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Sequential Investments with Stage Specific Risks and Drifts

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Sequential Investments with Stage Specific Risks and Drifts

Abstract

We provide a generalized analytical methodology for evaluating a real sequential investment opportunity, which does not rely on a multivariate distribution function, but which allows for stage specific risks and drifts. This model may be a useful capital budgeting and valuation tool for exploration and development projects, where risks change over the stages. We construct a stage threshold pattern whereby the final stage threshold exceeds the early stage threshold due to drift differentials between the project values at the various stages, value volatility differences, and correlation differentials, implying a rich menu of parameter values that may be suitable for a variety of projects. Governments seeking to motivate early final stage investments might lower final stage project volatility or specify project value decline over time, unless prospective owners are willing to pay the real option value (ROV) for concessions. In contrast, concession owners, more interested in ROV than thresholds that motivate early investments, may welcome final stage value escalation, or guarantees that reduce the correlation between project value and construction cost.

Introduction

We determine the project value thresholds that justify an investment at any stage for a project composed of multiple sequential investments, allowing for stage specific risks and drifts. This extends the Adkins and Paxson (2014) multi-factor model, which requires the project failure probabilities to decline at each stage approaching the final completion stage. It also complements Cassimon et al. (2011), which allows for stage specific project volatilities but not for stage specific value drifts or for other uncertainties regarding project failure or investment cost risks and drifts. Cassimon et al. (2011) builds on the Geske (1979) European compound option approach, so no thresholds are determined, and there is no flexibility in investment timing. Our solution requires some package of positive drift or volatility or correlation differentials between the final stage and the initial stage, (if there is no probability of complete project failure), which is a characteristic of many exploration and development (E&D), and infrastructure projects.

Suppose a real sequential investment opportunity consists of a set of distinct, ordered investments that have to be made before the project can be completed. The project can be interpreted as a collection of investment stages, such that no stage investment, except the first, can be started until the preceding stage has been completed. The project value is realized when all the stages have been successfully completed. The following four-stage E&D opportunity from Cortazar et al. (2003) provides an illustration: (i) undertaking geological research. (ii) drilling an exploratory well, (iii) drilling development wells and (iv) implementing the infrastructure for production.

Making an investment at any stage depends on whether the prevailing project value is of sufficient magnitude to economically justify making the stage investment. We are interested in the package of value and investment cost drifts, volatilities and correlations that result in a pattern of project value thresholds that justify making an immediate investment at each stage that increase as the stages near completion. Adkins and Paxson (2014) obtain this result through assuming that the investment opportunity at any stage is subject to a catastrophic failure that causes the option value to be entirely destroyed, so the project as an entity becomes irredeemably lost like in some R&D and venture capital activities. There are investment opportunities such as mineral leases where there is no doubt that the reservoir or relevant minerals exist but economic development depends on the prevailing commodity prices, and investment and operating costs. (Canadian tar sands are a primary example at low crude oil prices.) There are three sources of uncertainty, the stochastic project values and the stochastic investment costs, which are permitted to be correlated at each of the project stages. Because of value conservation, the option for any stage except completion is evaluated at the point where the investment required to continue is less than the value of the option created at the next stage. In this way, the sequential opportunity is a compound option specified over multiple stages. We formulate an analytical solution to a multiple compound option based on American perpetuities.

One of the first attempts to obtaining a solution for sequential investments based on American perpetuity options is formulated by Dixit and Pindyck (1994). They suggest a rule for a two-stage sequential investment with fixed investment costs, but their solution is seemingly implausible as it is identical to the one-stage result except for the accumulated costs. This problem is noted by Rodrigues (2009), who uses differential segment demand volatilities,

investment costs, and some other measures, to evaluate optimal timing among segments, with thresholds increasing with time and investment, under an endogenous regime-switching process. Kort et al. (2010) propose that the American perpetuity option value for a two-stage sequential investments is equal to the sum of the separate option values, but this formulation suffers the defect of a lack of compoundedness in the sense that the first and second stage option values are independent.

An alternative method for reaching a closed-form rule for sequential investment opportunities is to suspend the need for an elapse time between stages that is unknown and uncertain, and to replace it by one that is known and fixed. This relaxation converts the underlying option type from American to European. Cassimon et al. (2011) extends the Cassimon et al. (2004) European compound model and illustrates that the ROV of a mobile payments project with value volatilities declining from 54% to 35% in four stages approaching completion is €20.14m, compared to €22.10m with a constant volatility of 54%¹. Of course, these European options are not investment timing models.

Some authors eschew the reputed merits of closed-form European compound options and solve the sequential investment opportunity through the power of numerical techniques. A trinomial lattice formulation is used by Childs and Triantis (1999) to solve a multiple sequential investment model having cash-flow interaction. Schwartz and Moon (2000) provide a numerical solution for complex R&D options, with project failure that does not always decline as the

¹ They did not provide the stage specific ROVs, or (since European) the V thresholds that would justify making the required investments at each stage.

project approaches completion, but with constant asset volatility, drifts and investment cost volatility over four stages. Cortazar et al. (2003) assume the probability of success increases as the E&D stages near completion (production) with investment costs almost always increasing near completion. An implicit finite-difference numerical solution provides a value first without options, and then with operating, development and exploration options as a function of expected copper mine size. Cortelezzi and Villani (2009) use Monte Carlo simulation for valuing a R&D project characterized as an American sequential exchange option. Koussis et al. (2013) provide numerical solutions for multi-stages with multiple options. The shortcomings of these solution methods are the possibly onerous and not always transparent calculations.

The aim of this paper is to reformulate and solve analytically the sequential investment model by incorporating three distinct sources of stage specific uncertainties. The three sources are characterized by the uncertainty associated with the project value and the investment cost for each stage, and the correlation of value and cost. Based on an American perpetuity option framework, we find that the project value threshold that justifies investment at each stage is a recursive expression represented by a function of the investment cost threshold at the particular stage and those for all the subsequent stages. Further, we demonstrate that the option value for each stage is a homogenous degree-1 and convex function, in keeping with the Merton (1973) assertion. In contrast to Adkins and Paxson (2014), there is no requirement that the probability of catastrophic failure continually declines at each stages until completion. However, there are usually alternative requirements that, for instance, the expected project value (V) drift (as defined in EQ 1 or its risk neutral equivalent as in EQ 2) for the final stage exceeds that for the initial stage, or the stage one V_1 volatility exceeds the stage two V_2 volatility, or that the

correlation of V_1 and investment cost (K_1) is sufficiently lower than the V_2, K_1 correlation, and possibly other correlation differentials, which result in the threshold for the initial stage being lower than the final stage. While various combinations of these requirements may not be present in all projects, this approach offers a richer menu and package of requirements than the simple Adkins and Paxson (2014) project failure pattern.

The major analytical findings for this sequential investment model are developed in Section 2. Based on the three sources of uncertainty, the model is presented first for a one-stage opportunity, and then developed for a two-stage sequential investment opportunity. We develop closed-form solutions for whether or not to commence investment at a particular stage and for the option value at each stage. In Section 3, we obtain further insights into the behavior of the model through numerical illustrations. The last section summarizes some advantages and limitations of this model and suggests plausible extensions.

Sequential Investment Model

A monopolist is considering an investment project made up of a two sequential stages, each involving an individual non-zero instantaneous investment cost. The project as an entity is not fully implemented and the project value not realized until the two sequential stages have been successfully completed. Each successive investment stage relies on the successful completion of the investment made at the preceding stage. We order each investment stage by the number J of remaining stages, including the current one, until project completion, so the final stage is stage 1.

After an initial investment, there may be differences in expected project value drifts, volatilities, K drifts, volatilities and/or correlations, so that the thresholds that justify making any subsequent investments increase approaching completion. Generally this will be accompanied by an increase in the power parameter values nearing completion. It may be that these stage specific parameter values are due to the previous stage investments being made, or they may also be exogenous, perhaps due to the nature of the stage. Situations do arise when an investment can produce an innovative breakthrough and generate an unanticipated increase in the project value. We have ignored this possibility, but allow for an exogenous change in project value drift or volatility, possibly due to a change in or different segments of the term structure of forward commodity prices. Also, other forms of optionality, such as terminating a project before completion for its abandonment value, are not considered.

The value of the project is defined by V . This value cannot be realized until the ultimate investment at $J=1$ has been successfully completed. Both the project value and the set of investment expenditures are treated as stochastic. It is assumed that they are individually well described by the geometric Brownian motion process:

$$dX = \alpha_{X_j} X dt + \sigma_{X_j} X dz_{X_j}, \quad (1)$$

for $X \in \{V_j, K_j \forall J\}$, where α_{X_j} represent the respective drift parameters for each stage, σ_{X_j} the respective instantaneous volatility parameter, and dz_{X_j} the respective increment of a standard Wiener process. Dependence between any two of the factors is represented by the covariance term; so, for example, the covariance between the project value and the investment expenditure at stage J is specified by:

$$\text{Cov}[dV_J, dK_J] = \rho_{V_J K_J} \sigma_{V_J} \sigma_{K_J} dt.$$

The project value and investment cost drift, volatility and correlation parameter values are permitted to vary by stage, but are assumed to remain constant during the stage. It is assumed that the investment expenditure at each stage is instantaneous with a zero time-to-build and that sufficient internally generated or external funds are available on time to meet the financing needs at each stage.

1.1 One-Stage Model

The stage-1 model represents the investment opportunity for developing a project value V_1 following the investment cost K_1 . We develop the solution based on the two-factor solution of Adkins and Paxson (2011), which is extendable to dimensions greater than two. The value $ROV_1 = F_1$ of the investment opportunity at stage $J=1$ depends on the project value and the investment cost, so $F_1 = F_1(V_1, K_1)$. By Ito's lemma, the risk neutral valuation relationship is:

$$\begin{aligned} \frac{1}{2} \sigma_{V_1}^2 V_1^2 \frac{\partial^2 F_1}{\partial V_1^2} + \frac{1}{2} \sigma_{K_1}^2 K_1^2 \frac{\partial^2 F_1}{\partial K_1^2} + \rho_{V_1 K_1} \sigma_{V_1} \sigma_{K_1} V_1 K_1 \frac{\partial^2 F_1}{\partial V_1 \partial K_1} \\ + \theta_{V_1} V_1 \frac{\partial F_1}{\partial V_1} + \theta_{K_1} K_1 \frac{\partial F_1}{\partial K_1} - r F_1 = 0, \end{aligned} \quad (2)$$

where the θ_{X_j} for $X \in \{V_J, K_J, \forall J\}$ denote the respective risk neutral drift rate parameters² and r the risk-free rate treated as constant over all stages. The generic solution to (2) is the two-factor power function:

$$F_1 = A_1 V_1^{\beta_1} K_1^{\eta_1}, \quad (3)$$

² See Dixit and Pindyck (1994) for some theories relating the real world and risk-neutral drift. We assume for convenience that θ is not affected by changes in volatilities, or correlations.

where β_1 and η_1 denote the generic unknown parameters for the two factors, project value and investment cost, and A_1 denotes a generic unknown coefficient. In this notation, the first subscript for A_1 , β_1 and η_1 refers to the specific stage under consideration, while the second subscript refers to the stage specific power parameter value where appropriate. We expect $A_1 \geq 0$ since the option value is positive. A justified economic incentive to exercise the stage-1 option exists whenever the project value is sufficiently high and the investment cost is sufficiently low, and this incentive intensifies for project value increases and investment cost decreases. The threshold levels for the project value and the investment cost signaling the optimal exercise for the investment option at stage $J = 1$ are denoted by \hat{V}_1 and \hat{K}_1 , respectively. The value matching relationship describes the conservation equality at optimality that the option value $\hat{F}_1 = F_1(\hat{V}_1, \hat{K}_1)$ exactly compensates for the net asset value $\hat{V}_1 - \hat{K}_1$ obtained by spending the investment cost. Then:

$$A_1 \hat{V}_1^{\beta_1} \hat{K}_1^{\eta_1} = \hat{V}_1 - \hat{K}_1. \quad (4)$$

The first order condition for optimality is characterized by the two associated smooth pasting conditions, one for each factor, Samuelson (1965). These can be expressed as:

$$A_1 \hat{V}_1^{\beta_1} \hat{K}_1^{\eta_1} = \frac{\hat{V}_1}{\beta_1}. \quad (5)$$

$$A_1 \hat{V}_1^{\beta_1} \hat{K}_1^{\eta_1} = -\frac{\hat{K}_1}{\eta_1} \quad (6)$$

Since the option value is always non-negative, $A_1 \geq 0$, as expected. Together, (4,5,6) demonstrate Euler's result on homogeneity degree-1 functions, so $\beta_1 + \eta_1 = 1$. So the characteristic root equation satisfying (2) is:

$$Q_1(\beta_1, 1 - \beta_1) = \frac{1}{2} \sigma_1^2 \beta_1 (\beta_1 - 1) + \beta_1 (\theta_{V_1} - \theta_{K_1}) - (r - \theta_{K_1}) = 0, \quad (7)$$

where $\sigma_1^2 = \sigma_{V_1}^2 + \sigma_{K_1}^2 - 2\rho_{V_1, K_1} \sigma_{V_1} \sigma_{K_1}$. From (7), β_1 is obtained as the positive root solution for the quadratic characteristic equation, which is greater than 1. Further, the threshold levels are

related by:
$$\hat{V}_1 = \frac{\beta_1}{\beta_1 - 1} \hat{K}_1, \quad (8)$$

with $A_1 = \beta_1^{-\beta_1} (\beta_1 - 1)^{\beta_1 - 1}$.

