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Пособие посвящено гипераналитическим функциям, или функциям, аналитическим по Дуглису. Для них строится теория, аналогичная классической теории аналитических функций. Указаны приложения к эллиптическим уравнениям, плоской теории упругости и гидромеханике.

Предназначается для иностранных студентов и аспирантов математических специальностей университетов

Содержание

Introduction

Analytic functions of complex variable $\phi(z) = u(x, y) + iv(x, y)$ are defined as solutions to Cauchy-Riemann equation

$$\frac{\partial \phi}{\partial y} - i\frac{\partial \phi}{\partial x} = 0. \tag{0.1}$$

Substituting it by a more general equation

$$\frac{\partial \phi}{\partial y} - i\frac{\partial \phi}{\partial x} = a\phi + b\overline{\phi}$$

one is led to the theory of generalized analytic functions which was developed by I.N.Vekua, M.A.Lavrentiev, and L.Bers in fifties.

Another natural generalization of equation (0.1) is provided by the system

$$\frac{\partial \phi}{\partial y} - J\frac{\partial \phi}{\partial x} = 0 \tag{0.2}$$

for vector-function $\phi = (\phi_1, \ldots, \phi_s)$, where J is a constant $(s \times s)$ matrix having no real eigenvalues.

From this point of view equation (0.2) was investigated by A.Douglis under assumption that matrix J is a triangular Töplitz matrix, i.e. its elements only depend on the difference of indices.

Functions ϕ satisfying equation (0.2) were called *hyperanalytic* by A.Douglis [2]. This topic was further developed in [3], [4], [5], [6], [7], [8], [34] and so on. In particular, an analogue of the classical theory of analytic (holomorphic) functions was developed for solutions of equation (2), so they are sometimes called *analytic functions in the sense of Douglis*.

As is well known, solutions to Laplace equation

$$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{0.3}$$

can be described as the real parts of analytic functions. Analytic functions are also helpful for representing solutions of more general equations with real analytic coefficients.

A unified approach to the study of such representations was suggested by I.Vekua [10]. Later on A.Bitsadze [11] obtained representations of general solutions to elliptic systems through analytic vector-functions and their derivatives.

Recently, it turned out [12], [13] that the representations obtained by Bitsadze can be substantially simplified using hyperanalytic functions. One can say that hyperanalytic functions play the same role with respect to elliptic system s with constant coefficients as analytic functions do with respect to Laplace equation (3). Analogous statements were obtained by N.Zhura [14] for systems which are elliptic in the sense of Douglis-Nirenberg, and for systems which are hyperbolic in the sense of Leray and Petrovsky.

In the present paper we give an updated review of results in this direction. For reader's convenience, necessary results from the theory of matrices are included in §1. In §2 we develop an analog of analytic functions theory for equation (0.2). Elliptic systems and equations of arbitrary order are considered in §3. Main attention is given to representation of general solution to such systems in terms of hyperanalytic functions. At the end on this section some applications to boundary value problems are described. In §4 we study various concepts of ellipticity for elliptic system s of second order which are most important for the applications. The case of two equations of second order is considered in some detail in §5. The last section contains results of N.Zhura [15] on representation of general solution to elliptic system s in the sense of Douglis-Nirenberg. As an illustration we present an application to the linearized Stokes system of hydrodynamics.

1. Preliminaries from linear algebra

1.1. **Functions of matrices.** Let $\mathbb{C}^{s_1\times s_2}$ be the set of all complex $s_1 \times s_2$ matrices and let $\mathbb{R}^{s_1\times s_2}$ have the same sense for real matrices. Algebraic properties of matrices and determinants are well-known, in particular, $\mathbb{C}^{s\times s}$ is a $\mathbb{C}$-algebra [16]. Matrices can be written in block form: for $A \in \mathbb{C}^{l\times s}$ and $l = l_1 + \ldots + l_k$, $s = s_1 + \ldots + s_r$,, notation $A = (A_{ij})$ means that $A_{ij} \in \mathbb{C}^{l_i\times s_j}$. For $k = 1$ and $r = 1$ we get block-row and block-column, respectively, in which cases we write $A = (A_1, \ldots, A_r)$, with $A_j \in \mathbb{C}^{l\times s_j}$, or $A =\downarrow (A_1, \ldots, A_r)$, with $A_i \in \mathbb{C}^{l_j\times s}$, respectively. For $k = r$ block-matrix (A_{ij}) is a square-matrix. If $l_i = s_i$ for all i and $A_{ij} = 0$ for $i > j$ $(i < j)$, one gets upper(lower)-triangular matrix. If $A_{ij} = 0$ for $i \neq j$ this matrix is block-diagonal and we write $A = \operatorname{diag}(A_{11}, \ldots, A_{kk})$ (analogously to usual diagonal matrix). The determinant $\det A$ of a block-triangular matrix A is equal to $\det A_{ii}$, $i = 1, \ldots, n$. By 0 and 1 we denote, respectively, the zero and identity matrix. Number $z \in \mathbb{C}$ is identified with the scalar matrix $z \cdot 1 \in \mathbb{C}^{s\times s}$. If $\det A \neq 0$, then there exists the inverse matrix $B = A^{-1}$ such that $AB = BA = 1$.

Polynomial of degree s

$$\det (z - A) = \prod\nolimits_{j=1}^{n} (z - \nu_j)^{s_j} \tag{1.1}$$

is called the characteristic polynomial of A. In (1.1) is supposed that $\nu_i \neq \nu_j$ for $i \neq j$. Complex numbers ν_i are called eigenvalues of and constitute the spectrum $\sigma(A)$. Natural number s_j is called the multiplicity of eigenvalue ν_j.

Each polynomial $p(z) = a_0 + a_1 z + \cdots + a_k z^k$ with $a_j \in \mathbb{C}$ defines a matrix

$$p(A) = a_0 + a_1 A + \cdots + a_k A^k \tag{1.2}$$

which is called the value of p at A.

From (1.2) immediately follows that for arbitrary polynomials p_1, p_2 one has

$$\begin{aligned} &(p_1 + p_2)(A) = p_1(A) + p_2(A), \\ &(p_1 p_2)(A) = p_1(A) p_2(A), \\ &p_1[p_2(A)] = p(A),\ p(z) = p_1[p_2(z)]. \end{aligned} \tag{1.3}$$

This, in particular, implies that

$$\sigma(p(A)) = \{p(\nu), \nu \in \sigma(A)\}, \tag{1.4}$$

which is well known [16].

One has also

$$
\begin{aligned}
&B^{-1}p(A)B = p(B^{-1}AB),\\
&p(\mathrm{diag}\,(A_1,\dots,A_n)) = \mathrm{diag}\,(p(A_1),\dots,p(A_n)).
\end{aligned}
\tag{1.5}
$$

It is said that $p(z)$ annihilates A if $p(A) = 0$. A famous example of such polynomial is provided by Hamilton-Cayley theorem [16].

Theorem 1.1. *The characteristic polynomial of A annihilates A.*

Proof can be easily obtained by considering matrix $R(z)$ consisting of algebraic complements to elements of $z-A$, $z \in \mathbb{C}$. Indeed, by Cramer's rule, for each $z \in \mathbb{C}$, one has

$$(z-A)R(z) = R(z)(z-A) = X(z), \tag{1.6}$$

where $X(z)$ is the determinant of $z-A$, i.e. the characteristic polynomial of A. Moreover, one can write

$$R(z) = R_1 + R_2 z + \ldots + R_s z^{s-1},$$

where R_j commute with A in virtue of (1.6): $R_j A = AR_j$. Hence the second equality in (1.6), which reads

$$X(z) = (R_1 + R_2 z + \ldots + R_s z^{s-1})(z-A),$$

remains true if one substitutes A instead of z and the theorem is proven.

