(x) = 4\hbar^4\omega^2 x(p+1-x)\left(x^2 - (1+\alpha)x - \frac{\beta}{8} + \frac{\alpha^2}{4} + \frac{\alpha}{2} + \frac{1}{4}\right), \tag{2.42}$$

$$E = \hbar\omega\left(p + \frac{2}{3} - \frac{\alpha}{3}\right), \tag{2.43}$$

$$\Phi(x) = 4\hbar^4\omega^2 x(p + 1 - x)\left(x^2 - (3 + \alpha + 2p)x + p^2 + \alpha p + 3p - \frac{\beta}{8} + \frac{\alpha^2}{4} + \frac{\alpha}{2} + \frac{9}{4}\right), \tag{2.44}$$

$$E = -\hbar\omega\left(p + \frac{4}{3} + \frac{\alpha}{3}\right). \tag{2.45}$$

III. THIRD ORDER SHAPE INVARIANCE AND SUPERINTEGRABLE SYSTEMS

The concept of higher-derivative supersymmetric quantum mechanics (HSQM) was introduced by Andrianov *et al.*[49] HSQM is characterized by polynomial relations between supercharges and the Hamiltonian. Second order derivative supersymmetry was investigated in Ref. [50]. We will present in this section results not given in Ref. [51] but directly related to the potential given by Eq. (1.1) with $\epsilon = 1$. Let us recall some aspects of the particular case of third order shape invariance related to the potential with $\epsilon = -1$ obtained in Ref. [51]. In SUSYQM two superpartners are isospectral or almost isospectral, and if we know the spectrum and the eigenfunctions of one superpartner, we can obtain the spectrum and the eigenfunctions of the other superpartner. A special case occurs when the two superpartners $V_1(x, a_0)$ and $V_2(x, a_0)$ satisfy the relation $V_2(x, a_1) = V_1(x, a_0) + R(a_1)$, where $a_1 = f(a_0)$ and $R(a_1)$ do not depend on x. In this special case we can find directly the energy and the eigenfunctions. The superpartners are called shape invariant potentials. We consider the following particular case of shape invariance:

$$H_1 a^\dagger = a^\dagger(H_1 + 2\lambda), \tag{3.1}$$

where $a^\dagger$ and a are third order operators. This particular case of shape invariance can be constructed from a first order and second order supersymmetries given by the following interwining relations:

$$H_1 q^\dagger = q^\dagger(H_2 + 2\lambda), \quad H_1 M^\dagger = M^\dagger H_2, \tag{3.2}$$

where

$$H_i = P_x^2 + V_i(x), \tag{3.3}$$

$$q^\dagger = \partial + W(x), \quad q = -\partial + W(x), \tag{3.4}$$

$$M^\dagger = \partial^2 - 2h(x)\,\partial + b(x), \quad M = \partial^2 + 2h(x)\,\partial + b(x). \tag{3.5}$$

The key element in obtaining the equivalence between Eqs. (3.1) and (3.2) is to define the following third order operators a and $a^\dagger$ written as products of first order and second order supercharges:

$$a^\dagger = q^\dagger M, \quad a = M^\dagger q. \tag{3.6}$$

The third order shape invariance of the form given by Eq. (3.1) can be investigated using Eq. (3.2). The two interwining relations of Eq. (3.2) give, respectively, the following relations:

$$V_1 = W'(x) + W^2(x), \quad V_2 = -W'(x) + W^2(x) - 2\lambda, \tag{3.7}$$

and

$$V_{1,2} = \mp 2h'(x) + h^2(x) + \frac{h''(x)}{2h(x)} - \frac{h'^2(x)}{4h^2(x)} - \frac{d}{4h^2(x)} + \gamma, \tag{3.8}$$

$$b(x) = -h'(x) + h^2(x) - \frac{h''(x)}{2h(x)} + \frac{h'^2(x)}{4h^2(x)} + \frac{d}{4h^2(x)}. \tag{3.9}$$

Equation (3.7)–(3.9) impose that the potential V_1 should have the form

$$V_1 = -2h'(x) + 4h^2(x) + 4\lambda x h(x) + \lambda^2 x^2 - \lambda, \tag{3.10}$$

with

$$h''(x) = \frac{h'^2(x)}{2h(x)} + 6h^3(x) + 8\lambda x h^2(x) + 2(\lambda^2 x^2 - (\lambda + \gamma))h(x) + \frac{d}{2h(x)}, \tag{3.11}$$

$$W(x) = W_3(x) = -2h(x) - \lambda x. \tag{3.12}$$

As in the case of first order supersymmetry, we can define

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & M^\dagger \\ 0 & 0 \end{pmatrix}. \tag{3.13}$$

We get the following SUSY algebra:

$$[H,Q] = [H,Q^\dagger] = 0, \quad \{Q,Q\} = \{Q^\dagger, Q^\dagger\} = 0, \quad \{Q,Q^\dagger\} = (H - \gamma)^2 + d. \tag{3.14}$$