1.2 Two-Stage Model

At the preceding stage, $J = 2$, the firm examines the viability of committing an investment K_2 to acquire the option to invest F_1 by comparing the value of the compound option $ROV_2 = F_2$ with the net benefits $F_1 - K_2$. F_2 depends on the three factors V_2 , K_1 and K_2 , so $F_2 = F_2(V_2, K_1, K_2)$. By Ito's lemma, the risk neutral valuation relationship for F_2 is:

$$\begin{aligned} & \frac{1}{2} \sigma_{V_2}^2 V_2^2 \frac{\partial^2 F_2}{\partial V_2^2} + \frac{1}{2} \sigma_{K_1}^2 K_1^2 \frac{\partial^2 F_2}{\partial K_1^2} + \frac{1}{2} \sigma_{K_2}^2 K_2^2 \frac{\partial^2 F_2}{\partial K_2^2} \\ & + \rho_{V_2, K_1} \sigma_{V_2} \sigma_{K_1} V_2 K_1 \frac{\partial^2 F_2}{\partial V_2 \partial K_1} + \rho_{V_2, K_2} \sigma_{V_2} \sigma_{K_2} V_2 K_2 \frac{\partial^2 F_2}{\partial V_2 \partial K_2} + \rho_{K_1, K_2} \sigma_{K_1} \sigma_{K_2} K_1 K_2 \frac{\partial^2 F_2}{\partial K_1 \partial K_2} \\ & + \theta_{V_2} V_2 \frac{\partial F_2}{\partial V_2} + \theta_{K_2} K_2 \frac{\partial F_2}{\partial K_2} + \theta_{K_1} K_1 \frac{\partial F_2}{\partial K_1} - r F_2 = 0. \end{aligned} \quad (9)$$

We conjecture that the solution to (9) is a product power function, with generic form:

$$F_2 = A_2 V_2^{\beta_2} K_1^{\eta_{21}} K_2^{\eta_{22}}, \quad (10)$$

where β_2 , η_{21} and η_{22} denote the generic unknown parameters for the three factors, project value at stage two and investment expenditures envisioned as of stage two, and A_2 denotes an unknown coefficient. It is similarly expected that $A_2 \geq 0$. We conjecture that $\beta_2 \geq 0$ because

project value increases incentivize the option exercise, while $\eta_{21}, \eta_{22} < 0$ because investment cost increases inhibit the option exercise.

We specify that the stage-2 threshold levels signaling an optimal exercise are represented by \hat{V}_2 , \hat{K}_{21} and \hat{K}_{22} for V , K_1 and K_2 , respectively. The set $\{\hat{V}_2, \hat{K}_{21}, \hat{K}_{22}\}$ forms the boundary that discriminates between the “exercise” decision and the “wait” decision. This boundary is determined from establishing the relationship amongst \hat{V}_2 , \hat{K}_{21} and \hat{K}_{22} , or alternatively, from identifying the dependence of \hat{V}_2 with respect to \hat{K}_{21} and \hat{K}_{22} . A stage-2 option exercise occurs for the balance between the stage-2 option value $A_2 \hat{V}_2^{\beta_2} \hat{K}_{21}^{\eta_{21}} \hat{K}_{22}^{\eta_{22}}$ and the stage-1 option value $A_1 \hat{V}_1^{\beta_1} \hat{K}_{11}^{\eta_{11}}$ less the investment cost \hat{K}_{22} incurred in its acquisition. This equilibrium amongst the threshold levels is the value matching relation that is expressed as:

$$A_2 \hat{V}_2^{\beta_2} \hat{K}_{21}^{\eta_{21}} \hat{K}_{22}^{\eta_{22}} = A_1 \hat{V}_2^{\beta_1} \hat{K}_{21}^{1-\beta_1} - \hat{K}_{22}, \quad (11)$$

where A_1 and β_1 are known from the stage-1 evaluation. The three smooth pasting conditions associated with (11), one for each of the three factors V , K_1 and K_2 , respectively, can be

expressed as:
$$\beta_2 A_2 \hat{V}_2^{\beta_2} \hat{K}_{21}^{\eta_{21}} \hat{K}_{22}^{\eta_{22}} = \beta_1 A_1 \hat{V}_2^{\beta_1} \hat{K}_{21}^{1-\beta_1}, \quad (12)$$

$$\eta_{21} A_2 \hat{V}_2^{\beta_2} \hat{K}_{21}^{\eta_{21}} \hat{K}_{22}^{\eta_{22}} = (1 - \beta_1) A_1 \hat{V}_2^{\beta_1} \hat{K}_{21}^{1-\beta_1}, \quad (13)$$

$$\eta_{22} A_2 \hat{V}_2^{\beta_2} \hat{K}_{21}^{\eta_{21}} \hat{K}_{22}^{\eta_{22}} = -\hat{K}_{22}. \quad (14)$$

Since an option value is non-negative, then $A_2 \geq 0$ as expected. This implies that $\beta_2 \geq 0$ from (12), $\eta_{21} < 0$ from (13), and $\eta_{22} < 0$ from (14). If we specify $\phi_1 = \beta_1$ and $\phi_2 = \beta_2 / \beta_1 \geq 0$, then

$$\eta_{22} = 1 - \frac{\beta_2}{\beta_1}, \eta_{21} = \frac{1 - \beta_1}{\beta_1} \beta_2, \beta_2 + \eta_{21} + \eta_{22} = 1. \quad (15)$$

Also, the quadratic function Q_2 can be expressed as:

$$\begin{aligned}
Q_2 = & \frac{1}{2} \phi_2 (\phi_2 - 1) \sigma_2^2 \\
& + \phi_2 \left\{ (r - \theta_{K_2}) + \phi_1 (\theta_{V_2} - \theta_{V_1}) \right. \\
& \quad \left. + \frac{1}{2} \phi_1 (\phi_1 - 1) \left[(\sigma_{V_2}^2 - \sigma_{V_1}^2) - 2(\rho_{V_2 K_1} \sigma_{V_2} \sigma_{K_1} - \rho_{V_1 K_1} \sigma_{V_1} \sigma_{K_1}) \right] \right\} \\
& - (r - \theta_{K_2}) = 0.
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
\sigma_2^2 = & \phi_1^2 \sigma_{V_2}^2 + (1 - \phi_1)^2 \sigma_{K_1}^2 + \sigma_{K_2}^2 \\
& + 2\phi_1 (1 - \phi_1) \rho_{V_2 K_1} \sigma_{V_2} \sigma_{K_1} - 2\phi_1 \rho_{V_2 K_2} \sigma_{V_2} \sigma_{K_2} - 2(1 - \phi_1) \rho_{K_1 K_2} \sigma_{K_1} \sigma_{K_2}.
\end{aligned} \tag{17}$$

The parameter ϕ_2 , which is required to be greater than one, is evaluated as the positive root of the quadratic function Q_2 (16), knowing $\phi_1 = \beta_1$ from the previously calculated stage-1 solution.

The values of β_2 , η_{21} and η_{22} are then obtained from ϕ_2 and ϕ_1 . Subsequently, we show that $\phi_2 > 1$, so $\beta_2 > \beta_1$.

Because of (15), A_2 , \hat{V}_2 , \hat{K}_{21} and \hat{K}_{22} can be determined more conveniently by expressing F_2 , (10), as a function of the stage-1 option value F_1 , (3), and the stage-2 investment cost K_2 , and noting that F_2 is characterized as homogenous degree-1:

$$F_2 = F_{22}(F_1, K_2) = B_2 [F_1(V, K_1)]^{\phi_2} K_2^{1-\phi_2} = B_2 [A_1 V^{\beta_1} K_1^{\eta_{11}}]^{\phi_2} K_2^{1-\phi_2}, \tag{18}$$

where $B_2 = A_2 A_1^{-\phi_2}$. If the optimal stage-2 thresholds are, $\hat{F}_{12} = F_1(\hat{V}_2, \hat{K}_{21})$ for the stage-1 option, and \hat{K}_{22} , then the stage-2 value matching relationship (11).

$$B_2 \hat{F}_{12}^{\phi_2} \hat{K}_{22}^{1-\phi_2} = \hat{F}_{12} - \hat{K}_{22}. \tag{19}$$

Except for the change in variable, (19) is identical in form to (4), so $B_2 = (\phi_2 - 1)^{(\phi_2 - 1)} / \phi_2^{\phi_2}$, which implies:

$$A_2 = \frac{(\phi_2 - 1)^{(\phi_2 - 1)}}{\phi_2^{\phi_2}} \left[\frac{(\phi_1 - 1)^{\phi_1 - 1}}{\phi_1^{\phi_1}} \right]^{\phi_2},$$

so the stage-2 option value is specified by:

$$F_2(V_2, K_1, K_2) = \frac{(\phi_2 - 1)^{(\phi_2 - 1)}}{\phi_2^{\phi_2}} \left[\frac{(\phi_1 - 1)^{\phi_1 - 1}}{\phi_1^{\phi_1}} \right]^{\phi_2} V_2^{\phi_1 \phi_2} K_1^{(1 - \phi_1)\phi_2} K_2^{1 - \phi_2}. \quad (20)$$

The stage-2 option value function is homogenous degree-1 in the project value and investment costs. Also, the stage-2 option threshold level is given by:

$$\hat{F}_{12} = F_1(\hat{V}_2, \hat{K}_{12}) = A_1 \hat{V}_2^{\phi_1} K_{21}^{1 - \phi_1} = \frac{\phi_2}{\phi_2 - 1} \hat{K}_{22}. \quad (21)$$

Clearly, an economically meaningful solution to the stage-2 option threshold can only emerge provided $\phi_2 > 1$. Re-arranging (21) to yield the solution expressed in terms of the stage-2 project value threshold level:

$$\hat{V}_2 = \frac{\phi_1}{\phi_1 - 1} \left\{ \frac{\phi_2 (\phi_1 - 1)}{\phi_2 - 1} \right\}^{\frac{1}{\phi_1}} \hat{K}_{21}^{\frac{\phi_1 - 1}{\phi_1}} \hat{K}_{22}^{\frac{1}{\phi_1}}. \quad (22)$$

The project value threshold is a homogenous degree-1 function of the two investment cost thresholds, determined from their geometric mean with the $\hat{K}_{21}, \hat{K}_{22}$ power parameters dependent only on the stage-1 ϕ_1 . The relative magnitude of \hat{V}_1 and \hat{V}_2 is determined from comparing (8) and (22):

$$\frac{\hat{V}_2}{\hat{V}_1} = \left\{ \frac{\phi_2 (\phi_1 - 1) \hat{K}_2}{\phi_2 - 1 \hat{K}_1} \right\}^{\frac{1}{\phi_1}} \quad (23)$$

where $\hat{K}_1 = \hat{K}_{11} = \hat{K}_{12}$ and $\hat{K}_2 = \hat{K}_{22}$. Since $\phi_1 > 1$, then for stage-2 investment to be justified earlier than stage-1 investment, $\hat{V}_2 < \hat{V}_1$, the following lower bound LB must hold:

$$\frac{\hat{K}_1}{\hat{K}_2} > \frac{\phi_2(\phi_1 - 1)}{\phi_2 - 1} > 1. \quad (24)$$

The sequential investment model of Dixit and Pindyck (1994), Chapter 10, is based on identical stage-1 and -2 parameter values. It yields identical project value thresholds justifying immediate investment for the two sequential stages, so the two sequential stages effectively collapse to a single stage. For the current model, their assumption implies that all the stage-1 and -2 risk-neutral drifts, volatilities and correlations are equal, or $\theta_{V_2} = \theta_{V_1}$, $\sigma_{V_2}^2 = \sigma_{V_1}^2$ and $\rho_{V_2K_1} = \rho_{V_1K_1}$. Substituting this in (16) yields $\phi_2 = 1$, which produces an indefinite solution for the two-stage model. A meaningful economic solution also requires (24) to hold. This lower bound depends on the parameter values for the relevant stochastic factors at the two stages, the package of relative V_1, V_2 drifts and volatilities, K_1, K_2 drifts and volatilities, the covariance matrix, and the risk-free rate. We conjecture that possibly $\hat{V}_1 > \hat{V}_2$ if $\theta_{V_2} < \theta_{V_1}$, $\sigma_{V_2}^2 < \sigma_{V_1}^2$, $\rho_{V_2K_1} > \rho_{V_1K_1}$. There may be several other differentials that also result in $\hat{V}_1 > \hat{V}_2$, see the Appendix 2, also for an easy analytical solution. Note that \hat{V}_2 is dependent on both \hat{K}_1 and \hat{K}_2 , (22), whereas \hat{V}_1 is dependent on only \hat{K}_1 , (8).

3. Numerical Illustrations

To obtain additional insights into the behavior of the analytical framework, we conduct some numerical analyses on a two stage sequential investment project using the base case specification

in Table 1, which shows the standard input required, with $\theta_{V_1} = .02, \theta_{V_2} = 0$. Note a spreadsheet for the calculations is shown in Appendix 1.

Initially, the variances for the investment costs at the stages have been set to be equal and in the base case, the covariance terms between all of the factors equal zero. Some of these covariance terms are changed for the correlation sensitivity analysis. The sensitivity analyses show the impact of parametric changes on the option value and the exercise threshold for the two stages. A change in parameter value yields a corresponding variation in the lower bound conditions LB_j , which affects the option moneyness and the relative project value thresholds for the various stages. For consistent comparisons to be made across the various sensitivity analyses, first the magnitudes of the project value and the stage investment costs should result in the option values all being out-of-the money. To this end, all the analyses set the project value $V = 40$, and the

Table 1
Base Case Data

	INPUT			Stage 1	Stage 2	
Project value V	40		V^{\wedge}	216.7596	111.8724	
			ROV	7.0466	3.0136	
Risk-free rate	6%		ϕ	1.7100	1.3236	
Stage 1			LB		2.9043	
θ_{V_1}	0.02		Correlations	V_1	K_1	K_2
σ_{V_1}	20%		V_1	100%		
θ_{K_1}	0		K_1	0%	100%	
σ_{K_1}	5%		K_2	0%	0%	100%
Stage 2						
θ_{V_2}	0		Correlations	V_2	K_1	K_2
σ_{V_2}	20%		V_2	100%		
θ_{K_2}	0		K_1	0%	100%	
σ_{K_2}	5%		K_2	0%	0%	100%
			Threshold Levels	Factor Levels		
$K1^{\wedge}/K2^{\wedge}$	9.0000		$K1^{\wedge}$	90	K_1	90
-			$K2^{\wedge}$	10	K_2	10

Stage 1 model \hat{V}_1 is the solution for equations 7 and 8, ROV_1 is the result from equation 3, with the base data in Table 1. Stage 2 model \hat{V}_2 is the solution using equations 11-17, (or 16 and 22), ROV_2 is the result from equation 10. Note $\theta_{V_1} = .02, \theta_{V_2} = 0$. This drift differential results in $\hat{V}_1 > \hat{V}_2$ and $ROV_1 > ROV_2$. The restrictive lower bound LB is 2.9 which is exceeded by the ratio of \hat{K}_1 / \hat{K}_2 , equation 24.

stage investment costs $K_1 = 90$ and $K_2 = 10$. There is the requirement that the magnitudes of the stage investment cost threshold levels should result in the project value thresholds forming an ordered set with $\hat{V}_1 > \hat{V}_2$, which entails that the ratios of the consecutive investment cost thresholds obey the lower bound condition. For convenience, we set the threshold investment costs equal to their actual expected levels.

