

Reversible Combination of Inequivalent Kinds of Multipartite Entanglement

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We present a family of tripartite entangled states that, in an asymptotical sense, can be reversibly converted into Einstein-Podolsky-Rosen (EPR) states, shared by parties B and C , and tripartite Greenberger-Horne-Zeilinger (GHZ) states. Thus we show that a bipartite and a genuine tripartite entanglement can be reversibly combined in a tripartite state. For such states the corresponding fractions of GHZ and EPR states represent a complete quantification of their (asymptotical) entanglement resources. More generally, we show that AB , AC , and BC EPR entanglement and GHZ entanglement can be reversibly combined in a single tripartite state. Finally, we generalize this result to any number of parties.

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Understanding the inequivalent ways in which the parts of a composite system can be entangled is one of the central open questions of quantum information theory. When the system consists of only two parts, A and B , and it has been prepared in a pure state $|\psi\rangle_{AB}$, then its entanglement properties are qualitatively equivalent to those of an Einstein-Podolsky-Rosen (EPR) state [1],

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad (1)$$

in the following sense [2]. If two parties, Alice and Bob, share N copies of the system in state $|\psi\rangle_{AB}$ and are allowed to perform local operations assisted with classical communication (LOCC), then they can transform, reversibly in the large N limit, the state of their systems into $NE(\psi_{AB})$ copies of an EPR state (1), where $E(\psi_{AB})$ is the entropy of entanglement of state $|\psi\rangle_{AB}$, namely, the von Neumann entropy of the reduced density matrix describing either part A or B . Thus, reversibility of asymptotical conversions justifies that we regard all bipartite pure-state entanglement as equivalent and quantify it by means of $E(\psi_{AB})$.

It has been recently shown [3] that a Greenberger-Horne-Zeilinger (GHZ) state [4]

$$\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \quad (2)$$

of a tripartite system is inequivalent, even in this asymptotic sense, to EPR states distributed among the parties. This indicates that there is genuine tripartite entanglement. Similarly [3] (see also [5]), for any number n of parties sharing a system, the n -partite GHZ state

$$\frac{1}{\sqrt{2}}(|0^{\otimes n}\rangle + |1^{\otimes n}\rangle) \quad (3)$$

cannot be reversibly converted by means of LOCC into any distribution of entangled states, each one involving less than n parties. Here we will refer to entanglement of the form (3) as *canonical*. Thus, in the general case of a m -partite system, one can find at least $2^m - m - 1$ kinds of entanglement that are asymptotically inequivalent. They

correspond, for each $n = 2, \dots, m$, to all $m!/[n!(m-n)!]$ kinds of n -canonical entanglement, that is involving different subsets of n parties [6].

In this Letter we show that inequivalent kinds of multipartite entanglement can be reversibly combined into a pure state by means of LOCC. More specifically, we show that all kinds of n -canonical entanglement, $n = 2, \dots, m$, can be combined in a m -partite state, and then again reextracted, with asymptotically vanishing losses.

For instance, we will prove that N copies of some tripartite state

$$|\psi\rangle \equiv c_0|000\rangle + c_1|1\rangle \frac{1}{\sqrt{2}}(|11\rangle + |22\rangle) \quad (4)$$

can be reversibly transformed, in the limit $N \rightarrow \infty$, into copies of an EPR state shared by Bob and Claire, and copies of a GHZ state, that is,

$$|\psi\rangle^{\otimes N} \approx |\text{EPR}_{BC}\rangle^{\otimes Nl_{BC}} \otimes |\text{GHZ}\rangle^{\otimes Nl_{ABC}}. \quad (5)$$

This means that the asymptotical entanglement properties of $|\psi\rangle$ can be completely characterized by simply specifying the values of l_{BC} and l_{ABC} . Some other states $|\psi'\rangle$ of three parties will be reversibly converted into the three inequivalent kinds of bipartite EPR states and GHZ states, so that their entanglement can be characterized by the multi-component measure $L_{\psi'} \equiv (l_{AB}, l_{AC}, l_{BC}; l_{ABC})$. For an arbitrary number m of parties, a $(2^m - m - 1)$ -component measure will also quantify the entanglement properties of some other states $|\psi''\rangle$ by providing the amount of all inequivalent kinds of n -canonical entanglement ($n = 2, \dots, m$) that can be reversibly extracted from them.