Among all polynomials which annihilate A consider a monic (i.e. with the highest order coefficient equal to 1) polynomial $M(z)$ of the minimal possible degree r. If $p(z)$ annihilates A then p is divisible by M, in particular, polynomial M is uniquely defined and called the minimal polynomial of A.

By Theorem 1.1 polynomial (1.1) is divisible by M, and by (1.4) the set $\{M(\nu),\ \nu \in \sigma(A)\}$ consists of a single point $z = 0$ for each j. Hence

$$M(z) = \prod\nolimits_{j=1}^{n} (z-\nu_j)^{r_j}, \quad 1 \le r_j \le s_j, \tag{1.7}$$

so that $r = r_1 + \ldots r_n$. The number r_j is called the multiplicity of eigenvalue ν_j.

Using the minimal polynomial, formula (1.2) can be extended to functions $f(z)$ which are analytic in the neighbourhood of $\sigma(A)$. To this end with function f we associate vector $c = Lf \in \mathbb{C}^k$ by formula

$$Lf = \left(f^{(i)}(\nu_j),\ 0 \le i \le k_j - 1,\ j = 1,\dots,n\right). \tag{1.8}$$

Obviously, $Lf = 0$ means that $f(z)$ has zero of order not less than r_j in each point $z = \nu_j$, $j = 1,\dots,n$. This is equivalent to $f(z) = \tilde{f}(z)M(z)$, where $\tilde{f}$ is analytic in a neighbourhood of $\sigma(A)$.

Lemma 1.1. *On the set P_{k-1} of polynomials of degree not exceeding $r-1$ mapping L is one-to-one and has an inverse $L^{(-1)}:\ \mathbb{C}^r \to P_{r-1}$.*

Proof. If $Lp = 0$, $p \in P_{k-1}$, then $p(z)$ is divisible by $(z-\nu_j)^{r_j}$ for each j, hence divisible by $M(z)$. Since p does not exceed $k-1$ this is only possible if $p = 0$. Thus $L:\ P_{r-1} \to \mathbb{C}^r$ is one-to-one. Since dimension of P_{r-1} is r this mapping has an inverse $L^{(-1)}:\ \mathbb{C} \to P_{r-1}$.

For a given family $c = (c_{ij},\ 0 \le i \le r_j - 1,\ j = 1,\dots,n) \in \mathbb{C}^k$, polynomial $L^{(-1)}c \in P_{r-1}$ is called the interpolating polynomial (with respect to $(\nu_j, r_j),\ j = 1,\dots,n)$).

The lemma implies that for each function f analytic on $\sigma(A)$ there exists a function $\tilde{f}$, such that

$$f(z) = \tilde{f}(z)M(z) + q(z), \quad q = L^{(-1)}Lf. \tag{1.9}$$

Correspondingly, the value $f(A)$ is defined as $f(A) = q(A)$. The following result shows that this is a reasonable definition.

Theorem 1.2. *(a) Let sequence of analytic functions $f_n(z) \to f(z)$ converge uniformly in a neighbourhood of $\sigma(A)$. Then $f_n(A) \longrightarrow f(A)$ in the sense of convergence of each matrix entry. In particular, (1.3), (1.5) hold for analytic functions.*

(b) Let $g(z,t)$ be continuous on $G \times \Gamma$, where G is some neighbourhood of $\sigma(A)$, and Γ is some continuous curve. If $g(z,t)$ is analytic in G then the matrix-function $g(A,t)$ is continuous Γ u

$$g(A) = \int_\Gamma g(A,t)dt, \quad g(z) = \int_\Gamma f(z,t)dt. \tag{1.10}$$

Proof. (a) Let q_n be defined by f_n as in (1.9). Then $q_n \longrightarrow q$ coefficientwise. Hence $q_n(A) \longrightarrow q(A)$ as $n \longrightarrow \infty$. To show (1.3), (1.5) for analytic function s it suffices to write them for approximating polynomial s and pass to the limit.

(b) Denote by $g^{(i)}(z,t)$ the ith derivative with respect to z of function $g(z,t)$. As is known it is continuous on $G \times \Gamma$ and

$$f^{(i)}(z) = \int_\Gamma g^{(i)}(z,t)dt.$$

Hence if $p(z,t)$ and $q(z)$ are defined by, respectively , $g(z,t)$ и $f(z)$ as in (1.9) then

$$q(z) = \int_\Gamma p(z,t)dt.$$

This in turn implies (1.10).

Some corollaries are immediate. If function f develops in power series in a neighbourhood of $\sigma(A)$ as $a_0 + a_1 z + \cdots$, then by (a) we have a convergent matrix series

$$f(A) = a_0 + a_1 A + \cdots ,$$

which gives a natural generalization of (1.2). In particular, the series

$$\exp A = 1 + A + \frac{A^2}{2!} + \cdots$$

converges for any matrix A and defines the value of $e^z = \exp\ z$ at A. Analogously, if the absolute values of eigenvalues of A are less than 1, then the series

$$(1-A)^{-1} = 1 + A + A^2 + \cdots$$

converges and defines the inverse of $1 - A$.

From (1.3) follows that A^{-1} coincides with $f(A)$ for $f(z) = z^{-1}$. One just notices that $zf(z) = f(z)z = 1$. By the same reasoning matrix $f(A)$ commutes with A for any f.

From Theorem 1.2(b) follows an analog of Cauchy formula

$$f(A) = \frac{1}{2\pi i}\int_\Gamma f(t)(t-A)^{-1}dt, \tag{1.11}$$

where contour Γ contains $\sigma(A)$ in its interior and is oriented so that $\sigma(A)$ lies on the left of it. For the proof it is sufficient to notice that

$$f(z) = \frac{1}{2\pi i}\int_\Gamma f(t)(t-z)^{-1}dt$$

and apply (b) to $g(z,t) = f(t)(t-z)^{-1}$.

We present one more corollary which is important in the sequel.

Lemma 1.2. *Let $A(\tau)$ be a continuously differentiable matrix-function on $[0,1]$ and*

$$A'(\tau)A(\tau) = A(\tau)A'(\tau), \quad 0 \le \tau \le 1. \tag{1.12}$$

Let function f be analytic in an open set containing $\sigma(A(\tau))$ for all τ. Then matrix-function $B(\tau) = f(A(\tau))$ is continuously differentiable and

$$B'(\tau) = f'(A(\tau))A'(\tau). \tag{1.13}$$

Proof. Consider matrix-function $(t - A(\tau))^{-1}$, $t \in \Gamma$, $0 \le \tau \le 1$, where Γ is from (1.11). It is continuously differentiable with respect to τ and by (1.12) its derivative is

$$[(t - A(\tau))^{-1}]' = (t - A(\tau))^{-2}A'(\tau).$$

Hence we can differentiate (1.11) and get

$$B' = \left[\frac{1}{2\pi i}\int_\Gamma f(t)(t-A)^{-2}dt\right]A'.$$

In remains to notice that by theorem 1.2(b) applied to equality

$$f'(t) = \frac{1}{2\pi i}\int_\Gamma f(t)(t-z)^{-2}dt,$$

expression in square brackets coincides with $f'(A)$.