We first consider the results for the base case, and then explore the impact of key sensitivities on the model solution. We comment on the results from the viewpoint of the Chief Real Options Manager (CROM) who seeks a high ROV, and then from the viewpoint of a government (GOV) which seeks low V thresholds in order to encourage early investment. We also offer some guidelines for investors (INVEST) who believe that ROV implied by available stock market prices are different from evaluated ROV, based on investors particular expected volatilities and other parameter values.

Table 2 is designed to show how a change in one parameter value like θ_{v_i} affects the V thresholds and the ROV at each stage. We compare Panels B1 and C1 to A1 (the effect of changing one parameter value), and B2 to B1, C2 to C1 (the effect of increasing one parameter value, also shown in the Figures).

Table 2

Summary: θ, σ, ρ Differential Results

Table 2					
	Stage 1	Stage 2		Stage 1	Stage 2
Panel A1			Panel A2		
	θ_{V1}	0.02		θ_{V1}	0.04
V [^]	216.7596	111.8724	V [^]	393.9406	54.4155
ROV	7.0466	3.0136	ROV	15.6773	6.6525
ϕ	1.7100	1.3236	ϕ	1.2961	1.7484
Panel B1	θ_{V1}	0.02	Panel B2	θ_{V1}	0.02
	σ_{V1}	30%		σ_{V1}	40%
V [^]	286.6809	75.3040	V [^]	375.4515	56.8412
ROV	11.1436	4.2966	ROV	15.0114	6.2374
ϕ	1.4576	1.5549	ϕ	1.3153	1.7230
Panel C1	θ_{V1}	0.02	Panel C2	θ_{V1}	0.02
	ρ_{V1K1}	-50%		ρ_{V1K1}	-75%
	ρ_{V2K1}	50%		ρ_{V2K1}	50%
V [^]	231.6018	91.5822	V [^]	238.8103	87.9894
ROV	8.0101	2.8528	ROV	8.4591	2.9922
ϕ	1.6356	1.4351	ϕ	1.6048	1.5006

Stage 1 model \hat{V}_1 is the solution for equations 7 and 8, ROV_1 is the result from equation 3, with the base data in Table 1. Stage 2 model \hat{V}_2 is the solution using equations 11-17, ROV_2 is the result from equation 10. Note $\theta_{V_1} = .02, \theta_{V_2} = 0$, except for Panel A2.

3.1 Real Option Value and V Thresholds for $\theta_{V_1} > \theta_{V_2}$

In Panel A1 all parameter values are as in Table 1, but in Panel A2 θ_{V_1} is increased from .02 to .04. As a consequence the threshold that justifies immediate investment increases from 217 to 394 in stage 1, but decreases from 112 to 54 in stage 2. The ROV increases for both stages, from 7 to 16 in stage 1 and from 3 to 7 in stage 2. So just a difference in V drifts between the stages results in higher thresholds for the final stage 1, but lower for the initial stage 2 (as the final stage is more valuable, waiting for the V_1 escalation to proceed). The greater the drift difference, the

greater this effect. The CROM ought to favour greater drift differences with $\theta_{V_1} > \theta_{V_2}$, since the ROV increase. GOV should welcome increases in the V_1 drift only if the objective is to motivate early initial, but not final, stage investments.

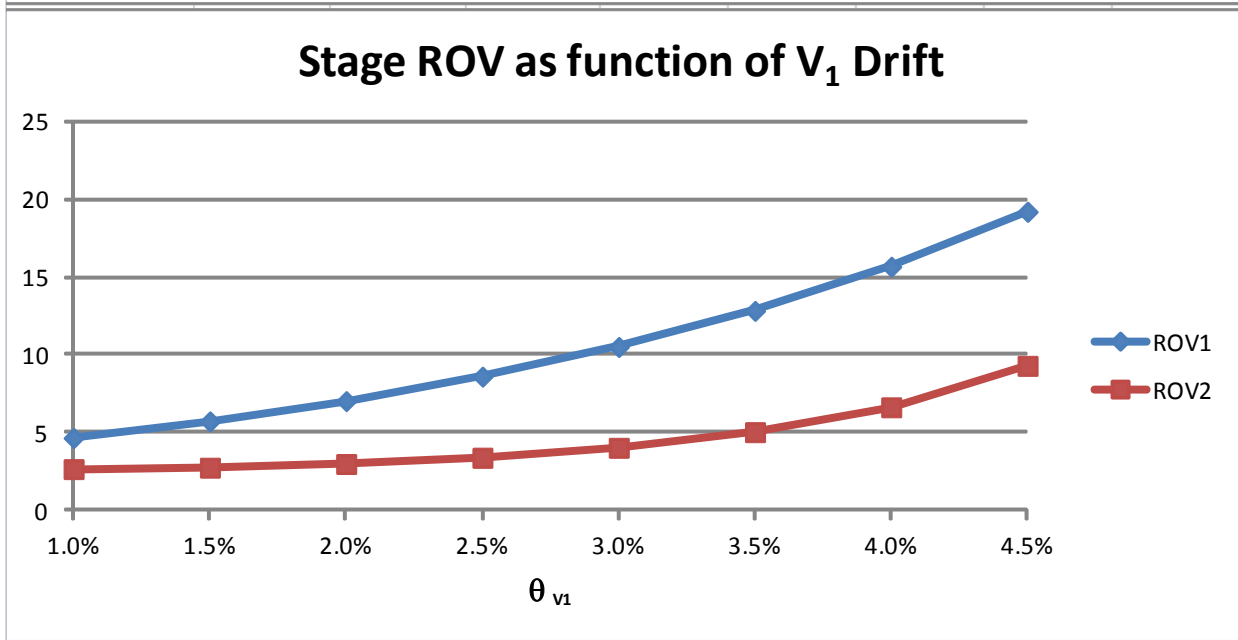
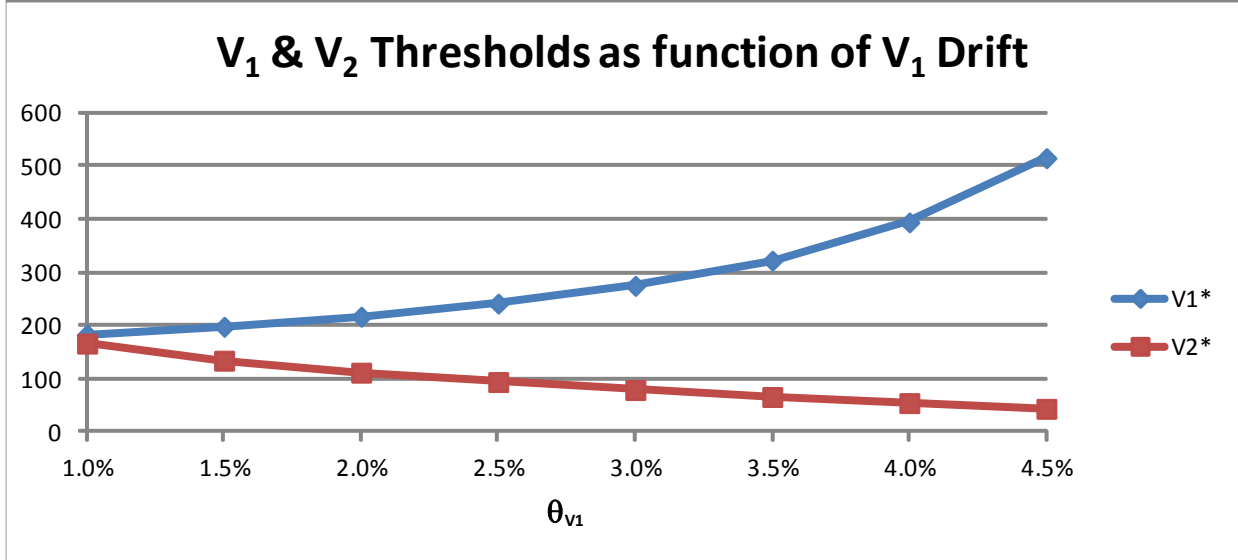
Figure 1 shows a stage specific divergence between the stage thresholds, and the stage ROVs. There is a dramatic divergence between the stage thresholds as the V_1 drift approaches the riskless interest rate, while the V_2 drift remains 0, and a gradual divergence between the ROVs for each stage.

The implications of these divergences are problematic in practice. For instance, a procedure where the allowed toll rate escalations on an infrastructure project such as a toll road or bridge are increased as the project approaches completion should lower the threshold for initial stages (although raises the threshold for the final stage, allowing for V_1 escalation to continue). Whether these V_1 drifts are exogenous or subject to government policy is also another interesting topic. We emphasise again that the CROM should always favour increases in value drift at all stages, especially at the completion stage. This is an obvious contribution to our understanding of growth options.

Figure 1

Effect of Drift Differentiation on Thresholds and ROV

θ_{v_1}	1.0%	1.5%	2.0%	2.5%	3.0%	3.5%	4.0%	4.5%
V_1^*	183.19	197.94	216.76	241.41	274.80	322.16	393.94	514.53
V_2^*	166.36	134.18	111.87	94.21	79.25	66.12	54.42	43.93
ROV_1	4.68	5.75	7.05	8.62	10.52	12.84	15.68	19.20
ROV_2	2.68	2.78	3.01	3.42	4.05	5.05	6.65	9.31



Stage 1 model \hat{V}_1 is the solution for equations 7 and 8, ROV_1 is the result from equation 3, with the base data in Table 1. Stage 2 model \hat{V}_2 is the solution using equations 11-17, ROV_2 is the result from equation 10. Note $\theta_{v_2} = 0, \theta_{v_1} = \text{range from 1\% to 4.5\%}$, so these drift differentials result in $\hat{V}_1 > \hat{V}_2$, and $ROV_1 > ROV_2$.

3.2 Real Option Value and Thresholds for $\sigma_{V1} > \sigma_{V2}$

For the single stage investment opportunity model, an increase in volatility is associated with increases in the option value and the project value threshold, *ceteris paribus*. We now set out to demonstrate whether this finding extends to the current multi-stage sequential investment model. We first consider the impact of project volatility changes. Generally an increase in project volatility σ_{V_1} leads to an increase in the overall stage volatility σ_j and this greater uncertainty is expected to be manifested in higher option values. When we consider the impact of stage specific project value volatility on the project value threshold at each stage, the expected result of a positive association is obtained.

In Table 2, Panel B1 all parameter values are as in Table 1 with $\theta_{V_1} = .02$ (for comparison with Panel A1) while σ_{V_1} is increased from 20% to 30%, and to 40% in Panel B2. As a consequence of the first volatility increase the threshold that justifies immediate investment increases from 217 to 287 in stage 1, but decreases from 112 to 75 in stage 2. With the additional volatility increase the threshold increases to 375 in stage 1, but decreases to 57 in stage 2. The stage 1 ROV₁ increases as the σ_{V_1} increases from 20% to 30%, and the stage 2 ROV₂ increases. The stage 1 ROV₁ increases as the σ_{V_1} increases from 30% to 40%, and the stage 2 ROV₁ increases. The greater the V₁ volatility, the greater the ROV₁, so the V₂ threshold to obtain the option for stage 1 is lower, but the threshold for exercising the stage 1 option is higher.

Figure 2 shows when the final stage value volatility exceeds the previous stage value volatility, for instance $\sigma_{V_1}=30\%$ while σ_{V_2} remains at 20%, the V₁ threshold exceeds the V₂ threshold, and

the $ROV_1 > ROV_2$. As the final stage value volatility increases, this pattern is maintained, but the level of both ROV increase, while the stage 1 and stage 2 threshold differences increase. Note \hat{V}_1 increases significantly as V_1 volatility increases as is logical and the ROV_1 increases, and \hat{V}_2 declines. From the value matching condition, \hat{V}_2 declines due to the ROV_1 increase at each stage.