Equation (3.11) can be transformed into the equation for the fourth Painlevé transcendent (1.4) by the following transformations:

$$h(x) = \frac{1}{2}\sqrt{\lambda}f(z), \quad z = \sqrt{\lambda}x, \quad \alpha = 1 + \frac{\gamma}{\lambda}, \quad \beta = \frac{2d}{\lambda^2}, \quad \lambda = \frac{\omega}{\hbar}, \tag{3.15}$$

and we obtain

$$\widetilde{V}_1 = \frac{\hbar^2}{2}V_1 = \frac{\omega^2}{2}x^2 - \frac{\omega\hbar}{2}f'\left(\sqrt{\frac{\omega}{\hbar}}x\right) + \frac{\omega\hbar}{2}f^2\left(\sqrt{\frac{\omega}{\hbar}}x\right) + \omega\sqrt{\omega\hbar}xf\left(\sqrt{\frac{\omega}{\hbar}}x\right) - \omega\hbar. \tag{3.16}$$

$\widetilde{V}_1(x)$ is the x part of the potential in (1.1) and coincides with $g_1(x)$ in Eq. (1.4) up to a constant. A particular case of third order shape invariance called «reducible» was considered in Ref. 51 by imposing further conditions. These conditions are $d \leq 0$ and the existence of real functions W_1 and W_2, such that

$$M^\dagger = (\partial + W_1(x))(\partial + W_2(x)), \quad W_{1,2} = -h(x) \pm \frac{h'(x) - \sqrt{-d}}{2h(x)} \tag{3.17}$$

(reducible means that $M^\dagger$ factorizes into product of two first order operators with real functions). The spectrum was obtained for cases where normalizable zero modes of the annihilation operator exist. Zero modes of the annihilation operator satisfy

$$a\psi_k^{(0)} = 0 \tag{3.18}$$

[we use the terminology of HSQM where «zero mode» refers to Eq. (3.18) so that zero modes may not have energy $E_0 = 0$]. The energies of zero modes were obtained by imposing the vanishing of the norm of $a\psi_k^{(0)}$ which involves the average of the operator product $a^\dagger a$,

$$a^\dagger a = q^\dagger M M^\dagger q = q^\dagger ((H_2 - \gamma)^2 + d) q = H_1((H_1 - \gamma - 2\lambda)^2 + d). \tag{3.19}$$

The energies of the zero modes are

$$E_1^{(0)} = 0, \quad E_2^{(0)} = \gamma + 2\lambda + \sqrt{-d}, \quad E_3^{(0)} = \gamma + 2\lambda - \sqrt{-d}. \tag{3.20}$$

The corresponding eigenfunctions $\psi_k^{(0)}$ can be calculated explicitly and are

$$\psi_1^0(x) = e^{\int^x W_3(x')dx'}, \tag{3.21}$$

$$\psi_2^0(x) = (W_2(x) - W_3(x))e^{-\int^x W_2(x')dx'}, \tag{3.22}$$

$$\psi_3^0(x) = (2\sqrt{-d} + (W_2(x) - W_3(x))(W_1(x) + W_2(x)))e^{-\int^x W_1(x')dx'}. \tag{3.23}$$

The creation operator can also have zero modes $\phi_k^{(0)}$ which correspond to a possible truncation of the sequence of excited levels. They were obtained by considering the following product:

$$aa^\dagger = (H_1 + 2\lambda)((H_1 - \gamma)^2 + d). \tag{3.24}$$

The energies of the zero modes are

$$E_1^{(0)} = \gamma - \sqrt{-d}, \quad E_2^{(0)} = \gamma + \sqrt{-d}, \quad E_3^{(0)} = -2\lambda, \tag{3.25}$$

with the corresponding eigenfunctions

$$\phi_1^0(x) = e^{\int^x W_1(x')dx'}, \tag{3.26}$$

$$\phi_2^0(x) = (W_1(x) + W_2(x))e^{\int^x W_2(x')dx'}, \tag{3.27}$$

$$\phi_3^0(x) = (\gamma + 2\lambda + \sqrt{-d} + (W_1(x) + W_2(x))(W_2(x) - W_3(x)))e^{-\int^x W_3(x')dx'}. \tag{3.28}$$

For nonsingular potentials it is not possible to have the negative energy $E_3^{(0)}$, and the total number of zero modes of the annihilation and creation operator cannot be more than three because of the asymptotics of the eigenfunctions. We can have three, two, or one infinite sequence of levels. These results coincide with those obtained as from the analysis of Fock-type representations of the cubic algebra of the superintegrable potential. When we apply the creation operator $a^\dagger$ on zero modes, we create eigenfunctions with 2λ more energy. These energies are corroborated (when we add a harmonic oscillator in the y direction) by those obtained using the cubic algebra and given by Eqs. (2.31), (2.33), and (2.35).