While it is not yet clear how many asymptotically inequivalent kinds of entanglement exist, not even whether there is only a finite number of them for the simplest non-trivial case (i.e., for tripartite systems), our results arguably help in the ongoing effort [3,5,7] to understand and classify multipartite quantum correlations, as they show for the first time that it is possible to quantify the entanglement of a given state by relating it to several inequivalent forms of entanglement.

We start by analyzing the asymptotic properties of the tripartite state $|\psi\rangle \in \mathcal{Q}^2 \otimes \mathcal{Q}^3 \otimes \mathcal{Q}^3$ given by Eq. (4), where $c_0, c_1 \geq 0$, $c_0^2 + c_1^2 = 1$. We will show how the parties can extract, from N copies of it and by means of LOCC, up to $Nc_1^2 + g_1(N)$ EPR pairs between Bob and Claire and $NS(c_0^2, c_1^2) + g_2(N)$ GHZ states, where $S(\{x_i\}) = -\sum_i x_i \log_2 x_i$ and both $g_1(x)/x$ and $g_2(x)/x$ vanish as $x \rightarrow \infty$. We will then prove that the same amounts of canonical entanglement—up to corrections that vanish as $N \rightarrow \infty$ —suffice to create the state $|\psi\rangle^{\otimes N}$ (actually a state as faithful to $|\psi\rangle^{\otimes N}$ as wished if N can be made arbitrarily large). Therefore we will have $l_{BC} = c_1^2$ and $l_{ABC} = S(c_0^2, c_1^2)$. Finally, we will then describe generalizations of this result to all possible kinds of canonical entanglement and to an arbitrary number of parties.

Let us consider, then, two copies of $|\psi\rangle$, which after some convenient relabeling of the local orthonormal basis in $\mathcal{H}^{N=2} = \mathcal{Q}^4 \otimes \mathcal{Q}^9 \otimes \mathcal{Q}^9$ can be written as

$$\begin{aligned} |\psi\rangle^{\otimes 2} = & c_1^2 |1^0\rangle \frac{1}{\sqrt{4}} (|1_1^0 1_1^0\rangle + |2_1^0 2_1^0\rangle + |3_1^0 3_1^0\rangle + |4_1^0 4_1^0\rangle) \\ & + c_0 c_1 |1^1\rangle \frac{1}{\sqrt{2}} (|1_1^1 1_1^1\rangle + |2_1^1 2_1^1\rangle) \\ & + c_0 c_1 |2^1\rangle \frac{1}{\sqrt{2}} (|1_2^1 1_2^1\rangle + |2_2^1 2_2^1\rangle) \\ & + c_0^2 |1^2\rangle |1_1^2 1_1^2\rangle \end{aligned} \quad (6)$$

By means of a local measurement the parties can project the state (6) into one of the three subspaces characterized by a constant coefficient $c_0^k c_1^{N-k}$ ($k = 0, 1, 2$). We point out the relevant fact that either Alice, Bob, or Claire could perform such a measurement locally because $|\psi\rangle$ is a linear combination $c_0|\phi_1\rangle + c_1|\phi_2\rangle$ of two locally orthogonal states $|\phi_i\rangle$, i.e.,

$$\text{Tr}[\rho_i^\alpha \rho_j^\alpha] = 0 \quad \forall i \neq j, \quad \alpha = A, B, C, \quad (7)$$

where ρ_i^α is the reduced density matrix of $|\phi_i\rangle$ for subsystem α , and this implies that the three subspaces of (6) are also locally orthogonal; in other words, the parties can manipulate locally each of these subspaces independently. If the result of the measurement corresponds to $k = 0$, then Bob and Claire will be sharing a 2^2 -level maximally entangled state, that is, two EPR pairs. If the outcome corresponds to $k = 1$, then the parties end up sharing an EPR_{BC} state and a GHZ state, as can be straightforwardly checked by expanding $|\text{EPR}_{BC}\rangle \otimes |\text{GHZ}\rangle$ as a linear combination of product states. Finally, an outcome related to the subspace $k = 2$ leaves the parties with a product state $|000\rangle$. This structure of outcomes easily generalizes to the case of N copies. Let us call block (N, k) , denoted by $|B_{N,k}\rangle$, the normalized projection of $|\psi\rangle^{\otimes N}$ into the subspace characterized by the coefficient $c_0^k c_1^{N-k}$, that is,

