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Superintegrability with third order integrals of motion, cubic algebras, and supersymmetric quantum mechanics. I. Rational function potentials

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We consider a superintegrable Hamiltonian system in a two-dimensional space with a scalar potential that allows one quadratic and one cubic integrals of motion. We construct the most general cubic algebra and we present specific realizations. We use them to calculate the energy spectrum. All classical and quantum superintegrable potentials separable in Cartesian coordinates with a third order integral are known. The general formalism is applied to quantum reducible and irreducible rational potentials separable in Cartesian coordinates in E_2. We also discuss these potentials from the point of view of supersymmetric and PT-symmetric quantum mechanics. © *2009 American Institute of Physics.* [DOI: 10.1063/1.3013804]

I. INTRODUCTION

In classical mechanics a Hamiltonian system with Hamiltonian H and integrals of motion X_a,

$$H=\frac{1}{2}g_{ik}p_ip_k+V(\vec{x},\vec{p}),\quad X_a=f_a(\vec{x},\vec{p}),\quad a=1,\ldots,n-1, \tag{1.1}$$

is called completely integrable (or Liouville integrable) if it allows n integrals of motion (including the Hamiltonian) that are well defined functions on phase space, are in involution $\{H,X_a\}_p=0$, $\{X_a,X_b\}_p=0$, $a,b=1,\ldots,n-1$, and are functionally independent ($\{,\}_p$ is a Poisson bracket). A system is superintegrable if it is integrable and allows further integrals of motion $Y_b(\vec{x},\vec{p})$, $\{H,Y_b\}_p=0$, $b=n,n+1,\ldots,n+k$ that are also well defined functions on phase space and the integrals $\{H,X_1,\ldots,X_{n-1},Y_n,\ldots,Y_{n+k}\}$ are functionally independent. A system is maximally superintegrable if the set contains $2n-1$ functions, quasimaximally superintegrable if it contains $2n-2$, and minimally superintegrable if it contains $n+1$ such integrals. The integrals Y_b are not required to be in evolution with $X_1,\ldots,X_{n-1}$ nor with each other. The same definitions apply in quantum mechanics, but $\{H,X_a,Y_b\}$ are well defined quantum mechanical operators, assumed to form an algebraically independent set.

Superintegrable systems appear in many domains of physics such as quantum chemistry, condensed matter, and nuclear physics. The most well known examples of (maximally) superintegrable systems are the Kepler–Coulomb[1,2] system $V(\vec{x})=\alpha/r$ and the harmonic oscillator $V(\vec{x})=\alpha r^2$.[3,4] A systematic search for superintegrable systems in two-dimensional Euclidean space E_2 was started some time ago.[5,6] In 1935 Drach[7] published two articles on two-dimensional Hamiltonian systems with third order integrals of motion and found ten such integrable classical potentials in complex Euclidean space $E_2(\mathbb{C})$. A systematic study of superintegrable classical and quantum systems with a third order integral is more recent.[8,9] All classical and quantum potentials with a second and a third order integral of motion that separate in Cartesian coordinates in the two-dimensional Euclidean space were found in Ref. 9. There are 21 quantum potentials and 8 classical potentials.

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The classical potentials were studied earlier.[10] In all eight cases of superintegrable systems, separating in Cartesian coordinates and allowing a third order integral of motion, the integrals of motion generate a cubic Poisson algebra. In many cases this polynomial algebra is reducible, which is a consequence of the existence of a simpler algebraic structure. We have also studied trajectories and have shown that bounded trajectories are always closed for these superintegrable potentials.

The quantum case is much richer: 21 superintegrable cases of the considered type exist, 13 of them irreducible. In this context we call a potential or a Hamiltonian "reducible" if the third order integral is the commutator (or Poisson commutator) of two second order integrals. The potentials are expressed in terms of rational functions in six cases, elliptic functions in two cases, and Painlevé transcendents[11] P_{I}, P_{II}, and P_{IV} in five cases.

The three reducible cases are

$$V=\frac{\omega^2}{2}(x^2+y^2), \quad V=\frac{\omega^2}{2}(x^2+y^2)+\frac{b}{x^2}+\frac{c}{y^2}, \quad V=\frac{\omega^2}{2}(4x^2+y^2)+\frac{b}{y^2}+cx.$$

The irreducible potentials with rational function are as follows:

- Potential 1:

$$V=\hbar^2\left[\frac{x^2+y^2}{8a^4}+\frac{1}{(x-a)^2}+\frac{1}{(x+a)^2}\right].$$

- Potential 2:

$$V=\frac{\omega^2}{2}(9x^2+y^2).$$

- Potential 3:

$$V=\frac{\omega^2}{2}(9x^2+y^2)+\frac{\hbar^2}{y^2}.$$

- Potential 4:

$$V=\hbar^2\left[\frac{9x^2+y^2}{8a^4}+\frac{1}{(y-a)^2}+\frac{1}{(y+a)^2}\right].$$

- Potential 5:

$$V=\hbar^2\left(\frac{1}{8a^4}\left[(x^2+y^2)+\frac{1}{y^2}+\frac{1}{(x+a)^2}+\frac{1}{(x-a)^2}\right]\right. .$$

- Potential 6:

$$V=\hbar^2\left[\frac{1}{8a^4}(x^2+y^2)+\frac{1}{(y+a)^2}+\frac{1}{(y-a)^2}+\frac{1}{(x+a)^2}+\frac{1}{(x-a)^2}\right].$$

It is well known that in quantum mechanics the operators commuting with the Hamiltonian form an $o(4)$ algebra for the hydrogen atom[3,4] and a $u(3)$ algebra for the harmonic oscillator. We can obtain from the algebra the energy spectrum. In many cases the algebra is no longer a Lie algebra and many examples of polynomial algebras were obtained in quantum mechanics.[12–22] Daskaloyannis[17] studied the case of the quadratic Poisson algebras of two-dimensional classical superintegrable systems and quadratic (associative) algebras of quantum superintegrable systems. He showed how the quadratic algebras provide a method to obtain the energy spectrum. He used realizations in terms of deformed oscillator algebras.[18] Potentials with a third order integral can be investigated using these techniques.

Supersymmetry (SUSY) was originally introduced in the context of grand unification theory in elementary particle physics in terms of quantum field theory involving a symmetry between bosons and fermions.[23] So far there is no experimental evidence of SUSY particles. At our energies we can distinguish bosons and fermions, and this symmetry should appear as a broken symmetry. Supersymmetric quantum mechanics (SUSYQM) was introduced by Witten[24] as a toy model to study SUSY breaking. This method is related to earlier investigation of spectral properties of Sturm–Liouville differential operators by Darboux[25] and Moutard[26] in the 19th century. SUSYQM is also related to the factorization method that was used by Schrodinger[27] in the context of the quantum harmonic oscillator. The factorization method was investigated more systematically later by Infeld and Hull.[28] SUSYQM is now an independent field with applications to atomic, nuclear, condensed matter, statistical physics, and quantum mechanics.[29] The relation with exactly solvable potentials has been discussed[30,31] and also with superintegrable potentials and quadratic algebras.[32]

This paper is organized in the following way. In Sec. II we give the general form of the cubic algebra for two-dimensional systems with a quadratic and a cubic operator that commute with the Hamiltonian. We give a realization of the cubic algebra in terms of parafermionic oscillator algebras. We study the finite-dimensional representations of the cubic algebra. In Sec. III we apply this method to the case of irreducible potentials separable in Cartesian coordinates in E_2 with a third order integral. In Sec. IV we investigate the irreducible potentials from the point of view of SUSYQM. In Sec. V we give the generating spectrum algebra of the irreducible potential 1. In Sec. VI we investigate the complexification of the irreducible potential 1. (All other cases can be obtained from potential 1.)

II. CUBIC AND PARAFERMIONIC ALGEBRAS

We consider a quantum superintegrable system with a quadratic Hamiltonian and one second order and one third order integrals of motion,

$$H = a(q_1,q_2)P_1^2 + 2b(q_1,q_2)P_1P_2 + c(q_1,q_2)P_2^2 + V(q_1,q_2),$$

$$A = d(q_1,q_2)P_1^2 + 2e(q_1,q_2)P_1P_2 + f(q_1,q_2)P_2^2 + g(q_1,q_2)P_1 + h(q_1,q_2)P_2 + Q(q_1,q_2),$$

$$\begin{aligned} B = {} & u(q_1,q_2)P_1^3 + 3v(q_1,q_2)P_1^2P_2 + 3w(q_1,q_2)P_1P_2^2 + x(q_1,q_2)P_2^3 + j(q_1,q_2)P_1^2 + 2k(q_1,q_2)P_1P_2 \\ & + l(q_1,q_2)P_2^2 + m(q_1,q_2)P_1 + n(q_1,q_2)P_2 + S(q_1,q_2), \end{aligned} \tag{2.1}$$

with

$$P_1 = -i\hbar\partial_1, \quad P_2 = -i\hbar\partial_2, \tag{2.2}$$

$$[H,A] = [H,B] = 0. \tag{2.3}$$

We assume that our integrals close in a cubic algebra. This is the quantum version of the cubic Poisson algebra obtained earlier[10] and the cubic generalization of the quadratic algebra studied by Daskaloyannis.[17] The most general form of such an algebra is

$$[A,B] = C,$$

$$[A,C] = \alpha A^2 + \beta\{A,B\} + \gamma A + \delta B + \epsilon,$$

$$[B,C] = \mu A^3 + \nu A^2 + \rho B^2 + \sigma\{A,B\} + \xi A + \eta B + \zeta, \tag{2.4}$$

where {,} denotes an anticommutator. The coefficients α, β, and μ are constants, but the other ones can be polynomials in the Hamiltonian H. The degrees of these polynomials are dictated by the fact that H and A are second order polynomials in the momenta and B is a third order one. Hence C can be a fourth order polynomial. The Jacobi identity $[A,[B,C]]=[B,[A,C]]$ implies $\rho=-\beta$, $\sigma=-\alpha$, and $\eta=-\gamma$. We obtain

$$[A,B]=C, \tag{2.5a}$$

$$[A,C]=\alpha A^2+\beta\{A,B\}+\gamma A+\delta B+\epsilon, \tag{2.5b}$$

$$[B,C]=\mu A^3+\nu A^2-\beta B^2-\alpha\{A,B\}+\xi A-\gamma B+\zeta. \tag{2.5c}$$

