The University of Bradford Institutional Repository

Link to publisher's version: *http://dx.doi.org/10.1088/1751-8113/47/48/485204*

Citation: Oladejo SO, Lei C and Vourdas A (2014) Partial ordering of weak mutually unbiased bases. Journal of Physics A: Mathematical and Theoretical. 47(48).

Partial ordering of weak mutually unbiased bases

S.O. Oladejo, C. Lei, A. Vourdas
Department of Computing,
University of Bradford,
Bradford BD7 1DP, United Kingdom

A quantum system $\Sigma(n)$ with variables in $\mathbb{Z}(n)$, where $n = \prod p_i$ (with p_i prime numbers), is considered. The non-near-linear geometry $G(n)$ of the phase space $\mathbb{Z}(n) \times \mathbb{Z}(n)$, is studied. The lines through the origin are factorized in terms of 'prime factor lines' in $\mathbb{Z}(p_i) \times \mathbb{Z}(p_i)$. Weak mutually unbiased bases (WMUB) which are products of the mutually unbiased bases in the 'prime factor Hilbert spaces' $H(p_i)$, are also considered. The factorization of both lines and WMUB is analogous to the factorization of integers in terms of prime numbers. The duality between lines and WMUB is discussed. It is shown that there is a partial order in the set of subgeometries of $G(n)$, isomorphic to the partial order in the set of subsystems of $\Sigma(n)$.

I. INTRODUCTION

There has been much work on finite quantum systems $\Sigma(n)$ with variables in $\mathbb{Z}(n)$ (the integers modulo n) [1–6], and mutually unbiased bases (MUB)[7–22]. When n is a prime number, $\mathbb{Z}(n)$ is a field and the number of MUB is $n+1$. When $n = p^e$ (a power of a prime number), the use of the Galois field $GF(p^e)$ leads to $n+1$ MUB. In the case that n is not a power of a prime number, $\mathbb{Z}(n)$ is a ring and it is a very difficult problem to find the number of MUB in this case. In view of this, refs.[23, 24] have introduced the concept of weak mutually unbiased bases (WMUB), where

$$|\langle \mathfrak{X}_\theta; \alpha | \mathfrak{X}_\phi; \beta \rangle|^2 = \frac{1}{m}; \quad m|n; \quad \alpha, \beta \in \mathbb{Z}(n). \tag{1}$$

Here $\mathfrak{X}_\theta$ is not a variable, but a label of the particular basis. The strict requirement in MUB that this overlap should be equal to $1/n$ is replaced here by the weaker requirement that the overlap should be $1/m$ where m is a divisor of n. For simplicity, n is taken in this paper to be the product of prime numbers to the power one:

$$n = p_1 \times ... \times p_N. \tag{2}$$

The $\mathbb{Z}(n)$ is isomorphic to $\mathbb{Z}(p_1) \times ... \times \mathbb{Z}(p_N)$ and the Chinese remainder theorem provides bijective maps between the two. Using these maps quantum states in $\Sigma(n)$ can be factorized in terms of quantum states in $\Sigma(p_1), ..., \Sigma(p_N)$[25, 26]. The WMUB are tensor products of MUB in each of the spaces $\Sigma(p_i)$ $(i = 1, ..., N)$. Since p_i is a prime number, there are $p_i + 1$ MUB in $\Sigma(p_i)$ and therefore the total number of WMUB is $\psi(n) = \prod(p_i + 1)$, where $\psi(n)$ is the Dedekind psi function. An important feature of the WMUB is that they are intimately linked to properties of lines in the phase space $\mathbb{Z}(n) \times \mathbb{Z}(n)$ of $\Sigma(n)$, which is a finite geometry. We note that most of the mathematical work on finite geometries [27–29] is on near-linear geometries, which have the axiom that two lines have at most one point in common. The $\mathbb{Z}(n) \times \mathbb{Z}(n)$ geometry is based on rings and it does not obey this axiom. Two lines have in common m points, where $m|n$, and we call this a 'subline'. It has been shown in [24] that there is a duality between the lines and WMUB. Other work on finite geometries for finite quantum systems has been presented in [30–33].

When $m|n$, $\mathbb{Z}(m)$ is a subgroup of $\mathbb{Z}(n)$, and the states of $\Sigma(m)$ can be embedded into the states of $\Sigma(n)$, as dicussed later. We express this by saying that $\Sigma(m)$ is subsystem of $\Sigma(n)$. We define the partial order 'subsystem' and the partial order 'subgeometry' and show an isomorphism between them. This shows a deeper connection between finite quantum systems and the geometries of their phase spaces, which includes a duality between lines and WMUB.

In section 2 we present the notation. In section 3 we discuss the relevant finite geometries, with emphasis on their partial ordering. In section 4, we discuss the WMUB in finite systems, with emphasis again on their partial ordering. We also prove the duality between lines in finite geometries and WMUB in the Hilbert spaces. We conclude in section 5, with a discussion of our results.

II. PRELIMINARIES

(1) $\mathbb{Z}(n)$ is the ring of integers modulo n. For simplicity, n is taken to be the product of prime numbers to the power one, as in Eq.(2).

(2) $\mathbb{Z}^*(n)$ is the reduced system of residues modulo n. Its cardinality is $\varphi(n)$ (the Euler totient function). $\mathbb{Z}^*(n)$ contains the invertible elements of $\mathbb{Z}(n)$, and $\mathbb{Z}(n) - \mathbb{Z}^*(n)$ the non-invertible elements.

(3) $\psi(n)$ is the Dedekind psi function. If $n = \prod p_i$ where p_i are primes different from each other, then $\psi(n) = \prod(p_i + 1)$.

(4) Throughout the paper, we have several partial orders and for simplicity we use the same symbol $\prec$ for all of them.

(5) $\mathbb{D}(n)$ is the set of divisors of n. $\mathbb{D}(n)$ is a poset with divisibility as partial order. We use the notation $a \prec b$ or $a|b$ for 'a divisor of b'. The cardinality of $\mathbb{D}(n)$ is the divisor function $\sigma_0(n)$.

(6) If $k \prec m$ then $\mathbb{Z}(k)$ is a subgroup of $\mathbb{Z}(m)$. Let

$$\mathfrak{Z}(n) = \{\mathbb{Z}(m) \mid m \in \mathbb{D}(n)\} \tag{3}$$

The set $\mathfrak{Z}(n)$ contains all subgroups of $\mathbb{Z}(n)$, and it is a poset with subgroup as partial order. The posets $\mathbb{D}(n)$ and $\mathfrak{Z}(n)$, are isomorphic to each other.

