CORE
Metadata, citation and similar papers at core.ac.uk
Provided by DSpace at Belgorod State University

Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 49, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

MATHEMATICAL MODELS OF A LIQUID FILTRATION FROM RESERVOIRS

ANVARBEK MEIRMANOV, NELLY ERYGINA, SALTANBEK MUKHAMBETZHANOV

Abstract. This article studies the filtration from reservoirs into porous media under gravity. We start with the exact mathematical model at the microscopic level, describing the joint motion of a liquid in reservoir and the same liquid and the elastic solid skeleton in the porous medium. Then using a homogenization procedure we derive the chain of macroscopic models from the poroelasticity equations up to the simplest Darcy's law in the porous medium and hydraulics in the reservoir.

1. Introduction

In article we consider a correct description of filtration from reservoirs into porous media under gravity. Our approach is based on the way suggested by Burridge and Keller [1] and Sanchez-Palencia [10]. We first describe the problem at the microscopic level using classical equations of continuum mechanics and after that derive all possible homogenized equations, describing the problem at the macroscopic level.

The problem in its simplest setting is modeled by two domains Ω_0 and Ω having a common boundary S^0. The domain Ω^0 models a reservoir and is occupied by liquid, and the domain Ω models a porous medium. Throughout this paper we impose the following constraints.

We will use the following assumptions:

(1) Let $\chi(\mathbf{y})$ be a 1-periodic function, $Y_s = \{\mathbf{y} \in Y : \chi(\mathbf{y}) = 0\}$ be the "solid part" of the unit cube $Y = (0,1)^3 \subset \mathbb{R}^3$, and let the "liquid part" $Y_f = \{\mathbf{y} \in Y : \chi(\mathbf{y}) = 1\}$ of Y be its open complement. We write $\gamma = \partial Y_f \cap \partial Y_s$ and assume that γ is a Lipschitz continuous surface.
(2) The domain E^ε_f is a periodic repetition in $\mathbb{R}^3$ of the elementary cell $Y^\varepsilon_f = \varepsilon Y_f$ and the domain E^ε_s is a periodic repetition in $\mathbb{R}^3$ of the elementary cell $Y^\varepsilon_s = \varepsilon Y_s$.
(3) The pore space $\Omega^\varepsilon_f \subset \Omega = \Omega \cap E^\varepsilon_f$ is a periodic repetition in Ω of the elementary cell εY_f, and the solid skeleton $\Omega^\varepsilon_s \subset \Omega = \Omega \cap E^\varepsilon_s$ is a periodic repetition in Ω of the elementary cell εY_s. The Lipschitz continuous boundary $\Gamma^\varepsilon = \partial\Omega^\varepsilon_s \cap \partial\Omega^\varepsilon_f$ is a periodic repetition in Ω of the boundary $\varepsilon\gamma$.
(4) Y_s and Y_f are connected sets.

2000 *Mathematics Subject Classification.* 35B27, 46E35, 76R99.
Key words and phrases. Lame's equation; Stokes equation; liquid filtration; homogenization; poroelasticity.
©2014 Texas State University - San Marcos.
Submitted January 6, 2014. Published February 19, 2014.

(5) The pore space Ω_f^ε and the solid skeleton Ω_s^ε are connected domains.

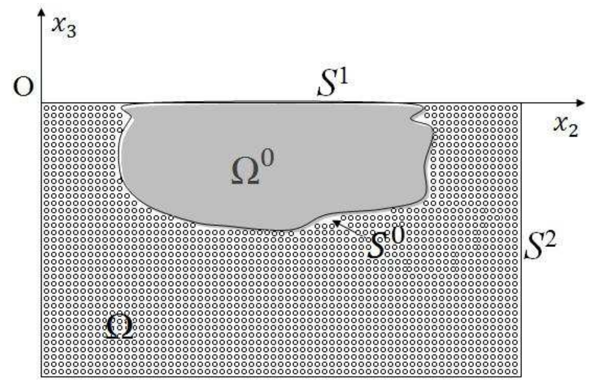

FIGURE 1. Filtration from reservoir

Let $Q = \Omega_0 \cup \Omega \cup S^0$, $S = \partial Q$, $\bar{S} = \bar{S}^1 \cup \bar{S}^2$.

We also assume that S^1 is a part of the plane $\{x_3 = 0\}$, $\mathbf{e} = -\mathbf{e}_3$, and that the domain Q is a subset of the half-space $\{x_3 < 0\}$. Moreover we suppose that S^2 is a C^2–smooth surface and in some small neighborhood of the plane $\{x_3 = 0\}$ it is represented by the equation $\Phi(x_1, x_2) = 0$.

The motion of the liquid in Ω^0 for $t > 0$ is governed by the dimensionless non-stationary Stokes system

$$\nabla \cdot \mathbf{w} = 0, \tag{1.1}$$

$$\alpha_\tau \varrho_f \frac{\partial^2 \mathbf{w}}{\partial t^2} = \nabla \cdot \mathbb{P}_f + \varrho_f \mathbf{e}, \quad \mathbb{P}_f = \alpha_\mu \mathbb{D}(x, \frac{\partial \mathbf{w}}{\partial t}) - p\,\mathbb{I}, \tag{1.2}$$

and the joint motion of the poroelastic media in Ω for $t > 0$ is governed by the model [7] consisting of the continuity equation (1.1) and the dimensionless momentum balance equation

$$\alpha_\tau \varrho^\varepsilon \frac{\partial^2 \mathbf{w}}{\partial t^2} = \nabla \cdot \mathbb{P} + \varrho^\varepsilon \mathbf{e}. \tag{1.3}$$

Here $\nabla \cdot \mathbf{w}$ is the divergence of the vector $\mathbf{w}$, $\nabla \cdot \mathbb{P}$ is the divergence of the tensor $\mathbb{P}$,

$$\mathbb{P} = \chi^\varepsilon \alpha_\mu \mathbb{D}\big(x, \frac{\partial \mathbf{w}}{\partial t}\big) + (1 - \chi^\varepsilon)\alpha_\lambda \mathbb{D}(x, \mathbf{w}) - p\,\mathbb{I}, \tag{1.4}$$

$$\mathbb{D}(x, \boldsymbol{v}) = \frac{1}{2}\big(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^*\big), \quad \varrho^\varepsilon = \varrho_f \chi^\varepsilon + \varrho_s(1 - \chi^\varepsilon),$$

$$\alpha_\tau = \frac{L}{g\tau^2}, \quad \alpha_\mu = \frac{2\mu}{\tau L g\,\rho^0}, \quad \alpha_\lambda = \frac{2\lambda}{L g\,\rho^0},$$

where $\mathbb{I}$ is the unit tensor, and L is the characteristic size of the physical domain in consideration, τ is the characteristic time of the physical process, ρ^0 is the mean density of water, g is acceleration due gravity, μ is the dynamic viscosity, λ is the elastic constant, ϱ_f and ϱ_s are the respective mean dimensionless densities of the liquid and the solid skeleton, correlated with the mean density of water ρ^0.

On the common boundary $S^0 = \partial\Omega \cap \partial\Omega^0$ for $t > 0$ the continuity conditions

$$\lim_{\mathbf{x} \to \mathbf{x}^0,\, \mathbf{x} \in \Omega^0} \mathbf{w}(\mathbf{x}, t) = \lim_{\mathbf{x} \to \mathbf{x}^0,\, \mathbf{x} \in \Omega} \mathbf{w}(\mathbf{x}, t), \tag{1.5}$$

$$\lim_{\mathbf{x}\to\mathbf{x}^0,\,\mathbf{x}\in\Omega^0}\mathbb{P}_f(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}^0)=\lim_{\mathbf{x}\to\mathbf{x}^0,\,\mathbf{x}\in\Omega}\mathbb{P}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}^0), \tag{1.6}$$

hold for displacements and for normal tensions. Here $\mathbf{n}(\mathbf{x}^0)$ is a normal vector to the boundary S^0 at $\mathbf{x}^0\in S^0$.