If has a single eigenvalue ν then (1.1) and (1.7) take the form

$$\det(z - A) = (z-\nu)^s, \ M(z) = (z-\nu)^m,$$

so that $(A-\nu)^k = 0$ and according to (1.8)

$$f(A) = \sum_{i=0}^{r-1} \frac{f^{(i)}(\nu)}{i!}(A-\nu)^i. \tag{1.14}$$

If $\nu = 0$ matrix is called nilpotent and satisfies $A^r = 0$.

As an illustration consider triangular matrix

$$A = \begin{pmatrix} \nu & 1 & 0 & \dots & 0 \\ 0 & \nu & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \nu \end{pmatrix}, \tag{1.15}$$

which is called Jordan block (or Jordan ν-block if one wishes to indicate its dependence on ν). Obviously, all elements of matrix $A - \nu$ vanish except the subdiagonal ones which are equal to 1. Thus according to (1.14)

$$f(A) = \begin{pmatrix} f(\nu) & f'(\nu) & \dots & f^{(s)}(\nu)/s! \\ 0 & f(\nu) & \dots & f^{(s-1)}(\nu)/(s-1)! \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f(\nu) \end{pmatrix}. \tag{1.16}$$

1.2. **Jordan normal form.** As usual matrix $A = (A_{ij})_1^s \in \mathbb{C}^{s\times s}$ can be considered as linear transformation of $X = \mathbb{C}^s$ which is denoted by the same letter and acts by formula

$$(Ax)_i = \sum_{i=1}^{s} A_{ij}x_j, \quad i = 1, \cdots, s.$$

This correspondence is a homomorphism. Moreover,

$$(AB)_{(i)} = AB_{(i)}, \quad (BC)_{(i)} = \sum_{j=1}^{s} C_{ji}B_{(j)}, \tag{1.17}$$

where $C = (C_{ij})_i^s$ and $B_{(i)}$ is the ith column of B considered as an element $(B_{1i}, \dots, B_{si})$ of $\mathbb{C}^s$.

Subspace $X_0 \subseteq \mathbb{C}^s$ is called A-invariant if $Ax \in X_0$ for $x \in X_0$. Such are the image Im $A = A(X)$ and kernel Ker $A = \{x, Ax = 0\}$ of . Spectrum $\sigma(A) = \{\nu_1, \dots, \nu_s\}$ is defined as the set of all ν, for which $\nu - A$ is not invertible or, equivalently, Ker $(\nu - A) \neq 0$.

Transformation P is called projector if $P^2 = P$ or, equivalently, if P is identical on $X_0 =$ Im P. This subspace is A-invariant if and only if A commutes with P, i.e. if $AP = PA$. Let $x_i, i = 1, \dots, s_0$, be a basis in X_0. Then A-invariance of X_0 means that

$$Ax_i = \sum_{j=1}^{s_0} \alpha_{ji}x_j, \quad i = 1, \dots, s_0. \tag{1.18}$$

Matrix $A_0 = (\alpha_{ij})_1^{s_0}$ is called the matrix of linear transformation $A : X_0 \to X_0$ in basis (x_i).

By (1.17) equation s (18) can be written in matrix form $AB_0 = B_0A_0$, where columns of $B_0 \in \mathbb{C}^{s\times s_0}$ are x_i.

Lemma 1.3. *Let $\mathbb{C}^s = X_1 \oplus \dots \oplus X_n$ and each X_i is A-invariant. Let matrix $B_i \in \mathbb{C}^{s\times s_i}$ have columns X_i and J_i be the matrix of A in this basis. Then*

$$AB = B\,\mathrm{diag}(J_1, \dots, J_n), \quad B = (B_1, \dots, B_n). \tag{1.19}$$

Proof. As mentioned above, $AB_i = B_iJ_i$ for each i. These relations in turn equivalent to (19).

Matrices A and $\tilde{A} = B^{-1}AB$ are called similar. Lemma 1.3 can be reversed: if $B^{-1}AB = \mathrm{diag}(J_1, \dots, J_n)$, $B = (B_1, \dots, B_n)$, where $B_i \in \mathbb{C}^{s_i\times s}$ and $J_i \in \mathbb{C}^{s_i\times s_i}$, then columns of B_i give a basis of an A-invariant subspace X_i. This implies the following spectral decomposition result [16].

Theorem 1.3. *To each eigenvalue ν_i of A corresponds an invariant subspace X_i, $i = 1, \ldots, n$, such that, for each i, transformation $A : X_i \to X_i$ has a single eigenvalue ν_i and $\mathbb{C}^s$ decomposes in direct sum of these subspaces. In particular, we have (19) with $\sigma(J_i) = \{\nu_i\}$.*

Subspace X_i is called the eigenspace of A corresponding to ν_i. Its dimension is equal to the multiplicity s_i of eigenvalue ν_i in (1.1). Indeed by (1.19) we have $B^{-1}(z - A)B = \operatorname{diag}(z - J_1, \ldots, z - J_n)$, from where

$$\det\ (z - A) = \prod\nolimits_i \det\ (z - J_i), \quad \det\ (z - J_i) = (z - \nu_i)^{s_i},$$

where s_i is the dimension of X_i.

If A is real then polynomial (1.1) takes real values at real points z. In particular, to each eigenvalue ν corresponds the complex conjugate eigenvalue $\overline{\nu}$ of the same multiplicity.

Lemma 1.4. *If $A \in \mathbb{R}^{s\times s}$ then matrix B in theorem* 1.3 *can be chosen so that $B_i = \overline{B_j}$ for $\nu_i = \overline{\nu_j}$.*

Proof. It suffices to show that $\nu_i = \overline{\nu_j}$ implies

$$X_i = \{\overline{x}, x \in X_j\}. \tag{1.20}$$

To this end put $f^*(z) = \overline{f(\overline{z})}$. If f is defined in a neighbourhood of $\sigma(A)$ then f^* is defined in a neighbourhood of $\sigma(\overline{A})$. If M is the minimal polynomial of A then M^* is the minimal polynomial of $\overline{A}$. Applying " $*$ " to (1.8) one derives that $\overline{f(A)} = f^*(\overline{A})$. In particular, $\overline{f(A)} = f^*(A)$ for real matrix A. Since $(p_i)^* = p_j$ for $\nu_i = \overline{\nu_j}$ we get $\overline{P_i} = P_j$, which completes proof of (1.20) and lemma.

A non-zero vector $x \in X_i$ satisfying equation $(A - \nu_i)^{r+1}\, x = 0$ is called *eigenvector* (for $(r = 0)$) or *adjoint vector* (for $(r \geq 1)$) associated with ν_i. Putting $x_j = (A - \nu_i)^{r-j}$, $j = 0, 1, \ldots, r$, we get a chain of eigenvectors and adjoint vectors satisfying

$$(A - \nu_i)x_0 = 0, \quad (A - \nu_i)x_1 = x_0, \ldots, (A - \nu_i)x_r = x_{r-1}. \tag{1.21}$$

It is easy to see that vectors x_j are linearly independent and generate an A-invariant subspace. Comparing (1.15) and (1.27) gives that the matrix of A in basis (x_j) is Jordan ν_i-block. The following fundamental theorem belongs to C.Jordan [16].

Theorem 1.4. *(C.Jordan) Each eigenspace X_i of A has a basis consisting of chains of eigenvectors and adjoint vectors. In particular, matrix B in theorem* 1.3 *can be chosen so that each J_i is a direct sum of Jordan ν_i-blocks. These blocks are uniquely determined up to a permutation.*

Matrix $J = \operatorname{diag}(J_1, \ldots, J_n)$ is called the Jordan (normal) form of A. It is also said that B brings A to Jordan form.

