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Type-I quantum superalgebras, q-supertrace, and two-variable link polynomials

Mark D. Gould and Jon R. Links
Department of Mathematics, University of Queensland, Brisbane, Qld 4072, Australia

Yao-Zhong Zhang[a)]
Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606, Japan

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A new general eigenvalue formula for the eigenvalues of Casimir invariants, for the type-I quantum superalgebras, is applied to the construction of link polynomials associated with *any* finite dimensional unitary irrep for these algebras. This affords a systematic construction of new two-variable link polynomials associated with any finite dimensional irrep (with a real highest weight) for the type-I quantum superalgebras. In particular infinite families of nonequivalent two-variable link polynomials are determined in fully explicit form. © *1996 American Institute of Physics.* [S0022-2488(96)00802-8]

I. INTRODUCTION

Following the celebrated discovery by Jones[1] of the so-called Jones' link polynomial, there has been considerable interest in recent years in modern knot theory, which has been found to be closely related, through the quantum Yang–Baxter equation (QYBE), to various areas of physics such as solvable models and quantum field theories.[2,3] With the equally important discovery of quantum algebras during the same period by Drinfeld[4] and Jimbo[5] following the initiatives of the St. Petersberg group, it was soon realized by Reshetikhin[6] and Turaev[7] that quantum algebras provided a useful tool in constructing link polynomials. This idea was further developed in Refs. 8–10 where the authors proposed a simple systematic procedure for the construction of link polynomials arising from quantum bosonic algebras.

There were many attempts (see, e.g., Refs. 2, 11, and 12) to construct new two- or multi-variable link polynomials since the work of HOMFLY[13] and Kauffman[14] concerning two-variable extensions of the Jones link polynomial. The two-variable HOMFLY and Kauffman link polynomials arise from the minimal representations of A_n and B_n, C_n, D_n quantum algebras, respectively.

Subsequently, link polynomials arising from quantum superalgebras have been addressed by various authors.[15–20] Among all quantum superalgebras those of type-I, $U_q[gl(m|n)]$ and $U_q[osp(2|2n)]$, are particularly interesting because they possess one-parameter families of finite-dimensional unitary irreps even for generic q. The freedom of having extra parameters in the irreps opens up new and exciting possibilities in physics.[21] For the current case, the link polynomials from such representations will then also depend on these extra parameters, thus naturally yielding multi-variable link polynomials. We remark however that such multi-variable link polynomials are not related[22] to those arising from "colored" braids.[11]

For the case of quantum superalgebras, the situation is much more complicated than the bosonic case. The fundamental difficulty is the zero q-supertrace problem over typical irreps, so that the usual techniques developed for computing the eigenvalues of Casimir invariants for quantum bosonic algebras fail in this case. Due to this problem, only very few isolated examples of multi-variable link polynomials for quantum superalgebras have so far been known. These include the two-variable link polynomials[23] based on $U_q[gl(2|1)]$ and multi-variable ones[22] for a

[a)]Electronic-mail: yzzhang@yukawa.kyoto-u.ac.jp; yzz@maths.uq.oz.au special class of representations of $U_q[gl(m|n)]$. In these examples, the authors only considered representations for which the above mentioned difficulty does not occur.

In this paper, we have succeeded in overcoming the above problem and obtain a well-defined q-supertrace formula which is applied to compute the eigenvalues of the Casimir invariants. These results are given in theorems 1, 2, and 3. However, the proof of the q-supertrace formulas are extremely lengthy and will be published in a separate paper.[32] Using these results, we are able to construct link polynomials associated with *any* finite dimensional unitary irrep of a type-I quantum superalgebra. Applied to one-parameter families of inequivalent finite dimensional irreps of $U_q[gl(m|n)]$ and $U_q[osp(2|2n)]$ for generic q, our method affords infinite families of nonequivalent two-variable link polynomials in fully explicit form.

This paper will be presented in the following order. After recalling some fundamentals in Sec. II, we give, in Sec. III an account of the atypicality indices and unitary irreps of $U_q(\mathcal{G})$. In Sec. IV we present three theorems concerning the computation of the q-supertraces and therefore the eigenvalues of Casimir invariants over typical irreps. Section V derives a spectral decomposition formula for the braid generator and its powers. A general method for constructing link polynomials is presented in Sec. VI and examples of two-variable link polynomials are illustrated in Sec. VII. In the last section, we give a brief discussion of our main results.

II. PRELIMINARIES

Let $\mathcal{G}$ be a type-I simple Lie superalgebra[24] with generators $\{e_i, f_i, h_i\}$ and let α_i, $i=0,1,\ldots,r$, be its simple roots with α_0 the unique odd simple root; here we choose the distinguished set of simple roots. (Superalgebras allow many inequivalent systems of simple roots. See Ref. 24. The relation between the different quantum superalgebras obtained by choosing different systems of simple roots is studied in Ref. 25.) Let (,) be a fixed invariant bilinear form on H^*, the dual of the Cartan subalgebra H of $\mathcal{G}$. The quantum superalgebra $U_q(\mathcal{G})$ has the structure of a $\mathbf{Z}_2$-graded quasi-triangular Hopf algebra. Throughout the paper we will assume that q is generic, i.e., not a root of unity. We will not give the full defining relations of $U_q(\mathcal{G})$ here but mention that the simple raising and lowering generators of $U_q(\mathcal{G})$ obey more relations than just the usual q-Serre relations known from quantum bosonic algebras.[26–29] These necessary extra relations are referred to as "extra q-Serre relations." $U_q(\mathcal{G})$ has a coproduct Δ and antipode S given by

$$\Delta(q^{\pm h_i})=q^{\pm h_i}\otimes q^{\pm h_i},$$

$$\Delta(e_i)=e_i\otimes q^{-h_i/2}+q^{h_i/2}\otimes e_i,$$

$$\Delta(f_i)=f_i\otimes q^{-h_i/2}+q^{h_i/2}\otimes f_i, \tag{1}$$

$$S(a)=q^{-h_\rho}\gamma(a)q^{h_\rho}, \quad a=e_i,f_i,h_i, \tag{2}$$

where γ is the principal anti-automorphism on $U_q(\mathcal{G})$ and ρ is the graded half-sum of positive roots of $\mathcal{G}$. We omit the formulas for the counit which are not needed here.

The algebra $U_q(\mathcal{G})$ is a quasitriangular graded Hopf algebra, which means the following. Let Δ' be the opposite coproduct so that $\Delta'=T\Delta$, where T is the graded twist map: $T(a\otimes b)=(-1)^{[a][b]}b\otimes a$, $\forall a,b\in U_q(\mathcal{G})$. Here $[a]\in\mathbf{Z_2}$ denotes the grading of element a: $[a]=0$ if a is even and $[a]=1$ if it is odd. Then Δ and Δ' are related by the universal R-matrix R in $U_q(\mathcal{G})\otimes U_q(\mathcal{G})$ satisfying, among others, the relations

$$R\Delta(a)=\Delta'(a)R, \quad \forall a\in U_q(\mathcal{G}), \tag{3}$$

$$(I\otimes\Delta)R=R_{13}R_{12}, \quad (\Delta\otimes I)R=R_{13}R_{23}, \tag{4}$$

where if $R=\Sigma a_t\otimes b_t$ then $R_{12}=\Sigma a_t\otimes b_t\otimes 1$, $R_{13}=\Sigma a_t\otimes 1\otimes b_t$, etc. It follows from (4) that R satisfies the QYBE:

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}. \tag{5}$$

Note that the multiplication rule for the tensor product is defined for homogeneous elements $a,b,c,d\in \mathrm{U}_q(\mathscr{G})$ by

$$(a\otimes b)(c\otimes d)=(-1)^{[b][c]}(ac\otimes bd). \tag{6}$$

It is a well established fact for quasitriangular Hopf algebras, that there exists a distinguished element[4]

$$u=\sum_t (-1)^{[t]}S(b_t)a_t, \tag{7}$$

where, as above, a_t and b_t are coordinates of the universal R-matrix. One can show that u has inverse

$$u^{-1}=\sum_t (-1)^{[t]}S^{-2}(b_t)a_t \tag{8}$$

and satisfies

$$S^2(a)=uau^{-1},\quad \forall a\in \mathrm{U}_q(\mathscr{G}),\quad \Delta(u)=(u\otimes u)(R^TR)^{-1}, \tag{9}$$

where $R^T=T(R)$. It is easy to check that

$$v=u^{-1}q^{-2h_\rho} \tag{10}$$

belongs to the center of $\mathrm{U}_q(\mathscr{G})$ and satisfies

$$\Delta(v)=(v\otimes v)(R^TR)^{-1}. \tag{11}$$

Moreover, on a finite dimensional irreducible module $V(\Lambda)$ with highest weight $\Lambda\in D^+$, the Casimir operator v takes the eigenvalue

$$\chi_\Lambda(v)=q^{(\Lambda,\Lambda+2\rho)}. \tag{12}$$

Note that the generators $\{e_i,f_i,q^{h_i},i=1,\dots,r\}$ form generators of the quantum group $\mathrm{U}_q(\mathscr{G}_0)$, where $\mathscr{G}_0$ is the "even subalgebra" of $\mathscr{G}$. Specifically,

$$\mathscr{G}_0=u(1)\oplus \mathrm{sl}(m)\oplus \mathrm{sl}(n),\quad \text{for } \mathscr{G}=\mathrm{sl}(m|n),\quad m,n\geqslant 2,$$

$$\mathscr{G}_0=u(1)\oplus \mathrm{sl}(n),\quad \text{for } \mathscr{G}=\mathrm{sl}(1|n),\quad n\geqslant 2,$$

$$\mathscr{G}_0=u(1)\oplus \mathrm{sp}(2n),\quad \text{for } \mathscr{G}=\mathrm{osp}(2|2n). \tag{13}$$

Throughout we let $V_0(\Lambda)$ denote the finite dimensional irreducible $\mathrm{U}_q(\mathscr{G}_0)$ module with highest weight $\Lambda\in D^+$. We call

$$D_q^0(\Lambda)=\prod_{\beta\in\Phi_0^+}\frac{[(\Lambda+\rho,\beta)]_q}{[(\rho,\beta)]_q} \tag{14}$$

the q-dimension of the $U_q(\mathscr{G}_0)$ irrep $V_0(\Lambda)$, where Φ_0^+ denotes the set of even positive roots of $\mathscr{G}$. Here and in what follows we will adopt the notation

$$[x]_q=\frac{q^x-q^{-x}}{q-q^{-1}}. \tag{15}$$

III. ATYPICALITY INDICES AND FINITE-DIMENSIONAL UNITARY IRREPS

Let $K(\Lambda)$ be the Kac-module associated to $V(\Lambda)$. $K(\Lambda)$ is not necessarily irreducible. If it is, we have $V(\Lambda)=K(\Lambda)$ and refer to Λ and $V(\Lambda)$ as "typical." Recall that Λ is typical iff$(\Lambda+\rho,\beta)\neq 0$, $\forall\beta\in\Phi_1^+$, where Φ_1^+ is the set of odd positive roots of $\mathscr{G}$.

Let us remark that for typical modules the dimensions are easily evaluated to be $\dim V(\Lambda)=2^d\cdot\dim V_0(\Lambda)$, where d, which is equal to mn for $gl(m|n)$ and $2n$ for $osp(2|2n)$, is the number of odd positive roots. This formula is particularly useful in determining tensor product decompositions of typical modules.

Definition 1: The integer

$$a_\Lambda=|\overline{\Phi_1^+}(\Lambda)|,\quad \overline{\Phi_1^+}(\Lambda)=\{\beta\in\Phi_1^+|(\Lambda+\rho,\beta)=0\} \tag{16}$$

is called the "atypicality index" of $\Lambda\in D^+$. In particular, $a_\Lambda=0$ iff Λ is typical.

The type-I quantum superalgebras admit two types of unitary representations which may be described as follows. We make the simplifying assumption that $q>0$ (i.e., q is real and positive) and define a conjugation operation on the $U_q(\mathscr{G})$ generators by $e_i^\dagger=f_i$, $f_i^\dagger=e_i$, $h_i^\dagger=h_i$ which is extended uniquely to all of $U_q(\mathscr{G})$ such that $(xy)^\dagger=y^\dagger x^\dagger$, $\forall x,y\in U_q(\mathscr{G})$. We call π_Λ type (1) unitary if

$$\pi_\Lambda(x^\dagger)=\overline{\pi_\Lambda(x)},\quad \forall x\in U_q(\mathscr{G}) \tag{17}$$

and type (2) unitary if

$$\pi_\Lambda(x^\dagger)=(-1)^{[x]}\overline{\pi_\Lambda(x)},\quad \forall x\in U_q(\mathscr{G}), \tag{18}$$

where the overline denotes Hermitian matrix conjugation. The two types of unitary representations are in fact related via duality.

Lemma 1: Such unitary representations have the property that they are always completely reducible and the tensor product of two irreducible unitary representations of the same type reduces completely into irreducible unitary representations of the same type. Moreover the atypicality indices of the irreps occurring in this decomposition are less than or equal to the atypicality index of either component.

The finite dimensional irreducible unitary representations for all type-I quantum superalgebras have been classified in Refs. 30 and 31. For completeness we cite these classification results below. Let us first of all introduce some notation. For $gl(m|n)$, we choose $\{\epsilon_i\}_{i=1}^m\cup\{\delta_j\}_{j=1}^n$ as a basis for H^* with $[\epsilon_i]=0$, $[\delta_j]=1$ and

$$(\epsilon_i,\epsilon_j)=\delta_{ij},\quad (\delta_i,\delta_j)=-\delta_{ij},\quad (\epsilon_i,\delta_j)=0. \tag{19}$$

Using this basis, any weight Λ may written as

$$\Lambda\equiv(\Lambda_1,\dots,\Lambda_m|\overline{\Lambda_1},\dots,\overline{\Lambda_n})\equiv\sum_{i=1}^m\Lambda_i\epsilon_i+\sum_{j=1}^n\overline{\Lambda_j}\delta_j \tag{20}$$

and the graded half-sum ρ of the positive roots is

$$2\rho=\sum_{i=1}^{m}(m-n-2i+1)\epsilon_i+\sum_{j=1}^{n}(m+n-2j+1)\delta_j. \tag{21}$$

For osp$(2|2n)$, choose $\{\epsilon_0\}\cup\{\epsilon_i\}_{i=1}^{n}$ as a basis for H^* with $[\epsilon_0]=1$, $[\epsilon_i]=0$ and

$$(\epsilon_0,\epsilon_0)=-1,\quad (\epsilon_i,\epsilon_j)=\delta_{ij},\quad \forall i,j=1,\ldots,n,\quad (\epsilon_0,\epsilon_i)=0. \tag{22}$$