When a potential allows only one infinite sequence of energies, this potential may also allow a singlet state or a doublet of states,

$$a^+\psi(x) = a^-\psi(x) = 0, \quad (a^+)^2\psi(x) = a^-\psi(x) = 0. \tag{3.29}$$

From an algebraic point of view these states correspond to trivial irreducible representations. Such a case was discussed in Ref. 19 for the potential $V = \hbar^2((x^2+y^2)/8a^4 + 1/(x-a)^2 + 1/(x+a)^2)$. This potential is a special case of the potential given by Eq. (1.1). The observed singlet state can now be naturally understood as a phenomenon of third order shape invariance.

All the results we presented apply to our potential for $\epsilon = -1$. We will present here the results that will be applicable to the case $\epsilon = 1$. We follow the same approach as in Ref. 51 and we consider the following potential:

$$V_2 = 2h'(x) + 4h^2(x) + 4\lambda x h(x) + \lambda^2 x^2 - \lambda. \tag{3.30}$$

Equations (3.9), (3.11), and (3.12) remain the same. We can define as for V_1 in Eq. (3.16) a potential $\widetilde{V}_2$ using the transformations of Eq. (3.15). The Hamiltonian H_2 of the form given by Eq. (3.3) with the potential given by Eq. (3.30) satisfies

$$H_2 a^\dagger = a^\dagger (H_2 + 2\lambda) \tag{3.31}$$

when we postulate

$$a^\dagger = M q^\dagger, \quad a = q M^\dagger. \tag{3.32}$$

We have the following product:

$$a^\dagger a = H_2((H_2 - \gamma)^2 + d), \quad aa^\dagger = (H_2 + 2\lambda)((H_2 + 2\lambda - \gamma)^2 + d). \tag{3.33}$$

The energies of the zero modes of the creation and annihilation operator are obtained by imposing the vanishing of their norm. This involves the average of the operator products $a^\dagger a$ and $aa^\dagger$ given by Eq. (3.33). The eigenfunctions of the zero modes are obtained directly by solving $a\psi_k^{(0)} = 0$ and $a^\dagger \phi_k^{(0)} = 0$. The energies of zero modes of the annihilation operator are

$$E_1^{(0)} = 0, \quad E_2^{(0)} = \gamma - \sqrt{-d}, \quad E_3^{(0)} = \gamma + \sqrt{-d}, \tag{3.34}$$

with the corresponding eigenfunctions

$$\psi_1^0(x) = (\gamma - \sqrt{-d} + (W_1(x) + W_2(x))(W_1(x) - W_3(x)))e^{\int^x W_3(x')dx'}, \tag{3.35}$$

$$\psi_2^0(x) = (W_1(x) + W_2(x))e^{-\int^x W_1(x')dx'}, \tag{3.36}$$

$$\psi_3^0(x) = e^{-\int^x W_2(x')dx'}. \tag{3.37}$$

The energies of zero modes of the creation operator are

$$E_1^{(0)} = -2\lambda, \quad E_2^{(0)} = \gamma - 2\lambda - \sqrt{-d}, \quad E_3^{(0)} = \gamma - 2\lambda + \sqrt{-d}, \tag{3.38}$$

with the corresponding eigenfunctions

$$\phi_1^0(x) = e^{-\int^x W_3(x')dx'}, \tag{3.39}$$

$$\phi_2^0(x) = (W_1(x) - W_3(x))e^{\int^x W_1(x')dx'}, \tag{3.40}$$

$$\phi_3^0(x) = (-2\sqrt{-d} + (W_1(x) - W_3(x))(W_1(x) + W_2(x)))e^{\int^x W_2(x')dx'}. \tag{3.41}$$

Again the total number of zero modes of the annihilation and creation operators cannot be more than 3 because of the asymptotics of the eigenfunctions. We can have three, two, or one infinite sequence of levels. When we apply the creation operator $a^\dagger$ on zero modes we create eigenfunctions with 2λ more energy. These energies are corroborated (when we add a harmonic oscillator in the y direction) by those obtained by the cubic algebra and given by Eqs. (2.13), (2.15), and (2.17). When a potential possess only one infinite sequence of energies, this potential may also possess a singlet state or a doublet states.

We will discuss the irreducible case that appears when $d > 0$. For $V_1(x)$ we get

$$E_1^{(0)} = 0, \quad \psi_0^0(x) = e^{\int^x W_3(x')dx'}. \tag{3.42}$$

For $V_2(x)$ we get

$$E_1^{(0)} = 0, \quad \psi_0^0(x) = (\gamma - \sqrt{d} + (W_1(x) + W_2(x))(W_1(x) - W_3(x)))e^{\int^x W_3(x')dx'}, \tag{3.43}$$

and W_1 and W_2 are not real functions.

IV. SPECIAL CASES

The fourth Painlevé transcendent satisfying Eq. (1.6) depends on two parameters and special solutions in terms of rational or classical special functions exist.[48] In this section, we will give the unitary representations, the corresponding energy spectra, and the eigenfunctions for some special cases.