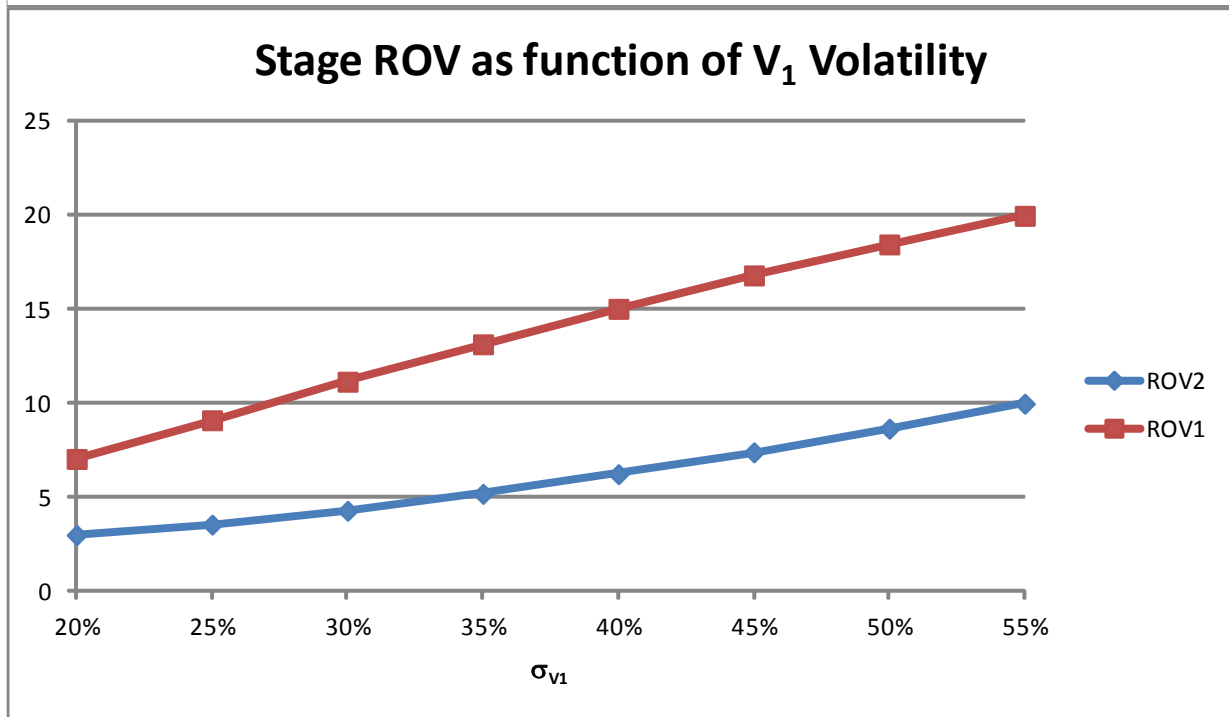
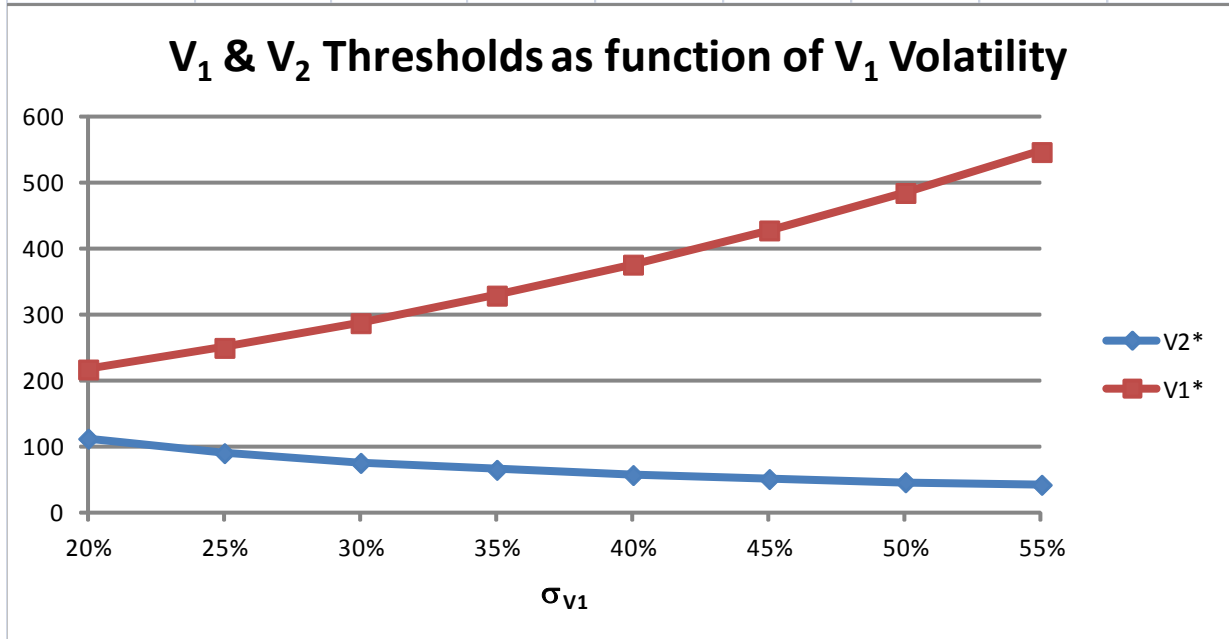
Because thresholds that justify investment diverge as the V_1 volatility increases, there is a greater incentive to make the initial stage investment, but defer the final stage investment. GOV policy may be ambiguous thus on the V_1 volatility, if project completion is the objective. A GOV which wants to encourage early stage investments might allow the expectation of high value volatility, but after that stage investment is completed, encourage final stage investment by offering price guarantees for the project value, which would lower the final stage value threshold. The CROM should favour increases in the V_1 volatility, since the ROV increases at all levels. But should the CROM seek to sell the project for the calculated ROV, there is apparently a greater incentive to defer selling the project until the final stage is reached. It is easy to see that INVEST may want to invest in early stage 2 projects if the implied stock market³ V_1 volatility is 25% (for a $ROV_2=3.56$, if $\sigma_{V_2}=20\%$) if they believe the V_1 volatility is 40% (for a $ROV_2=6.24$) (and hold through stage 1 when the $ROV_1=15.01$).

Figure 2

³ This assumes there are pure simple sequential investment opportunities in the stock market, not mixed with other operations, and that separate parameter values can be implied using a complex model. Currently accurate calibration of these parameter values is a challenge.

Effect of V Volatility Differentiation on Thresholds and ROV

σ_{V_1}	20%	25%	30%	35%	40%	45%	50%	55%
V_1^*	216.76	249.41	286.68	328.66	375.45	427.18	483.96	545.87
V_2^*	111.87	89.95	75.30	64.75	56.84	50.75	45.96	42.14
ROV_1	7.05	9.10	11.14	13.13	15.01	16.78	18.42	19.94
ROV_2	3.01	3.56	4.30	5.20	6.24	7.40	8.65	9.97



Stage 1 model \hat{V}_1 is the solution for equations 7 and 8, ROV_1 is the result from equation 3, with the base data in Table 1. Stage 2 model \hat{V}_2 is the solution using equations 11-17, ROV_2 is the result from equation 10. Note if $\sigma_{V_2} = 20\%$, while σ_{V_1} is higher at 25% to 55%, these volatility differentials also result in $\hat{V}_1 > \hat{V}_2$, and $ROV_1 > ROV_2$.

3.3 Correlation

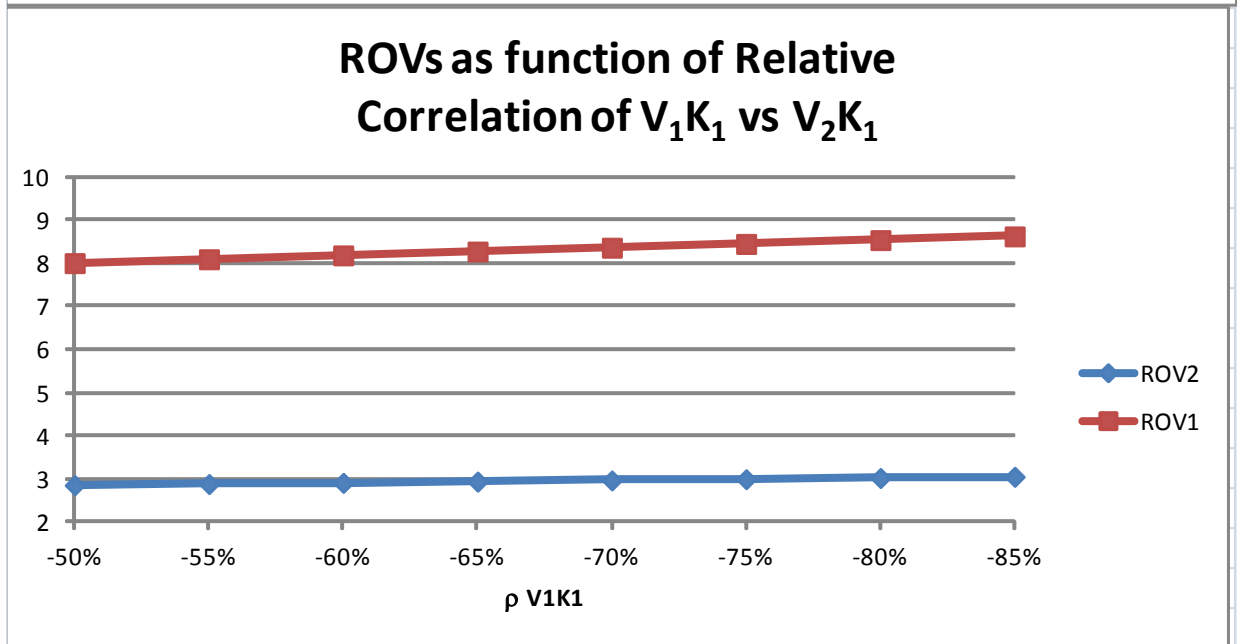
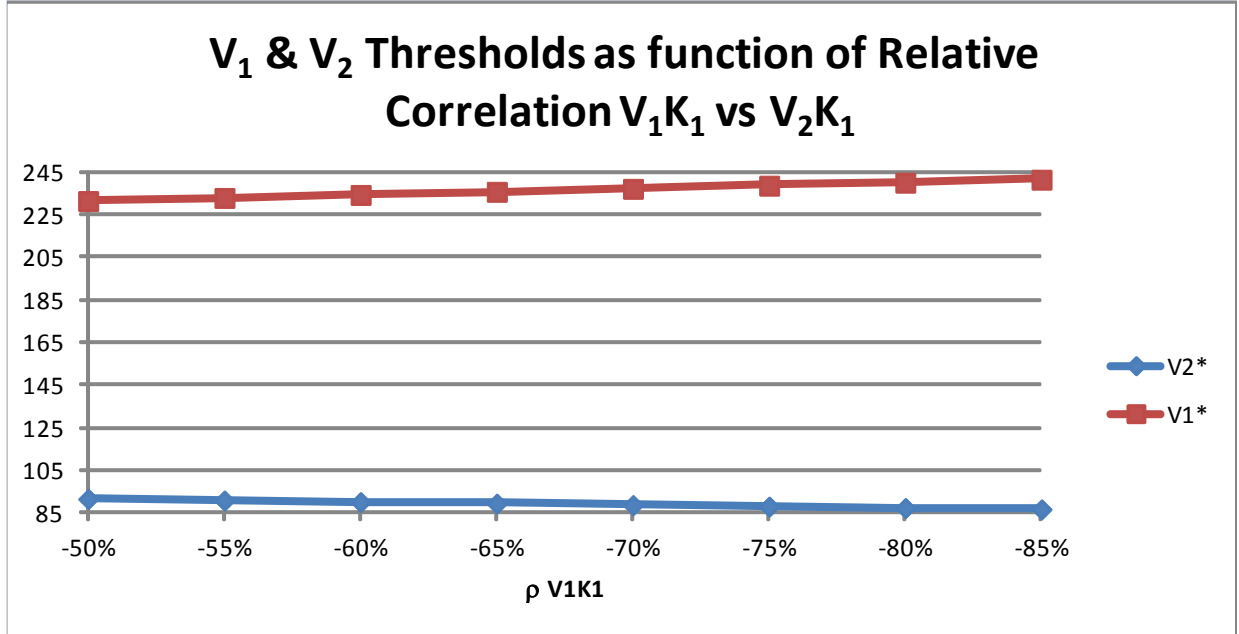
Changes in the correlation coefficients impact on the solution through the relevant stage volatilities, σ_j for $J=1,2$, which in turn influence the option values and the project value thresholds. We had conjectured that the relative correlation between V_1K_1 relative to V_2K_1 might be an alternative parameter value difference that could result in increasing thresholds nearing the final stage, but the implications for ROV appear to be limited.

For the two stages, Table 2 illustrates the effects of variations in the correlation between the project value and the investment cost K_1 on the option value and the project value threshold, respectively. In Table 2 Panel C1 all parameter values are as in Table 1 with $\theta_{V_1} = .02$, $\rho_{V_2K_1}$ is increased from 0 to 50%, and $\rho_{V_1K_1}$ is decreased from 0 to -50% in Panel C1, and to -75% in Panel C2. As a consequence of the first set of correlation changes the threshold that justifies immediate investment increases from 217 to 232 in stage 1, but decreases from 112 to 92 in stage 2. In comparison with C1, in C2 with $\rho_{V_1K_1} = -75\%$, the threshold increases to 239 in stage 1, but decreases to 88 in stage 2. The ROV_1 increases with the first set of correlation changes, and the ROV_2 decreases slightly, and in C2 the ROVs increase slightly. A wider correlation difference between V_1K_1 and V_2K_1 has the same effect as an increase in V_1 volatility. The greater the correlation difference, the greater the ROV_1 , so the V threshold to obtain the option for stage 1 is lower, but the threshold for exercising the stage 1 option is higher.

Figure 3

Effect of Correlation Differentiation on Thresholds and ROV

ρ_{V1K1}	-50%	-55%	-60%	-65%	-70%	-75%	-80%	-85%
$V1^*$	231.60	233.05	234.50	235.94	237.38	238.81	240.24	241.66
$V2^*$	91.58	90.83	90.09	89.37	88.67	87.99	87.32	86.67
ROV_1	8.01	8.10	8.19	8.28	8.37	8.46	8.55	8.63
ROV_2	2.85	2.88	2.91	2.94	2.96	2.99	3.02	3.05



Stage 1 model \hat{V}_1 is the solution for equations 7 and 8, ROV_1 is the result from equation 3, with the base data in Table 1. Stage 2 model \hat{V}_2 is the solution using equations 11-17, ROV_2 is the result from equation 10. Note that if $\rho_{V_2K_1} = 50\%$, while $\rho_{V_1K_1}$ ranges from -50% to -85%, these correlation differentials also result in $\hat{V}_1 > \hat{V}_2$, and $ROV_1 > ROV_2$.

GOV seeking early stage investment would encourage negative correlations at the final stage, but the result would be to slightly discourage investment at that final stage. The CROM would probably seek no correlation differences between V_1K_1 and V_2K_1 (indeed $\rho_{V_1K_1} > 0$ implies that the early stage value is somehow positively related to the final stage investment costs). Conceivably the V_1K_1 and V_2K_1 correlation differentials might be established through contracts for basing the final stage investment costs partially on the early stage project value, that is the greater the earlier stage value, the greater the final stage investment cost, an “afford to pay” scheme. Could negative correlation between final stage project value and final stage investment cost be a feasible type of milestone agreement between an early stage E&D firm and a Big Mining firm, that is the investment cost for final production stage would be reduced the higher the final stage project value?

Other parameter value changes that might also result in solutions that do not violate the stage investment cost boundary conditions, so that threshold increase as the project approaches completion, might be changes in stage 2 V volatility, differential investment cost drifts and volatilities, and changes in the general level of interest rates, as shown in Appendix 2.

Discussing these sensitivities indicates that changes in all of the parameter values should be considered in a package which meets the objectives of the regulator or government, or alternatively the project manager. Incentives are not always intuitive.

Are differences in V drifts, or correlation differences between V_1K_1 and V_2K_1 , or volatility for V_1 higher than for V_2 plausible? Cortazar et al. (2003) divide natural resource investments into early and final stages. The early exploration stage involves primarily geological-technical uncertainty following a zero-drift constant volatility Brownian motion process, independent of output prices. The final production stage involves primarily commodity price risk following a constant-convenience-yield Brownian motion process. So $\theta_{V_1} > \theta_{V_2} = 0$, the correlation with the investment cost over the two stages is not specified, and the V_1 volatility is greater than the V_2 volatility. In natural gas fracking in the U.S., there is evidence that the exploration-development stage has very limited failure risk or volatility (very few “dry holes”), while the production stage is exposed to the highly volatile U.S. natural gas prices, with a distinct convenience yield observable from the term structure of gas futures prices.

Conclusion

We provide an analytical solution for a multi-factor, multi-phase sequential investment process, where there is the real option of deferring investments at any stage. This model is particularly appropriate for real sequential E&D investment opportunities, such as infrastructure or natural resources that may have an initial development and then a final production investment stage.