We completement the problem with the Neumann boundary conditions

$$\mathbb{P}_f(\mathbf{x},t)\cdot\mathbf{n}=-p^0(\mathbf{x},t)\mathbf{n},\quad \mathbb{P}(\mathbf{x},t)\cdot\mathbf{n}=-p^0(\mathbf{x},t)\mathbf{n} \tag{1.7}$$

on the part $S^1_0=S^1\cap\overline{\Omega}_0$ of the outer boundary S of the domain $Q=\Omega^0\cup S^0\cup\Omega$ (which is also the part of the boundary $\partial\Omega^0$) for $\mathbb{P}_f$ and on the part $S^1_1=S^1\cap\overline{\Omega}$ of the outer boundary S of the domain Q (which is also the part of the boundary $\partial\Omega$) for $\mathbb{P}$, the Dirichlet boundary condition

$$\mathbf{w}(\mathbf{x},t)=0 \tag{1.8}$$

on the part $S^2=S\backslash\overline{S^1}$ of the outer boundary S for $t>0$, and initial conditions

$$\mathbf{w}(\mathbf{x},0)=\frac{\partial\mathbf{w}}{\partial t}(\mathbf{x},0)=0,\quad \mathbf{x}\in Q. \tag{1.9}$$

In (1.1)–(1.9) the characteristic function $\chi^\varepsilon(\mathbf{x})$ of the domain Ω^ε_f is given by the expression

$$\chi^\varepsilon(\mathbf{x})=\varsigma(\mathbf{x})\chi(\frac{\mathbf{x}}{\varepsilon}),$$

where $\varsigma(\mathbf{x})$ is the characteristic function of the domain Ω, $\chi(\mathbf{y})$ is the characteristic function of the liquid cell Y_f in the unit cube Y, and $\mathbf{e}$ is a unit vector in the direction of gravity. The given function p^0 is supposed to be smooth:

$$\int_{Q_T}\big(|\nabla p^0(\mathbf{x},t)|^2+|\nabla(\frac{\partial p^0}{\partial t})(\mathbf{x},t)|^2\big)\,dx\,dt=\mathfrak{P}^2<\infty. \tag{1.10}$$

The main problem in the macroscopic description of the physical problem is the boundary conditions on the common boundary for the solutions of homogenized equations. There are some particular results obtained by Jäger and A. Mikelić [3, 4, 5] for special geometry of pore space (disconnected solid skeleton) and only for domains in $\mathbb{R}^2$.

We study the complete problem in $\mathbb{R}^3$ for the arbitrary geometry of corresponding pore spaces. Namely, let

$$\lim_{\varepsilon\searrow 0}\alpha_\tau(\varepsilon)=\tau_0,\quad \lim_{\varepsilon\searrow 0}\alpha_\mu(\varepsilon)=\mu_0,\quad \lim_{\varepsilon\searrow 0}\alpha_\lambda(\varepsilon)=\lambda_0,\quad \lim_{\varepsilon\searrow 0}\frac{\alpha_\mu}{\varepsilon^2}=\mu_1.$$

To derive the desired homogenized problem for the case $\mu_0=0$, $\lambda_0=\infty$, and $0<\tau_0$, $\mu_1<\infty$ we use (1.1)–(1.9). First we fix $\tau_0>0$, pass to the limit as $\varepsilon\searrow 0$, and get two different systems

$$\tau_0\varrho_f\frac{\partial\mathbf{v}}{\partial t}+\nabla p=\varrho_f\,\mathbf{e},\quad \nabla\cdot\mathbf{v}=0 \tag{1.11}$$

and

$$\mathbf{v}^{(f)}=\int_0^t\mathbb{B}^{(f)}(\tau_0;t-\tau)\cdot\big(-\nabla p^{(f)}(\mathbf{x},\tau)+\varrho_f\mathbf{e}\big)d\tau, \tag{1.12}$$

$$\nabla\cdot\mathbf{v}^{(f)}=0 \tag{1.13}$$

for the velocity $\mathbf{v}$ and pressure p in Ω^0 and the velocity $\mathbf{v}^{(f)}$ and pressure $p^{(f)}$ in Ω.

These differential equations together with the boundary conditions

$$\lim_{\mathbf{x}\to\mathbf{x}^0\in S^0,\,\mathbf{x}\in\Omega} p^{(f)}(\mathbf{x},t) = \lim_{\mathbf{x}\to\mathbf{x}^0\in S^0,\,\mathbf{x}\in\Omega_0} p(\mathbf{x},t), \tag{1.14}$$

$$\lim_{\mathbf{x}\to\mathbf{x}^0\in S^0,\,\mathbf{x}\in\Omega} \mathbf{v}^{(f)}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}^0) = \lim_{\mathbf{x}\to\mathbf{x}^0\in S^0,\,\mathbf{x}\in\Omega_0} \mathbf{v}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}^0) \tag{1.15}$$

on the common boundary S^0 describe the liquid motion in the domain Q for $t>0$.

After that we pass to the limit as $\tau_0 \searrow 0$ and get the usual hydraulic equation

$$\nabla p = \varrho_f\,\mathbf{e}, \quad p(\mathbf{x},t) = p^0(t) - \varrho_f x_3 \equiv p_0(\mathbf{x},t) \tag{1.16}$$

in the domain Ω^0 and usual Darcy's system

$$\mathbf{v}^{(f)} = \frac{1}{\mu_1}\mathbb{B}\cdot\big(-\nabla p^{(f)} + \varrho_f\mathbf{e}\big), \quad \nabla\cdot\mathbf{v}^{(f)} = 0 \tag{1.17}$$

in the domain Ω, completed with the continuity condition (1.14) on the common boundary S^0.

So, if we need to take into account the water flow from reservoir, we have to use the first approximation (1.11)–(1.15). If we need the simplest model, we use the second approximation (1.16)–(1.17). Other homogenized models of (1.1)–(1.9) one may find in [7].

The notation of functional spaces and norm there are the same as in [6].

2. Main results

Definition 2.1. We say that the pair of functions $\{\mathbf{w}^\varepsilon, p^\varepsilon\}$, such that

$$p^\varepsilon \in L_2(Q_T), \mathbf{w}^\varepsilon, \frac{\partial \mathbf{w}^\varepsilon}{\partial t}, \big(\zeta + (1-\zeta)\chi^\varepsilon\big)\nabla\frac{\partial \mathbf{w}^\varepsilon}{\partial t}, \nabla\mathbf{w}^\varepsilon \in \mathbf{L}_2(Q_T),$$

is a weak solution of the problem (1.1)–(1.9), if it satisfies the continuity equation (1.1) almost everywhere in $Q_T = Q\times(0,T)$, the boundary condition (1.8), the initial condition (1.9) for the function $\mathbf{w}^\varepsilon$, and the integral identity

$$\begin{aligned} &\int_{Q_T}\Big(-\tau_0\,\tilde{\varrho}^{\,\varepsilon}\frac{\partial \mathbf{w}^\varepsilon}{\partial t}\cdot\frac{\partial\varphi}{\partial t} + \big(\zeta\mathbb{P}_f + (1-\zeta)\mathbb{P}\big):\mathbb{D}(x,\varphi)\Big)\,dx\,dt \\ &= \int_{Q_T}\big(\tilde{\varrho}^{\,\varepsilon}\mathbf{e}\cdot\varphi - \nabla\cdot(\varphi\,p^0)\big)\,dx\,dt \end{aligned} \tag{2.1}$$

for all smooth functions φ, such that $\varphi(\mathbf{x},t)=0$ at the boundary S_T^2, and $\varphi(\mathbf{x},T)=0$, $\mathbf{x}\in Q$.