In this case, any weight Λ may be expressed as

$$\Lambda\equiv(\overline{\Lambda}|\Lambda_1,\ldots,\Lambda_n)\equiv\overline{\Lambda}\epsilon_0+\sum_{i=1}^{n}\Lambda_i\epsilon_i \tag{23}$$

and the graded half-sum ρ of the positive roots is given by

$$\rho=\sum_{i=1}^{n}(n-i+1)\epsilon_i-n\epsilon_0. \tag{24}$$

Proposition 1: (I) A given $\mathrm{U}_q[\mathrm{gl}(m|n)]$*-module* $V(\Lambda)$*, with* $\Lambda\in D_+$*, is type* (1) *unitary* iff: *(i)* $(\Lambda+\rho,\epsilon_m-\delta_n)>0$; *or (ii) there exists an odd index* $\omega\in\{1,2,...,n\}$ *such that* $(\Lambda+\rho,\epsilon_m-\delta_\omega)=0=(\Lambda,\delta_\omega-\delta_n)$. *In the former case the given condition also enforces typicality on* $V(\Lambda)$*, while in the latter case all irreps are atypical.*
(II) *The* $\mathrm{U}_q[\mathrm{gl}(m|n)]$*-module* $V(\Lambda)$*, with* $\Lambda\in D_+$*, is type* (2) *unitary* iff: *(i)* $(\Lambda+\rho,\epsilon_1-\delta_1)<0$; *or (ii) there exists an even index* $k\in\{1,2,...,m\}$ *such that* $(\Lambda+\rho,\epsilon_k-\delta_1)=0=(\Lambda,\epsilon_1-\epsilon_k)$. *In the former case* $V(\Lambda)$ *is typical, while in the latter case it is atypical.*

Proposition 2: A given $\mathrm{U}_q[\mathrm{osp}(2|2n)]$*-module* $V(\Lambda)$ *is type* (1) *unitary* iff $(\Lambda,\alpha_0)\geqslant 0$, *where* α_0 *denotes the unique odd simple root, and type* (2) *unitary* iff *(i)* $(\Lambda+\rho,\epsilon_0+\epsilon_1)<0$; *or (ii) there exists an index* $k\in\{1,2,...,n\}$ *such that* $(\Lambda+\rho,\epsilon_0+\epsilon_k)=0=(\Lambda,\epsilon_1-\epsilon_k)$; *or (iii)* $\Lambda=0$.

IV. q-SUPERTRACE AND EIGENVALUES OF CASIMIR INVARIANTS

Throughout this section we assume $V(\Lambda)$ is a fixed but arbitrary finite dimensional irreducible $\mathrm{U}_q(\mathscr{G})$-module. Suppose $V(\nu)\subset V(\mu)\otimes V(\Lambda)$ is typical and let $P[\nu]$ denote the central projection of the tensor product module $V(\mu)\otimes V(\Lambda)$ onto its isotypic component $\overline{V}(\nu)\equiv m_\nu V(\nu)\subset V(\mu)\otimes V(\Lambda)$ [that is $\overline{V}(\nu)=V(\nu)\oplus\cdots\oplus V(\nu)$, m_ν copies]. We state the following q-supertrace formula:

Theorem 1: *For* μ, $\nu\in D^+$ *typical,*

$$(I\otimes\mathrm{str})(I\otimes\pi_\Lambda(q^{-2h_\rho}))P[\nu]=(-1)^{[\nu]}m_\nu\,\frac{\chi_\mu(\Gamma_0)}{\chi_\nu(\Gamma_0)}\cdot\frac{D_q^0(\nu)}{D_q^0(\mu)}, \tag{25}$$

where $[\nu]$ *modulo* 2 *is the degree of the weight* ν, Γ_0 *is a central element of* $\mathrm{U}_q(\mathscr{G}_0)$ *and* $\chi_\mu(\Gamma_0)$ *is the eigenvalue of* Γ_0 *on the* $\mathrm{U}_q(\mathscr{G}_0)$*-module* $V_0(\mu)$:

$$\chi_\mu(\Gamma_0)=\prod_{\beta\in\Phi_1^+}\frac{[(\mu+\rho,\beta)]_q}{[(\rho,\beta)]_q}. \tag{26}$$

The proof of this theorem is very lengthy and detailed, and will be published elsewhere.[32]

Proposition 3: If the operator $c\in U_q(\mathscr{G})\otimes\mathrm{End}\,V(\Lambda)$ *satisfies* $\Delta_\Lambda(a)c=c\Delta_\Lambda(a)$, $\forall a\in\mathrm{U}_q(\mathscr{G})$, *where* $\Delta_\Lambda=(I\otimes\pi_\Lambda)\Delta$*, then*

$$C_k^\Lambda=(I\otimes\mathrm{str})\{[I\otimes\pi_\Lambda(q^{-2h_\rho})]c^k\},\quad k\in\mathbf{Z}^+ \tag{27}$$

belong to the center of $U_q(\mathcal{G})$ *and thus form a family of Casimir invariants.*

An important example of c is given by

$$c = \frac{I \otimes I - R_\Lambda^T R_\Lambda}{q - q^{-1}}, \tag{28}$$

where $R_\Lambda = (I \otimes \pi_\Lambda) R$, with R the universal R-matrix.

Now assume $V(\mu) \otimes V(\Lambda)$ is completely reducible and write

$$V(\mu) \otimes V(\Lambda) = \bigoplus_\nu m_\nu V(\nu) \tag{29}$$

with now m_ν the multiplicity of the module $V(\nu)$ occurring in the tensor product. This always occurs when μ and Λ are unitary of the same type. Moreover, in such a case, each of the modules $V(\nu)$ is also unitary. If $c \in (I \otimes \pi_\Lambda)(Z \otimes Z)\Delta(Z)$ where Z is the center of $U_q(\mathcal{G})$, then one can deduce the following spectral decomposition for c and its powers c^k, $k \in \mathbf{Z}$:

$$c^k = \sum_\nu (\chi_\nu(c))^k P[\nu], \tag{30}$$

where $\chi_\nu(c)$ is the eigenvalue of c on $V(\nu) \subset V(\mu) \otimes V(\Lambda)$. Thus if c is given by the above example, then we have

$$\chi_\nu(c) = \frac{1 - q^{C(\mu) + C(\Lambda) - C(\nu)}}{q - q^{-1}}, \tag{31}$$

where $C(\Lambda) \equiv (\Lambda, \Lambda + 2\rho)$ denotes the eigenvalue of the second order Casimir invariant of $\mathcal{G}$.

With the aid of Theorem 1, we can determine the eigenvalues of the Casimir invariants C_k^Λ on a finite dimensional typical module $V(\mu)$ [notation as in Eq. (29)].

Theorem 2: *If μ,ν are all typical, then the eigenvalues of the Casimir invariants on $V(\mu)$ are given by*

$$\chi_\mu(C_k^\Lambda) = \sum_\nu (-1)^{[\nu]} m_\nu (\chi_\nu(c))^k \frac{\chi_\mu(\Gamma_0)}{\chi_\nu(\Gamma_0)} \cdot \frac{D_q^0(\nu)}{D_q^0(\mu)}, \quad k \in \mathbf{Z}. \tag{32}$$

Remark: Let $\{\lambda_i\}$ denote the set of distinct weights in $V(\Lambda)$ occurring with multiplicities m_{λ_i}. It can be shown that the above theorem may be extended to all finite dimensional modules $V(\mu)$, $\mu \in D^+$, by replacing ν, m_ν with $\mu + \lambda_i, m_{\lambda_i}$, respectively, and summing over λ_i. For more details see Ref. 33. The eigenvalue formula obtained in this way is referred to as the "extended eigenvalue formula" on $V(\mu)$, $\mu \in D^+$. Note that for generic μ, the extended eigenvalue formula determines a polynomial function on H^*. It is well defined if all $\mu + \lambda_i$ are typical but if some $\mu + \lambda_i$ is atypical it is necessary first to expand the right-hand side of the extended eigenvalue formula into a polynomial in order to avoid singularities.[33]

In the case of unitary $\mu \in D^+$ this latter problem can be overcome as follows. We set

$$\Phi_1^+(\lambda) = \{\beta \in \Phi_1^+ | (\lambda + \rho, \beta) \neq 0\} \tag{33}$$

so that

$$|\Phi_1^+(\lambda)| + a_\lambda = |\Phi_1^+|. \tag{34}$$

Then we have the following.