For the polynomials on the left and right sides of Eqs. (2.4) and (2.5) to have the same degree, we must have

$$\alpha=\alpha_0,\quad \beta=\beta_0,\quad \mu=\mu_0,$$

$$\gamma=\gamma_0+\gamma_1 H,\quad \delta=\delta_0+\delta_1 H,\quad \epsilon=\epsilon_0+\epsilon_1 H+\epsilon_2 H^2,$$

$$\nu=\nu_0+\nu_1 H,\quad \xi=\xi_0+\xi_1 H+\xi_2 H^2,$$

$$\zeta=\zeta_0+\zeta_1 H+\zeta_2 H^2+\zeta_3 H^3, \tag{2.6}$$

where $\alpha_0,\ldots,\zeta_3$ are constants. The Casimir operator of a polynomial algebra is an operator that commutes with all elements of the algebra:

$$[K,A]=[K,B]=[K,C]=0, \tag{2.7}$$

and this implies

$$K=C^2-\alpha\{A^2,B\}-\beta\{A,B^2\}+(\alpha\beta-\gamma)\{A,B\}+(\beta^2-\delta)B^2(+\beta\gamma-2\epsilon)B+\frac{\mu}{2}A^4+\frac{2}{3}(\nu+\mu\beta)A^3$$
$$+\left(-\frac{1}{6}\mu\beta^2+\frac{\beta\nu}{3}+\frac{\delta\mu}{2}+\alpha^2+\xi\right)A^2+\left(-\frac{1}{6}\mu\beta\delta+\frac{\delta\nu}{3}+\alpha\gamma+2\zeta\right)A. \tag{2.8}$$

Ultimately, the Casimir operator will be a function of the Hamiltonian alone. We construct a realization of the cubic algebra in terms of a deformed oscillator algebra[17,18] $\{b^t,b,N\}$ which satisfies the relation

$$[N,b^t]=b^t,\quad [N,b]=-b,\quad b^tb=\Phi(N),\quad bb^t=\Phi(N+1). \tag{2.9}$$

$\Phi(N)$ is called the "structure function." Following Daskaloyannis[17] we request $\Phi(N)$ to be a real function and impose $\Phi(0)=0$ and $\Phi(N)>0$ for $N>0$. We construct a Fock-type representation for the deformed oscillator algebra with a Fock basis $|n\rangle$, $n=0,1,2,\ldots$, satisfying

$$N|n\rangle=n|n\rangle,\quad b^t|n\rangle=\sqrt{\Phi(N+1)}|n+1\rangle, \tag{2.10}$$

$$b|0\rangle=0,\quad b|n\rangle=\sqrt{\Phi(N)}|n-1\rangle. \tag{2.11}$$

To obtain a finite-dimensional representation we request $\Phi(p+1)=0$.

Let us show that there exists a realization of the form

$$A=A(N),\quad B=b(N)+b^t\rho(N)+\rho(N)b. \tag{2.12}$$

The functions $A(N)$, $b(N)$, and $\rho(N)$ will be determined by the cubic algebra. We have by Eq. (2.5a)

$$C=[A,B]=b^{t}\Delta A(N)\rho(N)-\rho(N)\Delta A(N)b, \tag{2.13}$$

where $\Delta A(N)$ is defined to be $\Delta A(N)=A(N+1)-A(N)$. When we insert Eq. (2.12) into Eq. (2.5b) we obtain two equations that allow us to determine $A(N)$ and $b(N)$,

$$\Delta A(N)^2=\gamma(A(N+1)+A(N))+\epsilon,$$

$$\alpha A(N)^2+2\beta A(N)b(N)+\gamma A(N)+\delta b(N)+\epsilon=0. \tag{2.14}$$

Two distinct possibilities occur.

Case 1: $\beta\neq 0$. We find the following solution:

$$A(N)=\frac{\beta}{2}\left((N+u)^2-\frac{1}{4}-\frac{\delta}{\beta^2}\right),$$

$$b(N)=\frac{\alpha}{4}\left((N+u)^2-\frac{1}{4}\right)+\frac{\alpha\delta-\gamma\beta}{2\beta^2}-\frac{\alpha\delta^2-2\gamma\delta\beta+4\beta^2\epsilon}{4\beta^4}\frac{1}{(N+u)^2-\frac{1}{4}}. \tag{2.15}$$

The constant u will be determined below using the fact that we require that the deformed oscillator algebras should be nilpotent. Equation (2.5c) gives us

$$\begin{aligned}&2\Phi(N+1)\left(\Delta A(N)+\frac{\gamma}{2}\right)\rho(N)-2\Phi(N)\left(\Delta A(N-1)-\frac{\gamma}{2}\right)\rho(N-1)\\&\quad=\mu A(N)^3+\nu A(N)^2-\beta b(N)^2-2\alpha A(N)b(N)+\xi A(N)-\gamma b(N)+\zeta,\end{aligned} \tag{2.16}$$

and the Casimir operator is now realized as

$$\begin{aligned}K=&\Phi(N+1)(\beta^2-\delta-2\beta A(N)-\Delta A(N)^2)\rho(N)+\Phi(N)(\beta^2-\delta-2\beta A(N)-\Delta A(N-1)^2)\rho(N-1)\\&-2\alpha A(N)^2b(N)+(\beta^2-\delta-2\beta A(N))b(N)^2+2(\alpha\beta-\gamma)A(N)b(N)+(\beta\gamma-2\epsilon)b(N)+\frac{\mu}{2}A(N)^4\\&+\frac{2}{3}(\nu+\mu\beta)A(N)^3+\left(-\frac{1}{6}\mu\beta^2+\frac{\beta\nu}{3}+\frac{\delta\mu}{2}+\alpha^2+\epsilon\right)A(N)^2+\left(-\frac{1}{6}\mu\beta\delta+\frac{\delta a}{3}+\alpha\gamma+2\zeta\right)A(N).\end{aligned} \tag{2.17}$$

Finally the structure function is

$$\begin{aligned}\Phi(N)=&\frac{1}{\rho(N-1)\left(\Delta A(N-1)-\frac{\beta}{2}\right)(f)+\left(\Delta A(N)+\frac{\beta}{2}\right)(g)}\Bigg[\left(\Delta A(N)+\frac{\beta}{2}\right)\Bigg(K+2\alpha A(N)^2b(N)\\&-(\beta^2-\delta-2\beta A(N))b(N)^2-2(\alpha\beta-\gamma)A(N)b(N)-(\beta\gamma-2\epsilon)b(N)-\frac{\mu}{2}A(N)^4\\&-\frac{2}{3}(\nu+\mu\beta)A(N)^3-\left(-\frac{1}{6}\mu\beta^2+\frac{\beta\nu}{3}+\alpha^2+\xi\right)A(N)^2-\left(-\frac{1}{6}\mu\beta\delta+\frac{\delta\nu}{3}+\alpha\gamma+2\zeta\right)A(N)\Bigg)\\&-\frac{1}{2}(\beta^2-\delta-2\beta A(N)-\Delta A(N)^2)(\mu A(N)^3+\nu A(N)^2-\beta b(N)^2-2\alpha A(N)b(N)+\xi A(N)\\&-\gamma b(N)+\zeta)\Bigg],\end{aligned} \tag{2.18}$$

with

$$f = \beta^2 - \delta - 2\beta A(N) - \Delta A(N)^2, \quad g = \beta^2 - \delta - 2\beta A(N) - \Delta A(N-1)^2. \tag{2.19}$$

Thus the structure function depends only on the function ρ. This function can be arbitrarily chosen and does not influence the spectrum. We choose ρ to obtain a structure function that is a polynomial in N, namely, we set

$$\rho(N) = \frac{1}{3 \times 2^{12}\beta^8 (N+u)(1+N+u)(1+2(N+u))^2}. \tag{2.20}$$

From our expressions for $A(N)$, $b(N)$, and $\rho(N)$, the third relation of the cubic associative algebra, and the expression of the Casimir operator we find the structure function $\Phi(N)$. For Case 1 the structure function is a polynomial of order of 10 in N. The coefficients of this polynomial are functions of α, β, μ, γ, δ, ϵ, ν, ξ, and ζ. We give the formula in the Appendix.

Case 2: For $\beta=0$ and $\delta \neq 0$ we get the solution

$$A(N) = \sqrt{\delta}(N+u), \quad b(N) = -\alpha(N+u)^2 - \frac{\gamma}{\sqrt{\delta}}(N+u) - \frac{\epsilon}{\delta}. \tag{2.21}$$

We choose a trivial expression $\rho(N)=1$. The explicit expression of the structure function for this case is

$$\begin{aligned}\Phi(N) = &\left(\frac{K}{-4\delta} - \frac{\gamma\epsilon}{4\delta^{3/2}} - \frac{\zeta}{4\sqrt{\delta}} + \frac{\epsilon^2}{4\delta^2}\right) + \left(\frac{-\alpha\epsilon}{2\delta} - \frac{\xi}{4} - \frac{\gamma^2}{4\delta} + \frac{\gamma\epsilon}{2\delta^{3/2}} + \frac{\alpha\gamma}{4\sqrt{\delta}} + \frac{\zeta}{2\sqrt{\delta}} + \frac{\nu\sqrt{\delta}}{12}\right)(N+u) \\ &+ \left(\frac{-\nu\sqrt{\delta}}{4} - \frac{3\alpha\gamma}{4\sqrt{\delta}} + \frac{\gamma^2}{4\delta} + \frac{\epsilon\alpha}{2\delta} + \frac{\alpha^2}{4} + \frac{\xi}{4} + \frac{\mu\delta}{8}\right)(N+u)^2 \\ &+ \left(\frac{-\alpha^2}{2} + \frac{\gamma\alpha}{2\delta^{1/2}} + \frac{\nu\sqrt{\delta}}{6} - \frac{\mu\delta}{4}\right)(N+u)^3 + \left(\frac{\alpha^2}{4} + \frac{\mu\delta}{8}\right)(N+u)^4.\end{aligned} \tag{2.22}$$

We use a parafermionic realization in which the parafermionic number operator N and the Casimir operator K are diagonal. The basis of this representation is the Fock basis for the parafermionic oscillator. The vector $|k,n\rangle$, $n=0,1,2,\ldots$, satisfies the following relations:

$$N|k,n\rangle = n|k,n\rangle, \quad K|k,n\rangle = k|k,n\rangle. \tag{2.23}$$

The vectors $|k,n\rangle$ are also eigenvectors of the generator A.

$$A|k,n\rangle = A(k,n)|k,n\rangle,$$

$$A(k,n) = \frac{\beta}{2}\left((n+u)^2 - \frac{1}{4} - \frac{\delta}{\beta^2}\right), \quad \beta \neq 0,$$

$$A(k,n) = \sqrt{\delta}(n+u), \quad \beta = 0, \quad \delta \neq 0. \tag{2.24}$$

We have the following constraints for the structure function:

$$\Phi(0,u,k) = 0, \quad \Phi(p+1,u,k) = 0. \tag{2.25}$$

With these two relations we can find the energy spectrum. Many solutions for the system exist. Unitary representations of the deformed parafermionic oscillator obey the constraint $\Phi(x) > 0$ for $x=1,2,\ldots,p$. There are other conditions that should be imposed. The representations should be constrained by the differential character of the Hamiltonian and the integrals. For example, the mean energy should be greater than the minimum of the potential,

$$\langle H\rangle \geq \min V. \tag{2.26}$$

This restriction and possibly other ones coming from the differential operator character of the integrals should be taken into consideration to exclude spurious states.