The elements of $\mathbb{Z}(m)$ are embedded into $\mathbb{Z}(n)$ (for $m \prec n$), as follows

$$\mathbb{Z}(m) \ni \alpha \quad \rightarrow \quad \frac{n\alpha}{m} \in \mathbb{Z}(n) \tag{4}$$

In order to emphasize this embedding sometimes we say that $\alpha \in \frac{n}{m}\mathbb{Z}(m)$.

(7) $\mathrm{GCD}(\nu, \mu)$ denotes the greatest common divisor.

III. THE NON-NEAR-LINEAR GEOMETRY $G(n)$ AND ITS SUBGEOMETRIES

A finite geometry $G(n) = (P(n), L(n))$ is a pair comprised of the set of points $P(n)$ and the set of lines $L(n)$ (in this paper we are interested in the lines through the origin only). $P(n)$ is the set

$$P(n) = \{(a, b) \mid a, b \in \mathbb{Z}(n)\}. \tag{5}$$

A line through the origin is the set of points

$$\mathcal{L}(\nu, \mu) = \{(r\nu, r\mu) \mid r \in \mathbb{Z}(n)\}; \quad \nu, \mu \in \mathbb{Z}(n). \tag{6}$$

We call $L(n)$ the set of all lines through the origin. As we explained, our finite geometry is not a near-linear geometry. In near-linear geometries two lines have at most one point in common[27–29], and this is not true in our case. Only for prime n, $\mathbb{Z}(n)$ is a field, and then $G(n)$ is a near-linear geometry. Non-near-linear geometries have non-trivial subgeometries (in analogous way to non-prime numbers which have non-trivial factors). For $m|n$, $\mathbb{Z}(m)$ is a subgroup of $\mathbb{Z}(n)$, and $G(m)$ is a subgeometry of $G(n)$ (we denote this as $G(m) \prec G(n)$), in the sense of the following results [23, 24] which we summarize in the following proposition without proof.

Proposition III.1.

(1) In $G(n)$ there are $\psi(n)$ lines through the origin, with exactly n points, which we call maximal lines.

(2) If $\sigma \in \mathbb{Z}^*(n)$, then $\mathcal{L}(\sigma\nu, \sigma\mu) = \mathcal{L}(\nu, \mu)$. If $\sigma \in \mathbb{Z}(n) - \mathbb{Z}^*(n)$ then $\mathcal{L}(\sigma\nu, \sigma\mu) \subset \mathcal{L}(\nu, \mu)$. In this case we say that $\mathcal{L}(\sigma\nu, \sigma\mu)$ is a subline of $\mathcal{L}(\nu, \mu)$, and we denote this as $\mathcal{L}(\sigma\nu, \sigma\mu) \prec \mathcal{L}(\nu, \mu)$. It follows that $\mathcal{L}(\nu, \mu)$ is a maximal line in $G(n)$ if $\mathrm{GCD}(\nu, \mu) \in \mathbb{Z}^*(n)$, and a subline if $\mathrm{GCD}(\nu, \mu) \in \mathbb{Z}(n) - \mathbb{Z}^*(n)$.

(3) A line $\mathcal{L}(\nu, \mu)$ in $G(n)$, is also the line

$$\mathcal{L}(k\nu, k\mu) = \{(rk\nu, rk\mu)\ |r = 0, 1, ..., n-1\}, \tag{7}$$

in the bigger geometry $G(kn)$. The line $\mathcal{L}(k\nu, k\mu)$ in $G(kn)$ is a subline of the maximal line

$$\mathcal{L}(\nu, \mu) = \{(r'\nu, r'\mu)\ |r' = 0, 1, ..., kn-1\}. \tag{8}$$

(4) Two maximal lines through the origin in $G(n)$, have m points in common, where $m|n$. These m points form a subline $\mathcal{L}(\kappa, \lambda)$ where $\kappa, \lambda \in \frac{n}{m}\mathbb{Z}(m)$ (the elements of this are $0, \frac{n}{m}, ..., \frac{n(m-1)}{m}$). The subline $\mathcal{L}(\kappa, \lambda)$ is also a maximal line in the subgeometry $G(m)$. There are $\psi(m)$ maximal lines in $G(m)$ which are also sublines within the larger geometry $G(n)$.

We consider the matrices

$$s(\kappa, \lambda|\mu, \nu) \equiv \begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix}; \quad \kappa, \lambda, \mu, \nu \in \mathbb{Z}(n)$$
$$\det(s) = \kappa\nu - \lambda\mu = 1 \ (\mathrm{mod}\ n). \tag{9}$$

They form the $Sp(2, \mathbb{Z}(n))$ group. Acting $s(\kappa, \lambda|\mu, \nu)$ on all the points of a line $\mathcal{L}(\rho, \sigma)$ we get all the points of the line $\mathcal{L}(\kappa\rho + \lambda\sigma, \mu\rho + \nu\sigma)$, which we denote as $s(\kappa, \lambda|\mu, \nu)\mathcal{L}(\rho, \sigma)$.

In the case of prime n, acting with $s(0, 1| -1, \theta)$ on a fixed line $\mathcal{L}(0, 1)$, we get all $n+1$ lines (through the origin). We label these lines as

$$\begin{aligned} \theta = -1 \ &\rightarrow\ \Lambda(-1) = \mathcal{L}(0, 1) \\ \theta = 0, ..., n-1 \ &\rightarrow\ \Lambda(\theta) = s(0, 1| -1, \theta)\mathcal{L}(0, 1) = \mathcal{L}(1, \theta). \end{aligned} \tag{10}$$

Here (and later) we use the convention that in the case $\theta = -1$, the $s(0, 1| -1, \theta)$ is replaced by $\mathbf{1}$. It can be checked that if we use general symplectic transformations, the $s(\kappa, \lambda|\mu, \nu)\mathcal{L}(0, 1)$ is simply one of the lines $s(0, 1| -1, t)\mathcal{L}(0, 1)$.