Theorem 3: *The eigenvalues of the Casimir invariants on a unitary module $V(\mu)$ are given by*

$$\chi_\mu(C_k^\Lambda)=\sum_{\{\nu|a_\nu=a_\mu\}}(-1)^{[\nu]}m_\nu(\chi_\nu(c))^k\,\frac{\Pi_{\beta\in\Phi_1^+(\mu)}[(\mu+\rho,\beta)]_q}{\Pi_{\beta\in\Phi_1^+(\nu)}[(\nu+\rho,\beta)]_q}\cdot\frac{D_q^0(\nu)}{D_q^0(\mu)},\quad k\in\mathbf{Z}^+ \tag{35}$$

provided that $V(\Lambda)$, $V(\mu)$ *are unitary of the same type. Here the sum over* ν *is over* $V_0(\nu)\subset V_0(\mu)\otimes V(\Lambda)$ *and* m_ν *is the multiplicity of* $V_0(\nu)$ *in this space.*

For a given unitary module $V(\Lambda)$, the above formula is well defined for all unitary $\mu\in D^+$ of the same type.

V. DIAGONALIZATION OF THE BRAID GENERATOR

Let P be the graded permutation operator on $V(\Lambda)\otimes V(\Lambda)$ defined by $P(|x\rangle\otimes|y\rangle)=(-1)^{[x][y]}|y\rangle\otimes|x\rangle$, for all homogeneous $|x\rangle$, $|y\rangle\in V(\Lambda)$ and set

$$\sigma=PR\quad\in\mathrm{End}(V(\Lambda)\otimes V(\Lambda)). \tag{36}$$

Here and in what follows we regard elements of $\mathrm{U}_q(\mathscr{G})$ as operators on $V(\Lambda)$. Then (3) is equivalent to

$$\sigma\Delta(a)=\Delta(a)\sigma\quad\forall a\in\mathrm{U}_q(\mathscr{G}) \tag{37}$$

and (5) can be written as

$$(I\otimes\sigma)(\sigma\otimes I)(I\otimes\sigma)=(\sigma\otimes I)(I\otimes\sigma)(\sigma\otimes I). \tag{38}$$

It follows immediately that the operators $\sigma_i^\pm\in\mathrm{End}(V(\Lambda)^{\otimes M})$, $i=\{1,2,\dots,M-1\}$ defined by

$$\sigma_i^\pm=\underbrace{I\otimes\cdots\otimes I}_{i-1}\otimes\sigma^\pm\otimes\underbrace{I\otimes\cdots\otimes I}_{M-i-1}) \tag{39}$$

generate a nontrivial representation of the rank $(M-1)$ braid group B_M.

In the case when σ acts on $V(\Lambda)\otimes V(\Lambda)$ with $V(\Lambda)$ unitary, it can be shown that σ is self-adjoint and diagonalizable.[34] We remark however that only the type-I quantum superalgebras admit finite dimensional unitary irreps.

Similar to (29) we write,

$$V(\Lambda)\otimes V(\Lambda)=\bigoplus_\nu m_\nu V(\nu), \tag{40}$$

where again m_ν is the multiplicity of the module $V(\nu)$ occurring in the tensor product and each of the modules $V(\nu)$ is unitary. In view of the self-adjointness of σ, σ is diagonalizable on $V(\nu)\equiv m_\nu V(\nu)=V(\nu)\oplus\cdots\oplus V(\nu)$ (m_ν copies), regardless of the multiplicity. In fact it is possible to derive a spectral decomposition formula for σ, as in the case of quantum bosonic algebras.[10]

Recall that $\lim_{q\to1}\sigma=P$ and P is diagonalizable on $V(\Lambda)\otimes V(\Lambda)$ with eigenvalues ±1. Following Ref. 10, let $P[\pm]$ denote the projection operators defined by

$$P[\pm](V(\Lambda)\otimes V(\Lambda))=W_\pm, \tag{41}$$

where

$$W_\pm=\{w\in V(\Lambda)\otimes V(\Lambda)\,\big|\,\lim_{q\to1}(\sigma\mp1)w=0\}. \tag{42}$$

Since σ is an $U_q(\mathscr{G})$-invariant each subspace $W_\pm$ determines a $U_q(\mathscr{G})$-module and $P[\pm]$ commute with the action of $U_q(\mathscr{G})$. As above $P[\nu]$ denotes the projection operator onto the modules $V(\nu)$; then obviously

$$P[\nu,\pm]=P[\pm]P[\nu]=P[\nu]P[\pm] \tag{43}$$

is the projection onto the isotypic component $\bar{V}(\nu)$ consisting of eigenvectors of σ with parities ± 1, respectively [i.e., the component of $V(\nu)$ in $W_\pm$, respectively].

The diagonalizability of σ, together with the fact that

$$\sigma^2=PRP\cdot R=R^TR=(v\otimes v)\Delta(v^{-1}), \tag{44}$$

implies the following spectral decomposition for σ and its powers:

$$\sigma^k=q^{-kC(\Lambda)}\sum_\nu q^{(k/2)C(\nu)}(P[\nu,+]+(-1)^kP[\nu,-]), \quad k\in\mathbf{Z}, \tag{45}$$

where as before $C(\lambda)\equiv(\lambda,\lambda+2\rho)$. It follows in particular that σ satisfies the polynomial identity

$$\prod_\nu (\sigma-q^{\frac{1}{2}C(\nu)-C(\Lambda)})(\sigma+q^{\frac{1}{2}C(\nu)-C(\Lambda)})=0 \tag{46}$$

which leads to the generalized skein relations for the corresponding link polynomials investigated below.

VI. LINK POLYNOMIALS

Let $\theta\in B_M$ be a word in the generators $\sigma_i^\pm$, $1\leqslant i\leqslant M-1$ and let $\hat{\theta}$ denote the link obtained by closing the braid. For the construction of link polynomials, the Markov trace ϕ plays an essential role. It is defined by

$$\text{(i)} \quad \phi(\theta\eta)=\phi(\eta\theta), \quad \forall\,\theta,\eta\in B_M,$$

$$\text{(ii)} \quad \phi(\theta\sigma_{M-1})=z\phi(\theta), \quad \phi(\theta\sigma_{M-1}^{-1})=\bar{z}\phi(\theta), \quad \forall\,\theta\in B_{M-1}\subset B_M. \tag{47}$$

Given such a Markov trace, it is well-known that one can define a link polynomial $L(\hat{\theta})$ through

$$L(\hat{\theta})=(z\bar{z})^{(M-1)/2}(\bar{z}z^{-1})^{e(\theta)/2}\phi(\theta), \quad \theta\in B_M, \tag{48}$$

where $e(\theta)$ is the sum of the exponents of the σ_i's appearing in θ. The functional $L(\hat{\theta})$ enjoys the following properties:

$$\text{(i)} \quad L(\widehat{\theta\eta})=L(\widehat{\eta\theta}), \quad \forall\,\theta,\eta\in B_M,$$

$$\text{(ii)} \quad L(\widehat{\theta,\sigma_{M-1}^{\pm 1}})=L(\hat{\theta}), \quad \forall\,\theta\in B_{M-1}\subset B_M \tag{49}$$

and is an invariant of ambient isotopy.

Proposition 4: The functional $\phi(\theta)$ *defined by*

$$\phi(\theta)=\frac{(\mathrm{tr}\otimes\mathrm{str}^{\otimes(M-1)})(I\otimes\Delta^{(M-1)}(q^{-2h_\rho})\theta)}{\dim V(\Lambda)}, \tag{50}$$

where tr *and* str *denote the trace and supertrace over* $V(\Lambda)$, *respectively, qualifies as a Markov trace with*

$$z=q^{(\Lambda,\Lambda+2\rho)}, \quad \bar{z}=q^{-(\Lambda,\Lambda+2\rho)}. \tag{51}$$

Corollary 1: It follows that

$$L(\hat{\theta})=q^{-(\Lambda,\Lambda+2\rho)e(\theta)}\phi(\theta), \quad \theta\in B_M, \tag{52}$$

defines a link polynomial.