$$|\psi\rangle^{\otimes N} = \sum_{k=0}^N c_0^k c_1^{N-k} \sqrt{b_{N,k}} |B_{N,k}\rangle, \quad (8)$$

$b_{N,k} \equiv N!/[k!(N-k)!]$. A direct computation shows that $|B_{N,k}\rangle$ is of the form

$$\begin{aligned} |B_{N,k}\rangle = & \frac{1}{\sqrt{r!}} \{ |1^k\rangle (|1_1^k 1_1^k\rangle + |2_1^k 2_1^k\rangle + \dots + |r_1^k r_1^k\rangle) \\ & + |2^k\rangle (|1_2^k 1_2^k\rangle + |2_2^k 2_2^k\rangle + \dots + |r_2^k r_2^k\rangle) \\ & + |t^k\rangle (|1_t^k 1_t^k\rangle + |2_t^k 2_t^k\rangle + \dots + |r_t^k r_t^k\rangle) \}, \end{aligned} \quad (9)$$

where $r \equiv 2^{N-k}$, $t \equiv b_{N,k}$, and the local states satisfy $\langle i_a^k | j_b^{k'} \rangle = \delta_{i,j} \delta_{k,k'} \delta_{a,b}$ in both Bob and Claire and $\langle i^k | j^{k'} \rangle = \delta_{i,j} \delta_{k,k'}$ in Alice. Notice that (9) is equivalent to the tensor product of an r -level EPR state and a t -level GHZ state,

$$\begin{aligned} |B_{N,k}\rangle = & \left(\frac{1}{\sqrt{r}} \sum_{i=1}^r |ii\rangle_{BC} \right) \otimes \left(\frac{1}{\sqrt{t}} \sum_{i=1}^t |iii\rangle \right) \\ = & |r - \text{EPR}_{BC}\rangle \otimes |t - \text{GHZ}\rangle. \end{aligned} \quad (10)$$

Thus, by means of a local measurement projecting onto these blocks, the parties will obtain state (10) with probability $P_{N,k} \equiv b_{N,k} c_0^{2k} c_1^{2(N-k)}$. The expectation values

$$\begin{aligned} \left\langle \frac{\log_2 r}{N} \right\rangle &= \sum_{k=0}^N P_{N,k} \frac{N-k}{N}, \\ \left\langle \frac{\log_2 t}{N} \right\rangle &= \sum_{k=0}^N P_{N,k} \frac{\log_2 b_{N,k}}{N} \end{aligned} \quad (11)$$

correspond, respectively, to the fraction l_{BC} of EPR_{BC} states and to the fraction l_{ABC} of GHZ states that are obtained, on average and per copy of $|\psi\rangle$, from $|\psi\rangle^{\otimes N}$ [8]. Because of the smooth behavior of the functions $(N-k)/N$ and $(\log_2 b_{N,k})/N$ compared to the binomial distribution $P_{N,k}$, we can calculate the expectation values (11)—up to corrections that vanish in the limit $N \rightarrow \infty$ —by just evaluating the two functions at the peak of $P_{N,k}$, namely, at $k_{\max} \equiv Nc_0^2$, which gives us the announced amount of entanglement for each of the two canonical forms.