We easily find the following practical rule: If $\mathcal{L}(\nu, \mu)$ is a maximal line in $G(n)$, then it is also the subline $\mathcal{L}(k\nu, k\mu)$ in the larger geometry $G(kn)$. The maximal line in $G(kn)$ which has $\mathcal{L}(k\nu, k\mu)$ as a subline, is $\mathcal{L}(\rho, \sigma)$ with

$$\rho = \frac{\nu}{\mathrm{GCD}(\nu, \mu)}; \quad \sigma = \frac{\mu}{\mathrm{GCD}(\nu, \mu)}. \tag{11}$$

A. Factorization of lines in terms of prime factor lines

Based on the Chinese remainder theorem, Good [34] introduced two bijective maps between $\mathbb{Z}(n)$ and $\mathbb{Z}(p_1) \times ... \times \mathbb{Z}(p_N)$. The first map (which will be used for 'positions') is

$$m \leftrightarrow (m_1, ..., m_N); \quad m_i = m(mod\ p_i); \quad m = \sum m_i s_i. \tag{12}$$

The second map (which will be used for 'momenta') is

$$m \leftrightarrow (\overline{m}_1, ..., \overline{m}_N); \quad \overline{m}_i = mt_i = m_i t_i (\text{mod } p_i); \quad m = \sum \overline{m}_i r_i (\text{mod } n). \tag{13}$$

Here,

$$r_i = \frac{n}{p_i}; \quad t_i r_i = 1 \ (mod \ p_i); \quad s_i = t_i r_i \in \mathbb{Z}(n). \tag{14}$$

Using this factorization, we define the following bijective map between $\mathbb{Z}(n) \times \mathbb{Z}(n)$ and $[\mathbb{Z}(p_1) \times ... \times \mathbb{Z}(p_N)] \times [\mathbb{Z}(p_1) \times ... \times \mathbb{Z}(p_N)]$

$$(\rho, \sigma) \leftrightarrow (\rho_1, ..., \rho_N, \overline{\sigma}_1, ..., \overline{\sigma}_N). \tag{15}$$

We use here the map of Eq.(12) for ρ (which is related to positions) and the map Eq.(13) for σ (which is related to momenta). Then we can write an arbitrary line $\mathcal{L}(\rho, \sigma)$ as

$$\mathcal{L}(\rho, \sigma) = \mathcal{L}(\rho_1, \overline{\sigma}_1) \times ... \times \mathcal{L}(\rho_N, \overline{\sigma}_N) \tag{16}$$

The $\mathcal{L}(\rho_i, \overline{\sigma}_i)$ are 'prime factor lines' in the sense that they are in $\mathbb{Z}(p_i) \times \mathbb{Z}(p_i)$ where p_i is a prime number.

An example of this, which is need below, is that

$$\mathcal{L}(0, 1) = \mathcal{L}(0, t_1) \times ... \times \mathcal{L}(0, t_N). \tag{17}$$

The factorization of lines in Eq.(16) is analogous to the factorization of integers. The subline is a concept analogous to a divisor of an integer. The $\mathcal{L}(0, 0)$ (which contains only the $(0, 0)$ point) corresponds to the integer 1. If many lines have a subline in common, this is analogous to many integers having a common divisor.

The set $L(n)$ of all lines through the origin in $G(n)$ is a partially ordered set with subline as partial order. Its cardinality is

$$|L(n)| = \sum_{k \in \mathbb{D}(n)} \psi(k). \tag{18}$$

In the following lemma we express Eq.(16) in the notation of Eq.(10). We introduce this new notation in this paper because it is analogous to the notation in the WMUB later, and it will show clearly the isomorphism between the partial order of the geometries and the partial order of the quantum systems.

Lemma III.2. *The maximal lines in $G(n)$ are:*

(1) if all $\rho_i \neq 0$

$$\begin{aligned} &\mathcal{L}(\rho, \sigma) = \mathcal{L}(1, \rho_1^{-1} \overline{\sigma}_1) \times ... \times \mathcal{L}(1, \rho_N^{-1} \overline{\sigma}_N) = \Lambda(\theta_1) \times ... \times \Lambda(\theta_N) = \Lambda(\theta_1, ..., \theta_N) \\ &\theta_i = \rho_i^{-1} \overline{\sigma}_i; \quad \rho_i, \sigma_i \in \mathbb{Z}(p_i) \end{aligned} \tag{19}$$

(2) If $\rho_j = 0$ then the line $\mathcal{L}(0, \overline{\sigma}_j) = \mathcal{L}(0, 1) = \Lambda(-1)$ and in this case $\theta_j = -1$ in the above equation.

Proof. If $\rho_i \neq 0$ then $\mathcal{L}(\rho_i, \overline{\sigma}_i) = \mathcal{L}(1, \rho_i^{-1} \overline{\sigma}_i)$ (the ρ_i belongs in the field $\mathbb{Z}(p_i)$ and therefore the ρ_i^{-1} exists). Therefore, Eq.(16) can be written as in Eq.(19). If one (or more) $\rho_j = 0$ then corresponding factor line $\mathcal{L}(0, \overline{\sigma}_j) = \mathcal{L}(0, 1) = \Lambda(-1)$. For example, Eq.(17) is written as

$$\mathcal{L}(0, 1) = \Lambda(-1, ..., -1). \tag{20}$$

□

Other examples are discussed later (table I). Eq.(19) uses two different notations for the lines. The $\mathcal{L}(\rho, \sigma)$ is the 'natural notation' and the $\Lambda(\theta_1, ..., \theta_N)$ is used later to demonstrate the duality with the WMUB.

Given two maximal lines

$$\begin{aligned} \mathcal{L}(\rho_1, \overline{\sigma}_1) \times ... \times \mathcal{L}(\rho_N, \overline{\sigma}_N) = \Lambda(\theta_1, ..., \theta_N) \\ \mathcal{L}(\rho'_1, \overline{\sigma}'_1) \times ... \times \mathcal{L}(\rho'_N, \overline{\sigma}'_N) = \Lambda(\theta'_1, ..., \theta'_N) \end{aligned} \tag{21}$$

let I_1 be a subset of the set of indices $I = \{1, ..., N\}$ such that $\theta_i = \theta'_i$ for $i \in I_1$ or equivalently $\rho_i = k\rho'_i$ and $\sigma_i = k\sigma'_i$ with $k \in \mathbb{Z}^*(n)$. Then the $\mathcal{L}(r_1, \overline{s}_1) \times ... \times \mathcal{L}(r_N, \overline{s}_N)$ with

$$\begin{aligned} & r_i = \rho_i; \quad \overline{s}_i = \overline{\sigma}_i; \;\; \text{if} \;\; i \in I_1 \\ & r_i = \overline{s}_i = 0; \;\; \text{if} \;\; i \in I_2 = I - I_1 \end{aligned} \tag{22}$$

is a common subline of both of these lines. In other words, we find a common subline, by keeping some of the common 'factor lines' and replacing the rest of them with $\mathcal{L}(0, 0)$. This is analogous to finding a common divisor of two integers, by keeping some of the common factors and replacing the rest of the factors with 1.