Now consider the family of Casimir invariants

$$C_k^\Lambda=(I\otimes \mathrm{str})[I\otimes\pi_\Lambda(q^{-2h_\rho})]\sigma^k. \tag{53}$$

Let ξ_k^Λ denote the eigenvalues of the invariants C_k^Λ on $V(\Lambda)$. In view of (45) and Theorem 2, one can deduce, for Λ typical, that they are given explicitly by

$$\xi_k^\Lambda=q^{-kC(\Lambda)}\sum_\nu (-1)^{[\nu]}q^{(k/2)C(\nu)}(m_\nu^+ +(-1)^k m_\nu^-)\,\frac{\chi_\Lambda(\Gamma_0)}{\chi_\nu(\Gamma_0)}\cdot\frac{D_q^0(\nu)}{D_q^0(\Lambda)}, \tag{54}$$

where $m_\nu^\pm$ are the multiplicities of $V(\nu)$ in $W_\pm$, respectively, so that

$$m_\nu=m_\nu^+ + m_\nu^-. \tag{55}$$

Note: In the case that Λ is typical it necessarily follows that all $V(\nu)$ in the tensor product decomposition (40) are also typical so that (54) is always well defined (c.f. Lemma 1).

Theorem 4: *Consider the braid group B_M and a braid θ of the following general form:*

$$\theta=(\sigma_{i_1})^{k_1}(\sigma_{i_2})^{k_2}\cdots(\sigma_{i_{M-1}})^{k_{M-1}}, \quad k_i\in\mathbf{Z} \tag{56}$$

with $\{i_1,i_2,\ldots,i_{M-1}\}$ an arbitrary permutation of $\{1,2,...,M-1\}$. Then the following functional is a link polynomial

$$L(\hat{\theta})=q^{-(\Lambda,\Lambda+2\rho)\Sigma_{i=1}^{M-1}k_i}\prod_{i=1}^{M-1}\xi_{k_i}^\Lambda. \tag{57}$$

In the case that Λ is typical, ξ_k^Λ is given by (54).

VII. NEW TWO-VARIABLE LINK POLYNOMIALS

We will now apply the technique developed in previous sections to develop a general method for obtaining two-variable link polynomials corresponding to any real $\Lambda\in D^+$. Again we restrict to the type-I quantum superalgebras $\mathscr{G}=\mathrm{gl}(m|n)$ or $\mathscr{G}=\mathrm{osp}(2|2n)$.

Corresponding to any *real* $\Lambda\in D^+$ we have the one-parameter family of irreps

$$V(\Lambda_\alpha)\equiv V(\Lambda+\alpha\delta), \quad \alpha\in\mathbf{R},$$

$$\delta=\begin{cases}\sum_i \delta_i, & \text{for } \mathscr{G}=\mathrm{gl}(m|n)\\ \epsilon_0, & \text{for } \mathscr{G}=\mathrm{osp}(2|2n).\end{cases} \tag{58}$$

The module $V(\Lambda_\alpha)$ is typical and unitary for $|\alpha|$ sufficiently large. For example, for the case $\mathscr{G}=\mathrm{gl}(m|n)$, we have from Proposition 1 a type (1) unitary module for $\alpha>-(\Lambda+\rho,\epsilon_m-\delta_n)=n-1-(\Lambda,\epsilon_m-\delta_n)$, and a type (2) unitary module for $\alpha<-(\Lambda+\rho,\epsilon_1-\delta_1)=1-m-(\Lambda,\epsilon_1-\delta_1)$. Below we assume α belongs to this range (although the final formula for link polynomials should apply, by analytic continuation, to all real α).

Here we obtain a representation of the braid generator $\sigma\in\mathrm{End}[V(\Lambda+\alpha\delta)\otimes V(\Lambda+\alpha\delta)]$ and a formula for two variable link polynomials. Consider the $\mathrm{U}_q(\mathscr{G}_0)$-module direct sum decomposition

$$V_0(\Lambda)\otimes K(\Lambda)=\bigoplus_{\nu} m_\nu V_0(\nu), \tag{59}$$

where $\mathscr{G}_0$ is the even subalgebra of $\mathscr{G}$ and $V_0(\Lambda)$ the maximal $\mathbf{Z}$-graded component of $V(\Lambda)$. Then for $|\alpha|$ sufficient large (i.e., in the range considered above) we have the easily established decomposition

$$V(\Lambda+\alpha\delta)\otimes V(\Lambda+\alpha\delta)=\bigoplus_{\nu} m_\nu V(\nu+2\alpha\delta). \tag{60}$$

Note that this decomposition may be obtained solely from a knowledge of the $\mathrm{U}_q(\mathscr{G}_0)$ modules occurring in $K(\Lambda)$ and $\mathrm{U}_q(\mathscr{G}_0)$ tensor product rules. In principal this follows from the known characters of $K(\Lambda)$ and $V_0(\Lambda)$.

From our previous results we have the Casimir invariants

$$C_k^\Lambda=(I\otimes \mathrm{str})[I\otimes\pi_{\Lambda+\alpha\delta}(q^{-2h_\rho})]\sigma^k \tag{61}$$

which, from (54), take the following eigenvalues on $V(\nu+\alpha\delta)$:

$$\xi_k^\Lambda(q,\alpha)=q^{-kC(\Lambda+\alpha\delta)}\sum_\nu\,(-1)^{[\nu]}q^{(k/2)C(\nu+2\alpha\delta)}(m_\nu^+ +(-1)^k m_\nu^-)\,\frac{\chi_{\Lambda+\alpha\delta}(\Gamma_0)}{\chi_{\nu+2\alpha\delta}(\Gamma_0)}\cdot\frac{D_q^0(\nu)}{D_q^0(\Lambda)}, \tag{62}$$

where use has been made of the fact that $\alpha\delta$ is orthogonal to all even roots and $\Lambda+\alpha\delta$, $\nu+2\alpha\delta$ are all typical for α in the range considered.

Now for θ a braid of the general form (56), we arrive at at the link polynomial

$$L(\hat\theta)=q^{-(\Lambda+\alpha\delta,\Lambda+\alpha\delta+2\rho)\Sigma_{i=1}^{M-1}k_i}\prod_{i=1}^{M-1}\xi_{k_i}^\Lambda(q,\alpha) \tag{63}$$

with $\xi_k^\Lambda(q,\alpha)$ given by (62). In this way we obtain a two-variable link polynomial corresponding to any real $\Lambda\in D^+$.

A. Two-variable link polynomials from $\mathrm{U}_q[\mathrm{gl}(m|n)]$

Following Ref. 35, we assume $m\geqslant n$ and for $0\leqslant N\leqslant mn$ we call a Young diagram $[\lambda]=[\lambda_1,\lambda_2,\ldots,\lambda_t]$, $\lambda_1\geqslant\lambda_2\cdots\geqslant\lambda_t\geqslant 0$ for the permutation group S_N (i.e., $\lambda_1+\lambda_2+\cdots+\lambda_t=N$) allowable, if it has at most n columns and m rows; i.e., $t\leqslant m$, $\lambda_i\leqslant n$. Associated with each such Young diagram $[\lambda]$ we define a weight of $\mathrm{gl}(m|n)$

$$\Lambda_{[\lambda]}=(\dot 0_{m-t},-\lambda_t,\cdots,-\lambda_1|\underbrace{t,\cdots,t}_{\lambda_t},\underbrace{t-1,\cdots,t-1}_{\lambda_{t-1}-\lambda_t},\cdots,\underbrace{1,\cdots,1}_{\lambda_1-\lambda_2},\underbrace{0,\cdots,0}_{n-\lambda_1}). \tag{64}$$

Using the basis $\{\epsilon_i,\delta_j\}$, the weight $\Lambda_{[\lambda]}$ may be expressed as

$$\Lambda_{[\lambda]}=-\sum_{i=1}^{t}\lambda_i\epsilon_{m-i+1}+t\sum_{j=1}^{\lambda_t}\delta_j+\sum_{s=1}^{t}(t-s)\sum_{j=\lambda_{t-s+1}}^{\lambda_{t-s}}\delta_j. \tag{65}$$

Let us consider the one-parameter family of finite-dimensional irreducible $U_q(gl(m|n))$-modules $V(\Lambda_\alpha)$ with highest weights of the form $\Lambda_\alpha=(0,...,0|\alpha,...,\alpha)\equiv(0|\dot{\alpha})=\alpha\delta$. [That is the case $\Lambda=(0|0)$.] These irreps $V(\alpha\delta)$ are unitary of type (1) if $\alpha>n-1$ and unitary of type (2) if $\alpha<1-m$. As mentioned above we assume real α satisfying one of these conditions, in which case $V(\alpha\delta)$ is also typical of dimension 2^{mn}.