Let us look now at the inverse transformation. We start, for clearness sake, by showing that two EPR_{BC} and two GHZ states suffice to create state (6) locally and with certainty. Let us expand $|\text{EPR}_{BC}\rangle^{\otimes 2} \otimes |\text{GHZ}\rangle^{\otimes 2}$ as

$$\begin{aligned} \frac{1}{\sqrt{4}} \left\{ \frac{1}{\sqrt{4}} (|1_1^0 1_1^0\rangle + |2_1^0 2_1^0\rangle + |3_1^0 3_1^0\rangle + |4_1^0 4_1^0\rangle) \right. \\ + |1^1\rangle \frac{1}{\sqrt{4}} (|1_1^1 1_1^1\rangle + |2_1^1 2_1^1\rangle + |3_1^1 3_1^1\rangle + |4_1^1 4_1^1\rangle) \\ + |2^1\rangle \frac{1}{\sqrt{4}} (|1_2^1 1_2^1\rangle + |2_2^1 2_2^1\rangle + |3_2^1 3_2^1\rangle + |4_2^1 4_2^1\rangle) \\ \left. + |1^2\rangle \frac{1}{\sqrt{4}} (|1_1^2 1_1^2\rangle + |2_1^2 2_1^2\rangle + |3_1^2 3_1^2\rangle + |4_1^2 4_1^2\rangle) \right\}. \end{aligned} \quad (12)$$

In this expression [cf. Eq. (6)] we would like to give the first row a weight c_1^2 ; in both the second and third rows we should get rid of $|3_i^2 3_i^2\rangle$ and $|4_i^2 4_i^2\rangle$ and give each of the

rows a weight $c_0 c_1$; the fourth row should be reduced to a product state with weight c_0^2 . After these changes have been made, we will have state (6). We first note that the parties can transform, with certainty, the two GHZ states into a triorthogonal state with arbitrary relative weights,

$$\frac{1}{2} \sum_{i=1}^4 |iii\rangle \longrightarrow \sum_{i=1}^4 \lambda_i |iii\rangle, \quad (13)$$

by means of a local positive operator-valued measurement (POVM) $\{O_j\}$, $j = 1, \dots, 4$, on (any) one of the parties followed by an outcome-dependent, local unitary U_j applied once in each of the parties' subsystems. Here O_j is proportional to $\sum_i \lambda_i |i \oplus j\rangle \langle i \oplus j|$ [9] and U_j takes $|i \oplus j\rangle$ into $|i\rangle$ on each local subsystem. The tensor product of two EPR_{BC} states with the resulting state in (13) is equivalent to (12) but with row i having weight λ_i . Bob and Claire can now address each of the four rows locally and reduce their length at wish. Indeed, in order to shorten the fourth row into a product vector, one of them, say, Bob, can perform a POVM with four positive operators

$$Q_i = |i_1^2\rangle \langle i_1^2| + \frac{1}{2} \sum_{j=2,3,4} P_j, \quad i = 1, \dots, 4, \quad (14)$$

where P_j is a projector onto the local subspace supporting row j , e.g., $P_2 = \sum_i |i_1^1\rangle \langle i_1^1|$; then Bob and Claire need to relabel the surviving term $|i_1^2 i_1^2\rangle$ of the first row as $|1_1^2 1_1^2\rangle$. By means of similar POVMs followed by outcome-dependent local unitaries, Bob and Claire can also tailor the second and third rows so that they contain only two product terms each. Explicitly, a two-outcome POVM that reduces the second row reads

$$Q'_1 = |1_1^1 1_1^1\rangle \langle 1_1^1 1_1^1| + \frac{1}{\sqrt{2}} \sum_{j=1,3,4} P_j, \quad (15)$$

$$Q'_2 = |3_1^1 3_1^1\rangle \langle 3_1^1 3_1^1| + \frac{1}{\sqrt{2}} \sum_{j=1,3,4} P_j.$$

Notice that such measurements do not modify the relative weight of the rows. A proper choice of the coefficients λ_i in the first step of the transformation, namely $\lambda_1 = c_1^2$, $\lambda_2 = \lambda_3 = c_1 c_0$, and $\lambda_4 = c_0^2$, completes therefore the protocol for preparing two copies of $|\psi\rangle$.