A. Case $\alpha=5$, $\beta=-8$, $f(z)=4z(2z^2-1)(2z^2+3)/(2z^2+1)(4z^4+3)$, and $\epsilon=1$

We have with Eqs. (1.4) and (1.5)

$$V(x,y) = \frac{\omega^2}{2}(x^2 + y^2) - \frac{8\hbar^3\omega}{(2\omega x^2 + \hbar)^2} + \frac{4\hbar^2\omega}{(2\omega x^2 + \hbar)} + \frac{2\hbar\omega}{3}. \tag{4.1}$$

From the cubic algebra we get two unitary representations. The first unitary representation is given by Eq. (2.14) with the corresponding energy given by Eq. (2.15),

$$\phi(x) = 4\hbar^4\omega^2 x(p+1-x)(x+3)(x+2), \quad E = \hbar\omega\left(p + \tfrac{8}{3}\right), \tag{4.2}$$

The second solution is given by Eq. (2.12) or Eq. (2.20) with the corresponding energy spectrum,

$$\phi(x) = 4\hbar^4\omega^2 x(p+1-x)(x-3)(x-1), \quad E = \hbar\omega\left(p - \tfrac{1}{3}\right). \tag{4.3}$$

This representation is valid only for $p=0$.

We will also treat this systems using the results on supersymmetry. The eigenfunctions for the x part consist of an infinite sequence $\psi_n(x)$ starting from $psi_3^0(x)$ of Eq. (3.37) and a singlet state $\chi(x)$ given by Eqs. (3.35) and (3.40),

$$\psi_n(x) = N_n(a^\dagger)^n e^{-\omega x^2/2\hbar} \frac{x(3\hbar + 2\omega x^2)}{(\hbar + 2\omega x^2)}, \quad \chi(x) = C_0 \frac{e^{-\omega x^2/2\hbar}}{\hbar + 2\omega x^2}, \tag{4.4}$$

$$a\chi(x) = 0, \quad a^\dagger \chi(x) = 0. \tag{4.5}$$

The creation and annihilation operators are given by Eq. (3.32) with the following expressions for W_1, W_2, and W_3:

$$W_1 = \frac{-(-\hbar + 2\omega x^2)(9\hbar^3 + 27\hbar^2\omega x^2 + 12\hbar\omega^2 x^4 + 4\omega^3 x^6)}{\hbar x(3\hbar + 2\omega x^2)(3\hbar^2 + 4\omega^2 x^4)}, \tag{4.6}$$

$$W_2 = \frac{-(\hbar - 2\omega x^2)(3\hbar^2 + 3\hbar\omega x^2 + 2\omega^2 x^4)}{\hbar x(3\hbar^2 + 8\hbar\omega x^2 + 4\omega^2 x^4)}, \tag{4.7}$$

$$W_3 = \frac{-\omega x(-9\hbar^3 + 22\hbar^2\omega x^2 + 20\hbar\omega^2 x^4 + 8\omega^3 x^6)}{\hbar(\hbar + 2\omega x^2)(3\hbar^2 + 4\omega^2 x^4)}. \tag{4.8}$$

With the eigenfunctions for the y part of the Hamiltonian and the formula for the energies given by Eq. (3.34), we obtain the two following series of solutions:

$$\psi_{n,k} = \psi_n(x)e^{-\omega y^2/2\hbar}H_k\left(\sqrt{\frac{\omega}{\hbar}}y\right), \quad E = \hbar\omega\left(n + k + \frac{8}{3}\right), \tag{4.9}$$

$$\phi_m = \chi(x)e^{-\omega y^2/2\hbar}H_m\left(\sqrt{\frac{\omega}{\hbar}}y\right), \quad E_m = \hbar\omega\left(m - \frac{1}{3}\right). \tag{4.10}$$

B. Case $\alpha=5$, $\beta=-8$, $f(z)=4z(2z^2-1)(2z^2+3)/(2z^2+1)(4z^4+3)$, and $\epsilon=-1$

We have with Eqs. (1.4) and (1.5)

$$V(x,y) = \frac{\omega^2}{2}(x^2+y^2) - \frac{192\hbar^4\omega^2x^2}{(4\omega^2x^4+3\hbar^2)^2} + \frac{16\hbar^2\omega^2x^2}{4\omega^2x^4+3\hbar^2}. \tag{4.11}$$

From the cubic algebra we obtain three unitary representations. The first unitary representation is given by Eq. (2.32) with the corresponding energy spectrum Eq. (2.33),

$$\phi(x) = 4\hbar^4\omega^2x(p+1-x)(x+4)(x+2), \quad E = \hbar\omega(p+3), \tag{4.12}$$

The second solution is given by Eq. (2.38) with the corresponding energy spectrum given by Eq. (2.39),

$$\phi(x) = 4\hbar^4\omega^2x(p+1-x)(p-3-x)(p-x), \quad E = -\hbar\omega(p-1). \tag{4.13}$$

This representation is valid only for $p=0,1$.