Equality (1.19) can be written as $AB_i = B_iJ_i$, $i = 1, \ \ldots\ , n$. Obviously, columns of B_i are automatically linearly independent if J_i consists of one block. In such case eigenvalue ν_i is called simple. In terms of (1.1), (1.7), simplicity of ν_i is equivalent to equality $s_i = r_i$.

1.3. **Matrix polynomials.** Expression $z - A$ playing an important role in subsection 1.2, can be considered as first degree polynomial with matrix coefficient A. Consider more general matrix polynomial of such type:

$$P(z) = z^n - \sum\nolimits_{j=0}^{n-1} a_j z^j, \quad a_j \in \mathbb{C}^{l\times l}, \tag{1.22}$$

which already appeared in the proof of theorem 1.1. Matrix polynomial s are also called z-matrices [17]. They are usually investigated using so-called elementary transformations preserving the characteristic equation

$$\det P(z) = 0. \tag{1.23}$$

Following [18] let us consider these polynomials from the spectral point of view. Namely, let us refer to roots ν_j of equation (1.23) as eigenvalues (of multiplicity s_i) of polynomial $P(z)$. Matrix $P^{-1}(z)$ is a rational function with elements possibly having poles at ν_j. The highest order of those poles is called the order of ν_j. Finally, with polynomial $P(z)$ associate matrix $A \in \mathbb{C}^{nl\times nl}$ written in the $n \times n$-block form

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_0 & a_1 & a_2 & \dots & a_{n-1} \end{pmatrix}, \tag{1.24}$$

For $n = 1$ this matrix coincides with a_0.

Lemma 1.5. *Matrix A and matrix polynomial $P(z)$ have the same eigenvalues and the corresponding multiplicities and orders (for $\nu \neq 0$) coincide.*

Proof. For $n = 1$ it suffices to show that orders of eigenvalues of polynomial $z-A$ and matrix $A = a_0$ coincide. By (1.5), (1.19)

$$B^{-1}[(z-A)^{-1}]B = \mathrm{diag}\,[f(J_1), \dots, f(J_n)],$$

where $f(u) = (z-u)^{-1}$. Applying formula (1.14) to $f(J_i)$, with $r = r_i$ being the order of eigenvalue ν_i, we come to conclusion that r_i is equal to the order of eigenvalue of polynomial $z - A$.

For $n > 1$, everything follows from the matrix identity

$$(z-A)\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & z & z & \dots & z \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & z^{n-1} \end{pmatrix} = \begin{pmatrix} z & 0 & 0 & \dots & 0 \\ 0 & z^2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_0 & P_1 & P_2 & \dots & P \end{pmatrix}, \tag{1.25}$$

where $P_k(z) = a_0 + a_1 z + \ldots + a_k z^k$, $k = 0, 1, \ldots, k$. Correctness of this equality is verified by direct check.

If ν is an eigenvalue of $P(z)$ then, as in (1.21), vectors $x_0, x_1, \ldots, x_r \in \mathbb{C}^l$ defined by equalities

$$\begin{gathered} P(\nu)x_0 = 0, \quad P(\nu)x_1 + P'(\nu)x_0 = 0, \ldots, \\ P(\nu)x_r + P'(\nu)x_{r-1} + \cdots + \frac{P^{(r)}(\nu)}{r!}x_0 = 0, \end{gathered} \tag{1.26}$$

are called a chain of eigenvectors and adjoint vectors of polynomial $P(z)$ corresponding to eigenvalue ν.

Theorem 1.5. *Adopting notation* (24)*, equality $AB = B_0 J_0$, where $B_0 \in \mathbb{C}^{s\times s_0}$, $J_0 \in \mathbb{C}^{s_0\times s_0}$, is equivalent to*

$$B_0 =\downarrow (b_0, b_0 J, \ldots, b_0 J^{n-1}), \quad b_0 J^n = \sum\nolimits_{i=0}^{n-1} a_i b_0 J_0^i. \tag{1.27}$$

Here matrix J_0 is a Jordan block if and only if columns of $b_0 \in \mathbb{C}^{l\times s_0}$ form a chain of eigenvectors and adjoint vectors of polynomial $p(z)$.

Proof. In one direction this follows from the product rule for block matrices (1.24) and (1.27). Conversely, let

$$AB_0 = B_0 J_0, \quad B_0 =\downarrow (b_0, b_1, \ldots, b_{n-1}).$$

Then by (1.24)

$$b_{j+1} = b_j J_0,\ j = 0, 1, \ldots, n-1, \quad \sum\nolimits_{i=0}^{n-1} a_i b_i = b_{n-1} J_0,$$

which implies (1.27).

Let $J_0 = J$ be Jordan block (1.15) and $x_0, \ldots, x_r$ be columns of b_0. According to (1.16)

$$(J^p)_{ij} = \begin{cases} f_p^{(i-j)}(\nu)/(i-j)! & , \quad i \geq j, \\ 0 & , \quad i < j, \end{cases} \tag{1.28}$$

where $f_p(u) = u^p,\ i, j = 0, 1, \ldots, r$. Applying (1.17) to (1.27), one obtains:

$$\sum_j (J^n)_{ji}\, x_j = \sum_{p=0}^{n-1} a_p \sum_j (J^p)_{ji}\, x_j.$$

Substituting here expressions (1.28) and remembering that

$$P(z) = f_n(z) - \sum\nolimits_{p=0}^{n-1} a_p f_p(z)$$

we arrive to (1.26):

$$\sum_{j=0}^{i} \frac{1}{(i-j)!} P^{(i-j)}(\nu) x_j = 0, \quad i = 0, \ldots, r.$$

Since these considerations can be reversed, the theorem is proven.

If $l = 1$ write

$$P(z) = \prod\nolimits_{j=1}^{m} (z - \nu_j)^{s_j}, \quad s_1 + \ldots + s_m = n, \tag{1.29}$$

where $\nu_i \neq \nu_j$ for $i \neq j$, and let J_i denote Jordan ν_i-block of order s_i. Put

$$\begin{gathered} b_i = (1, 0, \ldots, 0) \in \mathbb{C}^{1\times s_i}, \quad B_i =\downarrow (b_i, b_i J, \ldots, b_i J^{n-1}) \in \mathbb{C}^{n\times s_i}, \\ B = (B_1, \ldots, B_m) \in \mathbb{C}^{n\times n}. \end{gathered} \tag{1.30}$$

As $P^{(j)}(\nu_i) = 0, \quad j = 0, 1, \ldots, s_i - 1$, according to (1.26) matrix b_i is composed of a chain of eigenvectors and adjoint vectors $x_k\mathbb{R}$. Hence by theorem 1.5 and remark at the end of subsection 1.2, matrix B is invertible and brings A to Jordan form

$$B^{-1}AB = J, \quad J = \operatorname{diag}(J_1, \ldots, J_n). \tag{1.31}$$

It is clear that eigenvalues ν_j of matrix A are simple. This follows from lemma 1.5 since for $l = 1$ the order and multiplicity of eigenvalue ν_j of polynomial (1.29) coincide and are equal to s_j.

Notice that by (1.28) matrix B_i in (1.30) is composed of columns

$$h^{(j)}(\nu_i)/j!, \quad i = 0, 1, \ldots, s_i - 1,$$

where $h(z) = \downarrow (1, z, \dots, z^{n-1})$.