In the case of a large number N of copies the parties start with several EPR_{BC} and GHZ states, and want to create a state

$$|N_{k_-}^{k_+}\rangle \equiv K \sum_{k=k_-}^{k_+} c_0^{N-k} c_1^k \sqrt{b_{N,k}} |B_{N,k}\rangle, \quad (16)$$

such that $F \equiv |\langle N_{k_-}^{k_+} | \psi^{\otimes N} \rangle|^2 = 1 - h(N)$, where $h(x \rightarrow \infty) = 0$, that is, a state which asymptotically cannot be distinguished from $|\psi\rangle^{\otimes N}$. We note that an arbitrary faithfulness can be achieved, asymptotically, by considering only the blocks $|B_{N,k}\rangle$ [cf. Eq. (9)] that correspond to k 's around $k_{\max} = N c_0^2$. Indeed, a straightforward computation of the fidelity shows that it suffices to take

$k_{\pm} = c_0^2 N \pm \alpha N^{\beta}$ for some $\alpha > 0$ and $1/2 < \beta < 1$: by noting that a binomial distribution is asymptotically equivalent to a normal (Gaussian) distribution, the fidelity F can be seen to be bounded from below by $\Phi(2\alpha N^{\beta-1/2})$, where $\Phi(x) \equiv 1/\sqrt{2\pi} \int_{-x}^x e^{-y^2/2} dy$. For our choice of α, β , we see that $F \rightarrow 1$ when $N \rightarrow \infty$. As with the $N = 2$ case, our plan is [starting from a reasonable amount of EPR_{BC} and GHZ states, which can be expanded in the fashion of (12)] (i) to modify conveniently the weight of each row in the expansion and (ii) to shorten the length of each row, in order to obtain the pattern of lengths given by the block structure of (16). Notice that a straightforward generalization of (13) provides row i with an arbitrary weight λ_i by locally manipulating the initial GHZ states, and we have also already seen how to shorten each row independently by means of a local POVM in either Bob's or Claire's side (see POVMs (14) and (15) as examples). Thus, the only question left concerns the amount of canonical entanglement required to produce all blocks $|B_{N,k}\rangle$ in (16). Since we have a mechanism to shorten but not to lengthen the rows, the number of EPR_{BC} states must allow one to obtain the longest rows, which are those of the block $|N, k_-\rangle$ and contain 2^{N-k_-} product terms each. That is, $N - k_-$ EPR_{BC} states will suffice. The total number of GHZ states required is the logarithm of the total number of rows in (16), and thus reads as follows: $\log_2(\sum_{k=k_-}^{k_+} b_{N,k})$. Let $k_0 \in [k_-, k_+]$ be the value that maximizes $b_{N,k}$ in this interval. Then the amount of GHZ states required to prepare (16) is bounded from above by $\log_2[(k_+ - k_- + 1)b_{N,k_0}]$. Finally, substitution of k_{\pm} and k_0 in this bound and the previous estimation for EPR_{BC} states shows that both amounts of canonical entanglement needed to prepare (16) are the expected ones, apart from corrections which scale sublinearly in N and that therefore become irrelevant for N sufficiently large. This concludes the proof that state (4) is asymptotically equivalent to canonical entanglement.

We can now generalize the previous example and reversibly combine the three kinds of bipartite entanglement and the canonical tripartite entanglement in a single state. Indeed, the tripartite state,

$$|\psi'\rangle \equiv c_0 |000\rangle + c_1 |1\rangle \frac{1}{\sqrt{2}} (|11\rangle + |22\rangle) \\ + c_2 \frac{1}{\sqrt{2}} (|233\rangle + |334\rangle) \\ + c_3 \frac{1}{\sqrt{2}} (|44\rangle + |55\rangle) |5\rangle, \quad (17)$$

can be transformed by means of LOCC into EPR states shared by two parties and into GHZ states, with the asymptotic ratios being $l_{AB} = c_1^2$, $l_{AC} = c_2^2$, and $l_{BC} = c_3^2$ for the bipartite entanglement and $l_{ABC} = S(\{c_i^2\})$ for the tripartite entanglement.