In (2.1) the convolution $\mathbb{A}:\mathbb{B}$ of two tensors $\mathbb{A}=(A_{ij})$ and $\mathbb{B}=(B_{ij})$ is defined as $\mathbb{A}:\mathbb{B} = \operatorname{tr}(\mathbb{A}\cdot\mathbb{B}) = \sum_{i,j=1}^3 A_{ij}B_{ji}$,

$$\mathbb{P}_f = \alpha_\mu\mathbb{D}(x,\frac{\partial \mathbf{w}^{\,\varepsilon}}{\partial t}) - p^{\,\varepsilon}\,\mathbb{I},$$

$$\mathbb{P} = \chi^\varepsilon\alpha_\mu\mathbb{D}\Big(x,\frac{\partial \mathbf{w}\chi^\varepsilon}{\partial t}\Big) + (1-\chi^\varepsilon)\alpha_\lambda\mathbb{D}(x,\mathbf{w}\chi^\varepsilon) - p^\varepsilon\chi^\varepsilon\,\mathbb{I},$$

$$\tilde{\varrho}^{\,\varepsilon} = \big(\zeta + (1-\zeta)\chi^\varepsilon\big)\varrho_f + (1-\zeta)(1-\chi^\varepsilon)\varrho_s$$

and $\zeta=\zeta(\mathbf{x})$ is the characteristic function of the domain Ω^0 in Q.

The equation (2.1) obviously contains equations (1.2) and (1.3), and boundary conditions (1.6) and (1.7).

The solution of the problem (1.1)–(1.9) possesses different smoothness in domains Ω_f^ε and Ω_s^ε. To preserve the best properties - which the solution possesses in the solid part - we extend the function $\mathbf{w}^\varepsilon$ from the solid part Ω_s^ε of the domain Ω onto the whole domain Ω. To do this we use the extension result (see [2, 8], and [7, Lemma B.4.2]): there exists an extension

$$\mathbf{w}_s^\varepsilon = \mathbb{E}_{\Omega_s^\varepsilon}\big(\mathbf{w}^\varepsilon\big), \quad \mathbb{E}_{\Omega_s^\varepsilon} : \mathbf{W}_2^1(\Omega_s^\varepsilon) \to \mathbf{W}_2^1(\Omega), \tag{2.2}$$

such that

$$\big(1-\chi^\varepsilon(\mathbf{x})\big)\big(\mathbf{w}^\varepsilon(\mathbf{x},t)-\mathbf{w}_s^\varepsilon(\mathbf{x},t)\big)=0, \quad \mathbf{x}\in\Omega,\ t\in(0,T), \tag{2.3}$$

and

$$\begin{gathered}\int_\Omega |\mathbf{w}_s^\varepsilon(\mathbf{x},t)|^2 dx \leqslant C_0 \int_{\Omega_s^\varepsilon} |\mathbf{w}^\varepsilon(\mathbf{x},t)|^2 dx, \\ \int_\Omega |\mathbb{D}\big(x,\mathbf{w}_s^\varepsilon(\mathbf{x},t)\big)|^2 dx \leqslant C_0 \int_{\Omega_s^\varepsilon} |\mathbb{D}\big(x,\mathbf{w}^\varepsilon(\mathbf{x},t)\big)|^2 dx, \quad t\in(0,T),\end{gathered} \tag{2.4}$$

where C_0 is independent of ε, and $t\in(0,T)$.

Theorem 2.2. *Let*

$$p^0 = p^0(t). \tag{2.5}$$

Then for all $\varepsilon > 0$ and for an arbitrary time interval $[0,T]$ there exists a unique generalized solution of problem (1.1)–(1.9) *and*

$$\begin{aligned}&\max_{0\le t\le T}\int_Q \Big(\alpha_\tau^2|\frac{\partial^2\mathbf{w}^\varepsilon}{\partial t^2}|^2 + \alpha_\tau|\frac{\partial\mathbf{w}^\varepsilon}{\partial t}|^2 + \alpha_\lambda(1-\zeta)(1-\chi^\varepsilon)|\mathbb{D}(x,\mathbf{w}^\varepsilon)|^2\Big)dx \\ &+ \int_{Q_T}\Big(|p^\varepsilon|^2 + \alpha_\mu\big(\zeta+(1-\zeta)\chi^\varepsilon\big)|\mathbb{D}(x,\frac{\partial\mathbf{w}^\varepsilon}{\partial t})|^2\Big)\,dx\,dt \leqslant C_0,\end{aligned} \tag{2.6}$$

here and in what follows, we denote as C_0 any constant independent of τ_0 and ε.

Theorem 2.3. *Under the conditions of Theorem 2.2 let*

$$\mu_0 = 0, \quad 0<\mu_1, \quad \tau_0<\infty, \quad \lambda_0=\infty$$

$\{\mathbf{w}^\varepsilon, p^\varepsilon\}$ be the weak solution of the problem (1.1)–(1.9) *and $\mathbf{w}_s^\varepsilon = \mathbb{E}_{\Omega_s^\varepsilon}\big(\mathbf{w}^\varepsilon\big)$ be an extension* (2.4) *from the domain Ω_s^ε onto the domain Ω.*

Then the sequences $\{\zeta p^\varepsilon\}$, $\{\zeta\mathbf{w}^\varepsilon\}$, $\{\zeta\frac{\partial\mathbf{w}^\varepsilon}{\partial t}\}$, $\{\zeta\frac{\partial^2\mathbf{w}^\varepsilon}{\partial t^2}\}$, $\{(1-\zeta)\chi^\varepsilon\,p^\varepsilon\}$, $\{(1-\zeta)\chi^\varepsilon\,\mathbf{w}^\varepsilon\}$, $\{(1-\zeta)\chi^\varepsilon\,\frac{\partial\mathbf{w}^\varepsilon}{\partial t}\}$ and $\{(1-\zeta)\chi^\varepsilon\,\frac{\partial^2\mathbf{w}^\varepsilon}{\partial t^2}\}$ converge weakly in $L_2(Q_T)$ and $\mathbf{L}_2(Q_T)$ to the functions $p\in W_2^{1,0}(\Omega_T^0)$, $\mathbf{w}$, $\zeta\mathbf{v}$, $\zeta\frac{\partial\mathbf{v}}{\partial t}$, $(1-\zeta)m\,p^{(f)}\in W_2^{1,0}(\Omega_T)$, $(1-\zeta)\mathbf{w}^{(f)}$, $(1-\zeta)\mathbf{v}^{(f)}$ and $(1-\zeta)\frac{\partial\mathbf{v}^{(f)}}{\partial t}$ respectively as $\varepsilon\to0$, and the sequence $\{\mathbf{w}_s^\varepsilon\}$ converges strongly in $\mathbf{W}_2^{1,0}(\Omega_T)$ to zero as $\varepsilon\to0$.