2. Hyperanalyticity in the sense of Douglis

2.1. **Basic concepts.** Suppose that matrix $J \in \mathbb{C}^{s\times s}$ is invertible and all its eigenvalues ν have positive imaginary parts, i.e. lie in the upper half-plane. Consider in a domain $D \subset \mathbb{C}$ a continuously differentiable vector-function $\phi(z) = (\phi_1(z), \dots, \phi_s(z))$ of complex variable $z = x + iy$. This function is called *hyperanalytic* (or *analytic in the sense of Douglis*) if its partial derivatives satisfy

$$\frac{\partial\phi}{\partial y} - J\frac{\partial\phi}{\partial x} = 0. \tag{2.1}$$

For $J = i$ this relation coincides with Cauchy-Riemann equations so this definition gives the usual analyticity. In general case, in order to emphasize dependence on J, function ϕ is called a J-analytic function.

If matrix $B \in \mathbb{C}^{s\times s}$ is invertible then substitution $\tilde\phi = B\phi$ transforms (2.1) into

$$\frac{\partial\tilde\phi}{\partial y} - \tilde J\frac{\partial\tilde\phi}{\partial x} = 0. \tag{2.1$^\sim$}$$

with a conjugate matrix $\tilde J = B^{-1}JB$. In other words, vector-function $\tilde\phi$ is $\tilde J$-analytic in D.

By Theorem 1.3 matrix $B = (B_1, \dots, B_n)$, $B_k \in \mathbb{C}^{s\times s_k}$ can be chosen so that

$$J = \mathrm{diag}(J_1, \dots, J_m), \quad \sigma(J_k) = \{\nu\}. \tag{2.2}$$

In this case by introducing the block form $(\phi_1, \dots, \phi_k)$ of vector ϕ system (2.1) splits into collection of systems

$$\frac{\partial\phi_k}{\partial y} - J_k\frac{\partial\phi_k}{\partial x} = 0, \quad k = 1, \dots, m.$$

In other words, block components ϕ_k are J_k-analytic functions.

By Theorem 1.4 one can also achieve that all matrices J_k in (2.2) are Jordan blocks (in this case ν_k may repeat and n is the total number of blocks in Jordan normal form of J). However in the sequel we do not impose on J additional conditions of the form (2.2).

As was mentioned, (2.1) can be considered as an analogue of Cauchy-Riemann equations. In the same spirit one can introduce the notion of *monogeneity* equivalent to the existence of complex derivative. To each complex number $t = t_1 + it_2 \in \mathbb{C}$ associate $s \times s$ -matrix

$$[t] = t_1 + t_2 J, \quad t_j \in \mathbb{R}. \tag{2.3}$$

Its eigenvalues are $t_1 + \nu t_2$, $\nu \in \sigma(J)$. In particular, $[t]$ is invertible for $t \neq 0$. The inverse $[t]^{-1}$ as a function of t is homogeneous of order -1, hence its norm in $\mathbb{C}^{s\times s}$ can be estimated as

$$|[t]^{-1}| \le C|t|^{-1}. \tag{2.4}$$

If ϕ is hyperanalytic in D then for fixed $z \in D$ the condition of differentiability reads

$$\phi(z + ty) - \phi(z) = t_1\phi_x(z) \; + \; t_2\phi_y(z) + o(t) \text{ as } t \to 0.$$

Taking into account (2.1), (2.3), (2.4) we get

$$[t]^{-1}\{\phi(z + t) - \phi(z)\} = \phi_x(z) + o(1) \,\mathrm{as}\, t \to 0.$$

Thus at each $z \in D$ there exists limit

$$\lim_{t\to 0}[t]^{-1}\{\phi(z_t) - \phi(z)\} = \phi'(z), \tag{2.5}$$

which coincides with the partial derivative ϕ_x at z. Conversely: suppose the limit (2.5) exists for each $z \in D$ and the resulting function ϕ' is continuous. Then putting $t_1 = 0$ and $t_2 = 0$ in (2.4) we get $\phi_x = \phi'$, $\phi_y = J\phi'$ for partial derivatives of ϕ, which means J-analyticity of ϕ.

Thus the concept of J-analyticity may be defined by the monogeneity condition (2.5). If $\phi \in C^{n+1}(D)$ then from its hyperanalyticity follows that all of its partial derivatives up to order n are also hyperanalytic. Moreover,

$$\phi^{(k)} = \frac{\partial^k \phi}{\partial x^k}$$

may be considered as consequent "complex"derivatives $\phi^{(0)} = \phi$, $\phi^{(1)} = \phi'$, $\phi^{(2)} = (\phi')'$ and so on. The rest partial derivatives according to (2.1) are given by

$$\frac{\partial^k \phi}{\partial x^{k-r}\partial y^r} = J^r \phi^{(k)}, \quad 0 \le r \le k. \tag{2.6}$$

In subsection 2.2 will be shown that J-analytic functions are indeed infinitely differentiable in their domains.

Along with s-vector functions ϕ equation (2.1) is considered for $s \times s$-matrix functions $F(z)$ under additional commutation condition: $F(z)J = JF(z)$ for all z. The last requirement guarantees invariance of $\phi \to F\phi$ in the class of hyperanalytic functions ϕ. Then one also has $(F\phi)' = F'\phi + F\phi'$.

Basic examples of hyperanalytic function s are constructed using functions of matrices. Let function f be analytic in an open set which, for each $\nu \in \sigma(J)$, contains the image of D under affine transformation

$$z = x + iy \to x + \nu y. \tag{2.7}$$

Then by (1.2) we get a matrix-function $F(z) = f([z])$ defined for all $z \in D$. Its values commute with J and by Lemma 1.2 its partial derivatives are $F_x = f'([z])$, $F_y = Jf'([z])$. Hence this function is hyperanalytic and

$$\{f([z])\}' = f'([z]). \tag{2.8}$$

Obviously $[z]$ can be substituted by $[z - z_0]$ with fixed z_0. In particular, for each integer k, matrix $[z - z_0]^k$ and vector $[z - z_0]^k c$, $c \in \mathbb{C}^s$, are hyperanalytic. Thus the finite sums

$$\sum\nolimits_k [z - z_0]^k c_k, \quad c_k \in \mathbb{C}^s \tag{2.9}$$

are rational J-analytic functions. For $k \ge 0$, these sums provide J-analytic polynomials.

There exists an invertible linear transformation sending analytic functions to J-analytic functions.

Theorem 2.1. *Let J have block-diagonal form* (2.2) *and correspondingly s-vectors have the form $f = (f_1, \dots, f_m)$ with s_k-vector function s f_k. Let ϕ be a $J-$analytic function in D with $\phi = (\phi_1, \dots, \phi_m) \in C^\infty(D)$ and D_k be the image of D under affine transformation $x = x + iy \to x + \nu_k y$. Then the formula*

$$\phi_k(x + iy) = \sum_{r=0}^{s-1} \frac{y^r}{r!}(J_k - \nu_k)^r \psi_k^{(r)}(x + \nu_k y), \quad k = 1, \dots, m \tag{2.10}$$

gives a one-to-one correspondence between ϕ and s-vector $\psi = (\psi_1, \ldots, \psi_m)$, where s_k-vector function ψ_k is analytic in D_k, $k = 1, \ldots, m$.

Proof. It suffices to consider the case when $m = 1$, i.e. when matrix J has a single eigenvalue ν. In such case matrix $J = \nu$ is nilpotent:

$$(J - \nu)^s = 0. \tag{2.11}$$

Introduce a linear operation in $C^\infty(D)$ by

$$\exp\left\{y(J-\nu)\frac{\partial}{\partial x}\right\} = \sum_{i\geq 0}\frac{y^i}{i!}(J-\nu)^i\frac{\partial^i}{\partial x^i}. \tag{2.12}$$

By (2.11) this sum is finite, it terminates at $i = s - 1$.