B. Partial ordering of the finite geometries

With the help of the maps (12),(13), the $Sp(2, \mathbb{Z}(n))$ is factorized as $Sp(2, \mathbb{Z}(p_1)) \times ... \times Sp(2, \mathbb{Z}(p_N))$ where

$$s(\kappa, \lambda|\mu, \nu) = \bigotimes_i s(\kappa_i, \lambda_i r_i|\overline{\mu}_i, \nu_i) \tag{23}$$

The $\kappa_i, \lambda_i, \nu_i$ are related to κ, λ, ν as in Eq.(12), and $\overline{\mu}_i$ is related to μ as in Eq.(13)[23, 24]. Therefore

$$s(0, 1| - 1, \theta) = \bigotimes_i s(0, r_i| - t_i, \theta_i); \qquad \theta_i = \theta (\mathrm{mod}\; p_i) \tag{24}$$

The set $\mathfrak{G}(n)$ of all subgeometries of $G(n)$ with the partial order 'subgeometry' is isomorphic to the partially ordered set $\mathbb{D}(n)$. The following proposition describes the embedding of smaller geometries into larger geometries and their partial ordering.

Proposition III.3.

(1) Let $I = \{1, ..., N\}$ be a set of indices, $I_1 \subseteq I$ and $I_2 = I - I_1$. Also let n be given by Eq.(2) and

$$m = \prod_{i \in I_1} p_i; \qquad \frac{n}{m} = \prod_{i \in I_2} p_i. \tag{25}$$

If in Eq.(16), $\mathcal{L}(\rho_i, \overline{\sigma}_i) = \mathcal{L}(0, 0)$ for all $i \in I_1$, then $\mathcal{L}(\rho, \sigma)$ is a subline in the subgeometry $G(\frac{n}{m})$ (the subgeometry corresponding to the set of primes p_i with $i \in I_2$).

(2) There are $\psi(m)$ maximal lines in $G(n)$ which have in common a subline with $\frac{n}{m}$ points.

Proof.

(1) We first point out that in the factorization of Eqs(12),(13), if $\rho_i = 0$ for all $i \in I_1$ then $\rho \in m\mathbb{Z}(\frac{n}{m})$. Conversely, if $\rho \in m\mathbb{Z}(\frac{n}{m})$, then $\rho_i = \overline{\rho}_i = 0$. The fact that $\mathcal{L}(\rho_i, \overline{\sigma}_i) = \mathcal{L}(0, 0)$ for all $i \in I_1$, implies precisely that $\rho_i = \overline{\rho}_i = 0$, and therefore the points of the line $\mathcal{L}(\rho, \sigma)$ belong to $m\mathbb{Z}(\frac{n}{m}) \times m\mathbb{Z}(\frac{n}{m})$. In other words the $\mathcal{L}(\rho, \sigma)$ is a subline in the subgeometry $G(\frac{n}{m})$.

(2) We consider lines $\Lambda(\theta_1, ..., \theta_N)$ which have all θ_i with $i \in I_2$ in common, and which differ in the θ_j with $j \in I_1$. There are

$$\prod_{i \in I_1} (p_i + 1) = \psi(m) \tag{26}$$

such lines. As explained earlier, in order to find their common subline we replace the $\mathcal{L}(\rho_i, \overline{\sigma}_i)$ for $i \in I_1$, with $\mathcal{L}(0, 0)$. Therefore the common subline has

$$\prod_{i \in I_2} p_i = \frac{n}{m} \tag{27}$$

points.

□

C. Examples

In fig.1 we present the Hasse diagram showing the subgeometries of $G(30)$. This is isomorphic to the Hasse diagram for the divisors $\mathbb{D}(30)$, with divisibility as partial order.

In table 1 we present the $\psi(30) = 72$ maximal lines in the geometry $G(30)$. We use both notations described in Eq.(19). We also present the $\psi(15) = 24$ maximal lines in the geometry $G(15)$, the $\psi(10) = 18$ maximal lines in the geometry $G(10)$, the $\psi(6) = 12$ maximal lines in the geometry $G(6)$, the $\psi(5) = 6$ maximal lines in the geometry $G(5)$, the $\psi(3) = 4$ maximal lines in the geometry $G(3)$, and the $\psi(2) = 3$ maximal lines in the geometry $G(2)$. The maximal lines in a subgeometry are also sublines in 'supergeometry'. For example, the maximal lines in $G(5)$ are also sublines in $G(10), G(15), G(30)$.

As examples, we consider in $G(30)$ the $\mathcal{L}(15, 15) \prec \mathcal{L}(5, 5) \prec \mathcal{L}(1, 1)$. The subline

$$\mathcal{L}(15, 15) = \{(0, 0), (15, 15)\} \tag{28}$$

in $G(30)$ is the maximal line

$$\mathcal{L}(1, 1) = \{(0, 0), (1, 1)\} \tag{29}$$

in $G(2)$. The subline

$$\mathcal{L}(5, 5) = \{(0, 0), (5, 5), (10, 10), (15, 15), (20, 20), (25, 25)\} \tag{30}$$

in $G(30)$ is the maximal line

$$\mathcal{L}(1, 1) = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\} \tag{31}$$

in $G(6)$.

IV. PARTIAL ORDERING OF THE SET OF QUANTUM SYSTEMS WITH VARIABLES IN $\mathbb{Z}(n)$

We consider a quantum system with positions and momenta in $\mathbb{Z}(n)$, which we denote as $\Sigma(n)$. If m is a divisor of n, $\mathbb{Z}(m)$ is a subgroup of $\mathbb{Z}(n)$. In this case, we say that $\Sigma(m)$ is a subsystem of $\Sigma(n)$.