III. IRREDUCIBLE RATIONAL FUNCTION POTENTIALS

In the case of the three reducible superintegrable potentials the cubic algebra is a direct consequence of a simpler algebraic structure. The first potential $V=(\omega^2/2)(x^2+y^2)$ is the well known isotropic harmonic oscillator. We can construct the quadratic or the cubic algebra from the Lie algebra as in the classical case.[10] The eigenfunctions of the harmonic oscillator are well known and are given in terms of the Hermite polynomials. The two other reducible potentials $V=(\omega^2/2)(x^2+y^2)+b/x^2+c/y^2$ and $V=(\omega^2/2)(4x^2+y^2)+b/y^2+cx$ are two of the four types of potentials found a long time ago.[6] There is no Lie algebra in these cases but a quadratic algebra,[17] and we can obtain the cubic algebra directly from this algebra. We obtain from the cubic algebra the same unitary representations that were obtained from the quadratic algebra.[17]

In this section we will apply to the irreducible quantum potentials the results of Sec. II and give all unitary representations and the corresponding energy spectra. Notice that in all cases we have $\beta=0$, so only Case 2 of Sec. II occurs.

Potential 1:

$$V=\hbar^2\left(\frac{x^2+y^2}{8a^4}+\frac{1}{(x-a)^2}+\frac{1}{(x+a)^2}\right).$$

This potential has the following two integrals:

$$A=P_x^2-P_y^2+2\hbar^2\left(\frac{x^2-y^2}{8a^4}+\frac{1}{(x-a)^2}+\frac{1}{(x+a)^2}\right), \tag{3.1}$$

$$\begin{aligned}B=&\frac{1}{2}\{L,P_x^2\}+\frac{1}{2}\hbar^2\left\{y\left(\frac{4a^2-x^2}{4a^4}-\frac{6(x^2+a^2)}{(x^2-a^2)^2)}\right),P_x\right\}\\&+\frac{1}{2}\hbar^2\left\{x\left(\frac{(x^2-4a^2)}{4a^4}-\frac{2}{x^2-a^2}+\frac{4(x^2+a^2)}{(x^2-a^2)^2}\right),P_y\right\}.\end{aligned} \tag{3.2}$$

The integrals A, B, and H give rise to the following cubic algebra and Casimir operators:

$$[A,B]=C,\quad [A,C]=\frac{4h^4}{a^4}B,$$

$$[B,C]=-2\hbar^2A^3-6\hbar^2A^2H+8\hbar^2H^3+6\frac{\hbar^4}{a^2}A^2+8\frac{\hbar^4}{a^2}HA-8\frac{\hbar^4}{a^2}H^2+2\frac{\hbar^6}{a^4}A-2\frac{\hbar^6}{a^4}H-6\frac{\hbar^8}{a^6}, \tag{3.3}$$

$$K=-16\hbar^2H^4+32\frac{\hbar^4}{a^2}H^3+16\frac{\hbar^6}{a^4}H^2-40\frac{\hbar^8}{a^6}H-3\frac{\hbar^{10}}{a^8}. \tag{3.4}$$

The structure function is given by the expression

$$\begin{aligned}\Phi(x)=&\left(\frac{-\hbar^8}{a^4}\right)\left(x+u-\left(\frac{-a^2E}{\hbar^2}-\frac{1}{2}\right)\right)\left(x+u-\left(\frac{a^2E}{\hbar^2}+\frac{1}{2}\right)\right)\left(x+u-\left(\frac{-a^2E}{\hbar^2}+\frac{3}{2}\right)\right)\\&\times\left(x+u-\left(\frac{-a^2E}{\hbar^2}+\frac{5}{2}\right)\right).\end{aligned} \tag{3.5}$$

There are three unitary representations. The first unitary solution is for $a = ia_0$, $a_0 \in \mathbb{R}$. From the condition $\Phi(0,u,k)=0$ we find $u = -a_0^2 E/\hbar^2 + 1/2$. The second constraint $\Phi(p+1,u,k)=0$ implies

$$E = \frac{\hbar^2(p+2)}{2a_0^2}, \quad \Phi(x) = \left(\frac{\hbar^8}{a_0^4}\right) x(p+1-x)(p+3-x)(p+4-x), \tag{3.6}$$

where $p \in \mathbb{N}$. We have $\Phi(p+1)=0$ which means that the unitary representations have dimension $p+1$. This is also the degeneracy of the energy levels. The second unitary solution is for $a = ia_0$, $a_0 \in \mathbb{R}$. We have $u = (a_0^2 E/\hbar^2) - 1/2$ and

$$E = -\frac{\hbar^2(p)}{2a_0^2}, \quad \Phi(x) = \left(\frac{\hbar^8}{a_0^4}\right) x(p+1-x)(3-x)(2-x), \tag{3.7}$$

valid only for $p=1,2$. We have

$$E \geq \min V = V(0,0) = \frac{-2\hbar^2}{a_0^2}, \tag{3.8}$$

so this is a physically meaningful solution. A third unitary solution exists this time for $a \in \mathbb{R}$. We have $u = (-a^2 E/\hbar^2) + 5/2$ and

$$E = \frac{\hbar^2(p+3)}{2a^2}, \quad \Phi(x) = \left(\frac{\hbar^8}{a^4}\right) x(p+1-x)(x+1)(x+3). \tag{3.9}$$

Potential 2:

$$V = \frac{\omega^2}{2}(9x^2 + y^2).$$

This potential has the two integrals

$$A = P_x^2 - P_y^2 + \omega^2(9x^2 - y^2),$$

$$B = \frac{1}{2}\{L, P_y^2\} + \frac{\omega^2}{6}\{y^3, P_x\} - \frac{3\omega^2}{2}\{xy^2, P_y\}. \tag{3.10}$$

The cubic algebra and Casimir operator of this system are

$$[A,B] = C, \quad [A,C] = 144\omega^2\hbar^2 B,$$

$$[B,C] = -2\hbar^2 A^3 + 6\hbar^2 H A^2 - 8\hbar^2 H^3 - 56\omega^2\hbar^4 A + 72\omega^2\hbar^4 H, \tag{3.11}$$

$$K = -16\hbar^2 H^4 + 64\omega^2\hbar^4 H^2 + 720\omega^4\hbar^6. \tag{3.12}$$

The structure function is

$$\Phi(x) = (-36\omega^2 h^4)\left(x + u - \left(\frac{-E}{6\omega\hbar} + \frac{1}{2}\right)\right)\left(x + u - \left(\frac{E}{6\omega\hbar} + \frac{1}{6}\right)\right)\left(x + u - \left(\frac{E}{6\omega\hbar} + \frac{1}{2}\right)\right) \times \left(x + u - \left(\frac{E}{6\omega\hbar} + \frac{5}{6}\right)\right). \tag{3.13}$$

We use the two constraints given by Eq. (2.25). We obtain $u = (-E/6\omega\hbar) + 1/2$ and three unitary representations with the corresponding energy spectra:

$$E=3\omega\hbar\left(p+\tfrac{2}{3}\right), \quad \Phi(x)=(36\omega^2h^4)x(p+1-x)\left(p+\tfrac{2}{3}-x\right)\left(p+\tfrac{1}{3}-x\right), \tag{3.14}$$

$$E=3\omega\hbar(p+1), \quad \Phi(x)=(36\omega^2h^4)x(p+1-x)\left(p+\tfrac{2}{3}-x\right)\left(p+\tfrac{4}{3}-x\right), \tag{3.15}$$

$$E=3\omega\hbar\left(p+\tfrac{4}{3}\right), \quad \Phi(x)=(36\omega^2h^4)x(p+1-x)\left(p+\tfrac{5}{3}-x\right)\left(p+\tfrac{4}{3}-x\right). \tag{3.16}$$

These results coincide with those obtained by solving the Schrödinger equation using separation of variable. The eigenfunctions are well known and given by

$$\phi_{k_1}(x)=\frac{1}{\sqrt{2^{k_1}k_1!}}\left(\frac{3\omega}{\pi\hbar}\right)^{1/4}e^{(-3\omega/2\hbar)x^2}H_{k_1}\left(\sqrt{\frac{3\omega}{\hbar}}x\right), \tag{3.17}$$

$$\phi_{k_2}(y)=\frac{1}{\sqrt{2^{k_2}k_2!}}\left(\frac{\omega}{\pi\hbar}\right)^{1/4}e^{(-\omega/2\hbar)y^2}H_{k_2}\left(\sqrt{\frac{\omega}{\hbar}}y\right), \tag{3.18}$$

where H_k are Hermite polynomials. The corresponding energy spectrum is

$$E=\omega\hbar(3k_1+k_2+2). \tag{3.19}$$

Potential 3:

$$V=\frac{\omega^2}{2}(9x^2+y^2)+\frac{\hbar^2}{y^2}.$$

The two integrals of this potential are

$$A=P_x^2-P_y^2+\omega^2(9x^2-y^2)-\frac{2\hbar^2}{y^2},$$

$$B=\frac{1}{2}\{L,P_y^2\}+\frac{1}{2}\left\{\frac{\omega^2y^3}{3}-\frac{\hbar^2}{y},P_x\right\}+\frac{1}{2}\left\{3x(-\omega^2y^2+\frac{\hbar^2}{y^2},P_y\right\}. \tag{3.20}$$