The limiting pressure p and the limiting velocity $\mathbf{v}$ of the liquid in the domain Ω_0 satisfy in Ω_0 for $t>0$ the system

$$\tau_0\varrho_f\frac{\partial\mathbf{v}}{\partial t} + \nabla p = \varrho_f\,\mathbf{e}, \quad \nabla\cdot\mathbf{v}=0 \tag{2.7}$$

In the domain Ω for $t>0$ limiting functions solve the homogenized system, consisting of the continuity equation

$$\nabla\cdot\,\mathbf{v}^{(f)}=0, \tag{2.8}$$

and the homogenized momentum balance equation

$$\mathbf{v}^{(f)} = \int_0^t \mathbb{B}^{(f)}(\tau_0;t-\tau)\cdot\big(-\nabla p^{(f)}\,(\mathbf{x},\tau)+\varrho_f\mathbf{e}\big)d\tau. \tag{2.9}$$

The problem is complemented with the continuity conditions

$$\lim_{\mathbf{x}\to\mathbf{x}^0\in S^0,\quad \mathbf{x}\in\Omega} p^{(f)}(\mathbf{x},t) = \lim_{\mathbf{x}\to\mathbf{x}^0\in S^0,\quad \mathbf{x}\in\Omega_0} p(\mathbf{x},t), \tag{2.10}$$

$$\lim_{\mathbf{x}\to\mathbf{x}^0\in S^0,\quad \mathbf{x}\in\Omega} \mathbf{v}^{(f)}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}^0) = \lim_{\mathbf{x}\to\mathbf{x}^0\in S^0,\quad \mathbf{x}\in\Omega_0} \mathbf{v}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}^0) \tag{2.11}$$

on the common boundary S^0*, the boundary condition*

$$p(\mathbf{x},t) = p^0(t) \tag{2.12}$$

on the part $S^1_0 = S^1 \cap \overline{\Omega}_0$ *of the outer boundary* S*, the boundary condition*

$$\mathbf{v}^{(f)}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}) = 0 \tag{2.13}$$

on the part S^2 *of the outer boundary* S*, the boundary condition*

$$p^{(f)}(\mathbf{x},t) = p^0(t) \tag{2.14}$$

on the part S^1_1 *of the outer boundary* S*, and homogeneous initial conditions*

$$\mathbf{v}(\mathbf{x},0) = 0, \quad \mathbf{x}\in\Omega_0. \tag{2.15}$$

In (2.7)–(2.15)*,* $\mathbf{n}(\mathbf{x})$ *is a unit normal to* S^0 *(or* S^2*) at* $\mathbf{x}\in S^0$ *(or* S^2*),*

$$\hat{\varrho} = m\,\varrho_f + (1-m)\,\varrho_s, \quad m = \int_Y \chi(\mathbf{y})dy,$$

and the symmetric matrix $\mathbb{B}^{(f)}(\tau_0;t)$ *is given by* (4.14) *below.*

Finally, for the solution $\mathbf{v}$*,* p*,* $\mathbf{v}^{(f)}$*, and* $p^{(f)}$ *of the problem* (2.7)–(2.15) *satisfy the estimate*

$$\begin{aligned} &\int_{\Omega^0_T} \big(\tau_0^2|\frac{\partial\mathbf{v}}{\partial t}|^2 + \tau_0|\mathbf{v}|^2 + |\nabla p|^2\big)\,dx\,dt \\ &+ \int_{\Omega_T} \big(\tau_0^2|\frac{\partial\mathbf{v}^{(f)}}{\partial t}|^2 + |\mathbf{v}^{(f)}|^2 + |\nabla p^{(f)}|^2\big)\,dx\,dt \leqslant C_0. \end{aligned} \tag{2.16}$$

Theorem 2.4. *Under the conditions of Theorem 2.3, let* $\{\mathbf{v}^{(f,k)}, p^{(f,k)}, p^k\}$ *be a solution of* (2.7)–(2.15) *with* $\tau_0 = \frac{1}{k}$*.*

Then the sequence $\{p^{(f,k)}\}$ *converges weakly in* $W^{1,0}_2(\Omega_T)$ *to the function* $p^{(f)}$*, the sequence* $\{\mathbf{v}^{(f,k)}\}$ *converges weakly in* $L_2(\Omega_T)$ *to the function* $\mathbf{v}^{(f)}$*, and the sequence* $\{p^k\}$ *converges strongly in* $W^{1,0}_2(\Omega^0_T)$ *to the function* $p_0(\mathbf{x},t) = p^0(t) - \varrho_f x_3$ *as* $k\to\infty$*.*

In the domain Ω *for* $t>0$ *limiting functions solve the homogenized system, consisting of the continuity equation*

$$\nabla\cdot\mathbf{v}^{(f)} = 0 \tag{2.17}$$

and Darcy's law

$$\mathbf{v}^{(f)} = \frac{1}{\mu_1}\mathbb{B}\cdot\big(-\nabla p^{(f)} + \varrho_f\mathbf{e}\big), \tag{2.18}$$

for the liquid component, completed with the boundary conditions (2.13)*,* (2.14)*, and the boundary condition*

$$p^{(f)} = p_0(\mathbf{x},t) \tag{2.19}$$

on the common boundary S^0 *for* $t>0$*.*

The symmetric constant matrix $\mathbb{B}$ *is given by* (5.3) *below.*

3. Proof of Theorem 2.2

The proof of this theorem is straightforward and is based on the energy equalities

$$
\begin{aligned}
&\frac{1}{2}\int_Q \Big(\alpha_\tau \tilde{\varrho}^{\,\varepsilon} |\frac{\partial \mathbf{w}^\varepsilon}{\partial t}(\mathbf{x},t)|^2 + \alpha_\lambda (1-\zeta)(1-\chi^\varepsilon)|\mathbb{D}\big(x,\mathbf{w}^\varepsilon(\mathbf{x},t)\big)|^2\Big)dx \\
&+ \alpha_\mu \int_0^t \int_Q \big(\zeta + (1-\zeta)\chi^\varepsilon\big)\mathbb{D}\big(x, \frac{\partial \mathbf{w}^\varepsilon}{\partial \tau}(\mathbf{x},\tau)\big)|^2 dx d\tau \\
&= \int_0^t \int_Q \tilde{\varrho}^{\,\varepsilon} \mathbf{e} \cdot \frac{\partial \mathbf{w}^\varepsilon}{\partial t}(\mathbf{x},\tau) dx d\tau,
\end{aligned} \tag{3.1}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\int_Q \Big(\alpha_\tau \tilde{\varrho}^{\,\varepsilon} |\frac{\partial^2 \mathbf{w}^\varepsilon}{\partial t^2}(\mathbf{x},t)|^2 + \alpha_\lambda (1-\zeta)(1-\chi^\varepsilon)|\mathbb{D}\big(x,\frac{\partial \mathbf{w}^\varepsilon}{\partial t}(\mathbf{x},t)\big)|^2\Big)dx \\
&+ \alpha_\mu \int_0^t \int_Q \big(\zeta + (1-\zeta)\chi^\varepsilon\big)|\mathbb{D}\big(x, \frac{\partial^2 \mathbf{w}^\varepsilon}{\partial \tau^2}(\mathbf{x},\tau)\big)|^2 dx d\tau \\
&= \frac{1}{2}\int_Q \alpha_\tau \tilde{\varrho}^{\,\varepsilon} |\frac{\partial^2 \mathbf{w}^\varepsilon}{\partial t^2}|^2(\mathbf{x},0) dx = I_0.
\end{aligned} \tag{3.2}
$$