Writing relation $e^{-z}e^z = 1$ in the form of series shows that this operation is invertible and the inverse is $\exp(-yJ_0\partial/\partial x)$, $J_0 = J - \nu$. By definition (2.12) we have:

$$\left(\frac{\partial}{\partial y} - J\frac{\partial}{\partial x}\right)\exp\left(yJ_0\frac{\partial}{\partial x}\right) = \sum_{k\geq 0}\frac{y^{k-1}}{(k-1)!}\left(J_0\frac{\partial}{\partial x}\right)^k +$$

$$+\sum_{k\geq 0}\frac{y^k}{k!}\left(J_0\frac{\partial}{\partial x}\right)^k\left(\frac{\partial}{\partial y} - J\frac{\partial}{\partial x}\right),$$

which gives

$$\left(\frac{\partial}{\partial y} - J\frac{\partial}{\partial x}\right)\exp\left(yJ_0\frac{\partial}{\partial x}\right) = \exp\left(yJ_0\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial y} - \nu\frac{\partial}{\partial x}\right).$$

Hence ϕ satisfies equation (2.1) if and only if vector-function $u = \exp(-yJ_0\partial/\partial x)\phi$ satisfies equation

$$\frac{\partial u}{\partial y} - \nu\frac{\partial u}{\partial x} = 0. \tag{2.13}$$

Under transformation (2.7) this equation turns into Cauchy-Riemann equation. In other words, function ψ defined by $u(x, y) = \psi(x + \nu y)$ is analytic in domain $\{x + \nu y \,|\, x + iy \in D\}$. Combined with relation $\phi = \exp(yJ_0\partial/\partial x)u$ this completes the proof.

Notice that summation in (2.12) is actually over $0 \leq i \leq r - 1$, where r is the order of eigenvalue ν. Correspondingly, in formula (2.10) summation goes over $0 \leq r \leq r_k - 1$, where r_k is the order of ν_k.

If $f = \psi$ is a scalar function then formula (2.10) defines a J-analytic matrix-function $F = \phi$. Taking into account (1.14) this matrix-function coincides with $f([z])$ introduced above.

As an illustration consider function $f(t) = t^\zeta p(\ln t)$ in some sector of complex plane with vertex $t = 0$, where $\zeta \in \mathbb{C}$ and p is a scalar polynomial. Write

$$\{t^\zeta p(\ln t)\}^{(k)} = t^{\zeta - k}p_k(\ln t), \quad k = 0, 1, \ldots, \tag{2.14}$$

where polynomial p_k has the same degree as p and is defined by

$$p_k(t) = \prod\nolimits_{i=0}^{k-1}(\zeta - i + d/dt)p(t).$$

Substituting this into (2.10) we get

$$[z]^\zeta p(\ln[z]) = \sum_{k\ge 0} \frac{y^k(x+\nu y)^{\zeta-k}}{k!} p_k(\ln(x+\nu y))(J-\nu)^k.$$

In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ this takes the form

$$[z]^\zeta p(\ln[z]) \;=\; r^\zeta \sum_{i\ge 0} \frac{\ln^i r}{i!}\, R_i(\theta),$$

$$R_i(\theta) \;=\; \sum_{k\ge 0} \frac{\sin^k\theta\,(\cos\theta+\nu\sin\theta)^{\zeta-k}}{k!}\, h_k^{(i)}[\ln(\cos\theta+\nu\sin\theta)](J-\nu)^k.$$

All sums above are finite since h_k has the same degree as h. Thus $R_i = 0$ for i bigger than this degree.

Notice also the following relation between analytic and J-analytic functions. Let $X(D)$ be a $\mathbb{R}$-linear space of functions continuous in D. Let us say that $X(D)$ is a *uniqueness class* for equation (2.13) if each solution from this class is constant.

For example, the class of functions $u \in C(\overline{D})$ vanishing on some arc $\Gamma \subseteq \partial D$ is a uniqueness class. Indeed, transformation (2.7) sends solutions of (2.13) into analytic functions in $\{x+\nu y, x+iy \in D\}$ so this follows from the classical uniqueness theorem for analytic functions.

Another example is as follows. Suppose D has piecewise-smooth boundary and $X(D)$ consists of continuous and bounded functions in $\overline{D}$ with the real parts vanishing on the boundary. Then $X(D)$ is a uniqueness class for (2.1), which can be shown in the same way as above. The following lemma is also proved along these lines.

Lemma 2.1. *Let J be triangular and $X(D)$ be a uniqueness class for equation (2.7) for each $\nu \in \sigma(J)$. Then $X(D)$ is a uniqueness class for equation (2.10) as well. In other words, if $\phi = (\phi_1, \dots, \phi_s)$ is J-analytic with all components $\phi_k \in X(D)$, then it is constant in D.*

Proof. Let matrix J be lower triangular, i.e. $J_{kr} = 0$ for $k < r$. Then diagonal elements $\nu_k = J_{kk}$ are eigenvalues of J and (2.1) can be written as

$$\frac{\partial\phi_1}{\partial y} - \nu_1\frac{\partial\phi_1}{\partial x} = 0, \quad \frac{\partial\phi_2}{\partial y} - \nu_2\frac{\partial\phi_2}{\partial x} = J_{21}\frac{\partial\phi_1}{\partial y}, \;\dots,$$

$$\frac{\partial\phi_s}{\partial y} - \nu_s\frac{\partial\phi_s}{\partial x} = \sum_{k=1}^{s-1} J_{sk}\frac{\partial\phi_k}{\partial y}.$$

By assumption $\phi_1 \in X(D)$, so that from the first equation follows that $\phi_1 =$ const. Hence the second equation transforms into (2.7) with $\nu = \nu_2$ and by the same argument $\phi_2 =$ const. Repeating this procedure we get $\phi =$ const in domain D.

It is often convenient to use complex derivatives

$$2\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}, \quad 2\frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}. \tag{2.15}$$

In this notation system (2.1) is written as

$$\frac{\partial\phi}{\partial\overline{z}} = Q\frac{\partial\phi}{\partial z}, \quad Q = (1-iJ)^{-1}(1+iJ). \tag{2.16}$$

Obviously, eigenvalues q of matrix Q satisfy $|q| < 1$.

Since $z + \overline{z}\, Q = 2(1-iJ)(x+Jy)$, all considerations can be made for this system by substituting everywhere $[z]$ by matrix $z_Q = z + \overline{z}\, Q$. In particular, (2.14) becomes

$$\phi(z) = \sum_{k\geq 0} \frac{\overline{z}^k}{k!}\,(Q-q)^k\,\psi^{(k)}(z+q\overline{z}), \quad \sigma(Q) = \{q\}.$$

The picture is especially simple when $\sigma(J) = \{i\}$ and matrix Q is nilpotent.

For example, the preceding formula takes the form

$$\phi(z) = \sum_{k\geq 0} \frac{\overline{z}^k}{k!}\,Q^k\,\psi^{(k)}(z), \quad \sigma(Q) = \{0\}. \tag{2.17}$$

2.2. **Cauchy integral.** To get Cauchy integral for hyperanalytic functions it is sufficient to substitute $dt/(t-z)$ by its matrix analogue $[t-z]^{-1}[dt]$, where the matrix differential $[dt] = d\xi + J\,d\eta$ has the same sense as in (2.3).