The cubic algebra and the Casimir operator are

$$[A,B]=C \quad [A,C]=144\omega^2\hbar^2B,$$

$$[B,C]=-2\hbar^2A^3+6\hbar^2HA^2-8\hbar^2H^3-8\omega^2\hbar^4A+72\omega^2\hbar^4H, \tag{3.21}$$

$$K=-16\hbar^2H^4+256\omega^2\hbar^4H^2-1008\omega^4\hbar^6. \tag{3.22}$$

The structure function is

$$\Phi(x)=(-36\omega^2h^4)\left(x+u-\left(\frac{-E}{6\omega\hbar}+\frac{1}{2}\right)\right)\left(x+u-\left(\frac{E}{6\omega\hbar}-\frac{1}{6}\right)\right)\left(x+u-\left(\frac{E}{6\omega\hbar}+\frac{1}{2}\right)\right)$$
$$\times\left(x+u-\left(\frac{E}{6\omega\hbar}+\frac{7}{6}\right)\right). \tag{3.23}$$

Using Eq. (2.25) we obtain $u=(-E/6\omega\hbar)+1/2$ and two unitary representations:

$$\Phi(x)=(36\omega^2\hbar^4)x\left(p+\tfrac{5}{3}-x\right)(p+1-x)\left(p+\tfrac{7}{3}-x\right), \quad E=3\omega\hbar\left(p+\tfrac{5}{3}\right), \tag{3.24}$$

$$\Phi(x)=(36\omega^2\hbar^4)x\left(p+\tfrac{1}{3}-x\right)\left(p+\tfrac{5}{3}-x\right)(p+1-x), \quad E=3\omega\hbar(p+1). \tag{3.25}$$

These results are corroborated by those obtained when we use separation of variable and solve the Schrödinger equation. The eigenfunctions are well known and are given by

$$\phi_{k_1}(x) = \frac{1}{\sqrt{2^{k_1}k_1!}}\left(\frac{3\omega}{\pi\hbar}\right)^{1/4} e^{(-3\omega/2\hbar)x^2} H_{k_1}\left(\sqrt{\frac{3\omega}{\hbar}}x\right), \tag{3.26}$$

$$\phi_{k_2}(y) = \left(\frac{\omega}{\hbar}\right)\left(\frac{k_2!\left(\left(\frac{\omega}{\hbar}\right)^{3/2}\right)}{\Gamma\left(k_2+\frac{5}{2}\right)}\right)^{1/2} e^{-(\omega/2\hbar)y^2} y^2 L_{k_2}^{3/2}\left(\frac{\omega}{\hbar}y^2\right). \tag{3.27}$$

where L_n^α is a Laguerre polynomial. The corresponding energy spectrum is

$$E = \omega\hbar(3k_1 + 2k_2 + 4). \tag{3.28}$$

Potential 4:

$$V = \hbar^2\left(\frac{9x^2+y^2}{8a^4} + \frac{1}{(y-a)^2} + \frac{1}{(y+a)^2}\right).$$

The two integrals are given by the formulas

$$A = P_x^2 - P_y^2 + 2\hbar^2\left(\frac{9x^2-y^2}{8a^4} + \frac{1}{(y-a)^2} + \frac{1}{(y+a)^2}\right), \tag{3.29}$$

$$B = \frac{1}{2}\{L,P_y^2\} + \frac{1}{2}\hbar^2\left\{y\left(\frac{y^2}{12a^4} - \frac{8a^2}{(y^2-a^2)^2} - \frac{2}{y^2-a^2}\right),P_y\right\} + \frac{1}{2}\hbar^2\left\{x\left(\frac{8(y^2+a^2)}{(y^2-a^2)^2} - \frac{y^2}{a^4}\right),P_y\right\}. \tag{3.30}$$

The cubic algebra and the Casimir operator are

$$[A,B] = C, \quad [A,C] = \frac{36h^4}{a^4}B,$$

$$[B,C] = -2\hbar^2A^3 - 6\hbar^2A^2H - 8\hbar^2H^3 + 10\frac{\hbar^6}{a^4}A + 18\frac{\hbar^6}{a^4}H - 24\frac{\hbar^8}{a^6}, \tag{3.31}$$

$$K = -16\hbar^2H^4 + 112\frac{\hbar^6}{a^4}H^2 + 96\frac{\hbar^8}{a^6}H - 171\frac{\hbar^{10}}{a^8}. \tag{3.32}$$

The structure function is

$$\begin{aligned}\Phi(x) = \frac{-9\hbar^6}{a^4}&\left(x+u-\left(\frac{a^2E}{3\hbar^2}-\frac{1}{2}\right)\right)\left(x+u-\left(\frac{-a^2E}{3\hbar^2}+\frac{1}{2}\right)\right)\left(x+u-\left(\frac{a^2E}{3\hbar^2}+\frac{5}{6}\right)\right)\\ &\times\left(x+u-\left(\frac{a^2E}{3\hbar^2}+\frac{7}{6}\right)\right).\end{aligned} \tag{3.33}$$

For the case $a=ia_0$, $a_0\in\mathbb{R}$, we get the three following unitary representations:

$$\Phi(x) = \frac{9\hbar^6}{a_0^4}x(p+1-x)\left(x-\frac{4}{3}\right)\left(x-\frac{5}{3}\right), \quad E = \frac{3\hbar^2(p)}{2a_0^2}, \tag{3.34}$$

$$\Phi(x) = \frac{36\hbar^6}{a_0^4} x(p+1-x)\left(x+\frac{4}{3}\right)\left(x-\frac{1}{3}\right), \quad E = \frac{3\hbar^2\left(p+\frac{4}{3}\right)}{2a_0^2}, \tag{3.35}$$

$$\Phi(x) = \frac{36\hbar^6}{a_0^4} x(p+1-x)\left(x+\frac{2}{3}\right)\left(x+\frac{1}{3}\right), \quad E = \frac{3\hbar^2\left(p+\frac{5}{3}\right)}{2a_0^2}. \tag{3.36}$$

For the case $a \in \mathbb{R}$ we get the three unitary representations:

$$\Phi(x) = \frac{9\hbar^6}{a^4} x(p+1-x)\left(p+\frac{7}{3}-x\right)\left(p+\frac{8}{3}-x\right), \quad E = \frac{3\hbar^2(p+2)}{2a^2}, \tag{3.37}$$

$$\Phi(x) = \frac{9\hbar^6}{a^4} x(p+1-x)\left(p+\frac{4}{3}-x\right)\left(p-\frac{1}{3}-x\right), \quad E = \frac{3\hbar^2\left(p+\frac{2}{3}\right)}{2a^2}, \tag{3.38}$$

$$\Phi(x) = \frac{9\hbar^6}{a^4} x(p+1-x)\left(p+\frac{2}{3}-x\right)\left(p-\frac{2}{3}-x\right), \quad E = \frac{3\hbar^2\left(p+\frac{1}{3}\right)}{2a^2}. \tag{3.39}$$

Potential 5:

$$V = \hbar^2\left(\frac{1}{8a^4}\left[(x^2+y^2) + \frac{1}{y^2} + \frac{1}{(x+a)^2} + \frac{1}{(x-a)^2}\right]\right),$$

and

potential 6:

$$V = \hbar^2\left[\frac{1}{8a^4}(x^2+y^2) + \frac{1}{(y+a)^2} + \frac{1}{(y-a)^2} + \frac{1}{(x+a)^2} + \frac{1}{(x-a)^2}\right],$$

are particular. Their integrals of motion A, B, and C do not close in a finite cubic algebra. Closure at a higher order remains to be investigated. In these cases, we have the separation of variables and the unidimensional parts are related to potentials 1 and 4 and their spectra. We will see also in Sec. IV that we can obtain information using SUSYQM.

IV. SUPERSYMMETRIC QUANTUM MECHANICS

In this section we will investigate a relation between SUSYQM (Ref. 24) and superintegrable systems with a third order integral of motion. Let us recall some aspects of SUSYQM. We define two first order operators,

$$A = \frac{\hbar}{\sqrt{2}}\frac{d}{dx} + W(x), \quad A^\dagger = -\frac{\hbar}{\sqrt{2}}\frac{d}{dx} + W(x). \tag{4.1}$$

We consider the following two Hamiltonians which are called "superpartners:"

$$H_1 = A^\dagger A = -\frac{\hbar^2}{2}\frac{d^2}{dx^2} + W^2 - \frac{\hbar}{\sqrt{2}}W', \quad H_2 = AA^\dagger = -\frac{\hbar^2}{2}\frac{d^2}{dx^2} + W^2 + \frac{\hbar}{\sqrt{2}}W'. \tag{4.2}$$

There are two cases. The first is $A\psi_0^{(1)} \neq 0$, $E_0^{(1)} \neq 0$, $A^\dagger\psi_0^{(2)} \neq 0$, and $E_0^{(2)} \neq 0$. We have

$$E_n^{(2)} = E_n^{(1)} > 0, \quad \psi_n^{(2)} = \frac{1}{\sqrt{E_n^{(1)}}}A\psi_n^{(1)}, \quad \psi_n^{(1)} = \frac{1}{\sqrt{E_n^{(2)}}}A^\dagger\psi_n^{(2)}. \tag{4.3}$$

and the two Hamiltonians are isospectral. This case corresponds to broken SUSY.