We may use, for example, Galerkin's method. This method shows that for any $t \geqslant 0$ and any divergent free function $\varphi \in W_2^1(Q)$, vanishing at $\mathbf{x} \in S^2$, the equality

$$
\begin{aligned}
&\int_Q \alpha_\tau \tilde{\varrho}^{\,\varepsilon} \frac{\partial^2 \mathbf{w}^\varepsilon}{\partial t^2}(\mathbf{x},t)\cdot \varphi(\mathbf{x}) dx + \int_Q \big(\zeta \mathbb{P}_f + (1-\zeta)\mathbb{P}\big)(\mathbf{x},t) : \mathbb{D}\big(x, \varphi(\mathbf{x})\big) dx \\
&= \int_Q \tilde{\varrho}^{\,\varepsilon} \mathbf{e} \cdot \varphi(\mathbf{x}) dx
\end{aligned}
$$

holds. For $t = 0$ $\mathbb{P}_f + (1-\zeta)\mathbb{P} = 0$, we have

$$
\int_Q \alpha_\tau \tilde{\varrho}^{\,\varepsilon} \frac{\partial^2 \mathbf{w}^\varepsilon}{\partial t^2}(\mathbf{x},0)\cdot \varphi(\mathbf{x}) dx = \int_Q \tilde{\varrho}^{\,\varepsilon} \mathbf{e} \cdot \varphi(\mathbf{x}) dx.
$$

In particular, Galerkin's method states that $\frac{\partial^2 \mathbf{w}^\varepsilon}{\partial t^2}(\mathbf{x},0)$ is a divergent free function in Q. Therefore, for

$$
\varphi(\mathbf{x}) = \frac{\partial^2 \mathbf{w}^\varepsilon}{\partial t^2}(\mathbf{x},0),
$$

$$
\int_Q \alpha_\tau \tilde{\varrho}^{\,\varepsilon} |\frac{\partial^2 \mathbf{w}^\varepsilon}{\partial t^2}(\mathbf{x},0)|^2 dx = \int_Q \tilde{\varrho}^{\,\varepsilon} \mathbf{e} \cdot \frac{\partial^2 \mathbf{w}^\varepsilon}{\partial t^2}(\mathbf{x},0) dx,
$$

which implies

$$
\int_Q \alpha_\tau \tilde{\varrho}^{\,\varepsilon} |\frac{\partial^2 \mathbf{w}^\varepsilon}{\partial t^2}(\mathbf{x},0)|^2 dx \leqslant \frac{C_0}{\alpha_\tau}.
$$

The above relation and (3.2) provide an estimate of the time derivative $\frac{\partial^2 \mathbf{w}^\varepsilon}{\partial t^2}$ in (2.6).

To estimate the right-hand side of (3.1) we use representations

$$
\tilde{\varrho}^{\,\varepsilon} = \varrho_f + (1-\zeta)(1-\chi^\varepsilon)(\varrho_s - \varrho_f), \quad \mathbf{e} = -\nabla x_3,
$$

integration by parts and the continuity equation (1.1),

$$
\varrho_f \int_Q \mathbf{e} \cdot \mathbf{w}^\varepsilon dx = -\varrho_f \int_Q (\nabla x_3)\cdot \mathbf{w}^\varepsilon dx = 0.
$$

So,

$$
\begin{aligned}
I &= \int_0^t \int_Q \tilde{\varrho}^{\,\varepsilon} \mathbf{e} \cdot \frac{\partial \mathbf{w}^\varepsilon}{\partial t} dx d\tau \\
&= -\varrho_f \int_Q (\nabla x_3) \cdot \mathbf{w}^\varepsilon dx + (\varrho_s - \varrho_f) \int_\Omega (1-\chi^\varepsilon) \mathbf{e} \cdot \mathbf{w}^\varepsilon dx \\
&= (\varrho_s - \varrho_f) \int_\Omega (1-\chi^\varepsilon) \mathbf{e} \cdot \mathbf{w}^\varepsilon dx.
\end{aligned}
$$

Next we apply the Hölder inequality,

$$
\begin{aligned}
I &\leqslant (\varrho_s - \varrho_f) \Big(\int_\Omega dx \Big)^{1/2} \Big(\int_\Omega (1-\chi^\varepsilon) |\mathbf{w}^\varepsilon|^2 dx \Big)^{1/2} \\
&\leqslant \frac{(\varrho_s - \varrho_f)^2}{2\delta} |\Omega| + \frac{\delta}{2} \int_\Omega (1-\chi^\varepsilon) |\mathbf{w}^\varepsilon|^2 dx
\end{aligned}
$$

and the extension $\mathbf{w}_s^\varepsilon = \mathbb{E}_{\Omega_s^\varepsilon}\big(\mathbf{w}^\varepsilon\big)$ (see [7, Appendix B, Lemma B.4.2]) from the domain Ω_s^ε onto the domain Q with Friedrichs-Poincaré's inequality:

$$
I_1 = \int_\Omega (1-\chi^\varepsilon) |\mathbf{w}^\varepsilon|^2 dx = \int_\Omega (1-\chi^\varepsilon) |\mathbf{w}_s^\varepsilon|^2 dx \leqslant C \int_\Omega (1-\chi^\varepsilon) |\nabla \mathbf{w}_s^\varepsilon|^2 dx,
$$

and Korn's inequality

$$
\begin{aligned}
\int_\Omega (1-\chi^\varepsilon) |\nabla \mathbf{w}_s^\varepsilon|^2 dx &\leqslant C \int_\Omega (1-\chi^\varepsilon) |\mathbb{D}(x, \mathbf{w}_s^\varepsilon)|^2 dx \\
&= C \int_Q (1-\chi^\varepsilon)(1-\zeta) |\mathbb{D}(x, \mathbf{w}^\varepsilon)|^2 dx.
\end{aligned}
$$

Finally one has

$$
I \leqslant \frac{(\varrho_s - \varrho_f)^2}{2\delta} |\Omega| + C \frac{\delta}{2} \int_Q (1-\chi^\varepsilon)(1-\zeta) |\mathbb{D}(x, \mathbf{w}^\varepsilon)|^2 dx,
$$

which together with (3.2) prove (2.6) for terms, containing in (3.1) and (3.2).

The pressure p is estimated as a linear functional ([7, Chapter 3, Theorem 3.1]).

4. Proof of Theorem 2.3

Here we use the method of two-scale convergence, suggested by Nguetseng [9]. This method states that any bounded in $L_2(Q_T)$ sequence $\{u_n\}$ contains two-scale convergent subsequence $\{u_{n_k}\}$, such that

$$
\int_{Q_T} u_{n_k}(\mathbf{x}, t) \varphi(\mathbf{x}, t, \frac{\mathbf{x}}{\varepsilon})\, dx\, dt \to \int_{Q_T} \Big(\int_Y U(\mathbf{x}, t, \mathbf{y}) \varphi(\mathbf{x}, t, \mathbf{y}) dy \Big)\, dx\, dt
$$

as $\varepsilon \to 0$ for any smooth 1–periodic in $\mathbf{y}$ function $\varphi(\mathbf{x}, t, \mathbf{y})$. The 1-periodic in $\mathbf{y}$ function $U(\mathbf{x}, t, \mathbf{y}) \in L_2(Q_T \times Y)$ is called a two-scale limit of the sequence $\{u_{n_k}\}$.