Let $|X_n;\alpha\rangle$ and $|P_n;\alpha\rangle$ be position and momentum states, respectively. Here the X_n, P_n are not variables but they simply indicate that they are position and momentum states, respectively, in the n-dimensional Hilbert space of this system $H(n)$. The Fourier transform is given by:

$$F = n^{-1/2}\sum_{\alpha,\beta}\omega(\alpha\beta)|X_n;\alpha\rangle\langle X_n;\beta|$$
$$\omega(\alpha) = \exp\left(i\frac{2\pi\alpha}{n}\right);\quad \alpha\in\mathbb{Z}(n). \tag{32}$$

Then

$$|P_n;\alpha\rangle = F|X_n;\alpha\rangle. \tag{33}$$

Refs [25, 26] used Eqs(12), (13) to factorize a system with variables in $\mathbb{Z}(n)$, where n is given in Eq.(2), in terms of N subsystems with variables in $\mathbb{Z}(p_i)$. There is a bijective map between $H(n)$ and the tensor product $\bigotimes H(p_i)$ where

$$|X_n;\alpha\rangle \leftrightarrow |X_1;\overline{\alpha}_1\rangle\otimes ...\otimes |X_N;\overline{\alpha}_N\rangle. \tag{34}$$

For simplicity we use here the notation $|X_i;\overline{\alpha}_i\rangle$ instead of $|X_{p_i};\overline{\alpha}_i\rangle$. Also

$$|P_n;\alpha\rangle \leftrightarrow |P_1;\alpha_1\rangle\otimes ...\otimes |P_N;\alpha_N\rangle. \tag{35}$$

For m a divisor of n, the $\Sigma(m)$ is a subsystem of $\Sigma(n)$. This means that the variables of $\Sigma(m)$ take values in a subgroup of the group of the variables of $\Sigma(n)$. The quantum states of $\Sigma(m)$ are embedded into $\Sigma(n)$, as follows:

$$\sum_{\alpha=0}^{m-1} s_\alpha|X_m;\alpha\rangle \quad\leftrightarrow\quad \sum_{\alpha=0}^{m-1} s_\alpha|X_n;\frac{n\alpha}{m}\rangle \tag{36}$$

This is analogous to Eq.(4).

Therefore, there is a partial order 'subsystem' in the set $\mathfrak{S}(n)$ of all subsystems of $\Sigma(n)$ (i.e. the set of all systems $\Sigma(m)$ with $m|n$). There is also a partial order 'subspace' in the set $\mathfrak{H}(n)$ of their Hilbert spaces (i.e., the set of all $H(m)$ with $m|n$). This is shown in fig.1, and enhances the concept of duality between geometries and quantum systems.

A. Mutually unbiased bases

There are well known differences in the formalism of finite quantum systems for the cases of even and odd dimension (e.g., [37–39]), and in this part of the paper we consider systems with odd dimension.

We consider the displacement operators

$$D(\gamma,\delta) = Z^\gamma\, X^\delta\, \omega(-2^{-1}\gamma\delta);\;\; Z^\gamma = \sum_{\alpha\in\mathbb{Z}(n)}\omega(\alpha\gamma)|X_n;\alpha\rangle\langle X_n;\alpha|;\;\; X^\delta = \sum_{\alpha\in\mathbb{Z}(n)}\omega(-\alpha\delta)|P_n;\alpha\rangle\langle P_n;\alpha|$$
$$X^\delta\, Z^\gamma = Z^\gamma\, X^\delta\, \omega(-\gamma\delta);\quad X^n = Z^n = \mathbf{1}. \tag{37}$$

$D(\gamma,\delta)\omega(\epsilon)$, where $\gamma,\delta,\epsilon\in\mathbb{Z}(n)$, form a representation of the Heisenberg-Weyl group. We note that $F\; X\; F^\dagger = Z$ and $F\; Z\; F^\dagger = X^{-1}$.

Symplectic transformations in the present context have been studied in [35, 36]. They obey the relations

$$\begin{aligned}
&S(\kappa,\lambda|\mu,\nu)\ X\ [S(\kappa,\lambda|\mu,\nu)]^\dagger = D(\lambda,\kappa)\\
&S(\kappa,\lambda|\mu,\nu)\ Z\ [S(\kappa,\lambda|\mu,\nu)]^\dagger = D(\nu,\mu)\\
&\kappa\nu-\lambda\mu=1;\qquad \kappa,\lambda,\mu,\nu\in\mathbb{Z}(n).
\end{aligned}\tag{38}$$

The $s(\kappa,\lambda|\mu,\nu)$ used earlier and the $S(\kappa,\lambda|\mu,\nu)$ used here, belong to different representations of the $Sp(2,\mathbb{Z}(n))$ group. A special case is $S(0,1|-1,0)=F$.

Mutually unbiased bases in systems with prime dimension p, are given by

$$\begin{aligned}
\theta=-1\ &\rightarrow\ |\mathfrak{X}_\theta;\alpha\rangle = |X_p;\alpha\rangle\\
\theta=0,...,p-1\ &\rightarrow\ |\mathfrak{X}_\theta;\alpha\rangle = S(0,1|-1,\theta)|X_p;\alpha\rangle.
\end{aligned}\tag{39}$$

In the special case $\theta=0$, we get $|\mathfrak{X}_0;\alpha\rangle=|P_p;\alpha\rangle$.

There are $p+1$ such bases labelled with $\theta=-1,...,p-1$. They obey the relation

$$|\langle\mathfrak{X}_\theta;\alpha|\mathfrak{X}_\phi;\beta\rangle|^2=\frac{1}{p}.\tag{40}$$

We label these bases, in analogous way to the lines, as

$$B(\theta)=\{|\mathfrak{X}_\theta;\alpha\rangle\};\qquad \theta=-1,...,p-1.\tag{41}$$

B. Duality between WMUB in $H(n)$ and lines in $G(n)$

We next introduce the concept of weak mutually unbiased bases (WMUB) studied in [23, 24]. They are products of mutually unbiased bases in each of the Hilbert spaces $H(p_i)$:

$$|\mathfrak{X}_{\theta_1,...,\theta_N};\overline{\alpha}_1,...,\overline{\alpha}_N\rangle = |\mathfrak{X}_{1\theta_1};\overline{\alpha}_1\rangle\otimes...\otimes|\mathfrak{X}_{N\theta_N};\overline{\alpha}_N\rangle\tag{42}$$

where $\alpha_i\in\mathbb{Z}(p_i)$ and $\theta_i=-1,...,p_i-1$. In the special case $\theta_1=...=\theta_N=-1$ we get

$$|\mathfrak{X}_{-1,...,-1};\overline{\alpha}_1,...,\overline{\alpha}_N\rangle = |\mathfrak{X}_{1,-1};\overline{\alpha}_1\rangle\otimes...\otimes|\mathfrak{X}_{N,-1};\overline{\alpha}_N\rangle = |X_1;\overline{\alpha}_1\rangle\otimes...\otimes|X_N;\overline{\alpha}_N\rangle\tag{43}$$

In the special case $\theta_1=...=\theta_N=0$ we get

$$|\mathfrak{X}_{0,...,0};\overline{\alpha}_1,...,\overline{\alpha}_N\rangle = |\mathfrak{X}_{10};\overline{\alpha}_1\rangle\otimes...\otimes|\mathfrak{X}_{N0};\overline{\alpha}_N\rangle = |P_1;\alpha_1\rangle\otimes...\otimes|P_N;\alpha_N\rangle\tag{44}$$