For the second case the SUSY is unbroken and we have $A\psi_0^{(1)}=0$, $E_0^{(1)}=0$, $A^\dagger\psi_0^{(2)}\neq 0$, and $E_0^{(2)}\neq 0$. Without loss of generality we take H_1 as the potential having the zero energy ground state. We have

$$E_n^{(2)}=E_{n+1}^{(1)}, \quad E_0^{(1)}=0, \quad \psi_n^{(2)}=\frac{1}{\sqrt{E_{n+1}^{(1)}}}A\psi_{n+1}^{(1)}, \quad \psi_{n+1}^{(1)}=\frac{1}{\sqrt{E_n^{(2)}}}A^\dagger\psi_n^{(2)}. \tag{4.4}$$

We can define the matrices

$$H=\begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}, \quad Q=\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad Q^t=\begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}. \tag{4.5}$$

We get the relations

$$[H,Q]=[H,Q^t]=0, \quad \{Q,Q\}=\{Q^\dagger,Q^\dagger\}=0, \quad \{Q,Q^\dagger\}=H. \tag{4.6}$$

We have an $sl(1|1)$ superalgebra and H_1 and H_2 are superpartners. By construction, all our potentials can be viewed as the sum of two one-dimensional potentials, $H=H_x+H_y$. The unidimensional parts of the three reducible potentials and the irreducible potentials 2 and 3 are known in SUSYQM. These potentials have the shape invariance property.[29,31] We will show that potentials 1 and 4–6 can be also discussed from the point of view of SUSY.

A. Potential 1

The Hamiltonian is

$$H=H_x+H_y=\frac{P_x^2}{2}+\frac{P_y^2}{2}+\hbar^2\left(\frac{x^2+y^2}{8a^4}+\frac{1}{(x-a)^2}+\frac{1}{(x+a)^2}\right). \tag{4.7}$$

Let us define the two operators,

$$b^\dagger=\frac{1}{\sqrt{2}}\left(-\hbar\frac{d}{dx}-\frac{\hbar}{2a^2}x-\hbar\left(\frac{-1}{x-a}+\frac{-1}{x+a}\right)\right), \tag{4.8}$$

$$b=\frac{1}{\sqrt{2}}\left(\hbar\frac{d}{dx}-\frac{\hbar}{2a^2}x-\hbar\left(\frac{-1}{x-a}+\frac{-1}{x+a}\right)\right). \tag{4.9}$$

For $a=ia_0$, $a_0\in\mathbb{R}$, we have

$$H_1=b^\dagger b=\frac{P_x^2}{2}+\frac{\hbar^2x^2}{8a_0^4}+\frac{\hbar^2}{(x-ia_0)^2}+\frac{\hbar^2}{(x+ia_0)^2}+\frac{3\hbar^2}{4a_0^2}, \tag{4.10}$$

$$H_2=bb^\dagger=\frac{P_x^2}{2}+\frac{\hbar^2x^2}{8a_0^4}+\frac{5\hbar^2}{4a_0^2}. \tag{4.11}$$

These two unidimensional Hamiltonians are almost isospectral. H_1 has a zero energy ground state. The SUSY is unbroken. This potential was discussed in Ref. 33. Nonsingular superpartners of the harmonic oscillator were discussed in Refs. 34 and 35. Coherent states of superpartners of the harmonic oscillator have also been studied.[36] Wee see that $H_1=H_x+3\hbar^2/4a_0^2$ is the Hamiltonian that we are interested in and its superpartner H_2 corresponds to a harmonic oscillator.

We apply results for the unbroken SUSY. The zero energy ground state satisfies $b\phi_0=0$ and is

$$\phi_0(x)=a_0^{3/2}\left(\frac{2}{\pi}\right)^{1/4}\frac{e^{-x^2/4a_0^2}}{a_0^2+x^2}. \tag{4.12}$$

The other eigenfunctions of H_1 are obtained by the equation $\phi_{n+1}^{(1)} = (1/\sqrt{E_n^{(2)}})b^\dagger \phi_n^{(2)}$. In this case $\psi_n^{(2)}$ are only the eigenfunctions of the harmonic oscillator (H_2) that are written in terms of Hermite polynomials. We get directly for H_1

$$\phi_{k_1+1}(x) = b^\dagger\left(\frac{1}{\sqrt{2^{k_1}k_1!}}\left(\frac{1}{2a_0^2\pi}\right)^{1/4} e^{(-1/4a_0^2)x^2} H_{k_1}\left(\sqrt{\frac{1}{2a_0^2}}x\right)\right)$$
$$= \frac{a_0}{\sqrt{(k_1+3)}}\left(\frac{1}{2a_0^2\pi}\right)^{1/4}\frac{1}{\sqrt{2^{k_1}k_1!}}e^{-x^2/4a_0^2}\left(\frac{(x^3+3xa_0^2)}{a_0^2(x^2+a_0^2)}H_{k_1} - \frac{2k_1}{\sqrt{2}a_0}\lambda H_{k_1-1}\right), \tag{4.13}$$

$\lambda = 1$ for $k_1 \geq 1$ and $\lambda = 0$ for $k_1 = 0$. With this expression we get for $k_1 = 0$

$$\phi_1 = \frac{1}{\sqrt{3}(2\pi)^{1/4}a_0^{3/2}}e^{-x^2/4a_0^2}\frac{x(3a_0^2+x^2)}{a_0^2+x^2}. \tag{4.14}$$

We have the following energy spectrum for H_1:

$$E_0^{(1)} = 0, \quad E_{k_1+1}^{(1)} = \frac{\hbar^2}{2a_0^2}(k_1+3). \tag{4.15}$$

We thus obtain the spectrum of H_x (the x part of the irreducible potential 1):

$$E_0^x = -\frac{3\hbar^2}{4a_0^2}, \quad E_{k_1+1}^x = \frac{\hbar^2}{2a_0^2}\left(k_1 + \frac{3}{2}\right). \tag{4.16}$$

If we add H_y to these results we get the energy spectrum and the eigenfunctions of potential 1. There are two families of solutions. The first corresponds to the energies

$$E = \frac{(k_1+k_2+2)\hbar^2}{2a_0^2} = \frac{(p+2)\hbar^2}{2a_0^2}, \tag{4.17}$$

with eigenfunctions

$$\phi_{k_1+1}(x) = \frac{a_0}{\sqrt{(k_1+3)}}\left(\frac{1}{2a_0^2\pi}\right)^{1/4}\frac{1}{\sqrt{2^{k_1}k_1!}}e^{-x^2/4a_0^2}\left(\frac{(x^3+3xa_0^2)}{a_0^2(x^2+a_0^2)}H_{k_1} - \frac{2k_1}{\sqrt{2}a_0}\lambda H_{k_1-1}\right), \tag{4.18}$$

$$\chi_{k_2}(y) = \frac{1}{\sqrt{2^{k_2}k_2!}}\left(\frac{1}{2a_0^2\pi}\right)^{1/4} e^{(-1/4a_0^2)y^2} H_{k_2}\left(\sqrt{\frac{1}{2a_0^2}}y\right) \tag{4.19}$$

and is also obtained from the cubic algebra. The second corresponds to the energies

$$E = \frac{\hbar^2(k_2-1)}{2a_0^2}, \tag{4.20}$$

with the corresponding eigenfunctions

$$\psi(x,y) = \phi_0(x)\chi_{k_2}(y), \quad \phi_0(x) = a_0^{3/2}\left(\frac{2}{\pi}\right)^{1/4}\frac{e^{-x^2/4a_0^2}}{a_0^2+x^2}, \tag{4.21}$$

and $\chi_{k_2}(y)$ as in Eq. (4.19).

The two states obtained from Eq. (3.8) are given by Eq. (4.20) for $k_2 = 0, 1$. For $k_3 \geq 3$ there are common eigenvalues given by Eqs. (4.17) and (4.20) and therefore the degeneracy is $p+2$.

Let us consider the case $a \in \mathbb{R}$. We have the following Hamiltonians:

$$H_1 = b^t b = \frac{P_x^2}{2} + \frac{\hbar^2 x^2}{8a^4} + \frac{\hbar^2}{(x-a)^2} + \frac{\hbar^2}{(x+a)^2} - \frac{3\hbar^2}{4a^2}, \tag{4.22}$$

$$H_2 = b b^t = \frac{P_x^2}{2} + \frac{\hbar^2 x^2}{8a^4} - \frac{5\hbar^2}{4a^2}. \tag{4.23}$$

This case is more complicated because of the singularities on the x axis for the Hamiltonian H_1. We have a regular Hamiltonian connected to a singular one and we have also for H_2 negative energy states. Such situations have attracted a lot of attention and many articles were devoted to such singular potentials. An important case is the one of Jevicki and Rodrigues.[37,38] The corresponding Hamiltonians are

$$H_- = \frac{d^2}{dx^2} + x^2 - 3, \quad H_+ = -\frac{d^2}{dx^2} + x^2 + \frac{2}{x^2} - 1. \tag{4.24}$$

Factorization of Hamiltonians H_1 and H_2 given by Eqs. (4.22) and (4.23) gives us an algebraic relation that does not take into account the presence of singularities or boundary conditions. The wave functions given in Eqs. (4.3) and (4.4) do not necessarily belong to the Hilbert space of square integrable functions. The potential in Eq. (4.22) has impenetrable barriers coming from the singularities. We can consider the superpartner to be the harmonic oscillator with two infinite barriers (at $x = \pm a$) to recover the SUSY.[39] In Ref. 39 a superpartner of the harmonic oscillator with one singularity was considered but the method can be extended to more singularities. The only case that was solved analytically and where the energy levels are equidistant is when the singularity was at the origin. In our case we were not able to solve analytically and we leave for future investigations these numerical calculations that appear interesting from a phenomenological point of view. Singular potentials were also investigated by Das and Pernice[40] by means of the regularization method. Znojil[41] discussed another method that consist in the complexification of the potential. In Sec. VI we will discuss the complexification of the irreducible quantum superintegrable potential 1.

B. Potential 4

We apply these results to the next irreducible potential,

$$V = \hbar^2 \left(\frac{9x^2 + y^2}{8a^4} + \frac{1}{(y-a)^2} + \frac{1}{(y+a)^2} \right). \tag{4.25}$$

We can also use SUSYQM because the y part is the same as the x part of potential 1. For the case $a = i a_0$, $a_0 \in \mathbb{R}$, we find with energy

$$E = \frac{\hbar^2}{2a_0^2}(3k_1 + k_2 + 3), \tag{4.26}$$

with the corresponding eigenfunctions

$$\chi_{k_1}(x) = \frac{1}{\sqrt{2^{k_1} k_1!}} \left(\frac{3}{2a_0^2 \pi} \right)^{1/4} e^{-(3/4a_0^2)x^2} H_{k_1}\left(\sqrt{\frac{3}{2a_0^2}} x \right), \tag{4.27}$$

$$\phi_{k_2+1}(y) = \frac{a_0}{\sqrt{(k_2+3)}} \left(\frac{1}{2a_0^2 \pi} \right)^{1/4} \frac{1}{\sqrt{2^{k_2} k_2!}} e^{-y^2/4a_0^2} \left(\frac{(y^3 + 3y a_0^2)}{a_0^2 (y^2 + a_0^2)} H_{k_2} - \frac{2k_2}{\sqrt{2} a_0} \lambda H_{k_2 - 1} \right), \tag{4.28}$$

and we get from the singlet state the energies

$$E = \frac{\hbar^2}{2a_0^2}(3k_1) \tag{4.29}$$

and eigenfunctions

$$\psi(x,y) = \phi_0(y)\chi_{k_1}(x), \quad \phi_0(y) = a_0^{3/2}\left(\frac{2}{\pi}\right)^{1/4}\frac{e^{-y^2/4a_0^2}}{a_0^2+y^2} \tag{4.30}$$

and $\chi_{k_1}(x)$ as in Eq. (4.27).