Let ϕ be continuous in a compact domain $\overline{D}$ with piecewise-smooth boundary Γ and J-analytic inside. Then applying the Green's formula to the left hand side of (2.1) we get

$$\int_\Gamma [dt]\phi(t) = 0, \tag{2.18}$$

which is an analogue of Cauchy theorem. Contour Γ is oriented so that D lies on the left of it.

Further, for each integer k and $z \in D$, one has

$$\frac{1}{2\pi i}\int_\Gamma [t-z]^{-1}[dt] = \begin{cases} 0, & k \neq -1, \\ 1, & k = -1. \end{cases} \tag{2.19}$$

Indeed, consider in $\operatorname{Im} u > 0$ analytic function

$$f(u) = \frac{1}{2\pi i}\int_{\Gamma_0} (\xi - x + u(\eta - y))^k (d\xi + u\,d\eta), \tag{2.20}$$

where $z = x+iy, t = \xi + i\eta$. Obviously, for a fixed $u = \nu$, transformation (2.7) leaves the real axis and upper half-plane invariant. Applying it to integral (2.20) we get

$$f(u) = \frac{1}{2\pi i}\int_{\tilde{\Gamma_0}} \left(\tilde{t} - \tilde{z}\right)^k\,\tilde{dt} = \begin{cases} 0 & , \quad k \neq -1, \\ 1 & , \quad k = -1. \end{cases}$$

By Theorem 1.2(b) this implies (2.19).

Using (2.19) it is easy to get Cauchy formula

$$\phi(z) = \frac{1}{2\pi i}\int_\Gamma [t-z]^{-1}[dt]\phi(t), \quad z \in D. \tag{2.21}$$

Indeed, by (2.19) it suffices to show this assuming that $\phi(z) = 0$ (point $z \in D$ is fixed). Applying formula (2.18) to function $\tilde{\Phi}(t) = [t-z]^{-1}\Phi(t)$ in domain

$D \cap \{t, |t-z| > \epsilon\}$, where $\epsilon > 0$ is sufficiently small, one gets:

$$\int_{\Gamma_0} [t-z]^{-1}[dt]\,\Phi(t) \;=\; \int_{|t-z|=\epsilon} [t-z]^{-1}[dt]\,\Phi(t) \;=$$

$$= \int_0^{2\pi} [e^{i\theta}]^{-1}\,[i\,e^{i\theta}]\,\Phi(z+\varepsilon\,e^{i\theta})\,d\theta.$$

The right hand side of this equality continuously depends on θ and vanishes for $\varepsilon \to 0$ since by assumption $\Phi(z) = 0$. This completes the proof of (2.21).

The integral in (2.21) defines a function in D which can be differentiated under the sign of integral using (2.8). Thus ϕ is infinitely differentiable and its complex derivatives $\phi^{(n)}$ are expressed as

$$\phi^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma [t-z]^{-n-1}[dt]\phi(t). \tag{2.22}$$

Cauchy formula has a number of corollaries some of which we delay till subsection 2.3 and the others are collected in the following statement which can be proven in a standard way.

Theorem 2.2. *(a) Let $\phi_k(z)$ be hyperanalytic in D and converge to ϕ uniformly on compact subsets of D. Then function ϕ is also hyperanalytic and $\phi_k^{(n)} \to \phi^{(n)}$ in the same sense.*

(b) Let function ϕ be continuous in D and hyperanalytic in $D\backslash\gamma$, where γ is a smooth arc. Then ϕ is hyperanalytic in the whole of D.

Proof. (a) Without restricting generality we may assume that domain D is bounded and its boundary Γ is piecewise smooth and function ϕ is continuous in $\overline{D}$.

Applying (2.21) to ϕ_k and taking the limit as $k \to \infty$, we see that ϕ is a hyperanalytic function in D. Using (2.22) we can analogously show that $\phi_k^{(n)} \longrightarrow \phi^{(n)}$ uniformly on compact subsets of D_0.

(b) Without restring generality we can assume that γ connects two points of Γ and entirely lies in D except its endpoints. It is sufficient to show that ϕ is a hyperanalytic function in D. Let D_0, D_1 be the two domains on which D is decomposed by arc γ. Then applying (2.18) and (2.21) to $D_k, k = 0, 1$, we have:

$$\frac{1}{2\pi i}\int_{\partial D_k} [t-z]^{-1}[dt]\phi(t) \;=\; \begin{cases} \phi(z) & , \quad z \in D_k, \\ 0 & , \quad z \in D_{1-k}. \end{cases}$$

Adding these equalities together and taking into account that the integrals over the common arc cancel, we come to formula (2.21) which holds for all $z \in D$. This implies that ϕ is J-analytic in D.

2.3. **Taylor and Laurent series.** By analogy with the classical case series (2.9) over all integer k can be called Laurent series while its part with $k \geq 0$ can be called Taylor series. If its partial sums converge uniformly on compact subsets then the sum is hyperanalytic.

For studying convergence it is convenient to define norms in $\mathbb{C}^s$ and $\mathbb{C}^{s\times s}$ as the sum of absolute values of all components. Then

$$|A\,x| \leq |A|\,|x|, \quad |A\,B| \leq |A|\,|B| \tag{2.23}$$

for all $x \in \mathbb{C}^s, \quad A, B \in \mathbb{C}^{s\times s}$.

Transformation (2.7) sends the unit circle onto ellipse with half-axes a_ν and b_ν. Thus

$$b_\nu \le |x+\nu y| \le a_\nu, \quad x^2+y^2=1. \tag{2.24}$$

Taking $x=1,\ y=0$ we get inequalities $b_\nu \le 1 \le a_\nu$. Put

$$a = \max_{\nu\in\sigma(J)} a_\nu, \quad b = \min_{\nu\in\sigma(J)} b_\nu, \quad q = \frac{a}{b}.$$

Lemma 2.2. *One has the estimates*

$$\begin{aligned} |[z]^k| &\le C\, a^k |z|^k, \quad k \ge 0, \\ |[z]^k| &\le C\, b^k |z|^k, \quad k \le 0, \end{aligned} \tag{2.25}$$

where constant $C>0$ only depends on J.

Proof. Let $B^{-1}JB = \tilde{J}$ and $[\tilde{z}]$ is determined by $\tilde{J}$ as in (2.1). Then $B^{-1}[z]^kB = [\tilde{z}]^k$. From this and (2.23) we get

$$|[z]^k| \le |B|\,|B^{-1}|\,|[\tilde{z}]^k|.$$

Hence without restricting generality matrix J can be taken in the form (2.2) with Jordan blocks J_k. Working blockwise we can assume that J is a Jordan block (1.15). In this case explicit expression for matrix $[z]^k = f_k(J), \quad f_k(u) = (x+uy)^k$, is given by (1.16). Combining this with (2.24) we get estimates (2.25).

Theorem 2.3. *(a) Let ϕ be hyperanalytic in domain D which contains the circle $\{|z-z_0| \le qR\}$. Then in $\{|z-z_0|\le R\}$ function ϕ is developable in an absolutely and uniformly convergent Taylor series*

$$\phi(z) = \sum_{k=0}^{\infty} \frac{1}{k!}[z-z_0]^k\, \phi^{(k)}(z_0). \tag{2.26}$$

(b) Let function ϕ be hyperanalytic in domain D which contains the annulus $\{q^{-1}R_0 \le |z-z_0| \le qR_1\}$. Then in $\{R_0 \le |z-z_0| \le R_1\}$ function ϕ is developable in Laurent series

$$\phi(z) = \sum_{k=-\infty}^{+\infty} [z-z_0]^k\, c_k, \quad c_k = \frac{1}{2\pi i}\int_\Gamma [t-z_0]^{-k-1}[dt]\,\phi(t), \tag{2.27}$$

where Γ is the circle $\{|z-z_0| = R\}$, $R_0 \le R \le R_1$, oriented counterclockwise. Series $\sum_{k\ge 0}$ and $\sum_{k\le 0}$ converge absolutely and uniformly in domains $|z| \le R_1$ and $|z| \ge R_0$, respectively.