Here

$$|\langle\mathfrak{X}_{\theta_1,...,\theta_N};\overline{\alpha}_1,...,\overline{\alpha}_N|\mathfrak{X}_{\phi_1,...,\phi_N};\overline{\beta}_1,...,\overline{\beta}_N\rangle|^2=\frac{1}{k},\tag{45}$$

where k is a divisor of n. In this way we get $\psi(n)=\prod(p_i+1)$ weak mutually unbiased bases, which we label in analogous way to the lines, as

$$B(\theta_1,...,\theta_N)=\{|\mathfrak{X}_{\theta_1,...,\theta_N};\overline{\alpha}_1,...,\overline{\alpha}_N\rangle\}\tag{46}$$

Proposition IV.1. *There is a duality between the lines in Eq.(19) and the WMUB:*

$$\Lambda(\theta_1,...,\theta_N)\ \ \leftrightarrow\ \ B(\theta_1,...,\theta_N)\tag{47}$$

Proof. Lemma III.2 for the factorized lines, makes the duality self-evident. □

In this duality:

(1) Maximal lines in $G(n)$ correspond to WMUB in $H(n)$. There are $\psi(n)$ maximal lines in $G(n)$ and $\psi(n)$ WMUB in $H(n)$. Each maximal line has n points and each WMUB has n vectors.

(2) The subgeometries of $G(n)$ correspond to the subsystems $\Sigma(n)$, and also to the divisors of n. There are $\sigma_0(n)$ (the divisor function) divisors of n, and therefore $\sigma_0(n)$ subgeometries of $G(n)$ and $\sigma_0(n)$ subsystems of $\Sigma(n)$.

This is summarized in table 2.

V. DISCUSSION

We have studied as a partially ordered set, the set $\mathfrak{G}(n)$ of subgeometries of $G(n)$, related to the quantum system $\Sigma(n)$. These geometries are not near-linear geometries and consequently they have subgeometries, and there is a partial order which we have studied in this paper. This is analogous to non-prime numbers, which have non-trivial divisors. The lines through the origin have been factorized in terms of 'prime factor lines' in $\mathbb{Z}(p_i) \times \mathbb{Z}(p_i)$. We have explained, that this factorization of lines is analogous to the factorization of integers in terms of prime factors. We labelled the lines in two different ways in Eq.(19). The first is a natural notation for lines, and the second is used to demonstrate the duality with the WMUB.

We have also studied the set $\mathfrak{S}(n)$ of subsystems of $\Sigma(n)$ as a partially ordered set, and we have shown that it is isomorphic to $\mathfrak{G}(n)$. WMUB which are products of the mutually unbiased bases in the 'prime factor Hilbert spaces' $H(p_i)$, have also been discussed.

There is an intimate link between geometry (lines) and physics (quantum systems) which is formalized in this paper. The work also demonstrates the relationship between smaller and bigger structures, in both geometry and physics .

[1] A. Vourdas, Rep. Prog. Phys. 67, 1 (2004)
[2] M. Kibler, J. Phys. A42, 353001 (2009)
[3] N. Cotfas, J.P. Gazeau, J.Phys. A43, 193001(2010)
[4] T. Durt, B.G. Englert, I. Bengtsson, K. Zyczkowski, Int. J. Quantum Comp. 8, 535 (2010)
[5] P. Stovicek, J. Tolar, Rep. Math. Phys. 20, 157 (1984)
[6] J. Tolar, G. Chadzitaskos, J. Phys. A42, 245306 (2009)
[7] W. Wootters, B.D. Fields, Ann. Phys. (NY), 191, 363 (1989)
[8] S. Chaturvedi, Phys. Rev. A65, 044301 (2002)
[9] S. Bandyopadhyay, P.O. Boykin, V.Roychowdhury, F. Vatan, Algorithmica 34, 512 (2002)
[10] K. Gibbons, M.J. Hoffman, W. Wootters, Phys. Rev. A70, 062101 (2004)
[11] A. Klappenecker, M. Rotteler, Lect. Notes Comp. Science 2948, 137 (2004)
[12] A. Klimov, L. Sanchez-Soto, H. de Guise, J. Phys. A38, 2747 (2005)
[13] J.L. Romero, G. Bjork, A.B. Klimov, L.L. Sanchez-Soto, Phys. Rev. A72, 062310 (2005)
[14] M.R. Kibler, M. Planat, Intern. J. Mod. Phys. B20, 1802 (2006)
[15] M. Saniga, M. Planat, J. Phys. A39, 435 (2006)
[16] P. Sulc, J. Tolar, J. Phys. A40, 15099 (2007)
[17] I. Bengtssson, W. Brudza, A. Ericsson, J.A. Larson, W. Tadej, K. Zyczkowski, J. Math. Phys. 48, 052106 (2007)
[18] S. Brierley, S. Weigert, Phys. Rev. A78, 042312 (2008)
[19] S. Brierley, S. Weigert, Phys. Rev. A79, 052316 (2009)

[20] O. Albouy, J. Phys. A42, 072001 (2009)
[21] M. Planat, J. Phys. A44, 045301 (2011)
[22] M. Kibler, Entropy 15, 1726 (2013)
[23] M. Shalaby, A. Vourdas, J. Phys. A45, 052001 (2012)
[24] M. Shalaby, A. Vourdas, Ann. Phys. 337, 208 (2013)
[25] A. Vourdas, C. Bendjaballah, Phys. Rev. A47, 3523 (1993)
[26] A. Vourdas, J. Phys. A36, 5645 (2003)
[27] L.M. Batten, 'Combinatorics of finite geometries', Cambridge Univ. Press, Cambridge, 1997
[28] J.W.P. Hirchfeld, 'Projective geometries over finite fields' (Oxford Univ. Press, Oxford, 1979)
[29] J.W.P. Hirchfeld, J.A. Thas, 'General Galois geometries' (Oxford Univ. Press, Oxford, 1991)
[30] M. Planat, M. Saniga, M. R. Kibler, SIGMA, 2, 66 (2006)
[31] H. Havlicek, M. Saniga, J. Phys. A40, F943 (2007)
[32] H. Havlicek, M. Saniga, J. Phys. A41, 015302 (2008)
[33] M. Korbelar, J. Tolar, J. Phys. A43, 375302 (2010)
[34] I.J. Good, IEEE Trans. Computers, C-20, 310 (1971)
[35] H.G. Feichtinger, M. Hazewinkel, N. Kaiblinger, E. Matusiak, M. Neuhauser, Quartely J. Math.,59, 15 (2008)
[36] A. Vourdas, C. Banderier, J. Phys. A43, 042002 (2010)
[37] T. Durt, J. Phys. A38, 5267 (2005)
[38] J. Zak, J. Phys. A44, 345305 (2011)
[39] J. Zak, J. Math. Phys.53, 103514 (2012)