C. Potential 5

The potential is

$$V = \hbar^2\left[\frac{1}{8a^4}\left[(x^2+y^2) + \frac{1}{y^2} + \frac{1}{(x+a)^2} + \frac{1}{(x-a)^2}\right]\right] . \tag{4.31}$$

For the case $a=ia_0$, $a_0 \in \mathbb{R}$, we have

$$E = \frac{(k_1+2k_2+5)}{2a_0^2}\hbar^2, \tag{4.32}$$

with the eigenfunctions given by

$$\phi_{k_1}(x) = \frac{a_0}{\sqrt{(k_1+3)}}\left(\frac{1}{2a_0^2\pi}\right)^{1/4}\frac{1}{\sqrt{2^{k_1}k_1!}}e^{-x^2/4a_0^2}\left(\frac{(x^3+3xa_0^2)}{a_0^2(x^2+a_0^2)}H_{k_1} - \frac{2k_1}{\sqrt{2}a_0}\lambda H_{k_1-1}\right), \tag{4.33}$$

$$\chi_{k_2}(y) = \left(\frac{1}{2a_0^2}\right)^{1/4}\left(\frac{k_2\,!\left(\frac{1}{2a_0^2}\right)^{3/2}}{\Gamma\left(k_2+\frac{5}{2}\right)}\right)^{1/2} e^{-y^2/4a_0^2}y^2L_{k_2}^{3/2}\left(\frac{y^2}{2a_0^2}\right), \tag{4.34}$$

where $L_k^{\nu}(z)$ are Laguerre polynomials. We have also the energies

$$E = \frac{\hbar^2(2k_2+2)}{2a_0^2}, \tag{4.35}$$

with the corresponding eigenfunctions

$$\psi(x,y) = \chi_{k_2}(y)\phi_0(x), \quad \phi_0(x) = a_0^{3/2}\left(\frac{2}{\pi}\right)^{1/4}\frac{e^{-x^2/4a_0^2}}{a_0^2+x^2} \tag{4.36}$$

and $\chi_{k_2}(y)$ as in Eq. (4.34).

D. Potential 6

We consider the potential

$$V = \hbar^2\left[\frac{1}{8a^4}(x^2+y^2) + \frac{1}{(y+a)^2} + \frac{1}{(y-a)^2} + \frac{1}{(x+a)^2} + \frac{1}{(x-a)^2}\right]. \tag{4.37}$$

For the case $a=ia_0$, $a_0 \in \mathbb{R}$, we have the energies

$$E=\frac{(k_1+k_2+3)}{2a_0^2}\hbar^2, \tag{4.38}$$

with the eigenfunctions given by

$$\phi_{k_1+1}(x)=\frac{a_0}{\sqrt{(k_1+3)}}\left(\frac{1}{2a_0^2\pi}\right)^{1/4}\frac{1}{\sqrt{2^{k_1}k_1!}}e^{-x^2/4a_0^2}\left(\frac{(x^3+3xa_0^2)}{a_0^2(x^2+a_0^2)}H_{k_1}-\frac{2k_1}{\sqrt{2}a_0}\lambda H_{k_1-1}\right), \tag{4.39}$$

$$\chi_{k_2+1}(y)=\frac{a_0}{\sqrt{(k_2+3)}}\left(\frac{1}{2a_0^2\pi}\right)^{1/4}\frac{1}{\sqrt{2^{k_2}k_2!}}e^{-y^2/4a_0^2}\left(\frac{(y^3+3ya_0^2)}{a_0^2(y^2+a_0^2)}H_{k_2}-\frac{2k_2}{\sqrt{2}a_0}\lambda H_{k_2-1}\right). \tag{4.40}$$

The singlet state in the x part of the Hamiltonian gives the energies

$$E=\frac{\hbar^2(k_2)}{2a_0^2}, \tag{4.41}$$

with eigenfunctions

$$\psi(x,y)=\chi_{k_2}(y)\phi_0(x), \quad \phi_0(x)=a_0^{3/2}\left(\frac{2}{\pi}\right)^{1/4}\frac{e^{-x^2/4a_0^2}}{a_0^2+x^2} \tag{4.42}$$

and $\chi_{k_2}(y)$ as in Eq. (4.40).

We also obtain another kind of solution from the singlet state in the y part. The energies are

$$E=\frac{\hbar^2(k_1)}{2a_0^2}, \tag{4.43}$$

with the corresponding eigenfunctions

$$\psi(x,y)=\phi_{k_1}(x)\chi_0(y), \quad \chi_0(y)=a_0^{3/2}\left(\frac{2}{\pi}\right)^{1/4}\frac{e^{-y^2/4a_0^2}}{a_0^2+y^2} \tag{4.44}$$

and $\phi_{k_1}(x)$ as in Eq. (4.39) and a state coming from the singlet state in the two parts with energies

$$E=-\frac{3\hbar^2}{2a_0^2}, \tag{4.45}$$

with the following expression for the eigenfunctions:

$$\phi_0(x)=a_0^{3/2}\left(\frac{2}{\pi}\right)^{1/4}\frac{e^{-x^2/4a_0^2}}{a_0^2+x^2}, \quad \chi_0(y)=a_0^{3/2}\left(\frac{2}{\pi}\right)^{1/4}\frac{e^{-y^2/4a_0^2}}{a_0^2+y^2}. \tag{4.46}$$

V. GENERATING SPECTRUM ALGEBRA

The SUSY allows us to find the creation and annihilation operators of the x part of the irreducible potential 1. They are given by

$$M=b^{\dagger}cb, \quad M^{\dagger}=b^{\dagger}c^{\dagger}b, \tag{5.1}$$

where c and $c^{\dagger}$ are annihilation and creation operators of the superpartner H_2, that is, a harmonic oscillator. We have

$$c=\frac{\hbar}{2a^2}\left(x+2a^2\frac{d}{dx}\right),\quad c^\dagger=\frac{\hbar}{2a^2}\left(x-2a^2\frac{d}{dx}\right) \tag{5.2}$$

and

$$M=\frac{1}{\sqrt{2}}\left(-\hbar\frac{d}{dx}-\frac{\hbar}{2a^2}x+\hbar\left(\frac{1}{x-a}+\frac{1}{x+a}\right)\right)\frac{\hbar}{2a^2}\left(x+2a^2\frac{d}{dx}\right)\frac{1}{\sqrt{2}}$$
$$\times\left(\hbar\frac{d}{dx}-\frac{\hbar}{2a^2}x+\hbar\left(\frac{1}{x-a}+\frac{1}{x+a}\right)\right), \tag{5.3}$$

$$M^\dagger=\frac{1}{\sqrt{2}}\left(-\hbar\frac{d}{dx}-\frac{\hbar}{2a^2}x+\hbar\left(\frac{1}{x-a}+\frac{1}{x+a}\right)\right)\frac{\hbar}{2a^2}\left(x-2a^2\frac{d}{dx}\right)\frac{1}{\sqrt{2}}$$
$$\times\left(\hbar\frac{d}{dx}-\frac{\hbar}{2a^2}x+\hbar\left(\frac{1}{x-a}+\frac{1}{x+a}\right)\right). \tag{5.4}$$

The zero energy ground state given by Eq. (4.12) is annihilated not only by the annihilation operator but also by the creation operator,

$$M\phi_0(x)=M^\dagger\phi_0=0. \tag{5.5}$$

The creation and annihilation operators for the y part (H_y) of potential 1 are

$$L=\frac{\hbar}{2a^2}\left(y+2a^2\frac{d}{dy}\right),\quad L^\dagger=\frac{\hbar}{2a^2}\left(y-2a^2\frac{d}{dy}\right). \tag{5.6}$$

We have the commutators

$$[M,M^\dagger]=\frac{3}{4}\left(H+\frac{1}{2}A\right)^2-\frac{\hbar^2}{a^2}\left(H+\frac{1}{2}A\right)-\frac{3\hbar^4}{16a^4},\quad [L,L^\dagger]=1. \tag{5.7}$$

We consider the following operators:[16]

$$E_+=M^\dagger L^\dagger,\quad E_-=ML,\quad F_+=(M^\dagger)^2,\quad F_-=M^2,\quad G_+=(L^\dagger)^2,\quad G_-=L^2. \tag{5.8}$$