Decisions relating to the sequential investment opportunity are affected by three distinct sources of stage specific uncertainty, arising from a stochastic project value, stochastic investment costs and correlations of value and cost. Amongst the three sources of uncertainty, the most crucial for obtaining a meaningful solution with differences in stage 1 and stage 2 thresholds are the relative

value drifts and volatilities. The primary condition in Adkins and Paxson (2014) that the failure probabilities for the various stages have to obey the constraint of declining nearing completion is not necessary, although sufficient. However, there is still the constraint that the ratio of consecutive investment cost thresholds exceeds the lower bound LB_j .

The closed-form solution to the multi-stage sequential investment opportunity is formulated on American-perpetuity options. It yields an analytical solution that is straightforward conceptually and less onerous to evaluate compared with some previous models. However, its solution now relies on a package of conditions, with a menu that is rich and varied. Some real world projects may allow for these conditions, but necessarily not all. For the exceptions, the conditions will need to be loosened in some way, possibly by a mixture of European and American options or through the presence of an abandonment alternative at each stage.

The American-perpetuity option model applied to the parsimonious design of a sequential investment opportunity yields an elegant analytical solution that can be implemented in a simple spreadsheet, useful for practical capital budgeting. The solution method may be extendable to cases where the realized opportunity has embedded options, where there is a required time to build, where there is more than a single project opportunity, where there are several stages, or where the value can be partially realized at each stage. The assumption of a monopolist player can be loosened by considering the comparative strategies of say two rivals both engaged in the same E&D or infrastructure battle.

For future research it will be interesting to show analytically the partial derivatives of the ROV at each stage to changes in V , V volatility (real stage specific deltas and vegas) and drift, and failure probability, some of which are indicated in Appendix 2. An investor seeking “bang for the buck” expecting increases in stage value volatility might want to select an investment in an early stage rather a final stage. Various capital funds investing in E&D may indicate a preference for early stage participation, others (like Big Mining Companies) in later stages.

Finally, the multi-stage American-perpetuity model could be applied to valuing equity as a compound call option on an asset unlikely to disappear or fail completely either partly funded by debt or itself with embedded options.

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Appendix 1 Spreadsheet Solution for a Set of Equations

For $J=1$, the two-factor one-stage investment model is evaluated, assuming $\hat{K}_1=K_1$. Then according to the sequential order $J=2$, the model is evaluated by solving simultaneously five equations for five unknowns, assuming that $\hat{K}_1=K_1$, $\hat{K}_2=K_2$ and $\eta_{22}=1-\beta_2-\eta_{21}$.

	A	B	C	D	E
1	SSRD BASIC SPREADSHEET				
2	INPUT	STAGE ONE	STAGE TWO		
3	V	40.00	40.00		
4	K1	90.00	90.00		
5	K2	10.00	10.00		
6	$\sigma V1$	0.20	0.20		
7	$\sigma V2$		0.20		
8	$\sigma K1$	0.05	0.05		
9	$\sigma K2$		0.05		
10	$\rho V1K1$	0.00	0.00		
11	$\rho V1K2$				
12	$\rho V2K1$		0.00		
13	$\rho V2K2$		0.00		
14	$\rho K1K2$		0.00		
15	r	0.06	0.06		
16	$\theta V1$	0.02	0.02		
17	$\theta V2$		0.00		
18	$\theta K1$	0.00	0.00		
19	$\theta K2$		0.00		
20					
21	OUTPUT				
22	$\sigma^{\wedge}2$	0.0425	0.1207	$(B31^2)*(C7^2)+(1-B31)^2*(C8^2)+(C9^2)+2*B31*(1-B31)*C12*C6^*C8-2*B31^*C13^*C6^*C9-2*(1-B31)^*C14^*C8^*C9$	EQ 17
23	$Q(\beta, \eta)$	0.0000	0.0000	$0.5*(C22)^*C38^*(C38-1)+C38*((C15-C19)+B38*(C17-C16)+0.5*B38*(B38-1)*((C7^2-C6^2)-2*(C12^*C7^*C9-C10^*C6^*C8)))-(C15-C19)$	EQ 7 & 16
24	SP1	0.0000	0.0000	$C31^*C30^*(C33^*(C31))^*(C35^*C32)^*(C36^*C34)-B31^*B30^*(C33^*(B31))^*(C35^*(1-B31))$	EQ 5 & 12
25	SP2	0.0000	0.0000	$C32^*C30^*(C33^*C31)^*(C35^*(C32-1))^*(C36^*C34)-(1-B31)^*B30^*(C33^*B31)^*(C35^*(B31))$	EQ 6 & 13
26	VM1/SP3	0.0000	0.0000	$C34^*C30^*(C33^*C31)^*(C35^*C32)^*(C36^*(C34-1))+1$	EQ 14
27	B23-B24-VM2	0.0000	0.0000	$C30^*(C33^*C31)^*(C35^*C32)^*(C36^*C34)-B30^*(C33^*B31)^*(C35^*(1-B31))+C36$	EQ 4 & 11
28	SOLVER: SET D29=0, CHANGING B30:C33				
29	SOLVER	0.0000	0.0000		0.00000
30	A2	0.3133	0.1031		
31	β_2	1.7100	2.2633		
32	η_{21}	-0.7100	-0.9397		
33	V1*	216.7596	111.8724		
34	η_{22}		-0.3236	$1-C31-C32$	
35	K1*	90.0000	90.0000		
36	K2*		10.0000		
37	VOLATILITY	0.2062	0.3475		
38	ϕ_1, ϕ_2	1.7100	1.3236	$C31/B38$	
39	ROV1, ROV2	7.0466	3.0136	$IF(C3<C33,C30^*(C3^*C31)^*(C4^*C32)^*(C5^*C34),C33-C35)$	EQ 3 & 10
40	PDE2, PDE9	0.0000	0.0000		EQ 2 & 9
41	$\Delta V1$	0.3012		$B31^*B30^*(B3^*(B31-1))^*(B4^*B32)$	
42	$\Delta V2$		0.1705	$C31^*C30^*(C3^*(C31-1))^*(C4^*C32)^*(C5^*C34)$	
43	$\Delta K1$	-0.0556	-0.0315	$C32^*C30^*(C3^*C31)^*(C4^*(C32-1))^*(C5^*C34)$	
44	$\Delta K2$		-0.0975	$C34^*C30^*(C3^*C31)^*(C4^*C32)^*(C5^*(C34-1))$	
45	$\Gamma V1$	0.0053		$B31^*(B31-1)^*B30^*(B3^*(B31-2))^*(B4^*B32)$	
46	$\Gamma V2$		0.0054	$C31^*(C31-1)^*C30^*(C3^*(C31-2))^*(C4^*C32)$	
47	$\Gamma K1$	0.0011	0.0007	$C32^*(C32-1)^*C30^*(C3^*C31)^*(C4^*(C32-2))^*(C5^*C34)$	
48	$\Gamma K2$		0.0129	$C34^*(C34-1)^*C30^*(C3^*C31)^*(C4^*C32)^*(C5^*(C34-2))$	
49	PDE 2	$0.5*(B6^2)^*(B3^2)^*B45+0.5*(B8^2)^*(B4^2)^*B47+B16^*B3^*B41+B18^*B4^*B43-B15^*B39$			
50	PDE 9	$0.5*(C7^2)^*(C3^2)^*C46+0.5*(C8^2)^*(C4^2)^*C47+0.5*(C9^2)^*(C5^2)^*C48+C17^*C3^*C42+C18^*C4^*C43+C19^*C5^*C44-C15^*C39$			
51	V2*	111.8724	$(B38/(B38-1))*(((C38*(B38-1)/(C38-1))^(1/B38))^*(C35^*((B38-1)/B38)))*((C36^*(1/B38)))$		EQ 22

This spreadsheet is available from the authors.

Appendix 2 An Easy Quadratic Solution and Some Additional Sensitivities

The Q_1 function (7) is a quadratic expressed as:

$$\begin{aligned} Q_1(\beta_1, 1-\beta_{11}) &= \frac{1}{2}\beta_1(\beta_1-1)\sigma_1^2 + \beta_1(\theta_{V_1} - \theta_{K_1}) - (r - \theta_{K_1}) \\ &= \frac{1}{2}\beta_1^2\sigma_1^2 + \beta_1(\theta_{V_1} - \theta_{K_1} - \frac{1}{2}\sigma_1^2) - (r - \theta_{K_1}) = 0, \end{aligned} \quad (A1)$$

with the solution
$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (A2)$$

where a, b and c are
$$\frac{1}{2}\sigma_1^2, \quad (\theta_{V_1} - \theta_{K_1} - \frac{1}{2}\sigma_1^2), \quad -(r - \theta_{K_1}) \quad (A2A)$$

The Q_2 function (16) is a quadratic expressed as:

$$\begin{aligned} Q_2 &= \phi_2^2 \left\{ \frac{1}{2}\phi_1^2\sigma_{V_2}^2 + \frac{1}{2}(\phi_1-1)^2\sigma_{K_1}^2 + \frac{1}{2}\sigma_{K_2}^2 \right. \\ &\quad \left. + \phi_1(1-\phi_1)\rho_{V_2K_1}\sigma_{K_1}\sigma_{V_2} - \phi_1\rho_{V_2K_2}\sigma_{K_2}\sigma_{V_2} + (\phi_1-1)\rho_{K_1K_2}\sigma_{K_1}\sigma_{K_2} \right\} \\ &+ \phi_2 \left\{ \phi_1\theta_{V_2} + (1-\phi_1)\theta_{K_1} - \theta_{K_2} \right. \\ &\quad \left. - \frac{1}{2}\phi_1\sigma_{V_2}^2 - \frac{1}{2}(1-\phi_1)\sigma_{K_1}^2 - \frac{1}{2}\sigma_{K_2}^2 + (1-\phi_1)\rho_{K_1K_2}\sigma_{K_1}\sigma_{K_2} + \phi_1\rho_{V_2K_2}\sigma_{K_2}\sigma_{V_2} \right. \\ &\quad \left. - \{r - \theta_{K_2}\} \right\} = 0 \end{aligned} \quad (A3)$$

Now specify:

$$\begin{aligned} \sigma_2^2 &= \phi_1^2\sigma_{V_2}^2 + (1-\phi_1)^2\sigma_{K_1}^2 + \sigma_{K_2}^2 \\ &\quad + 2\phi_1(1-\phi_1)\rho_{V_2K_1}\sigma_{V_2}\sigma_{K_1} - 2\phi_1\rho_{V_2K_2}\sigma_{V_2}\sigma_{K_2} - 2(1-\phi_1)\rho_{K_1K_2}\sigma_{K_1}\sigma_{K_2}. \end{aligned} \quad (17)$$

and
$$\sigma_{21}^2 = \sigma_{V_2}^2 + \sigma_{K_1}^2 - 2\rho_{V_2K_1}\sigma_{V_2}\sigma_{K_1} \quad (A4)$$

so
$$Q_2 = \frac{1}{2}\sigma_{21}^2\phi_2^2 + \phi_2 \left\{ \phi_1\theta_{V_2} + (1-\phi_1)\theta_{K_1} - \theta_{K_2} + \frac{1}{2}\phi_1(\phi_1-1)\sigma_{21}^2 - \frac{1}{2}\sigma_{21}^2 \right\} - \{r - \theta_{K_2}\} = 0 \quad (A5)$$

The coefficients for this quadratic are:

$$\frac{1}{2}\sigma_{21}^2, \quad \phi_1\theta_{V_2} + (1-\phi_1)\theta_{K_1} - \theta_{K_2} + \frac{1}{2}\phi_1(\phi_1-1)\sigma_{21}^2 - \frac{1}{2}\sigma_{21}^2, \quad -(r - \theta_{K_2}). \quad (A6)$$

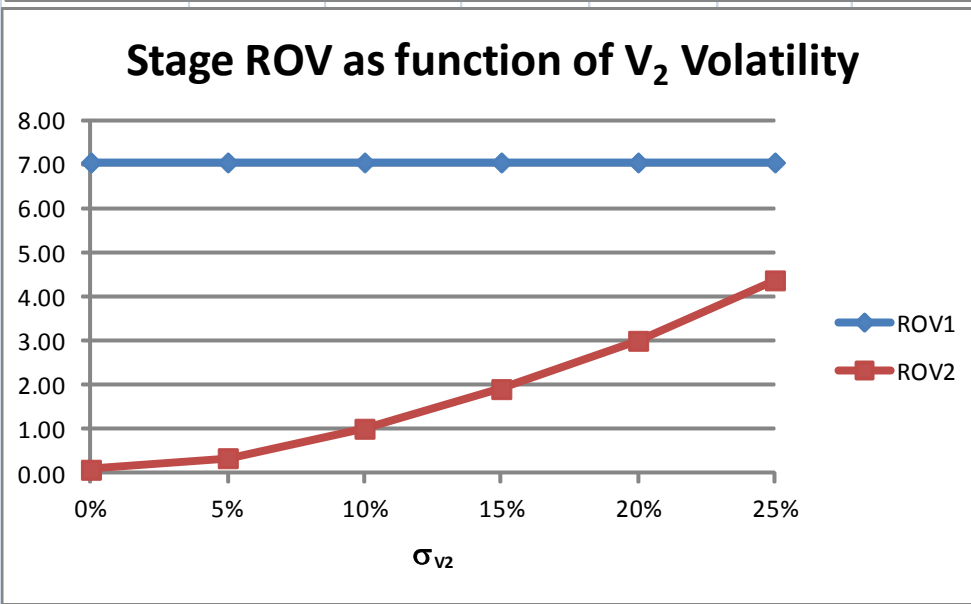
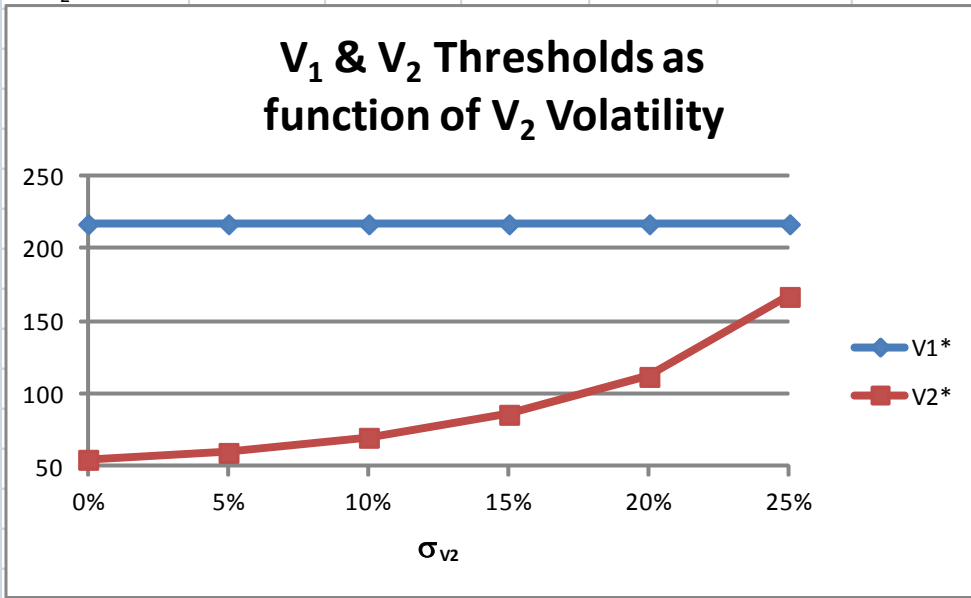
An Alternative Quadratic Solution for Thresholds and ROV