Figure 1: The Hasse diagram showing the geometry $G(30)$ and its subgeometries, and also the Hilbert spaces of the subsystems of $\Sigma(30)$

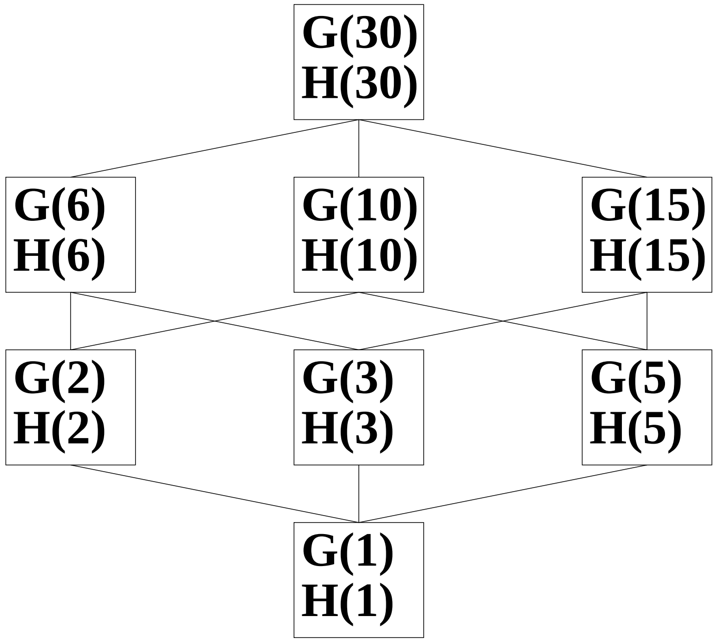

Table I: The maximal lines in $G(30)$ and its subgeometries, presented in both notation according to Eq.(19). The first is the 'natural notation' for lines, and the second is used to demonstrate the duality with the WMUB.