We have the following decomposition of $V(\alpha\delta)$ into irreps of the even subalgebra $gl(m)\oplus gl(n)$:

$$V(\alpha\delta)=\bigoplus_{N=0}^{mn}\bigoplus_{[\lambda]\in S_N}V_0(\Lambda_{[\lambda]}+\alpha\delta), \tag{66}$$

where the summation is over allowed N-box Young diagrams. Note that the index N gives the **Z**-graded level of the irrep concerned. Alternatively we may simply write

$$V(\alpha\delta)=\bigoplus_{[\lambda]}V_0(\Lambda_{[\lambda]}+\alpha\delta). \tag{67}$$

The number of boxes N_λ in the Young diagram $[\lambda]$ then gives the level. We can deduce the tensor product decomposition

$$V(\alpha\delta)\otimes V(\alpha\delta)=\bigoplus_{[\lambda]}V(\Lambda_{[\lambda]}+2\alpha\delta). \tag{68}$$

The parity of the module $V(\Lambda_{[\lambda]}+2\alpha\delta)$ is $(-1)^{N_\lambda}$. The eigenvalue of the second-order Casimir on the irrep $V(\Lambda_{[\lambda]}+2\alpha\delta)$ can be shown to be

$$C(\Lambda_{[\lambda]}+2\alpha\delta)=2\sum_{i=1}^{t}\lambda_i(\lambda_i+1-2\alpha-2i)-2\alpha n(2\alpha+m),$$

$$C(\alpha\delta)=-\alpha n(\alpha+m). \tag{69}$$

Introduce the notation

$$\gamma_\alpha[\lambda]\equiv\frac{1}{2}C(\Lambda_{[\lambda]}+2\alpha\delta)-C(\alpha\delta)=2\sum_{i=1}^{t}\lambda_i(\lambda_i+1-2\alpha-2i)-\alpha n(3\alpha+m). \tag{70}$$

For θ a braid of the general form (56) we arrive at the two variable link polynomial

$$L(\hat{\theta})=q^{-n\alpha(\alpha+m)\Sigma_{i=1}^{M-1}k_i}\prod_{i=1}^{M-1}\xi_{k_i}(q,\alpha), \tag{71}$$

where now

$$\xi_k(q,\alpha)=\sum_{[\lambda]}(-1)^{(k-1)N_\lambda}q^{k\gamma_\alpha[\lambda]}\frac{\chi_{\alpha\delta}(\Gamma_0)}{\chi_{\Lambda_{[\lambda]}+2\alpha\delta}(\Gamma_0)}\cdot\frac{D_q^0(\Lambda_{[\lambda]}+2\alpha\delta)}{D_q^0(\alpha\delta)}. \tag{72}$$

In this formula, the sum is again over all allowable Young diagrams. This formula can be made fully explicit if we make use of the easily established result (which takes a bit of algebra)

$$\chi_{\alpha\delta}(\Gamma_0)\cdot\prod_{\beta\in\Phi_1^+}[(\rho,\beta)]_q=\prod_{i=1}^{m}\prod_{j=1}^{n}[i-j+\alpha]_q,$$

$$\chi_{\Lambda_{[\lambda]}+2\alpha\delta}(\Gamma_0)\cdot\prod_{\beta\in\Phi_1^+}[(\rho,\beta)]_q=\prod_{i=1}^{m}\prod_{j=1}^{n}[i-j-\lambda_i+2\alpha]_q\prod_{l=1}^{t}\frac{[\lambda_i-i-2\alpha+1-l]_q}{[\lambda_l+\lambda_i-i-2\alpha-l+1]_q},\tag{73}$$

where, in this last formula, it is implicitly understood that $\lambda_i=0$ for $m\geqslant i>t$. We thus obtain

$$\xi_k(q,\alpha)=\sum_{[\lambda]}(-1)^{(k-1)N_\lambda}q^{k\gamma_\alpha([\lambda])}\chi_\alpha([\lambda])D_q^0(\Lambda_{[\lambda]}),\tag{74}$$

where

$$\chi_\alpha([\lambda])\equiv\frac{\chi_{\alpha\delta}(\Gamma_0)}{\chi_{\Lambda_{[\lambda]}+2\alpha\delta}(\Gamma_0)}=\prod_{i=1}^{m}\prod_{j=1}^{n}\frac{[i-j+\alpha]_q}{[i-j-\lambda_i+2\alpha]_q}\prod_{l=1}^{t}\frac{[\lambda_l+\lambda_i-i-2\alpha+1-l]_q}{[\lambda_i-i-2\alpha+1-l]_q}.\tag{75}$$

As an illustration, let us consider some specific cases in the remaining part of this subsection.
Example (1): $U_q[gl(2|2)]$
The tensor product decomposition is

$$\begin{aligned}V(\alpha\delta)\otimes V(\alpha\delta)=&V(0,0|2\alpha,2\alpha)\oplus V(0,-1|2\alpha+1,2\alpha)\oplus V(-1,-1|2\alpha+2,2\alpha)\oplus V(0,-2|2\alpha\\&+1,2\alpha+1)\oplus V(-1,-2|2\alpha+2,2\alpha+1)\oplus V(-2,-2|2\alpha+2,2\alpha+2).\end{aligned}\tag{76}$$

We have in this case (using the Young diagram notation)

$$\begin{aligned}&\gamma_\alpha(\cdot)=-2\alpha^2,\quad \gamma_\alpha(\square)=-2\alpha(\alpha+1),\\&\gamma_\alpha(\square\!\square)=-2(\alpha^2+2\alpha-1),\quad \gamma_\alpha\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)=-2(\alpha+1)^2,\\&\gamma_\alpha\left(\begin{smallmatrix}\square\!\square\\\square\;\end{smallmatrix}\right)=-2\alpha(\alpha+3),\quad \gamma_\alpha\left(\begin{smallmatrix}\square\!\square\\\square\!\square\end{smallmatrix}\right)=-2\alpha(\alpha+4),\\&D_q^0(\cdot)=1,\quad D_q^0(\square)=[2]_q^2,\quad D_q^0(\square\!\square)=[3]_q,\\&D_q^0\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)=[3]_q,\quad D_q^0\left(\begin{smallmatrix}\square\!\square\\\square\;\end{smallmatrix}\right)=[2]_q^2,\quad D_q^0\left(\begin{smallmatrix}\square\!\square\\\square\!\square\end{smallmatrix}\right)=1,\end{aligned}\tag{77}$$

while the χ_α factors read

$$
\begin{aligned}
\chi_\alpha(\cdot) &= \frac{[\alpha]_q^2[\alpha-1]_q[\alpha+1]_q}{[2\alpha]_q^2[2\alpha-1]_q[2\alpha+1]_q},\\
\chi_\alpha(\ydiagram{1}) = \chi_\alpha(\ydiagram{2,1}) &= \frac{[\alpha]_q^2[\alpha-1]_q[\alpha+1]_q}{[2\alpha]_q^2[2\alpha-2]_q[2\alpha+2]_q},\\
\chi_\alpha(\ydiagram{2}) = \chi_\alpha(\ydiagram{1,1}) &= \frac{[\alpha]_q^2[\alpha-1]_q[\alpha+1]_q}{[2\alpha+1]_q[2\alpha-1]_q[2\alpha-2]_q[2\alpha+2]_q},\\
\chi_\alpha(\ydiagram{2,2}) = \chi_\alpha(\cdot) &= \frac{[\alpha]_q^2[\alpha-1]_q[\alpha+1]_q}{[2\alpha]_q^2[2\alpha-1]_q[2\alpha+1]_q}.
\end{aligned}
\tag{78}
$$