	A	B	C	D
1	Quadratic Solution			
2	INPUT	STAGE ONE		EQ
3	V	40.00		
4	K1	90.00		
5	K2			
6	σV1	0.20		
7	σV2			
8	σK1	0.05		
9	σK2			
10	ρ V1K1	0.00		
11	ρ V2K1			
12	ρ V2K2			
13	ρ K1K2			
14	r	0.06		
15	θV1	0.02		
16	θV2			
17	θK1	0.00		
18	θK2			
19	STAGE TWO			
20	V	40.00		
21	K1	90.00		
22	K2	10.00		
23	σV1	0.20		
24	σV2	0.20		
25	σK1	0.05		
26	σK2	0.05		
27	ρ V1K1	0.00		
28	ρ V2K1	0.00		
29	ρ V2K2	0.00		
30	ρ K1K2	0.00		
31	r	0.06		
32	θV1	0.02		
33	θV2	0.00		
34	θK1	0.00		
35	θK2	0.00		
36	OUTPUT			
37	σ ²	0.0425	$B6^2 + B8^2 - 2 * B10 * B6 * B8$	7A
38	b1	-0.0013	$B15 - B17 - 0.5 * B37$	A2A
39	φ1	1.7100	$(-B38 + \text{SQRT}((B38^2) - 4 * 0.5 * B37 * (-B14 - B17))) / (2 * 0.5 * B37)$	A2
40	A1	0.3133	$(B39^(-B39)) * ((B39 - 1)^(B39 - 1))$	8A
41	V1*	216.7596	$(B39 / (B39 - 1)) * B4$	8
42	K1*	90.0000		
43	K2*			
44	β2			
45	η21	-0.7100	1 - B39	
46	ROV1	7.0466	$\text{IF}(B3 < B41, B40 * (B3^B39) * (B4^B45), B41 - B42)$	3
47	σ ²	0.1207	$(B39^2) * (B24^2) + ((1 - B39)^2) * (B25^2) + 2 * B39 * (1 - B39) * B28 * B24 * B25 - 2 * B39 * B29 * B24 * B26 - 2 * (1 - B39) * B30 * B25 * B26$	17
48	σ21 ²	0.0425	$B24^2 + B25^2 - 2 * B28 * B24 * B25$	A4A
49	b2	-0.0346	$((-B35) + B39 * B33 + 0.5 * B39 * (B39 - 1) * B48 - 0.5 * B47)$	A6
50	φ2	1.3236	$(-B49 + \text{SQRT}((B49^2) - 4 * 0.5 * B47 * (-B31 - B35))) / (2 * 0.5 * B47)$	A2
51	A2	0.1031	$((B50 - 1)^(B50 - 1)) / ((B50^B50) * (((B39 - 1)^(B39 - 1)) / (B39^B39)))^B50$	19A
52	V2*	111.8724	$(B39 / (B39 - 1)) * (((B50 * (B39 - 1)) / (B50 - 1))^(1 / B39)) * (B53^((B39 - 1) / B39)) * ((B54^(1 / B39)))$	22
53	K1*	90.0000		
54	K2*	10.0000		
55	β2	2.2633	$B50 * B39$	
56	η21	-0.9397	$(1 - B39) * B55 / B39$	
57	η22	-0.3236	$1 - B55 - B56$	
58	ROV2	3.0136	$\text{IF}(B20 < B52, B51 * (B20^B55) * (B21^B56) * (B22^B57), B52 - B54)$	10

	A	B	C	D
59	PDE1	0.0000	$0.5*(B6^2)*(B3^2)*B64+0.5*(B8^2)*(B4^2)*B66+B15*B3*B60+B17*B4*B62-B14*B47$	2
60	ΔV1	0.3012	$B39*B40*(B3^(B39-1))*(B4^B45)$	
61	ΔV2			
62	ΔK1	-0.0556	$B45*B40*(B3^B39)*(B4^(B45-1))$	
63	ΔK2			
64	ΓV1	0.0053	$B39*(B39-1)*B40*(B3^(B39-2))*(B4^B45)$	
65	ΓV2			
66	ΓK1	0.0011	$B45*(B45-1)*B40*(B3^B39)*(B4^(B45-2))$	
67	ΓK2			
68	PDE2	0.0000	$0.5*(B24^2)*(B20^2)*B74+0.5*(B25^2)*(B21^2)*B75+0.5*(B26^2)*(B22^2)*B76+B33*B20*B70+B34*B21*B71+B35*B22*B72-B31*B58$	9
69	ΔV1			
70	ΔV2	0.1705	$B55*B51*(B20^(B55-1))*(B21^B56)*(B22^B57)$	
71	ΔK1	-0.0315	$B56*B51*(B20^B55)*(B21^(B56-1))*(B22^B57)$	
72	ΔK2	-0.0975	$B57*B51*(B20^B55)*(B21^B56)*(B22^(B57-1))$	
73	ΓV1			
74	ΓV2	0.0054	$B55*(B55-1)*B51*(B20^(B55-2))*(B21^B56)*(B22^B57)$	
75	ΓK1	0.0007	$B56*(B56-1)*B51*(B20^B55)*(B21^(B56-2))*(B22^B57)$	
76	ΓK2	0.0129	$B57*(B57-1)*B51*(B20^B55)*(B21^B56)*(B22^(B57-2))$	

Figure A1

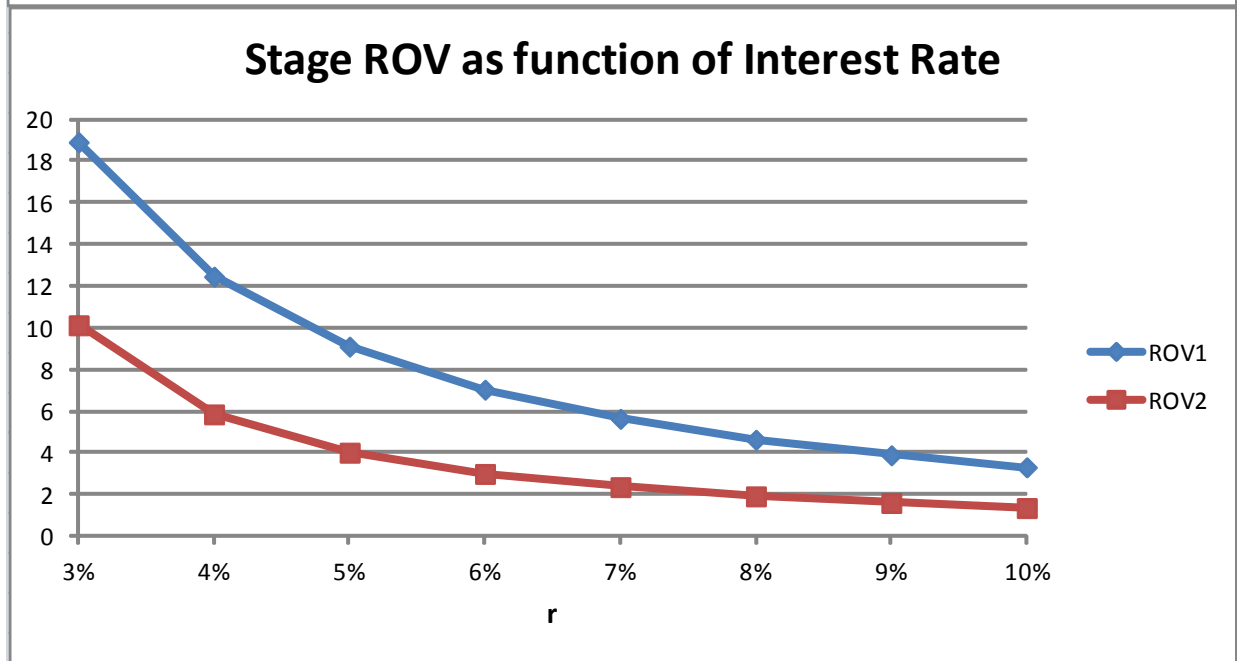
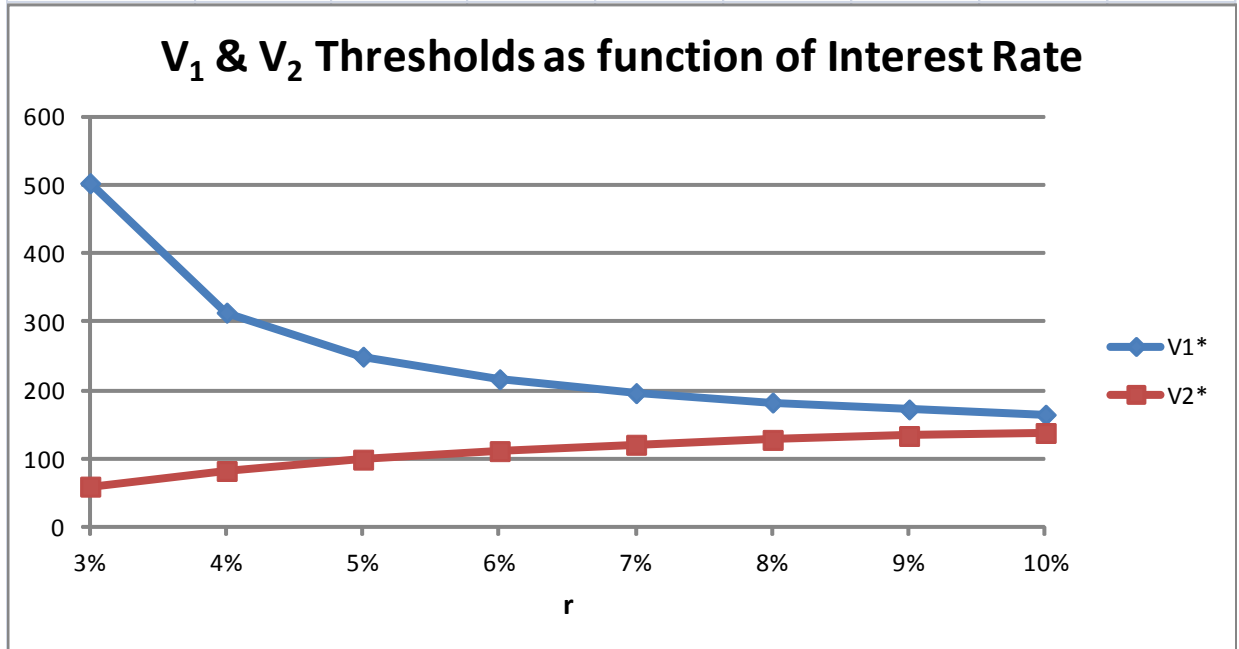
σ_{V_2}	0%	5%	10%	15%	20%	25%
V_1^*	216.76	216.76	216.76	216.76	216.76	216.76
V_2^*	54.89	59.66	70.03	85.82	111.87	166.82
ROV_1	7.05	7.05	7.05	7.05	7.05	7.05
ROV_2	0.09	0.36	1.02	1.92	3.01	4.38



$\hat{V}_1 > \hat{V}_2$ decreases as σ_{V_2} increases from 0 to 25% if σ_{V_1} remains at 20%, but beyond that the LB no longer holds, based on the Table 1 parameter values.

Figure A2

r	3%	4%	5%	6%	7%	8%	9%	10%
V1*	502.93	314.04	249.68	216.76	196.56	182.79	172.75	165.07
V2*	60.12	82.99	99.58	111.87	121.17	128.33	133.95	138.41
ROV ₁	18.91	12.47	9.11	7.05	5.65	4.65	3.90	3.33
ROV ₂	10.13	5.86	4.03	3.01	2.37	1.93	1.61	1.37