The third solution is given by Eq. (2.30) with the corresponding energy spectrum given by Eq. (2.31),

$$\phi(x) = 4\hbar^4\omega^2x(p+1-x)(x-3)(x-2), \quad E = \hbar\omega(p-1). \tag{4.14}$$

This representation is valid only for $p=0,1$.

We investigate this system using the results on supersymmetry. The eigenfunctions for the x part consist of an infinite sequence $\psi_n(x)$ starting from $\psi_0^2(x)$ given by Eq. (3.22) and doublet states $\chi_1(x)$ and $\chi_2(x)$ given by Eqs. (3.22), (3.21), and (3.26),

$$\psi_n(x) = N_n(a^\dagger)^n e^{-\omega x^2/2\hbar}\frac{(-9\hbar^3 + 18\hbar^2\omega x^2 + 12\hbar\omega^2x^4 + 8\omega^3x^6)}{(3\hbar^2+4\omega^2x^4)}, \tag{4.15}$$

$$\chi_1(x) = C_1 e^{-\omega x^2/2\hbar}\frac{(\hbar+2\omega x^2)}{(3\hbar^2+4\omega^2x^4)}, \quad \chi_2(x) = C_2 e^{-\omega x^2/2\hbar}\frac{x(3\hbar+2\omega x^2)}{(3\hbar^2+4\omega^2x^4)}. \tag{4.16}$$

$$a\chi_1(x) = 0, \quad a^\dagger\chi_1(x) = \chi_2(x), \quad a^\dagger\chi_2(x) = 0. \tag{4.17}$$

The creation and annihilation operators are given by Eq. (3.6) with W_1, W_2, and W_3 as in Eqs. (4.5)–(4.7).

With the eigenfunctions in the y part and the formula for the energies given by Eq. (3.20), we obtain the three following kinds of solutions:

$$\psi_{n,k} = \phi_n(x)e^{-\omega y^2/2\hbar}H_k\left(\sqrt{\frac{\omega}{\hbar}}y\right), \quad E = \hbar\omega(n+k+3), \tag{4.18}$$

$$\phi_{m_1} = \chi_1(x)e^{-\omega y^2/2\hbar}H_{m_1}\left(\sqrt{\frac{\omega}{\hbar}}y\right), \quad E_{m_1} = \hbar\omega(m_1-1), \tag{4.19}$$

$$\phi_{m_2} = \chi_2(x)e^{-\omega y^2/2\hbar}H_{m_2}\left(\sqrt{\frac{\omega}{\hbar}}y\right), \quad E_{m_2} = \hbar\omega m_2. \tag{4.20}$$

C. Case $\alpha=0$, $\beta=-\frac{2}{9}$, $f(z)=-\frac{2}{3}z$, and $\epsilon=1$

We have

$$V(x,y) = \frac{\omega^2}{2}\left(\frac{1}{9}x^2 + y^2\right). \tag{4.21}$$

From the cubic algebra we get three cases with unitary representations, and using Eqs. (2.12) and (2.17), we have

$$\phi(x) = 4\hbar^4\omega^2 x(p+1-x)\left(x+\tfrac{1}{3}\right)\left(x+\tfrac{2}{3}\right), \quad E = \hbar\omega\left(p+\tfrac{4}{3}\right), \tag{4.22}$$

$$\phi(x) = 4\hbar^4\omega^2 x(p+1-x)\left(x-\tfrac{1}{3}\right)\left(x+\tfrac{1}{3}\right), \quad E = \hbar\omega(p+1), \tag{4.23}$$

$$\phi(x) = 4\hbar^4\omega^2 x(p+1-x)\left(x-\tfrac{2}{3}\right)\left(x-\tfrac{1}{3}\right), \quad E = \hbar\omega\left(p+\tfrac{2}{3}\right). \tag{4.24}$$

We apply the results coming from supersymmetry. We get the following known eigenfunctions from Eqs. (3.35), (3.37), and (3.36), respectively, and the corresponding energy with Eq. (3.34),

$$\psi_{n_1,k_1} = N_{n_1 k_1}(a^\dagger)^{n_1} e^{-\omega x^2/6\hbar}(-3\hbar + 2\omega x^2)e^{-\omega y^2/2\hbar}H_{k_1}\left(\sqrt{\frac{\omega}{\hbar}}y\right), \tag{4.25}$$

$$E_1 = \hbar\omega\left(n_1 + k_1 + \tfrac{4}{3}\right), \tag{4.26}$$

$$\psi_{n_2,k_2} = N_{n_2 k_2}(a^\dagger)^{n_2} e^{-\omega x^2/6\hbar}x e^{-\omega y^2/2\hbar}H_{k_2}\left(\sqrt{\frac{\omega}{\hbar}}y\right), \tag{4.27}$$

$$E_2 = \hbar\omega(n_2 + k_2 + 1), \tag{4.28}$$

$$\psi_{n_3,k_3} = N_{n_3 k_3}(a^\dagger)^{n_3} e^{-\omega x^2/6\hbar} e^{-\omega y^2/2\hbar}H_{k_3}\left(\sqrt{\frac{\omega}{\hbar}}y\right), \tag{4.29}$$

$$E_2 = \hbar\omega\left(n_2 + k_2 + \tfrac{2}{3}\right). \tag{4.30}$$

The creation and annihilation operators are given by Eq. (3.32) with the following expression for W_1, W_2, and W_3:

$$W_1 = \frac{1}{x} + \frac{\omega x}{3\hbar}, \quad W_2 = -\frac{1}{x} + \frac{\omega x}{3\hbar}, \quad W_3 = -\frac{\omega x}{3\hbar}. \tag{4.31}$$