On the basis of estimates (2.6) and Nguetseng's theorem we conclude that as $\varepsilon \to 0$,

$$
\begin{gathered}
p^\varepsilon \to p(\mathbf{x}, t) \quad \text{weakly and two-scale in } L_2(\Omega_T^0), \\
p^\varepsilon \chi^\varepsilon \to p^{(f)}(\mathbf{x}, t) \chi(\mathbf{y}) \quad \text{two-scale in } L_2(\Omega_T), \\
p^\varepsilon \chi^\varepsilon \rightharpoonup m p^{(f)}(\mathbf{x}, t) \quad \text{weakly in } L_2(\Omega_T),\ m = \int_Y \chi(\mathbf{y}) dy,
\end{gathered}
$$

$$\begin{gathered}
\frac{\partial \mathbf{w}^\varepsilon}{\partial t} \rightharpoonup \mathbf{v} \quad \text{weakly in } \mathbf{L}_2(Q_T),\\
\frac{\partial^2 \mathbf{w}^\varepsilon}{\partial t^2} \rightharpoonup \frac{\partial \mathbf{v}}{\partial t}(\mathbf{x},t) \quad \text{weakly in } \mathbf{L}_2(Q_T),\\
\frac{\partial \mathbf{w}^\varepsilon}{\partial t} \to \mathbf{V}(\mathbf{x},t,\mathbf{y}) \quad \text{two-scale in } \mathbf{L}_2(Q_T),\\
\frac{\partial^2 \mathbf{w}^\varepsilon}{\partial t^2} \to \frac{\partial \mathbf{V}}{\partial t}(\mathbf{x},t,\mathbf{y}) \quad \text{two-scale in } \mathbf{L}_2(Q_T),\\
\varepsilon \mathbb{D}\big(x, \frac{\partial \mathbf{w}^\varepsilon}{\partial t}\big) \to \mathbb{D}\big(y, \mathbf{V}(\mathbf{x},t,\mathbf{y})\big) \quad \text{two-scale in } \mathbf{L}_2(Q_T),\\
\mathbf{w}_s^\varepsilon \to 0 \quad \text{strongly in } \mathbf{W}_2^{1,0}(\Omega_T).
\end{gathered}$$

Moreover,

$$\int_{Q_T}\int_Y \big(\tau_0|\mathbf{V}|^2 + \tau_0^2|\frac{\partial \mathbf{V}}{\partial t}|^2 + |\mathbb{D}\big(y,\mathbf{V}\big)|^2\big)dy\,dx\,dt \leqslant C_0, \tag{4.1}$$

where C_0 is independent of τ_0.

A general theory says that the two-scale limit usually depends on all variables $\mathbf{x}, t$, and $\mathbf{y}$. A detailed analysis for the case $\mu_0 = 0$ ([7, Chapter 1], [10]) shows that the two-scale limit of pressures does not depend on the variable $\mathbf{y}$ in Ω_T^0 and has a special form in Ω_T.

The similar analysis shows that

$$\zeta\,\mathbf{V} = \zeta\,\mathbf{v}(\mathbf{x},t). \tag{4.2}$$

In fact, the two-scale limit in (2.1) as $\varepsilon \to 0$ with test functions $\varphi = \varphi_0(\frac{\mathbf{x}}{\varepsilon})\psi(\mathbf{x},t)$, where $\varphi_0(\mathbf{y})$ is 1-periodic in $\mathbf{y}$ function with $\operatorname{supp}\varphi_0 \subset Y_f$, such that $\nabla_y \cdot \varphi_0 = 0$ for $\mathbf{y} \in Y$ and ψ is a smooth function with $\operatorname{supp}\psi \subset \Omega_T^0$ results in

$$\int_{\Omega_T^0} \big((A - \varrho_f\,\mathbf{e}\cdot\mathbf{a})\,\psi - p\,\mathbf{a}\cdot\nabla\psi\big)\,dx\,dt = 0,$$

where

$$A = \int_Y \tau_0\varrho_f \frac{\partial \mathbf{V}}{\partial t}\cdot\varphi_0 dy = \langle \frac{\partial \mathbf{V}}{\partial t}\cdot\varphi_0\rangle.$$

Following [7, Appendix B, Lemma B.5.3] for any unit vector $\mathbf{a}$ there exists a smooth function $\varphi_0(\mathbf{y})$ with $\operatorname{supp}\varphi_0 \subset Y_f$, such that $\nabla_y \cdot \varphi_0 = 0$ for $\mathbf{y} \in Y$ and $\langle\varphi_0\rangle = \mathbf{a}$.

Due to arbitrary choice of the test function ψ and vector $\mathbf{a}$ the last identity means the existence of $\nabla p \in L_2(\Omega_T^0)$ and it may be rewritten as

$$\int_{\Omega_T^0} \big(A - (\varrho_f\,\mathbf{e} - \nabla p)\cdot\mathbf{a}\big)\,\psi\,dx\,dt = 0.$$

After reintegration with respect to variables $(\mathbf{x},t)$ we arrive to the integral identity

$$\int_Y (\tau_0\varrho_f \frac{\partial \mathbf{V}}{\partial t} - \varrho_f\,\mathbf{e} + \nabla p)\cdot\varphi_0 dy = 0,$$

which is equivalent to the differential equation

$$\tau_0\varrho_f \frac{\partial \mathbf{V}}{\partial t}(\mathbf{x},t,\mathbf{y}) = \varrho_f\,\mathbf{e} - \nabla p(\mathbf{x},t).$$

It proves our statement and, at the same time, the first equation in (2.7).

Using this fact and the strong convergence of the sequence $\{\frac{\partial \mathbf{w}_s^\varepsilon}{\partial t}\}$ to zero we obtain that

$$\begin{aligned}\frac{\partial \mathbf{w}^\varepsilon}{\partial t} &= \zeta\frac{\partial \mathbf{w}^\varepsilon}{\partial t} + (1-\zeta)\chi^\varepsilon\frac{\partial \mathbf{w}^\varepsilon}{\partial t} + (1-\zeta)(1-\chi^\varepsilon)\frac{\partial \mathbf{w}^\varepsilon}{\partial t}\\ &\to \zeta\mathbf{v}(\mathbf{x},t) + (1-\zeta)\chi(\mathbf{y})\mathbf{V}(\mathbf{x},t,\mathbf{y})\end{aligned}$$

two-scale in $L_2(Q_T)$ as $\varepsilon \to 0$, or

$$\mathbf{V}(\mathbf{x},t,\mathbf{y}) = \zeta\mathbf{v}(\mathbf{x},t) + (1-\zeta)\chi(\mathbf{y})\mathbf{V}(\mathbf{x},t,\mathbf{y}). \tag{4.3}$$

Thus, $\mathbf{V} = 0$ in Y_s and we may apply Friedrichs-Poincaré's inequality, which together with (4.1) imply

$$\int_{Q_T}\int_Y |\mathbf{V}|^2 dy\,dx\,dt \leqslant C_0\int_{Q_T}\int_Y |\mathbb{D}\big(y,\mathbf{V}\big)|^2\big)\,dx\,dt \leqslant C_0. \tag{4.4}$$