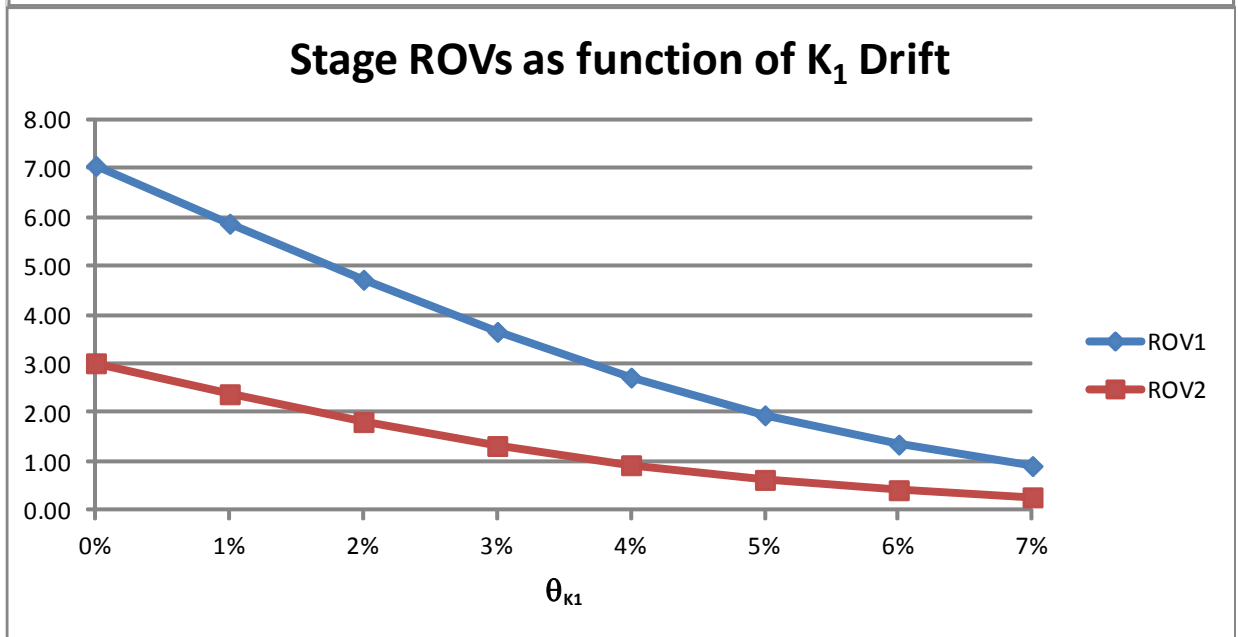
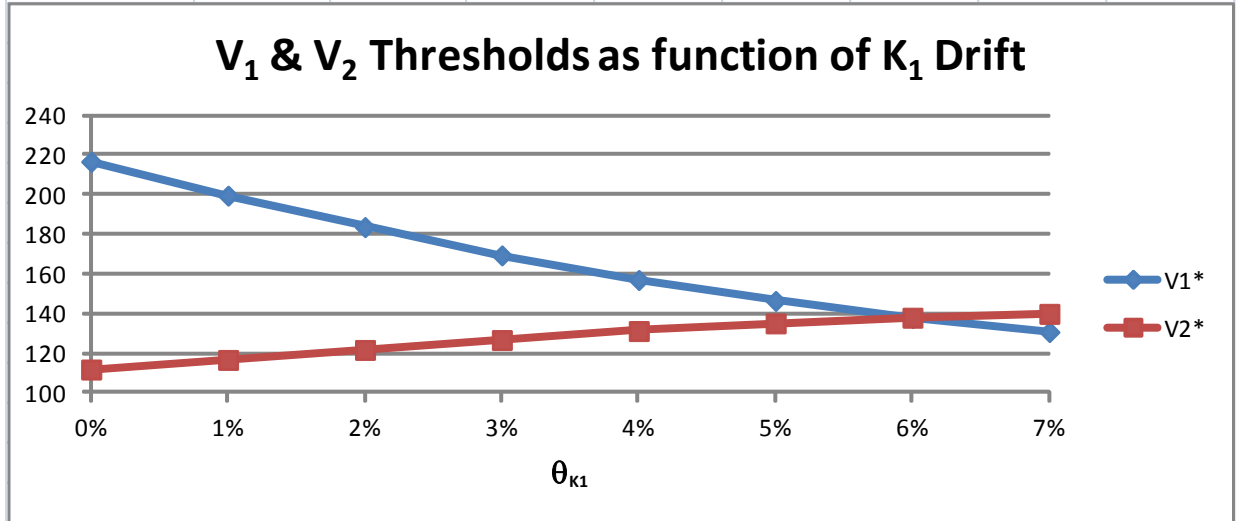


$\hat{V}_1 > \hat{V}_2$ decreases as r increases from 3% to 10%, but beyond that the LB no longer holds, based on the Table 1 parameter values.

Figure A3

θ_{K1}	0%	1%	2%	3%	4%	5%	6%	7%
V1*	216.76	199.58	183.72	169.46	157.02	146.49	137.81	130.79
V2*	111.87	116.75	121.79	126.75	131.30	135.14	138.05	139.98
ROV ₁	7.05	5.87	4.72	3.66	2.72	1.95	1.35	0.91
ROV ₂	3.01	2.38	1.81	1.32	0.93	0.63	0.42	0.27
LB	2.90	3.39	4.02	4.84	5.92	7.30	9.04	11.19
K1/K2	9.00	9.00	9.00	9.00	9.00	9.00	9.00	9.00

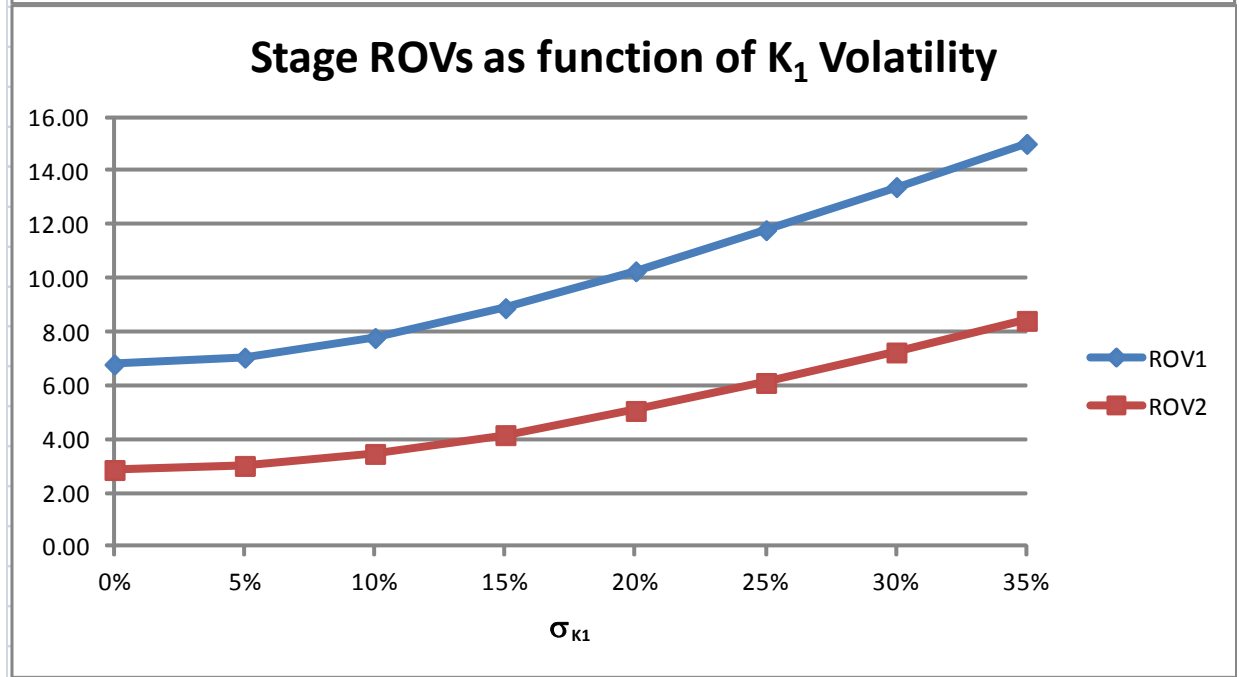
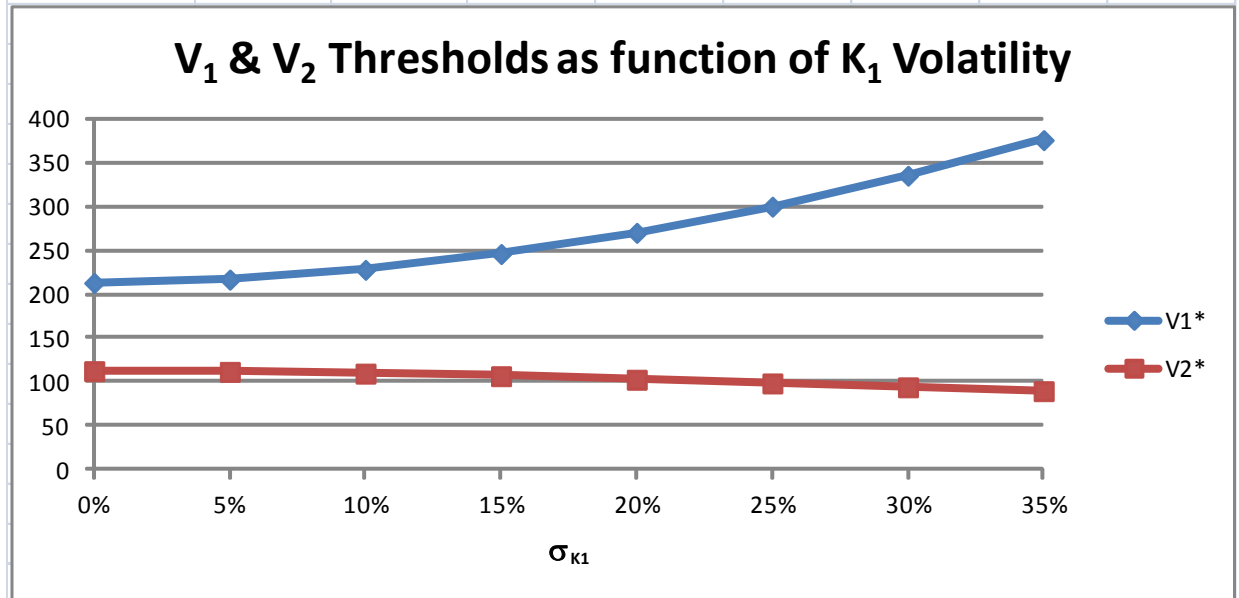
ViolateLB ViolateLB



$\hat{V}_1 > \hat{V}_2$ decreases as K₁ drift increases from 0 to almost 6% = r, but beyond that the LB no longer holds, based on the Table 1 parameter values.

Figure A4

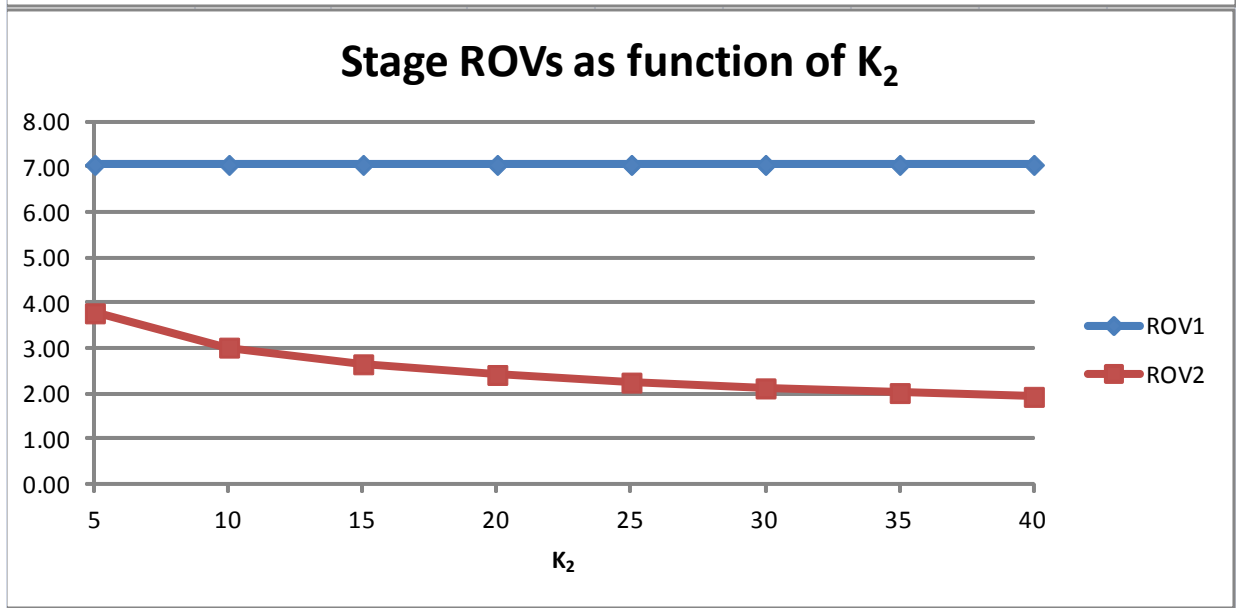
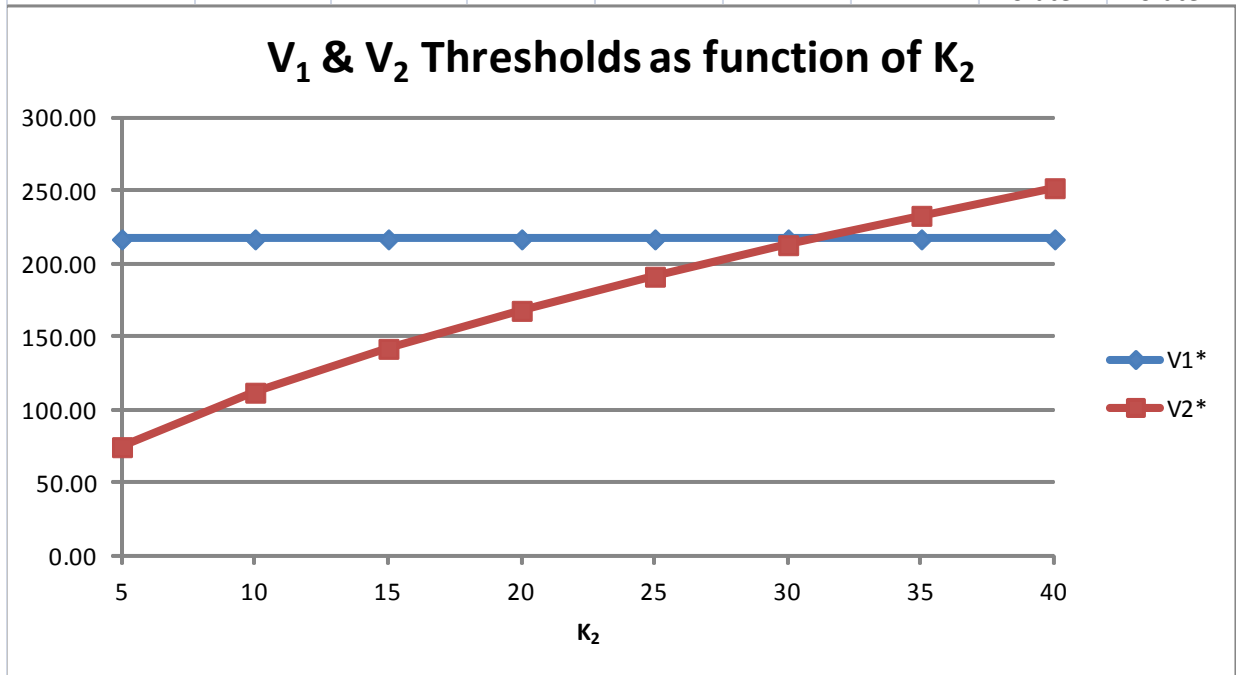
σ_{K1}	0%	5%	10%	15%	20%	25%	30%	35%
$V1^*$	212.94	216.76	227.95	245.90	270.00	299.78	334.98	375.45
$V2^*$	112.56	111.87	109.88	106.79	102.92	98.62	94.18	89.83
ROV_1	6.79	7.05	7.78	8.89	10.26	11.80	13.40	15.01
ROV_2	2.86	3.01	3.46	4.16	5.06	6.11	7.24	8.42



$\hat{V}_1 > \hat{V}_2$ decreases as σ_{K_1} increases from 0 to 35% showing it pays to wait at stage 1 if the final stage investment cost is increasingly volatile.