It follows that

$$
\begin{aligned}
\xi_k(q,\alpha) &= q^{k\gamma_\alpha(\cdot)}\chi_\alpha(\cdot)D_q^0(\cdot) + q^{k\gamma_\alpha(\ydiagram{2,2})}\chi_\alpha(\ydiagram{2,2})D_q^0(\ydiagram{2,2})\\
&\quad -(-1)^k\left[q^{k\gamma_\alpha(\ydiagram{1})}\chi_\alpha(\ydiagram{1})D_q^0(\ydiagram{1}) + q^{k\gamma_\alpha(\ydiagram{2,1})}\chi_\alpha(\ydiagram{2,1})D_q^0(\ydiagram{2,1})\right]\\
&\quad + q^{k\gamma_\alpha(\ydiagram{2})}\chi_\alpha(\ydiagram{2})D_q^0(\ydiagram{2}) + q^{k\gamma_\alpha(\ydiagram{1,1})}\chi_\alpha(\ydiagram{1,1})D_q^0(\ydiagram{1,1})\\
&= q^{-2k\alpha(\alpha+2)}\frac{(q^{4k\alpha}+q^{-4k\alpha})[\alpha+1]_q[\alpha-1]_q}{(q^\alpha+q^{-\alpha})^2[2\alpha+1]_q[2\alpha-1]_q}\\
&\quad -(-1)^k q^{-2k\alpha(\alpha+2)}\frac{(q^{2k\alpha}+q^{-2k\alpha})[2]_q^2}{(q^\alpha+q^{-\alpha})^2(q^{\alpha-1}+q^{-\alpha+1})(q^{\alpha+1}+q^{-\alpha-1})}\\
&\quad +\frac{q^{-2k(\alpha^2+\alpha+1)}(q^{2k\alpha}+q^{-2k\alpha})[3]_q(q^\alpha+q^{-\alpha})^2}{(q^{2\alpha-1}-q^{-2\alpha+1})(q^{2\alpha+1}-q^{-2\alpha-1})(q^{\alpha-1}+q^{-\alpha+1})(q^{\alpha+1}+q^{-\alpha-1})}.
\end{aligned}
\tag{79}
$$

Example (2): $U_q[gl(m|1)]$
We have the tensor product decomposition

$$
V(\alpha\delta)\otimes V(\alpha\delta) = V(\dot{0}|2\alpha)\oplus V(\dot{0},-1|2\alpha+1)\oplus V(\dot{0},-1,-1|2\alpha+2)\oplus\cdots\oplus V(-\dot{1}|2\alpha+m).
\tag{80}
$$

In this case $D_q^0(\Lambda_{[\lambda]})$ reads

$$
D_q^0(\Lambda_{[\lambda]}) = \prod_{i=1}^{t}\frac{[m+1-i]_q}{[t+1-i]_q} \equiv \frac{[m]_q!}{[m-t]_q![t]_q!}
\tag{81}
$$

and $\gamma_\alpha[\lambda]$, $\chi_\alpha([\lambda])$ reduce to, respectively,

$$
\begin{aligned}
\gamma_\alpha[\lambda] &= -t(t-1)-\alpha(\alpha+2t),\\
\chi_\alpha([\lambda]) &= \prod_{i=1}^{m}\frac{[i+\alpha-1]_q}{[i+2\alpha+t-1-\lambda_i]_q}.
\end{aligned}
\tag{82}
$$

The $\xi_k(q,\alpha)$ have the following form,

$$\xi_k(q,\alpha)=\sum_{t=0}^{m} (-1)^{(k-1)t}q^{-k[t(t-1)+\alpha(\alpha+2t)]}\prod_{i=1}^{t}\frac{[m+1-i]_q[i+\alpha-1]_q}{[t+1-i]_q[i+2\alpha+t-2]_q}\prod_{i>t}^{m}\frac{[i+\alpha-1]_q}{[i+2\alpha+t-1]_q}. \tag{83}$$

B. Two-variable link polynomials from adjoint representation of $U_q[gl(2|1)]$

As another illustration of how the general formalism works it is instructive to consider the case $\Lambda=\psi$, $\psi=(1,0|-1)$ the highest weight of the adjoint representation of gl(2|1). This example is of interest since it affords the simplest example of a two-variable link polynomial in which a multiplicity occurs in the tensor product space.

First note that in this case $\epsilon_1-\epsilon_2$ is the single even positive root and $\epsilon_1-\delta_1,\epsilon_2-\delta_1$ are the two odd positive roots, from which we deduce that for any $\Lambda=(\Lambda_1,\Lambda_2|\overline{\Lambda}_1)$

$$D_q^0[\Lambda]=[\Lambda_1-\Lambda_2+1]_q,\quad \chi_\Lambda(\Gamma_0)=[\Lambda_1+\overline{\Lambda}_1+1]_q[\Lambda_2+\overline{\Lambda}_1]_q. \tag{84}$$

For the Kac-module $K(\psi)$ we have the $U_q(\mathscr{G}_0)$-module ($\mathscr{G}_0=gl(2)\oplus u(1)$) decomposition (illustrated in terms of $\mathbf{Z}$-graded levels):

$$K(\psi)=V_0(1,0|-1)\oplus V_0(1,-1|0)\oplus V_0(0,0|0)\oplus V_0(0,-1|1) \tag{85}$$

which is easily seen to be $2^2\cdot 2=8$ dimensional as required. Thus

$$\begin{aligned}V_0(\psi)\otimes K(\psi)&=V_0(1,0|-1)\otimes V_0(1,0|-1)\oplus V_0(1,0|-1)\otimes[V_0(1,-1|0)\oplus V_0(0,0|0)]\\&\quad\oplus V_0(1,0|-1)\otimes V(0,-1|1)\\&=V_0(2,0|-2)\oplus V_0(1,1|-2)\oplus V_0(2,-1|-1)\oplus 2V_0(1,0|-1)\\&\quad\oplus V_0(1,-1|0)\oplus V_0(0,0|0)\end{aligned} \tag{86}$$

which yields the tensor product decomposition:

$$\begin{aligned}V(\psi+\alpha\delta)\otimes V(\psi+\alpha\delta)&=V(2,0|2\alpha-2)\oplus V(1,1|2\alpha-2)\oplus V(2,-1|2\alpha-1)\oplus 2V(1,0|2\alpha-1)\\&\quad\oplus V(1,-1|2\alpha)\oplus V(0,0|2\alpha).\end{aligned} \tag{87}$$

It is seen that $V(1,0|2\alpha-1)$ occurs twice in the tensor product space. From the above $\mathbf{Z}$ gradation on $V_0(\psi)\otimes K(\psi)$ we obtain

$$(-1)^{[\nu]}=\begin{cases}-1, & \text{for } \nu=(2,-1|2\alpha-1),\quad (1,0|2\alpha-1)\\ 1, & \text{otherwise.}\end{cases} \tag{88}$$