D. $\alpha=-1$, $\beta=-\frac{32}{9}$, $f(z)=-2z/3-(2z^2-3)/z(2z^2+3)$, and $\epsilon=1$

We have

$$V(x,y) = \frac{\omega^2}{2}\left(\frac{1}{9}x^2 + y^2\right) - \frac{24\hbar^3\omega}{(2\omega x^2 + 3\hbar)^2} + \frac{4\hbar^2\omega}{(2\omega x^2 + 3\hbar)}. \tag{4.32}$$

From the cubic algebra we get the three cases with unitary representations from Eq. (2.12) to Eq. (2.17),

$$\phi(x) = 4\hbar^4\omega^2 x(p+1-x)\left(x+\tfrac{1}{3}\right)\left(x+\tfrac{5}{3}\right), \quad E = \hbar\omega\left(p+\tfrac{5}{3}\right), \tag{4.33}$$

$$\phi(x) = 4\hbar^4\omega^2 x(p+1-x)\left(x-\tfrac{1}{3}\right)\left(x+\tfrac{4}{3}\right), \quad E = \hbar\omega\left(p+\tfrac{4}{3}\right), \tag{4.34}$$

$$\phi(x) = 4\hbar^4\omega^2 x(p+1-x)\left(x-\tfrac{5}{3}\right)\left(x-\tfrac{4}{3}\right), \quad E = \hbar\omega(p). \tag{4.35}$$

From the supersymmetry we obtain the following eigenfunctions from Eqs. (3.35)–(3.37), respectively, and the energy with Eq. (3.34),

$$\psi_{n_1,k_1} = N_{n_1 k_1}(a^\dagger)^{n_1} e^{-\omega x^2/6\hbar} x \frac{(-45\hbar^2 + 4\omega^2 x^4)}{(3\hbar + 2\omega x^2)} e^{-\omega y^2/2\hbar} H_{k_1}\left(\sqrt{\frac{\omega}{\hbar}}y\right), \tag{4.36}$$

$$E_1 = \hbar\omega\left(n_1 + k_1 + \tfrac{5}{3}\right), \tag{4.37}$$

$$\psi_{n_2,k_2} = N_{n_2 k_2}(a^\dagger)^{n_2} e^{-\omega x^2/6\hbar} \frac{(9\hbar^2 - 12\hbar\omega x^2 - 4\omega^2 x^4)}{(3\hbar + 2\omega x^2)} e^{-\omega y^2/2\hbar} H_{k_2}\left(\sqrt{\frac{\omega}{\hbar}}y\right), \tag{4.38}$$

$$E_2 = \hbar\omega\left(n_2 + k_2 + \tfrac{4}{3}\right), \tag{4.39}$$

$$\psi_{n_3,k_3} = N_{n_3 k_3}(a^\dagger)^{n_3} \frac{e^{-\omega x^2/6\hbar}}{(3\hbar + 2\omega x^2)} e^{-\omega y^2/2\hbar} H_{k_3}\left(\sqrt{\frac{\omega}{\hbar}}y\right), \tag{4.40}$$

$$E_3 = \hbar\omega(n_3 + k_3). \tag{4.41}$$

The creation and annihilation operators are given by Eq. (3.32) with the following expressions for W_1, W_2, and W_3:

$$W_1 = \frac{-27\hbar^3 + 27\hbar^2\omega x^2 + 48\hbar\omega^2 x^4 + 4\omega^3 x^6}{27\hbar^3 x - 36\hbar^2\omega x^3 - 12\hbar\omega^2 x^5}, \tag{4.42}$$

$$W_2 = \frac{351\hbar^3\omega x + 126\hbar^2\omega^2 x^3 + 12\hbar\omega^3 x^5 - 8\omega^4 x^7}{81\hbar^4 - 54\hbar^3\omega x^2 - 106\hbar^2\omega^2 x^4 - 24\hbar\omega^3 x^6}, \tag{4.43}$$

$$W_3 = \frac{-9\hbar^2 - 3\hbar\omega x^2 + 2\omega^2 x^4}{9\hbar^2 x + 6\hbar\omega x^3}. \tag{4.44}$$