We add to these operators the Hamiltonian and the integrals of motion A, B, and C [Eqs. (3.1)–(3.3)]. We have the following quintic algebra that contains 45 relations where the cubic algebra appears as a subalgebra:

$$[H,A]=0,\quad [H,B]=0,\quad [H,C]=0,\quad [H,E_\pm]=\pm\left(\frac{\hbar^2}{a^2}\right)E_\pm,$$

$$[H,F_\pm]=\pm\left(\frac{\hbar^2}{a^2}\right)F_\pm,\quad [H,G_\pm]=\pm\left(\frac{\hbar^2}{a^2}\right)G_\pm,$$

$$[A,B]=C,\quad [A,C]=\frac{4\hbar^4}{a^4}B,\quad [A,E_\pm]=0,\quad [A,F_\pm]=\pm\left(\frac{2\hbar^2}{a^2}\right)F_\pm,$$

$$[A,G_\pm]=\mp\left(\frac{2\hbar^2}{a^2}\right)G_\pm,\quad [B,C]=-2\hbar^2A^3-6\hbar^2A^2H+8\hbar^2H^3+6\frac{\hbar^4}{a^2}A^2+8\frac{\hbar^4}{a^2}HA-8\frac{\hbar^4}{a^2}H^2$$
$$+2\frac{\hbar^6}{a^4}A-2\frac{\hbar^6}{a^4}H-6\frac{\hbar^8}{a^6},$$

$$[B,E_-]=-2i\hbar F_-+\frac{3i\hbar}{2}\left(H+\frac{1}{2}A\right)^2G_- -\frac{2i\hbar^3}{a^2}\left(H+\frac{1}{2}A\right)G_- -\frac{3i\hbar^5}{8a^4}G_-,$$

$$[B,E_+]=-2i\hbar F_+ +\frac{3i\hbar}{2}\left(H+\frac{1}{2}A\right)^2G_+ -\frac{2i\hbar^3}{a^2}\left(H+\frac{1}{2}A\right)G_+ -\frac{3i\hbar^5}{8a^4}G_+,$$

$$[B,F_-]=3i\hbar\left(H+\frac{1}{2}A\right)^2E_- -\frac{7i\hbar^3}{a^2}\left(H+\frac{1}{2}A\right)E_- +\frac{11i\hbar^5}{4a^4}E_-,$$

$$[B,F_+]=3i\hbar\left(H+\frac{1}{2}A\right)^2E_+ -\frac{i\hbar^3}{a^2}\left(H+\frac{1}{2}A\right)E_+ -\frac{5i\hbar^5}{4a^4}E_+,$$

$$[B,G_+]=-4i\hbar E_+,\quad [B,G_-]=-4i\hbar E_-,$$

$$[C,E_-]=\frac{4i\hbar^3}{a^3}F_- +\frac{3i\hbar^3}{a^2}\left(H+\frac{1}{2}A\right)^2G_- -\frac{4i\hbar^5}{a^4}\left(H+\frac{1}{2}A\right)G_- -\frac{3i\hbar^7}{4a^6}G_-,$$

$$[C,E_+]=\frac{-4i\hbar^3}{a^3}F_+ -\frac{3i\hbar^3}{a^2}\left(H+\frac{1}{2}A\right)^2G_+ +\frac{4i\hbar^5}{a^4}\left(H+\frac{1}{2}A\right)G_+ +\frac{3i\hbar^7}{4a^6}G_+,$$

$$[C,F_-]=\frac{6i\hbar^3}{a^2}\left(H+\frac{1}{2}A\right)^2E_- -\frac{14i\hbar^5}{a^4}\left(H+\frac{1}{2}A\right)E_- +\frac{11i\hbar^7}{2a^6}E_-,$$

$$[C,F_+]=-\frac{6i\hbar^3}{a^2}\left(H+\frac{1}{2}A\right)^2E_+ +\frac{2i\hbar^5}{a^4}\left(H+\frac{1}{2}A\right)E_+ +\frac{5i\hbar^7}{2a^6}E_+,$$

$$[C,G_\pm]=\mp\frac{8i\hbar^3}{a^2}E_\pm,\quad [E_\pm,F_\pm]=0,\quad [E_\pm,G_\pm]=0,$$

$$[E_-,E_+]=\frac{-a^2\hbar^2}{16}A^3+\frac{3a^2\hbar^2}{4}AH^2+a^2\hbar^2H^3+\frac{\hbar^4}{8}A^2-\frac{\hbar^4}{2}AH-\frac{3\hbar^4}{2}H^2+\frac{\hbar^6}{16a^2}A-\frac{\hbar^6}{4a^2}H+\frac{3\hbar^8}{8a^4},$$

$$[E_+,F_-]=\frac{-3ia^2\hbar}{16}C\left(H+\frac{1}{2}A\right)^2+\frac{3i\hbar^3}{8}B\left(H+\frac{1}{2}A\right)^2+\frac{7i\hbar^3}{16}C\left(H+\frac{1}{2}A\right)-\frac{7i\hbar^5}{8a^2}B\left(H+\frac{1}{2}\right)$$
$$-\frac{11i\hbar^5}{64a^2}C+\frac{11i\hbar^7}{32a^4}B,$$

$$[E_-,F_+]=\frac{3ia^2\hbar}{16}C\left(H+\frac{1}{2}A\right)^2+\frac{3i\hbar^3}{8}B\left(H+\frac{1}{2}A\right)^2-\frac{i\hbar^3}{16}C\left(H+\frac{1}{2}A\right)-\frac{i\hbar^5}{8a^2}B\left(H+\frac{1}{2}\right)-\frac{5i\hbar^5}{64a^2}C$$
$$-\frac{5i\hbar^7}{32a^4}B,$$

$$[E_-,G_+]=\frac{ia^2\hbar}{4}C-\frac{i\hbar^3}{2}B,\quad [E_+,G_-]=\frac{-ia^2\hbar}{4}C-\frac{i\hbar^3}{2}B,$$

$$[F_-,F_+]=\frac{3a^2\hbar^2}{4}\left(H+\frac{1}{2}A\right)^5-\frac{5\hbar^4}{2}\left(H+\frac{1}{2}A\right)^4+\frac{25\hbar^6}{8a^2}\left(H+\frac{1}{2}A\right)^3-\frac{5\hbar^8}{4a^4}\left(H+\frac{1}{2}A\right)^2-\frac{53\hbar^{10}}{64a^6}\left(H+\frac{1}{2}A\right)+\frac{15\hbar^{12}}{32a^8},$$

$$[F_\pm,G_\pm]=[F_\pm,G_\mp]=0,\quad [G_-,G_+]=4a^2\hbar^2\left(H+\frac{1}{2}A\right). \tag{5.9}$$

This polynomial algebra is the spectrum-generating algebra.

VI. COMPLEXIFICATION OF SUPERINTEGRABLE POTENTIALS

In quantum mechanics textbooks the Hermiticity of the Hamiltonian is often presented as a condition for the energy spectrum to be real. There exist other requirements that can be chosen without losing essential features of quantum mechanics. One requirement that appears more physical is the space-time reflection symmetry, i.e., the Hamiltonian is invariant under the PT transformation,[42] i.e., the simultaneous reflections $P:x\to -x$, $p\to -p$ and $\tau:x\to x$, $p\to -p$, $i\to -i$. For potentials invariant under such transformations the energy spectrum can also consist of complex-conjugate pairs of eigenvalues. The PT symmetry is thus said to be broken. The notion of pseudo-Hermiticity was introduced by Mostafazadeh.[43] He showed also that every Hamiltonian with a real spectrum is pseudo-Hermitian and that all PT-symmetric Hamiltonians studied belong to the class of pseudo-Hermitian Hamiltonian. The replacement of the condition that the Hamiltonian is Hermitian by a weaker condition allows us to study many new kinds of Hamiltonians that would have been excluded and from a phenomenological point of view may describe physics phenomena. The case $H=p^2+x^2(ix)^\delta$ was studied in detail by Bender[42] in 1998.

Complexification has been proposed as a natural way to regularize singular potentials.[41] It consists in a transformation of the type $x\to x-i\epsilon$ applied to a potential. The harmonic oscillator and the Smorodinsky–Winternitz potential are PT-symmetric Hamiltonian after a complexification.[41]

We will consider the complexification of the Hamiltonian

$$H=H_x+H_y=\frac{P_x^2}{2}+\frac{P_y^2}{2}+\hbar^2\left(\frac{(x-i\epsilon)^2+(y-i\epsilon)^2}{8a^4}+\frac{1}{(x-i\epsilon-a)^2}+\frac{1}{(x-i\epsilon+a)^2}\right). \tag{6.1}$$

The complex harmonic oscillator Hamiltonian H_y is known to be PT symmetric. Its energy spectrum is real, namely,

$$E=\frac{\hbar^2}{2a^2}\left(m+\frac{1}{2}\right). \tag{6.2}$$

The eigenfunctions are

$$\phi_m(y)=N_m e^{-(y-i\epsilon)^2/4a^2}H_m\left(\frac{(y-i\epsilon)}{\sqrt{2}a}\right) \tag{6.3}$$

(here and below N_m is a normalization constant).

To get the energy spectrum and the eigenfunctions of H_x we complexify the operators given by Eqs. (4.8) and (4.9). We get two PT-symmetric Hamiltonians. This transformation allows to regularize H_x when $a\in\mathbb{R}$. The (real) energy levels and eigenfunctions of the Hamiltonian H_2 are known,

$$H_1=b'b=\frac{P_x^2}{2}+\frac{\hbar^2(x-i\epsilon)^2}{8a^4}+\frac{\hbar^2}{(x-i\epsilon-a)^2}+\frac{\hbar^2}{(x-i\epsilon+a)^2}-\frac{3\hbar^2}{4a^2}, \tag{6.4}$$

$$H_2 = bb' = \frac{P_x^2}{2} + \frac{\hbar^2(x-i\epsilon)^2}{8a^4} - \frac{5\hbar^2}{4a^2}. \tag{6.5}$$

The Darboux transformation is still valid for non-Hermitian Hamiltonians but SUSY is replaced by pseudo-SUSY.[44] We have $b'\psi_{gr}=0$ that correspond to the zero energy state of H_2,

$$\psi_{gr} = N_{gr} e^{-(x-i\epsilon)^2/4a^2}(a^2-(x-i\epsilon)^2). \tag{6.6}$$

We can obtain the eigenfunction of H_1 by applying b' on the other state of H_2 given in terms of Hermite polynomials. We get

$$\phi_n = N_n e^{-(x-i\epsilon)^2/4a^2}\left(\frac{2(x-i\epsilon)}{(x-i\epsilon)^2-a^2}H_{n+3}\left(\frac{(x-i\epsilon)}{\sqrt{2}a}\right) - \frac{2(n+3)}{\sqrt{2}a}H_{n+2}\left(\frac{(x-i\epsilon)}{\sqrt{2}a}\right)\right). \tag{6.7}$$

Let us give the explicit expression for the ground state and the first excited state,

$$\phi_0 = N_0 e^{-(x-i\epsilon)^2/4a^2}\frac{(3a^4+(x-i\epsilon)^4)}{(a^2-(x-i\epsilon)^2)}, \tag{6.8}$$

$$\phi_1 = N_1 e^{-(x-i\epsilon)^2/4a^2}\frac{(3a^4+2a^2i(x-i\epsilon)+(x-i\epsilon)^4)(x-i\epsilon)}{(a^2-(x-i\epsilon)^2)}. \tag{6.9}$$

The probabilistic interpretation of the wave function of non-Hermitian quantum systems[45] is given by a pseudonorm that is not positive definite,

$$\int_{-\infty}^{\infty} dx\,\psi^*(-x)\psi(x) = \sigma, \quad \sigma = \pm 1. \tag{6.10}$$

The corresponding energy spectrum is given by

$$E_n = \frac{(n+1)\hbar^2}{2a^2}. \tag{6.11}$$

We obtain for the complexified superintegrable potential the energy spectrum

$$E = \frac{(n+m+3)\hbar^2}{2a^2} = \frac{(p+3)\hbar^2}{2a^2}, \tag{6.12}$$

with eigenfunction given by Eqs. (6.3) and (6.7).