The two-scale limit in (2.1) as $\varepsilon \to 0$ with test functions $\varphi = \varphi_0(\frac{\mathbf{x}}{\varepsilon})\psi(\mathbf{x},t)$, where $\varphi_0(\mathbf{y})$ is 1-periodic in $\mathbf{y}$ function, such that $\nabla_y\cdot\varphi_0 = 0$ for $\mathbf{y}\in Y$, and $\operatorname{supp}\varphi_0 \subset Y_f$ and ψ is a smooth function, vanishing at $t = T$ and at S^2, results in

$$\begin{aligned}&\int_{Q_T}\mathbf{a}\cdot\nabla(\,p^0\,\psi)\,dx\,dt + \int_{\Omega_T^0}\big((\tau_0\varrho_f\frac{\partial \mathbf{v}}{\partial t} - \varrho_f\,\mathbf{e})\cdot\mathbf{a}\,\psi - p\,\mathbf{a}\cdot\nabla\psi)\big)\,dx\,dt\\ &+\int_{\Omega_T}\Big(\tau_0\varrho_f\frac{\partial \mathbf{v}^{(f)}}{\partial t} - \hat{\varrho}\,\mathbf{e})\cdot\mathbf{a}\,\psi + B\,\psi - p^{(f)}\,\mathbf{a}\cdot\nabla\psi\Big)\,dx\,dt = 0.\end{aligned} \tag{4.5}$$

Here

$$\mathbf{a} = \langle\varphi_0\rangle,\quad \mathbf{v}^{(f)} = \langle\mathbf{V}\chi\rangle,\quad B = \mu_1\langle\mathbb{D}\big(y,\mathbf{V}\big):\mathbb{D}\big(y,\varphi_0\big)\rangle.$$

As before, we conclude that $\nabla p \in L_2(\Omega_T^0)$, $\nabla p^{(f)} \in L_2(\Omega_T)$:

$$\int_{\Omega_T^0}|\nabla p(\mathbf{x},t)|^2\,dx\,dt + \int_{\Omega_T}|\nabla p^{(f)}(\mathbf{x},t)|^2\,dx\,dt \leqslant C_0, \tag{4.6}$$

hold true the continuity condition (2.10) on the common boundary S^0, boundary conditions (2.12) and (2.14) on the outer boundary S, and initial condition (2.15).

For the function ψ with a compact support in Ω_T (4.5) implies the integral identity

$$\int_{Y_f}\big((\tau_0\varrho_f\frac{\partial \mathbf{V}}{\partial t} + \nabla p^{(f)} - \hat{\varrho}\,\mathbf{e})\cdot\varphi_0 + \mu_1\mathbb{D}\big(y,\mathbf{V}\big):\mathbb{D}\big(y,\varphi_0\big)\big)dy = 0. \tag{4.7}$$

The reintegration of this identity results in the differential equation

$$\tau_0\varrho_f\frac{\partial \mathbf{V}}{\partial t} - \mu_1\nabla_y\cdot\mathbb{D}\big(y,\mathbf{V}\big) = -\nabla_y\,\Pi - \nabla p^{(f)} + \hat{\varrho}\,\mathbf{e} \tag{4.8}$$

in the domain Y_f for $t > 0$ and for almost all $\mathbf{x}\in\Omega$.

The term $\nabla_y\,\Pi$ has appeared virtue of the condition $\nabla_y\cdot\varphi_0 = 0$ on an arbitrary function φ_0.

To derive the limiting continuity equations we rewrite (1.1) in its equivalent form as an integral identity

$$\int_{Q_T}\nabla\xi\,\cdot\,\Big(\zeta\,\frac{\partial \mathbf{w}^\varepsilon}{\partial t} + (1-\zeta)\big(\chi^\varepsilon\frac{\partial \mathbf{w}^\varepsilon}{\partial t} + (1-\chi^\varepsilon)\,\frac{\partial \mathbf{w}_s^\varepsilon}{\partial t})\Big)\,dx\,dt = 0, \tag{4.9}$$

which holds true for any smooth function ξ vanishing at the part S^1 of the boundary S.

The limit in (4.9) as $\varepsilon \to 0$ results in the integral identity

$$\int_{Q_T} \nabla \xi \cdot \big(\zeta\, \mathbf{v} + (1-\zeta)\mathbf{v}^{(f)}\big)\, dx\, dt = 0, \tag{4.10}$$

which is equivalent to the continuity equation in (2.7), continuity equation (2.8), continuity condition (2.11) on the common boundary S^0, and boundary condition (2.13).

Finally, the limit in (4.9) as $\varepsilon \to 0$ with test function ξ in the form $\xi = \varepsilon\xi_0(\mathbf{x}, t)\xi_1(\frac{\mathbf{x}}{\varepsilon})$, where $\xi_1(\mathbf{y})$ is 1-periodic smooth function and $\operatorname{supp}\xi_0 \subset \Omega_T$, leads to the integral identity

$$\int_{Q_T} \xi_0(\mathbf{x}, t) \big(\int_{Y_f} \nabla_y \xi_1(\mathbf{y}) \cdot \mathbf{V}(\mathbf{x}, t, \mathbf{y}) dy \big)\, dx\, dt = 0,$$

and, consequently, to the differential equation

$$\nabla_y \cdot \mathbf{V} = 0 \tag{4.11}$$

in the domain Y_f.

The representation (4.3) and the smoothness of $\mathbf{V}$ evidently imply the boundary condition

$$\mathbf{V}(\mathbf{x}, t, \mathbf{y}) = 0, \quad \mathbf{y} \in \gamma = \partial Y_f \cap Y_s. \tag{4.12}$$

To find correctly $\mathbf{V}$ we complete differential equations (4.8) and (4.11) and boundary condition (4.12) with initial condition

$$\mathbf{V}(\mathbf{x}, 0, \mathbf{y}) = 0, \;\; \mathbf{y} \in Y_f. \tag{4.13}$$

Problem (4.8), (4.11)–(4.13) for $\tau_0 = 1$ has been solved in [7, Chapter 3, Theorem 3.5]. Therefore, we simply formulate the result.

Lemma 4.1. *For almost all $\mathbf{x} \in \Omega$ the function $\mathbf{v}^{(f)} = \langle \chi\, \mathbf{V} \rangle$ satisfies equation* (2.9)*, where*

$$\mathbb{B}^{(f)}(\tau_0; t) = \sum_{i=1}^{3} \big(\int_{Y_f} \mathbf{V}_i^{(f)}(\mathbf{y}, t) dy \big) \otimes \mathbf{e}_i, \tag{4.14}$$

and $\mathbf{V}_i^{(f)}$, $i = 1, 2, 3$, are solutions to the periodic initial boundary value problem

$$\tau_0 \varrho_f \frac{\partial \mathbf{V}_i^{(f)}}{\partial t} = \mu_1 \nabla_y \cdot \mathbb{D}\big(y, \mathbf{V}_i^{(f)}\big) - \nabla_y \Pi_i^{(f)}, \quad (\mathbf{y}, t) \in Y_f \times (0, T), \tag{4.15}$$

$$\nabla_y \cdot \mathbf{V}_i^{(f)} = 0, \;\; (\mathbf{y}, t) \in Y_f, \quad t > 0, \tag{4.16}$$

$$\tau_0 \varrho_f \mathbf{V}_i^{(f)}(\mathbf{y}, 0) = \mathbf{e}_i, \quad \mathbf{y} \in Y_f, \tag{4.17}$$

$$\mathbf{V}_i^{(f)}(\mathbf{y}, t) = 0, \;\; \mathbf{y} \in \gamma, \;\; t > 0. \tag{4.18}$$

In (4.14) $\mathbf{a} \otimes \mathbf{b}$ is a second order tensor, such that

$$(\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$$

for any vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. The estimate (2.16) follows from (4.1), (4.4), and (4.6).