$G(30)$	$\mathcal{L}(0,1)=\Lambda(-1,-1,-1)$,	$\mathcal{L}(1,0)=\Lambda(0,0,0)$,	$\mathcal{L}(1,1)=\Lambda(1,1,1)$,
	$\mathcal{L}(1,2)=\Lambda(0,2,2)$,	$\mathcal{L}(1,3)=\Lambda(1,0,3)$,	$\mathcal{L}(1,4)=\Lambda(0,1,4)$,
	$\mathcal{L}(1,5)=\Lambda(1,2,0)$,	$\mathcal{L}(1,6)=\Lambda(0,0,1)$,	$\mathcal{L}(1,7)=\Lambda(1,1,2)$,
	$\mathcal{L}(1,8)=\Lambda(0,2,3)$,	$\mathcal{L}(1,9)=\Lambda(1,0,4)$,	$\mathcal{L}(1,10)=\Lambda(0,1,0)$,
	$\mathcal{L}(1,11)=\Lambda(1,2,1)$,	$\mathcal{L}(1,12)=\Lambda(0,0,2)$,	$\mathcal{L}(1,13)=\Lambda(1,1,3)$,
	$\mathcal{L}(1,14)=\Lambda(0,2,4)$,	$\mathcal{L}(1,15)=\Lambda(1,0,0)$,	$\mathcal{L}(1,16)=\Lambda(0,1,1)$,
	$\mathcal{L}(1,17)=\Lambda(1,2,2)$,	$\mathcal{L}(1,18)=\Lambda(0,0,3)$,	$\mathcal{L}(1,19)=\Lambda(1,1,4)$,
	$\mathcal{L}(1,20)=\Lambda(0,2,0)$,	$\mathcal{L}(1,21)=\Lambda(1,0,1)$,	$\mathcal{L}(1,22)=\Lambda(0,1,2)$,
	$\mathcal{L}(1,23)=\Lambda(1,2,3)$,	$\mathcal{L}(1,24)=\Lambda(0,0,4)$,	$\mathcal{L}(1,25)=\Lambda(1,1,0)$,
	$\mathcal{L}(1,26)=\Lambda(0,2,1)$,	$\mathcal{L}(1,27)=\Lambda(1,0,2)$,	$\mathcal{L}(1,28)=\Lambda(0,1,3)$,
	$\mathcal{L}(1,29)=\Lambda(1,2,4)$,	$\mathcal{L}(2,1)=\Lambda(-1,2,3)$,	$\mathcal{L}(2,3)=\Lambda(-1,0,4)$,
	$\mathcal{L}(2,5)=\Lambda(-1,1,0)$,	$\mathcal{L}(2,7)=\Lambda(-1,2,1)$,	$\mathcal{L}(2,9)=\Lambda(-1,0,2)$,
	$\mathcal{L}(2,11)=\Lambda(-1,1,3)$,	$\mathcal{L}(2,13)=\Lambda(-1,2,4)$,	$\mathcal{L}(2,15)=\Lambda(-1,0,0)$,
	$\mathcal{L}(2,17)=\Lambda(-1,1,1)$,	$\mathcal{L}(2,19)=\Lambda(-1,2,2)$,	$\mathcal{L}(2,21)=\Lambda(-1,0,3)$,
	$\mathcal{L}(2,23)=\Lambda(-1,1,4)$,	$\mathcal{L}(2,25)=\Lambda(-1,2,0)$,	$\mathcal{L}(2,27)=\Lambda(-1,0,1)$,
	$\mathcal{L}(2,29)=\Lambda(-1,1,2)$,	$\mathcal{L}(3,1)=\Lambda(1,-1,2)$,	$\mathcal{L}(3,2)=\Lambda(0,-1,4)$,
	$\mathcal{L}(3,4)=\Lambda(0,-1,3)$,	$\mathcal{L}(3,5)=\Lambda(1,-1,0)$,	$\mathcal{L}(3,7)=\Lambda(1,-1,4)$,
	$\mathcal{L}(3,8)=\Lambda(0,-1,1)$,	$\mathcal{L}(3,10)=\Lambda(0,-1,0)$,	$\mathcal{L}(3,13)=\Lambda(1,-1,1)$,
	$\mathcal{L}(3,16)=\Lambda(0,-1,2)$,	$\mathcal{L}(3,19)=\Lambda(1,-1,3)$,	$\mathcal{L}(5,1)=\Lambda(1,2,-1)$,
	$\mathcal{L}(5,2)=\Lambda(0,1,-1)$,	$\mathcal{L}(5,3)=\Lambda(1,0,-1)$,	$\mathcal{L}(5,4)=\Lambda(0,2,-1)$,
	$\mathcal{L}(5,6)=\Lambda(0,0,-1)$,	$\mathcal{L}(5,11)=\Lambda(1,1,-1)$,	$\mathcal{L}(6,1)=\Lambda(-1,-1,1)$,
	$\mathcal{L}(6,5)=\Lambda(-1,-1,0)$,	$\mathcal{L}(6,7)=\Lambda(-1,-1,2)$,	$\mathcal{L}(6,13)=\Lambda(-1,-1,3)$,
	$\mathcal{L}(6,19)=\Lambda(-1,-1,4)$,	$\mathcal{L}(10,1)=\Lambda(-1,1,-1)$,	$\mathcal{L}(10,3)=\Lambda(-1,0,-1)$,
	$\mathcal{L}(10,11)=\Lambda(-1,2,-1)$,	$\mathcal{L}(15,1)=\Lambda(1,-1,-1)$,	$\mathcal{L}(15,2)=\Lambda(0,-1,-1)$
$G(15)$	$\mathcal{L}(0,1)=\Lambda(-1,-1)$,	$\mathcal{L}(1,0)=\Lambda(0,0)$,	$\mathcal{L}(1,1)=\Lambda(2,2)$,
	$\mathcal{L}(1,2)=\Lambda(1,4)$,	$\mathcal{L}(1,3)=\Lambda(0,1)$,	$\mathcal{L}(1,4)=\Lambda(2,3)$,
	$\mathcal{L}(1,5)=\Lambda(1,0)$,	$\mathcal{L}(1,6)=\Lambda(0,2)$,	$\mathcal{L}(1,7)=\Lambda(2,4)$,
	$\mathcal{L}(1,8)=\Lambda(1,1)$,	$\mathcal{L}(1,9)=\Lambda(0,3)$,	$\mathcal{L}(1,10)=\Lambda(2,0)$,
	$\mathcal{L}(1,11)=\Lambda(1,2)$,	$\mathcal{L}(1,12)=\Lambda(0,4)$,	$\mathcal{L}(1,13)=\Lambda(2,1)$,
	$\mathcal{L}(1,14)=\Lambda(1,3)$,	$\mathcal{L}(3,1)=\Lambda(-1,4)$,	$\mathcal{L}(3,2)=\Lambda(-1,3)$,
	$\mathcal{L}(3,4)=\Lambda(-1,1)$,	$\mathcal{L}(3,5)=\Lambda(-1,0)$,	$\mathcal{L}(3,8)=\Lambda(-1,2)$,
	$\mathcal{L}(5,1)=\Lambda(1,-1)$,	$\mathcal{L}(5,2)=\Lambda(2,-1)$,	$\mathcal{L}(5,3)=\Lambda(0,-1)$,
$G(10)$	$\mathcal{L}(0,1)=\Lambda(-1,-1)$,	$\mathcal{L}(1,0)=\Lambda(0,0)$,	$\mathcal{L}(1,1)=\Lambda(1,3)$,
	$\mathcal{L}(1,2)=\Lambda(0,1)$,	$\mathcal{L}(1,3)=\Lambda(1,4)$,	$\mathcal{L}(1,4)=\Lambda(0,2)$,
	$\mathcal{L}(1,5)=\Lambda(1,0)$,	$\mathcal{L}(1,6)=\Lambda(0,3)$,	$\mathcal{L}(1,7)=\Lambda(1,1)$,
	$\mathcal{L}(1,8)=\Lambda(0,4)$,	$\mathcal{L}(1,9)=\Lambda(1,2)$,	$\mathcal{L}(2,1)=\Lambda(-1,4)$,
	$\mathcal{L}(2,3)=\Lambda(-1,2)$,	$\mathcal{L}(2,5)=\Lambda(-1,0)$,	$\mathcal{L}(2,7)=\Lambda(-1,3)$,
	$\mathcal{L}(2,9)=\Lambda(-1,1)$,	$\mathcal{L}(5,1)=\Lambda(1,-1)$,	$\mathcal{L}(5,2)=\Lambda(0,-1)$,
$G(6)$	$\mathcal{L}(0,1)=\Lambda(-1,-1)$,	$\mathcal{L}(1,0)=\Lambda(0,0)$,	$\mathcal{L}(1,1)=\Lambda(1,2)$,
	$\mathcal{L}(1,2)=\Lambda(0,1)$,	$\mathcal{L}(1,3)=\Lambda(1,0)$,	$\mathcal{L}(1,4)=\Lambda(0,2)$,
	$\mathcal{L}(1,5)=\Lambda(1,1)$,	$\mathcal{L}(2,1)=\Lambda(-1,1)$,	$\mathcal{L}(2,3)=\Lambda(-1,0)$,
	$\mathcal{L}(2,5)=\Lambda(-1,2)$,	$\mathcal{L}(3,1)=\Lambda(1,-1)$,	$\mathcal{L}(3,2)=\Lambda(0,-1)$,
$G(5)$	$\mathcal{L}(0,1)=\Lambda(-1)$,	$\mathcal{L}(1,0)=\Lambda(0)$,	$\mathcal{L}(1,1)=\Lambda(1)$,
	$\mathcal{L}(1,2)=\Lambda(2)$,	$\mathcal{L}(1,3)=\Lambda(3)$,	$\mathcal{L}(1,4)=\Lambda(4)$,
$G(3)$	$\mathcal{L}(0,1)=\Lambda(-1)$,	$\mathcal{L}(1,0)=\Lambda(0)$,	$\mathcal{L}(1,1)=\Lambda(1)$,
	$\mathcal{L}(1,2)=\Lambda(2)$		
$G(2)$	$\mathcal{L}(0,1)=\Lambda(-1)$,	$\mathcal{L}(1,0)=\Lambda(0)$,	$\mathcal{L}(1,1)=\Lambda(1)$
$G(1)$	$\mathcal{L}(0,0)=\Lambda(0)$		

Table II: Duality between lines and WMUB

$G(n)$	$H(n)$
$\Lambda(\theta_1, ..., \theta_N)$ (Eq.(19)) $\psi(n)$ maximal lines	$B(\theta_1, ..., \theta_N)$(Eq.(46)) $\psi(n)$ WMUB
n points in each maximal line	n orthogonal vectors in each WMUB
subgeometries $G(m)$ $(m\|n)$	subsystems $\Sigma(m)$ $(m\|n)$ with Hilbert space $H(m)$