Figure A5

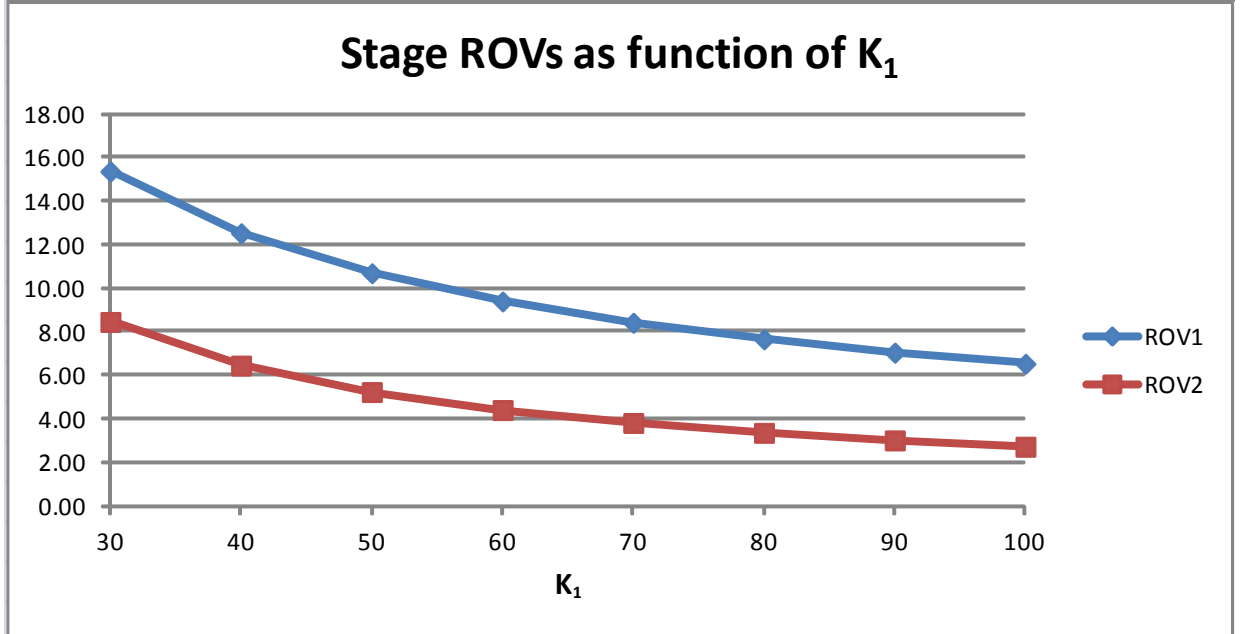
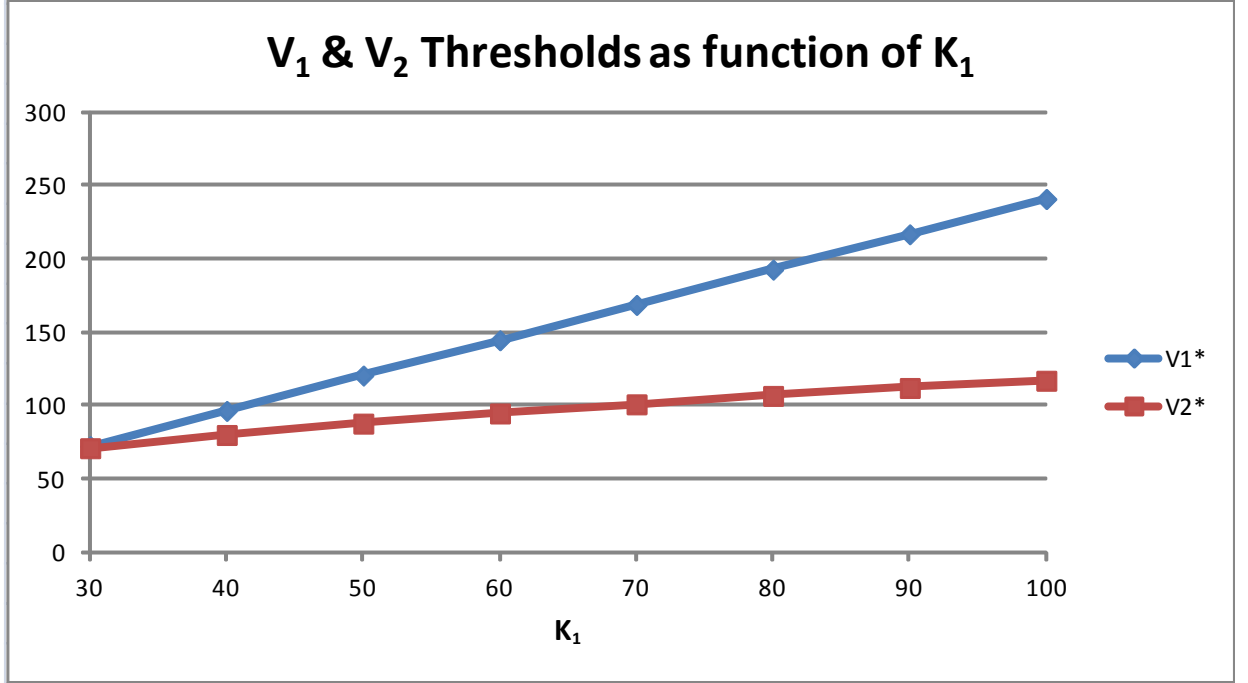
K2*	5	10	15	20	25	30	35	40
V1*	216.76	216.76	216.76	216.76	216.76	216.76	216.76	216.76
V2*	74.59	111.88	141.83	167.83	191.24	212.77	232.87	251.80
ROV ₁	7.05	7.05	7.05	7.05	7.05	7.05	7.05	7.05
ROV ₂	3.77	3.01	2.64	2.41	2.24	2.11	2.01	1.93
LB	2.90	2.90	2.91	2.91	2.91	2.91	2.91	2.91
K1/K2	18.00	9.00	6.00	4.50	3.60	3.00	2.57	2.25
							Violate LB	Violate LB



$\hat{V}_1 > \hat{V}_2$ decreases as K₂ increases because \hat{V}_2 increases naturally and ROV₂ decreases.

Figure A6

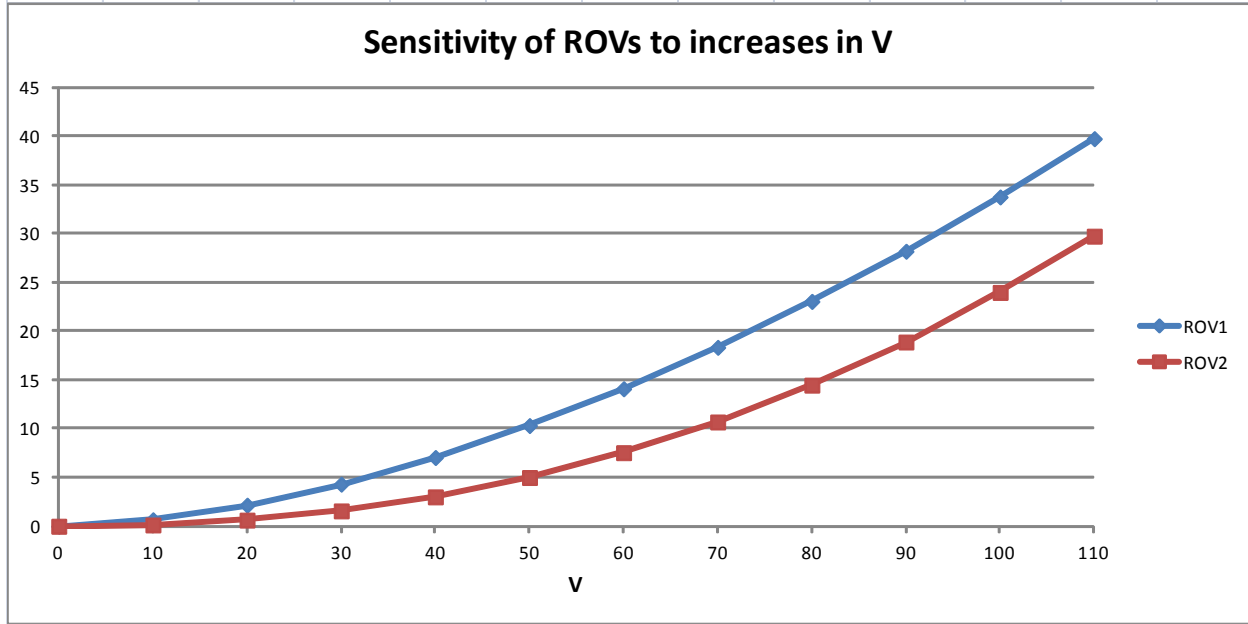
K1*	30.00	40.00	50.00	60.00	70.00	80.00	90.00	100.00
V1*	72.25	96.34	120.42	144.51	168.59	192.68	216.76	240.84
V2*	70.90	79.90	87.66	94.56	100.82	106.58	111.93	116.94
ROV ₁	15.37	12.53	10.70	9.40	8.42	7.66	7.05	6.54
ROV ₂	8.46	6.46	5.24	4.41	3.82	3.37	3.02	2.73
LB	2.90	2.90	2.91	2.91	2.91	2.91	2.91	2.91
K1/K2	3.00	4.00	5.00	6.00	7.00	8.00	9.00	10.00



$\hat{V}_1 > \hat{V}_2$ increases as K_1 increases because \hat{V}_1 increases naturally and both ROV decrease.

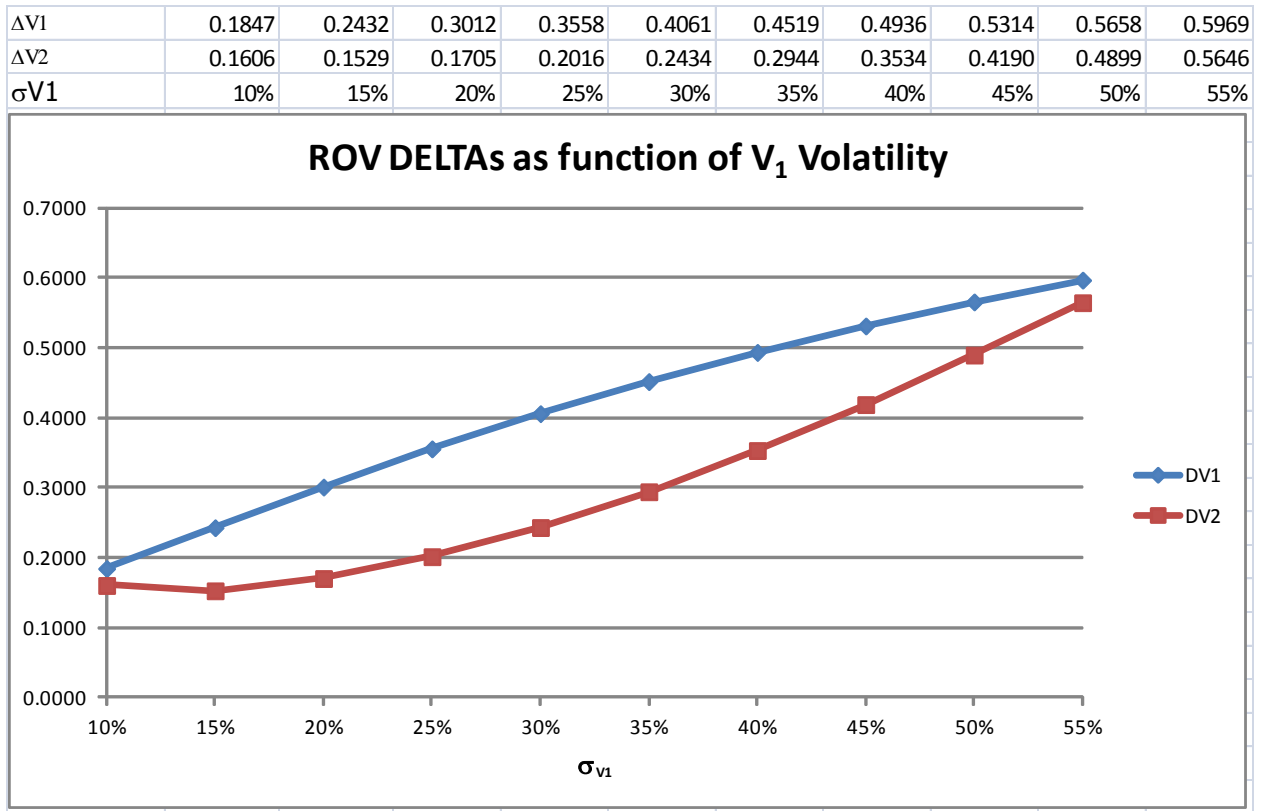
Figure A7

V	0.00	10.00	20.00	30.00	40.00	50.00	60.00	70.00	80.00	90.00	100.00	110.00
ROV ₁	0.00	0.66	2.15	4.31	7.05	10.32	14.10	18.35	23.05	28.20	33.76	39.74
ROV ₂	0.00	0.13	0.63	1.57	3.01	4.99	7.54	10.69	14.47	18.89	23.97	29.75
ΔV_1	0.00	0.11	0.18	0.25	0.30	0.35	0.40	0.45	0.49	0.54	0.58	0.62
ΔV_2	0.00	0.03	0.07	0.12	0.17	0.23	0.28	0.35	0.41	0.47	0.54	0.61



ROVs increase and the spread $ROV_1 > ROV_2$ increases as V increases, following EQ 3 and 10.

Figure A8



The ROV deltas increase and the spread $\Delta ROV_1 > \Delta ROV_2$ decreases as V_1 volatility increases past 25% .

Figure A9

ρ V2K1	0%	-10%	-20%	-30%	-40%	-50%	-60%	-70%
V1*	216.76	216.76	216.76	216.76	216.76	216.76	216.76	216.76
V2*	111.87	114.54	117.36	120.35	123.52	126.90	130.51	134.38
ROV ₁	7.05	7.05	7.05	7.05	7.05	7.05	7.05	7.05
ROV ₂	3.01	3.10	3.20	3.29	3.39	3.49	3.59	3.69

ρ K1K2	0%	-10%	-20%	-30%	-40%	-50%	-60%	-70%
V1*	216.76	216.76	216.76	216.76	216.76	216.76	216.76	216.76
V2*	111.87	111.78	111.69	111.59	111.50	111.41	111.31	111.22
ROV ₁	7.05	7.05	7.05	7.05	7.05	7.05	7.05	7.05
ROV ₂	3.01	3.01	3.01	3.00	3.00	3.00	2.99	2.99

Changes in the correlation of V₂ and K₁, and K₂ and K₁ do not significantly affect \hat{V}_2 or ROV₂.