VII. CONCLUSION

The main result of this article is that we have constructed a Fock-type representation for the most general cubic algebra generated by a second order and a third order order integral of motion by means of parafermionic algebras. We present in detail the cubic algebra for all irreducible quantum superintegrable potentials, the unitary representations, and the corresponding energy spectra. All cases with finite cubic algebras belong to Case 2 of Sec. II. Thus they correspond to $\beta=0$ in Eq. (2.5) and the structure function is given by Eq. (2.22). In two cases of irreducible potentials, the integrals of motion do not close in a finite-dimensional cubic algebra. It could be interesting to see what kind of algebraic structure is involved in these cases. Comparing with an earlier article[10] we can see from this article how the cubic Poisson algebra is deformed into a cubic algebra in quantum mechanics.

The method that we use to find energy spectra with the cubic algebra is independent of the choice of coordinate systems. We could apply these results in the future to systems with a third order integral that are separable in polar, elliptic, or parabolic coordinates. The method is also independent of the metric and could be applied to superintegrable systems in other spaces. The methods developed in this article could be applied to other physical systems. One such system is a Schrödinger equation with a position dependent mass;[32] others arise in the context of SUSYQM.

Potential 3 is also a special case of the following potential:[46,47]

$$V = \frac{\omega^2}{2}(k^2x^2 + m^2y^2) + \frac{\lambda_1}{x^2} + \frac{\lambda_2}{y^2}. \tag{7.1}$$

In general, this system has integrals of motion of order greater than 3 and the more complicated polynomial algebra should be studied.

All the potentials considered in this article can also be viewed as the sum of two one-dimensional potentials, $H=H_x+H_y$. We have investigated each of these unidimensional potentials in terms of SUSYQM. The superintegrability of these two-dimensional potentials seems to be related to the SUSY property. Using the SUSY we have obtained the energy spectra and the eigenfunctions. We have compared the results with those obtained using the cubic algebras. One particular feature is the appearance of singlet states. For potential 1 there is an additional degeneracy that is not obtained by the algebraic method using the cubic algebra.

It was shown that many well known potentials such as the Dirac delta and Poschl–Teller display a hidden SUSY where the reflection (parity) operator plays the role of the grading operator.[48] Potentials with elliptic functions can also be discussed from this point of view.[49] Potentials with elliptic functions appear in Ref. 9. These cases are not truly superintegrable since there exists a syzygy between the Hamiltonian, second order integral, and third order integral of motion but it has been shown that the third order integral can be used to obtain the eigenfunctions and the spectrum.[50] We leave quantum potentials involving Painlevé transcendents for a future article.

Superintegrable potentials and their integrals of motion can be complexified and investigated from the point of view of PT-symmetric quantum mechanics. The complexification appears also as a natural way to regularize the singular potentials.

It would be interesting to investigate the relation of pseudo-Hermitian Hamiltonians and SUSY with superintegrable systems.

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APPENDIX: STRUCTURE FUNCTION FOR THE CASE $\beta \neq 0$

$$\begin{aligned}
\Phi(N) = {} & 384\mu\beta^{10}N^{10} - 1920\mu\beta^{10}N^9 + (-1536\delta\mu\beta^8 + 1024\nu\beta^9 + 3040\mu\beta^{10} - 2304\beta^8\alpha^2)N^8 \\
& + (6144\delta\mu\beta^8 - 4096\nu\beta^9 - 640\mu\beta^{10} + 6144\beta^8\alpha^2)N^7 + (2304\delta^2\mu\beta^6 - 3072\nu\delta\beta^7 3072\xi\beta^8 \\
& - 7680\delta\mu\beta^8 + 5120\nu\beta^9 - 2512\mu\beta^{10} - 3072\delta\beta^6\alpha^2 + 2304\beta^8\alpha^2 + 3072\beta^7\alpha\gamma)N^6 \\
& + (-6912\delta^2\mu\beta^6 + 9216\nu\delta\beta^7 - 9216\xi\beta^8 + 1536\delta\mu\beta^8 - 1024\nu\beta^9 + 1712\mu\beta^{10} \\
& + 9216\delta\beta^6\alpha^2 - 7680\beta^8\alpha^2 - 9216\beta^7\alpha\gamma)N^5 + (-1536\delta^3\mu\beta^4 + 3072\nu\delta^2\beta^5 - 6144\xi\delta\beta^6 \\
& + 6336\delta^2\mu\beta^6 - 8448\nu\delta\beta^7\alpha\gamma + 8448\xi\beta^8 + 3264\delta\mu\beta^8 - 2176\nu\beta^9 + 428\mu\beta^{10} \\
& + 4608\delta^2\beta^4\alpha^2 - 8448\delta\beta^6\alpha^2 + 672\beta^8\alpha^2 - 9216\delta\beta^5\alpha\gamma + 8448\beta^7 + 3072\beta^6\gamma^2 \\
& + 6144\beta^6\alpha\epsilon + 12\,288\beta^7\zeta)N^4 + (3072\delta^3\mu\beta^4 - 6144\nu\delta^2\beta^5 + 12\,288\xi\delta\beta^6 - 1152\delta^2\mu\beta^6 \\
& + 1536\nu\delta\beta^7 - 1536\xi\beta^8 - 1920\delta\mu\beta^8 + 1280\nu\beta^9 - 616\mu\beta^{10} - 12\,288\delta^2\beta^4\alpha^2
\end{aligned}$$

$$+ 1536\delta\beta^6\alpha^2 + 2688\beta^8\alpha^2 + 24\,576\delta\beta^5\alpha\gamma - 1536\beta^7\alpha\gamma - 6144\beta^6\gamma^2 - 24\,576\beta^6\alpha\epsilon$$

$$- 24\,576\beta^7\zeta)N^3 + \Bigg(384\delta^4\mu\beta^2 - 1024\nu\delta^3\beta^3 + 3072\xi\delta^2\beta^4 - 1792\delta^3\mu\beta^4 + 3584\nu\delta^2\beta^5$$

$$- 9216\xi\delta\beta^6 - 784\delta^2\mu\beta^6 - 1728\xi\beta^8 + 1728\nu\delta\beta^7 - 96\delta\mu\beta^8 + 64\nu\beta^9 + \frac{119}{2}\mu\beta^{10}$$

$$- 12\,288\beta^6K - 3072\delta^3\beta^2\alpha^2 + 6912\delta^2\beta^4\alpha^2 + 1728\delta\beta^6\alpha^2 - 624\beta^8\alpha^2 + 9216\delta^2\beta^3\alpha\gamma$$

$$- 13\,824\delta\beta^5\alpha\gamma - 1728\beta^7\alpha\gamma - 6144\delta\beta^4\gamma^2 + 1536\beta^6\gamma^2 - 12\,288\delta\beta^4\alpha\epsilon + 9216\beta^6\alpha\epsilon$$

$$+ 12\,288\beta^5\gamma\epsilon - 12\,288\delta\beta^5\zeta + 18\,432\beta^7\zeta\Bigg)N^2 + \Bigg(- 384\delta^4\mu\beta^2 + 1024\nu\delta^3\beta^3$$

$$- 3072\xi\delta^2\beta^4 + 256\delta^3\mu\beta^4 - 512\nu\delta^2\beta^5 + 3072\xi\delta\beta^6 + 208\delta^2\mu\beta^6 - 960\nu\delta\beta^7 + 960\xi\beta^8$$

$$+ 288\delta\mu\beta^8 - 192\nu\beta^9 + \frac{129}{2}\mu\beta^{10}12\,288\beta^6K + 3072\delta^3\beta^2\alpha^2 - 960\delta\beta^6\alpha^2 - 288\beta^8\alpha^2$$

$$- 9216\delta^2\beta^3\alpha\gamma + 960\beta^7\alpha\gamma + 6144\delta\beta^4\gamma^2 + 1536\beta^6\gamma^2 12\,288\delta\beta^4\alpha\epsilon + 6144\beta^6\alpha\epsilon$$

$$- 12\,288\beta^5\gamma\epsilon + 12\,288\delta\beta^5\zeta - 6144\beta^7\zeta\Bigg)N + \Bigg(96\delta^4\mu\beta^2 - 256\nu\delta^3\beta^3 + 768\xi\delta^2\beta^4$$

$$+ 32\delta^3\mu\beta^4 - 64\nu\delta^2\beta^5 - 384\xi\delta\beta^6 + 20\delta^2\mu\beta^6 + 144\nu\delta\beta^7 - 144\xi\beta^8 - 54\delta\mu\beta^8 + 36\nu\beta^9$$

$$- \frac{117}{8}\mu\beta^{10} - 3072\beta^6K + 768\delta^4\alpha^2 - 768\delta^3\beta^2\alpha^2 - 480\delta^2\beta^4\alpha^2 + 144\delta\beta^6\alpha + 87\beta^8\alpha^2$$

$$- 3072\delta^3\beta\alpha\gamma + 2304\delta^2\beta^3\alpha\gamma + 960\delta\beta^5\alpha\gamma - 144\beta^7\alpha\gamma + 3072\delta^2\beta^2\gamma^2 - 1536\delta\beta^4\gamma^2$$

$$- 576\beta^6\gamma^2 + 6144\delta^2\beta^2\alpha\epsilon - 3072\delta\beta^4\alpha\epsilon - 2688\beta^6\alpha\epsilon - 12\,288\delta\beta^3\gamma\epsilon + 3072\beta^5\gamma\epsilon$$

$$+ 12\,288\beta^4\epsilon^2 - 3072\delta\beta^5\zeta + 768\beta^7\zeta. \tag{A1}$$

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