5. Proof of Theorem 2.4

Estimates (2.16), (4.1), (4.4), and (4.6) imply the strong convergence of the sequence $\{p^k\}$ in $W_2^{1,0}(\Omega_T^0)$, the weak compactness of the sequence $\{p^{(f,k)}\}$ in $W_2^{1,0}(\Omega_T)$, the weak compactness of the sequence $\{\mathbf{v}^{(f,k)}\}$ in $L_2(\Omega_T)$, and the weak compactness of the sequences $\{\mathbf{V}^k\}$ and $\{\nabla_y \mathbf{V}^k\}$ in $L_2(\Omega_T \times Y_f)$.

Let p, $p^{(f)}$, and $\mathbf{v}^{(f)}$ be the limits of above mentioned sequences. First of all note that in virtue of the continuity condition (2.10) functions $\tilde{p}^k$, where $\tilde{p}^k = p^k$ in Ω_T^0 and $\tilde{p}^k = p^{f,k}$ in Ω_T, belongs to $W_2^{1,0}(Q_T)$ and uniformly bounded there with respect to k. Therefore, the weak limit $\tilde{p}$ of the sequence $\{\tilde{p}^k\}$ in $W_2^{1,0}(Q_T)$ coincides with p in Ω_T^0 and with $p^{(f)}$ in Ω_T. It means that these functions p and $p^{(f)}$ still satisfy the continuity condition (2.10) on the common boundary S^0 and boundary conditions (2.12) and (2.14) on the outer boundary S.

Now we note that due to strong convergence of $\{\frac{1}{k}\mathbf{v}^k\}$ to zero in $L_2(\Omega_T^0)$ the function p satisfies in Ω_T^0 hydraulic's equation

$$\nabla p = \varrho_f \, \mathbf{e},$$

which together with the boundary condition (2.12) result in the equality

$$p = p_0(\mathbf{x}, t) = p^0(t) - \varrho_f x_3,$$

and the boundary condition (2.19).

We do not know, how to pass to the limit as $k \to \infty$ in the equation (2.9) to get Darcy's law, but we may do it using the integral identity (4.7), if we rewrite it as

$$\int_0^T \int_{Y_f} \Big(\big(\frac{1}{k} \varrho_f \mathbf{V}^k \frac{\partial \psi}{\partial t} + \psi (\nabla p^{(f,k)} - \hat{\varrho}\, \mathbf{e}) \big) \cdot \varphi_0 + \psi \mu_1 \mathbb{D}\big(y, \mathbf{V}^k\big) : \mathbb{D}\big(y, \varphi_0\big) \Big) dy dt = 0 \tag{5.1}$$

with test function $\psi(t)$ vanishing at $t = 0$ and $T = 0$.

The limit as $k \to \infty$ results in integral identities

$$\int_0^T \psi \Big(\int_{Y_f} \big((\nabla p^{(f)} - \hat{\varrho}\, \mathbf{e}) \cdot \varphi_0 + \psi \mu_1 \mathbb{D}\big(y, \mathbf{V}\big) : \mathbb{D}\big(y, \varphi_0\big) \big) dy \Big) dt = 0,$$

$$\int_{Y_f} \big((\nabla p^{(f)} - \hat{\varrho}\, \mathbf{e}) \cdot \varphi_0 + \psi \mu_1 \mathbb{D}\big(y, \mathbf{V}\big) : \mathbb{D}\big(y, \varphi_0\big) \big) dy = 0,$$

and the differential equation

$$\mu_1 \nabla_y \cdot \mathbb{D}\big(y, \mathbf{V}\big) = -\nabla_y \Pi - \nabla p^{(f)} + \hat{\varrho}\, \mathbf{e}. \tag{5.2}$$

It is obvious that the continuity equation (4.11) and boundary condition (4.12) for functions $\mathbf{V}^k$ will remain valid for the limit function $\mathbf{V}$.

Problem (5.2), (4.11), and (4.12) is well-known (see [10], [7]) and its solution $\mathbf{v}^{(f)} = \langle \chi \, \mathbf{V} \rangle$ is given by (2.18), where

$$\mathbb{B} = \sum_{i=1}^{3} \big(\int_{Y_f} \boldsymbol{V}^{(i)}(\mathbf{y}) dy \big) \otimes \boldsymbol{e}_i = \sum_{i=1}^{3} \langle \chi \boldsymbol{V}^{(i)} \rangle \otimes \boldsymbol{e}_i, \tag{5.3}$$

and 1-periodic functions $\boldsymbol{V}^{(i)}(\mathbf{y})$, $i = 1, 2, 3$, solve the periodic boundary value problem

$$\mu_1 \nabla_y \cdot \mathbb{D}\big(y, \boldsymbol{V}^{(i)}\big) - \nabla \Pi^{(i)} = -\boldsymbol{e}_i, \tag{5.4}$$

$$\nabla \cdot \boldsymbol{V}^{(i)} = 0, \quad \mathbf{y} \in Y_f, \tag{5.5}$$

$$\boldsymbol{V}^{(i)} = 0, \quad \mathbf{y} \in \gamma. \tag{5.6}$$

Acknowledgments. This research is partially supported by Grants N0 0691/GF and N0 0751/GF from the Ministry of Education and Science of Kazakhstan.

References

[1] R. Burridge, J. B. Keller; *Poroelasticity equations derived from microstructure*, Journal of Acoustic Society of America 70, No. 4 (1981) 1140-1146.

[2] C. Conca; *On the application of the homogenization theory to a class of problems arising in fluid mechanics*, Math. Pures et Appl., 64 (1985) 31-75.

[3] W. Jäger, A. Mikelić; *On the flow conditions at the boundary between a porous medium and an impervious solid.* In Progress in PDE: the Metz surveys 3. Pitman reseach Notes in Mathematics, N 314, Longman Scintific and Technical, London (1994).

[4] W. Jäger, W., A. Mikelić; *On the boundary conditions at the contact interface between a porous medium and a free fluid.* Ann. Sc. norm. Super. Pisa, Cl, Sci.-Ser. IV. **XXIII**, Fasc. 3, 403–465 (1996).

[5] W. Jäger, W., A. Mikelić; *On the boundary conditions at the contact interface between two porous media.* In PDE, Theory and numerical solution. Chapman and Hall/CRC Research notes in math., N 406, CRC Press, London (1999).

[6] O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Uraltseva; *Linear and Quasilinear Equations of Parabolic Type*, Providence, Rhode Island, 1968.

[7] A. Meirmanov; *Mathematical models for poroelastic flows*, Atlantis Press, Paris, 2013.

[8] G. Nguetseng; *Asymptotic analysis for a stiff variational problem arising in mechanics*, SIAM J. Math. Anal., 21 (1990) 1394-1414.

[9] G. Nguetseng; *A general convergence result for a functional related to the theory of homogenization*, SIAM J. Math. Anal., 20 (1989) 608-623.

[10] E. Sanchez-Palencia; *Non-Homogeneous Media and Vibration Theory*, Lecture Notes in Physics, Vol. 129, Springer-Verlag, New York, 1980.

Anvarbek Meirmanov
Department of higher mathematics and cybernetics, Kazakh-British Technical University, Toli Bi 59, 050000, Almaty, Kazakhstan
E-mail address: anvarbek@list.ru

Nelly Erygina
Department of mathematics, Belgorod State University, ul.Pobedi 85, 308015 Belgorod, Russia
E-mail address: eryginan@bsu.edu.ru

Saltanbek Mukhambetzhanov
Department of differential equations, Al-Farabi Kazakh National University, Al-Farabi 71, 050000, Almaty, Kazakhstan
E-mail address: mukhambetzhanov@mail.ru