E. Case $\alpha=0$, $\beta=-2$, $f(z)=-2z-\Psi(z)$, and $\epsilon=1$

We have

$$\Psi(z) = \frac{\psi'(z)}{\psi(z)}, \quad \psi(z) = 1 - tE_c(z). \tag{4.45}$$

$E_c(z)$ is the complementary error function and is given by

$$E_c(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt. \tag{4.46}$$

We have with Eqs. (1.4) and (1.5)

$$V(x,y) = \frac{\omega^2}{2}(x^2 + y^2) - \frac{2}{3}\hbar\omega + \frac{4e^{-2\omega x^2/h}t\hbar\omega}{\pi\left(1 - tE_c\left(\sqrt{\frac{\omega}{\hbar}}x\right)\right)^2} + \frac{4e^{-\omega x^2/h} + \omega\sqrt{\hbar\omega}x}{\pi\left(1 - tE_c\left(\sqrt{\frac{\omega}{\hbar}}x\right)\right)}. \tag{4.47}$$

The cubic algebra provides two unitary representations. The first unitary representation is given by Eq. (2.12) with the corresponding energy given by Eq. (2.13). Equations (2.14) and (2.15) give the same unitary representation and energy spectrum, and we have

$$\phi(x) = 4\hbar^4\omega^2 x(p + 1 - x)x(x + 1), \quad E = \hbar\omega\left(p + \frac{4}{3}\right). \tag{4.48}$$

The second solution is given by Eq. (2.16) with the corresponding energy spectrum given by Eq. (2.17),

$$\phi(x) = 4\hbar^4\omega^2 x(p + 1 - x)(x - 1)^2, \quad E = \hbar\omega\left(p + \frac{1}{3}\right). \tag{4.49}$$

This unitary representation is valid only for $p=0$.

We also use supersymmetry to treat this system. The eigenfunctions for the x part consist of an infinite sequence $\psi_n(x)$ starting from ψ_3^0 given by Eq. (3.37) and a singlet state $\chi(x)$ given by Eqs. (3.36) and (3.39),

$$\psi_n(x) = N_n(a^\dagger)^n e^{-3\omega x^2/2\hbar} \frac{\left(-t\sqrt{\hbar\omega} - e^{\omega x^2/\hbar}\sqrt{\pi}\omega x + e^{\omega x^2/\hbar}\sqrt{\pi}t\omega x E_c\left(\sqrt{\frac{\omega}{\hbar}}x\right)\right)}{\omega\left(-1 + tE_c\left(\sqrt{\frac{\omega}{\hbar}}x\right)\right)}, \tag{4.50}$$

$$\chi(x) = C_0 \frac{e^{-\omega x^2/2\hbar}}{\sqrt{\pi\hbar}\left(1 - tE_c\left(\sqrt{\frac{\omega}{\hbar}}x\right)\right)}, \tag{4.51}$$

$$a\chi(x) = 0, \quad a^\dagger\chi(x) = 0. \tag{4.52}$$

The creation and annihilation operators are given by Eq. (3.32) with the following expressions for W_1, W_2, and W_3:

$$W_1 = \frac{\left(-t\omega x - e^{\omega x^2/\hbar}\sqrt{\pi}\sqrt{\frac{\omega}{\hbar}}(\hbar + \omega x^2) + e^{\omega x^2/\hbar}\sqrt{\pi}t\sqrt{\frac{\omega}{\hbar}}(\hbar + \omega x^2)E_c\left(\sqrt{\frac{\omega}{\hbar}}x\right)\right)}{\hbar\left(-t - e^{\omega x^2/\hbar}\sqrt{\pi}\sqrt{\frac{\omega}{\hbar}}x + e^{\omega x^2/\hbar}\sqrt{\pi} + \sqrt{\frac{\omega}{\hbar}}xE_c\left(\sqrt{\frac{\omega}{\hbar}}x\right)\right)}, \tag{4.53}$$

$$W_2 = \frac{A}{B}, \tag{4.54}$$

$$A = e^{-\omega x^2/\hbar}\left(2t^2\sqrt{\omega\hbar} + 3e^{\omega x^2/\hbar}\sqrt{\pi}t\omega x - e^{2\omega x^2/\hbar}\pi\sqrt{\frac{\omega}{\hbar}}(h - \omega x^2) + \left(-3e^{\omega x^2/h}\sqrt{\pi}t^2\omega x\right.\right.$$
$$\left.\left. + 2e^{2\omega x^2/\hbar}\pi t\sqrt{\frac{\omega}{\hbar}}(\hbar - \omega x^2)\right)E_c\left(\sqrt{\frac{\omega}{\hbar}}x\right) - e^{2\omega x^2/\hbar}\pi t^2\sqrt{\frac{\omega}{\hbar}}(\hbar - \omega x^2)\left(E_c\left(\sqrt{\frac{\omega}{\hbar}}x\right)\right)^2\right), \tag{4.55}$$

$$B = \hbar\sqrt{\pi}\left(-1 + tE_c\left(\sqrt{\frac{\omega}{\hbar}}x\right)\right)\left(-t - e^{\omega x^2/\hbar}\sqrt{\pi}\sqrt{\frac{\omega}{\hbar}}x + e^{\omega x^2/\hbar}\sqrt{\pi}t\sqrt{\frac{\omega}{\hbar}}xE_c\left(\sqrt{\frac{\omega}{\hbar}}x\right)\right),$$