In the $q\to 1$ limit the above tensor product module decomposes into symmetric and antisymmetric components (which determine the parities):

$$V(\psi+\alpha\delta)\otimes V(\psi+\alpha\delta)=W_+\oplus W_- \tag{89}$$

with

$$\begin{aligned}W_-&=V(1,1|2\alpha-2)\oplus V(2,-1|2\alpha-1)\oplus V(1,0|2\alpha-1)\oplus V(0,0|2\alpha),\\W_+&=V(2,0|2\alpha-2)\oplus V(1,0|2\alpha-1)\oplus V(1,-1|2\alpha).\end{aligned} \tag{90}$$

Note that there is one copy of $V(1,0|2\alpha-1)$ in each of these spaces. For the Casimirs we have

$$\frac{1}{2}C(\nu+2\alpha\delta)-C(\psi+\alpha\delta)=\begin{cases}-\alpha(\alpha+2), & \nu=(2,0|-2),(1,-1|0)\\ -(\alpha^2+1), & \nu=(1,0|-1)\\ -(\alpha^2+2\alpha+2), & \nu=(0,0|0)\\ -(\alpha^2-2\alpha+2), & \nu=(1,1|-2)\\ -\alpha^2+2, & \nu=(2,-1|-1).\end{cases} \tag{91}$$

Collecting together all of this information and substituting into (62) we arrive at

$$\begin{aligned}\xi_k^{\psi}(q,\alpha)=&\,q^{-k\alpha(\alpha+2)}\frac{[\alpha+1]_q[\alpha-1]_q[3]_q}{[2\alpha+1]_q[2\alpha-2]_q[2]_q}+q^{-k\alpha(\alpha+2)}\frac{[\alpha+1]_q[\alpha-1]_q[3]_q}{[2\alpha+2]_q[2\alpha-1]_q[2]_q}+(-1)^k\\&\times q^{-k(\alpha^2+2\alpha+2)}\frac{[\alpha+1]_q[\alpha-1]_q}{[2\alpha+1]_q[2\alpha]_q[2]_q}+(-1)^kq^{-k(\alpha^2-2\alpha+2)}\frac{[\alpha+1]_q[\alpha-1]_q}{[2\alpha]_q[2\alpha-1]_q[2]_q}-\\&(-1)^kq^{-k(\alpha^2-2)}\frac{[\alpha+1]_q[\alpha-1]_q[4]_q}{[2\alpha+2]_q[2\alpha-2]_q[2]_q}-(1+(-1)^k)q^{-k(\alpha^2+1)}\\&\times\frac{[\alpha+1]_q[\alpha-1]_q}{[2\alpha+1]_q[2\alpha-1]_q}.\end{aligned} \tag{92}$$

C. Two-variable link polynomials from $U_q[osp(2|2n)]$

Consider the one-parameter family of 2^{2n}- dimensional irreducible $U_q[osp(2|2n)]$-modules $V(\Lambda_\alpha)$ with highest weights of form $\Lambda_\alpha=(\alpha|0,...,0)\equiv\alpha\epsilon_0$ [and with lowest weight $\Lambda_\alpha^-=(\alpha-2n)\epsilon_0$]. $V(\alpha\epsilon_0)$ is unitary and typical provided that $\alpha<0$ or $\alpha>2n$. We therefore consider the tensor product module $V(\alpha\epsilon_0)\otimes V(\alpha\epsilon_0)$ with $\alpha<0$ or $\alpha>2n$ which decomposes as

$$V(\alpha\epsilon_0)\otimes V(\alpha\epsilon_0)=\bigoplus_{c=0}^{n}\bigoplus_{d=0}^{n-c}V(\Lambda_{c,d}) \tag{93}$$

with

$$\Lambda_{c,d}=(2\alpha-c-2d)\epsilon_0+\lambda_c,\quad \lambda_c=\sum_{i=1}^{c}\epsilon_i. \tag{94}$$

The decomposition (93) is obtained from known character formulae [c.f. Eq. (59)].

From the **Z** gradation on $V(\alpha\epsilon_0)$ we can deduce that the level of the module $V(\Lambda_{c,d})$ is equal to $c+2d$. Thus the parity of the module $V(\Lambda_{c,d})$ is $([1])^{c+2d}$. The Casimir eigenvalues read

$$\begin{aligned}C(\Lambda_{c,d})&=4(\alpha-d)(n+c+d-\alpha)-2c(c-1),\\ C(\alpha\epsilon_0)&=\alpha(2n-\alpha).\end{aligned} \tag{95}$$

For θ a braid of the general form (56) we thus arrive at the two variable link polynomial

$$L(\hat{\theta})=q^{-\alpha(2n-\alpha)\Sigma_{i=1}^{M-1}k_i}\prod_{i=1}^{M-1}\xi_{k_i}(q,\alpha), \tag{96}$$

where the $\xi_k(q,\alpha)$'s are given by

$$\xi_k(q,\alpha)=\sum_{c=0}^{n}\sum_{d=0}^{n-c}(-1)^{(k-1)(c+2d)}q^{k\gamma_\alpha}\frac{\chi_{\alpha\epsilon}(\Gamma_0)}{\chi_{\Lambda_{c,d}}(\Gamma_0)}\cdot\frac{D_q^0(\Lambda_{c,d})}{D_q^0(\alpha\epsilon)} \tag{97}$$

with

$$\gamma_\alpha\equiv\tfrac{1}{2}C(\Lambda_{c,d})-C(\alpha\epsilon). \tag{98}$$

After a bit algebra, we end up with

$$\chi_{\alpha\epsilon}(\Gamma_0)\cdot\prod_{\beta\in\Phi_1^+}[(\rho,\beta)]_q=\prod_{i=1}^{n}[2n+1-i-\alpha]_q[i-\alpha-1]_q,$$

$$\chi_{\Lambda_{c,d}}(\Gamma_0)\cdot\prod_{\beta\in\Phi_1^+}[(\rho,\beta)]_q=\prod_{i=1}^{n}[c+2d+2n+1-i-2\alpha-\delta_{i\leqslant c}]_q\cdot[c+2d+i-2\alpha-1+\delta_{i\leqslant c}]_q, \tag{99}$$

where $\delta_{i\leqslant c}$ equals 1 for $i\leqslant c$ and zero otherwise. We thus obtain

$$\xi_k(q,\alpha)=\sum_{c=0}^{n}\sum_{d=0}^{n-c}(-1)^{(k-1)(c+2d)}q^{k\gamma_\alpha}\chi_\alpha(c,d)\cdot D_q^0(\lambda_c), \tag{100}$$

where

$$\chi_\alpha(c,d)\equiv\frac{\chi_{\alpha\delta}(\Lambda_0)}{\chi_{\Lambda_{c,d}}(\Lambda_0)}=\prod_{i=1}^{n}\frac{[2n+1-i-\alpha]_q[i-\alpha-1]_q}{[c+2d+2n+1-i-2\alpha-\delta_{i\leqslant c}]_q[c+2d+i-2\alpha-1+\delta_{i\leqslant c}]_q},$$

$$D_q^0(\lambda_c)=\prod_{i<j}^{c}\frac{[2(n+2)-i-j]_q}{[2(n+1)-i-j]_q}\prod_{l=1}^{c}\frac{[2(n+2-l)]_q}{[2(n+1-l)]_q}. \tag{101}$$

VIII. DISCUSSION

We have demonstrated how link polynomials can be constructed associated with any finite-dimensional unitary irrep of a type-I quantum superalgebra. This is achieved by successfully overcoming a fundamental problem in computing the eigenvalues of Casimir invariants for the quantum superalgebras. Applying our results to one-parameter families of inequivalent irreps, we have been able to construct infinite families of nonequivalent two-variable link polynomials. Such two-variable link polynomials were previously known only for some isolated cases. For a class of braids, we have computed the link polynomials in fully explicit form.

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