This result follows from considering analogous transformations to the ones described above. The expansion

of the state of N copies of $|\psi'\rangle$ into locally orthogonal subspaces such as in (8) depends now on three independent indices, with the weights defining the block structure being $c_0^{k_0} c_1^{k_1} c_2^{k_2} c_3^{N-k_1-k_2-k_3}$. The binomial probability distribution is replaced by a multinomial distribution, centered at $k_i = Nc_i^2$, and each one of the new blocks $|B_{N,k_0,k_1,k_2}\rangle$ is equivalent to the tensor product of some number of GHZ, EPR_{AB} , EPR_{AC} , and EPR_{BC} states. A local measurement onto the blocks leads, for sufficiently large N , to the desired expectation values for the fractions of canonical entanglement. Conversely, these amounts of entanglement suffice to create all of the relevant blocks $|B_{N,k_0,k_1,k_2}\rangle$. This is done by introducing some weights λ_i in the initial GHZ states and by locally tailoring the (now multidimensional) rows in the pertinent expansion, as we explained in the previous example.

More generally, let the m -partite state

$$|\psi''\rangle \equiv \sum_{i=0}^l c_i |\phi_i\rangle \quad (18)$$

be a linear combination of locally orthogonal states [see Eq. (7)] such that each one is the tensor product of a canonical state $|\tau_i\rangle$ (3) for n_i of the parties, and a product vector for the remaining $m - n_i$ parties [10]. Then N copies of the state $|\psi''\rangle$ are asymptotically equivalent to Nc_0^2 copies of $|\tau_0\rangle, \dots, Nc_l^2$ copies of $|\tau_l\rangle$ and to $NS(\{c_i^2\})$ copies of a m -canonical state, i.e.,

$$|\psi''\rangle^{\otimes N} \approx \left[\bigotimes_{i=0}^l |\tau_i\rangle^{\otimes Nc_i^2} \right] \otimes (|0^{\otimes m}\rangle + |1^{\otimes m}\rangle)^{\otimes NS(\{c_i^2\})}. \quad (19)$$

In this Letter we have provided examples of multipartite states whose entanglement properties can be classified and quantified in relation to the set of canonical states (3). The criteria underlying such classification is the asymptotical equivalence of states under LOCC. We have shown that at least in some cases, namely, for states of the form (18), this criteria brings a significant simplification to the general problem of classifying entanglement. Indeed, our results show that the states (18), which depend on the set of continuous nonlocal parameters $\{c_i\}$, contain only a finite set of inequivalent forms of entanglement. We have gone further and quantified the amount of each form of entanglement contained in state (18), which gives rise to a multicomponent measure. We do not know to what extent the coefficients, as well as the reference states of this measure, are essentially unique.

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 [6] In [3] it has been shown that m -canonical entanglement is asymptotically inequivalent to any distribution of different kinds of canonical entangled states if each of them involves less than m parties. We can easily argue that also n -canonical entanglement ($n < m$) involving parties $\alpha_1, \dots, \alpha_n$ is actually inequivalent to a distribution of any other kinds of canonical entanglement. Indeed, we need only notice that, for the transformation to be possible, the distribution of final states can only contain entanglement between parties $\alpha_1, \dots, \alpha_n$. But in this case the result of [3] applies, because our original state is the n -canonical entangled state in a subsystem of n parties and the rest of the candidates in the final distribution must involve less than n parties.
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 [8] An r -level n -canonical state is an n -partite state of the form $\frac{1}{\sqrt{r}} \sum_{i=1}^r |i^{\otimes n}\rangle$. For any integer $l \leq \log_2 r$, this state can always be transformed, locally and with certainty, into l copies of a (two-level) n -canonical state (3) by means, for instance, of a local POVM on one of the parties consisting of 2^l -dimensional positive operators O_j , $j = 1, \dots, r$ proportional to $\sum_{i=1}^{2^l} |i \oplus_r j\rangle \langle i \oplus_r j|$ [9] followed by the outcome-dependent local unitary $|i \oplus_r j\rangle \rightarrow |i\rangle$ performed by each party.
 [9] We use $i \oplus_r j$ to denote $i + j$ modulo r .
 [10] For instance, for $m = 5$, we could have $n_0 = 3$ and $|\phi_0\rangle = |0\rangle 1/\sqrt{2} (|000\rangle + |111\rangle) |0\rangle = |0\rangle |\tau_0\rangle |0\rangle$.