(4.56)

$$W_3 = \frac{\omega x}{\hbar} + \frac{2e^{-\omega x^2/\hbar} + \sqrt{\dfrac{\omega}{\hbar}}}{\sqrt{\pi}\left(1 - tE_c\left(\sqrt{\dfrac{\omega}{\hbar}}x\right)\right)}.$$

(4.57)

With the eigenfunctions for the y part of the Hamiltonian and the formula for the energies given by Eq. (3.34), we obtain the two following families of solutions:

$$\psi_{n,k} = \psi_n(x)e^{-\omega y^2/2\hbar}H_k\left(\sqrt{\frac{\omega}{\hbar}}y\right), \quad E = \hbar\omega\left(n + k + \frac{4}{3}\right),$$

(4.58)

$$\phi_m = \chi(x)e^{-\omega y^2/2\hbar}H_m\left(\sqrt{\frac{\omega}{\hbar}}y\right), \quad E_m = \hbar\omega\left(m + \frac{1}{3}\right).$$

(4.59)

F. Case $\alpha=0$, $\beta=-2$, $f(z)=-2z-\Psi(z)$, and $\epsilon=-1$

This case gives the harmonic oscillator. The zero mode is given by Eq. (3.22) is the well known ground state of the harmonic oscillator. There is other special solutions in terms of the complementary error function exist.

Many special solutions of the fourth Painlevé equation will give us singular Hamiltonians. These potentials can be regularized in several manners.[19,56–58]

V. CONCLUSION

The main results of this article are that we have constructed the cubic algebra, Fock-type presentations, and the corresponding energy spectrum for the potential given by Eqs. (1.4) and (1.5). Other superintegrable potentials written in term of Painlevé transcendents are known,[16] namely,

$$V_1 = \hbar^2(\omega_1^2 P_I(\omega_1 x) + \omega_2^2 P_I(\omega_2 y)),$$

(5.1)

$$V_2 = ay + \hbar^2\omega^2 P_I(\omega x),$$

(5.2)

$$V_3 = bx + ay + (2\hbar b)^{2/3}P_{II}^2\left(\left(\frac{2b}{\hbar^2}\right)^{1/3}x, 0\right),$$

(5.3)

$$V_4 = ay + (2\hbar^2 b^2)^{1/3}\left(P'_{II}\left(\left(\frac{-4b}{\hbar^2}\right)^{1/3}x, k\right) + P_{II}^2\left(\left(\frac{-4b}{\hbar^2}\right)^{1/3}x\right), k\right).$$

(5.4)

These potentials together with that in Eqs. (1.4) and (1.5) were also obtained as one-dimensional potentials in the context of higher and conditional symmetries by Fushchych and Nikitin.[46] For these four superintegrable potentials the simplest underlying structure of the type (2.5) is actually a finite dimensional Lie algebra that does not allow us to find the energy spectrum. Let us present these algebras.

For V_1 we have

$$[A,B] \equiv C = -i\hbar^5 \omega_1^5 \omega_2^5, \quad [A,C] = 0, \quad [B,C] = 0. \tag{5.5}$$

For V_2 we have

$$[A,B] \equiv C = -i\hbar^5 \omega^5, \quad [A,C] = 0, \quad [B,C] = 0. \tag{5.6}$$

For V_3 and V_4 we have

$$[A,B] \equiv C = 4abi\hbar\left(H + \tfrac{1}{2}A\right), \quad [A,C] = 0, \quad [B,C] = 8a^2b^2\hbar^2\left(H + \tfrac{1}{2}A\right). \tag{5.7}$$

These algebras coincide with the classical Poisson algebras presented earlier.[17] In these four cases we have a triplet of commuting operators. The x part of the potential V_4 was also obtained in the context of supersymmetric quantum mechanics.[55] The methods of this article are not directly applicable in that case, but it may be possible to generalize them.

The question of how these aspects of SUSYQM, shape invariance, and superintegrability are related is interesting and will require more study. Supersymmetry could also be a tool for the classification of superintegrable potentials. Higher order supersymmetry could be a suitable approach to treat these potentials. The search for superintegrable systems with higher order integrals of motion is thus closely related to the subject of polynomial algebras and higher order supersymmetric quantum mechanics.

The search for a grand unifying theory in particle physics is an important problem of comtemporary physics. One model that is envisaged as a candidate is string theory. The x part of the potential given by Eq. (1.1) appears also in the context of string theory[59] where supersymmetry is used as a method for constructing exact solutions. A more recent article[60] discusses how supersymmetric quantum mechanics can be used to construct solutions in string theory.

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