Proof. (a) For small $\varepsilon>0$, the circle $\Gamma : \{|z-z_0| = q(R+\varepsilon)\}$ lies in D so by Cauchy formula

$$\phi(z) = \frac{1}{2\pi i}\int_\Gamma [t-z]^{-1}[dt]\,\phi(t), \quad |z-z_0| \le R. \tag{2.28}$$

Write $[t-z] = [t-z_0]\,(1 - [t-z_0]^{-1}[z-z_0])$ and use Lemma 2.2 to get that

$$[t-z]^{-1} = [t-z_0]^{-1} \sum\nolimits_{k\ge 0} [t-z_0]^{-k}[z-z_0]^k$$

converges absolutely and uniformly with respect to $t\in\Gamma, \quad |z-z_0| \le R$. Substituting this into (2.28) we get (2.27), where $k\ge 0$. Applying formula (2.22) to the integrals defining c_k one gets (2.26).

(b) Let $\varepsilon > 0$ be so small that $\Gamma_1 : |z - z_0| = q(R_1 + \varepsilon)$ and $\Gamma_0 : |z - z_0| = q^{-1}(R_0 - \varepsilon)$ lie in D. Then by Cauchy formula

$$\phi(z) = \frac{1}{2\pi i} \int_{\Gamma_1} [t - z]^{-1} [dt]\, \phi(t) - \frac{1}{2\pi i} \int_{\Gamma_0} [t - z]^{-1} [dt]\, \phi(t), \tag{2.29}$$

where $R_0 \leq |z - z_0| \leq R_1$. Acting as in (a) we come to

$$\phi_1(z) = \frac{1}{2\pi i} \int_{\Gamma_1} [t - z]^{-1} [dt]\, \phi(t) = \sum_{k \geq 0} [z - z_0]^k\, c_k,$$

where c_k is defined as in (2.27) for Γ_1. By cauchy theorem the latter circle can be changed to Γ.

For the second integral use

$$[t - z]^{-1} = -[z - z_0]^{-1} \sum\nolimits_{k \leq 0} [t - z_0]^{-k}\, [z - z_0]^k,$$

which converges absolutely and uniformly for $t \in \Gamma_0, \quad |z - z_0| \geq R_0$. As above we get

$$\Phi_0(z) = \frac{1}{2\pi i} \int_{\Gamma_0} [t - z]^{-1} [dt]\, \phi(t) = -\sum\nolimits_{k \leq -1} [z - z_0]^k\, c_k,$$

where c_k is defined as in (2.27) for Γ_0, which again can be changed by Γ. Substituting this into (2.29) we get (2.27).

Point z_0 is called an isolated singular point of function ϕ, if domain D contains some punctured neighbourhood $\{z,\ 0 < |z - z_0| < R\}$. We say that ϕ is of order r in z_0 if function $|\phi(z)||z - z_0|^{-r}$ is bounded in a neighbourhood of z_0.

Theorem 2.3 permits to give another version of this definition. Namely, function ϕ is of order r at z_0 if and only if $c_k = 0$ for $k < r$ in (2.17) or, equivalently, function $[z - z_0]^{-r}\phi(z)$ is hyperanalytic in a neighbourhood of z_0. In particular, for $r \geq 0$ the singular point z_0 is removable. Indeed, if ϕ has order r at z_0, then by (2.25), (2.27) we have estimate

$$|c_k| \leq M\, R^{r-k}, \tag{2.30}$$

where constant $M > 0$ depends only on k and $R_0 \leq R \leq R_1$. Since R_0 can be taken arbitrarily small this implies that $c_k = 0$ for $k < r$. The converse is evident.

Another corollary of Theorem 2.3 is the following uniqueness result: if $\phi(z_n) = 0$ and sequence $\{z_n\}$ has an accumulation point in D then $\phi \equiv 0$ in D. Combined with Theorem 2.2(b) this gives another version of uniqueness theorem: if function $\phi \in \mathbb{C}(\overline{D})$ is hyperanalytic in D and vanishes on a sub-arc of the boundary then $\phi \equiv 0$ in D. The proof is the same as in the analytic case [19].

As usual one may add ∞ to $\mathbb{C}$ and get the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with the natural topology. The concept of isolated singular point is also applicable for $z_0 = \infty$. The order r at ∞ is determined by the boundedness at ∞ of the function $|z|^{-r}|\phi(z)|$. Taking in (2.30) the radius R to tend to ∞ we see that this is equivalent to $c_k = 0$ in (2.27). If $r \leq 0$ point ∞ is called a removable singularity. For $r = -1$, it is usually said that ϕ vanishes at ∞ as in the case of analytic functions.

Similarly, one gets the Liouville theorem: if ϕ is hyperanalytic and bounded in the whole plane then ϕ is constant. Indeed, one has the estimate (2.20) with $r = 0$. Taking $R \to 0$ for $k < 0$ and $R \to \infty$ for $k > 0$, we get $c_k = 0,\ k \neq 0$.

As a corollary we get decomposition into simple fractions. Namely, if ϕ is hyperanalytic on the Riemann sphere except the points $z_j,\ j = 1, \dots, n-1,\ z_n = \infty$, where it has finite orders $r_j \leq 0,\ j = 1, \dots, n-1,\ r_n \geq 0$, then

$$\phi(z) = \sum_{j=1}^{n} \sum_{k=0}^{r_j} [z - z_j]^k \, c_{kj}.$$

In conclusion we remark that estimates for the radii in Theorem 2.3 are not exact. Suppose J has a single eigenvalue ν. If $\nu = i$ then $a = b = 1$ in (2.24) and situation with the convergence radii is analogous to the analytic case.

In general, put $\tilde{\phi}(\tilde{z}) = \phi(z), \quad \tilde{z} = x + \nu y$. Function $\tilde{\phi}$ is defined and $\tilde{J}$-analytic in $\tilde{D}$ which is the image of D under transformation $z \to \tilde{z}$, where $\tilde{J} = i\nu^{-1}J$. Taking into account (2.24) we conclude that exact conditions on radii are that the circle $\{|\tilde{z} - \tilde{z}_0| \leq R\,a\}$ in Theorem 2.3(a) and the annulus $\{R_0 b \leq |\tilde{z} - \tilde{z}_0| \leq R_1\, a\}$ in Theorem 2.3(b) are contained in D.

2.4. **Indefinite integral.** In the class of functions hyperanalytic in D consider equation

$$\phi^{(k)} = \psi, \tag{2.31}$$

defined by the operator of k-th derivative. For $\psi = 0$, all solutions are J-analytic polynomials

$$p(z) = \sum_{j=0}^{k-1} [z]^j \, c_j$$

of degree not exceeding $k-1$ as follows from Theorem 2.3(a). This set of polynomials is denoted by P_{k-1}.

Theorem 2.4. *If D is simply connected then the integral*

$$\phi(z) = \frac{1}{(k-1)!} \int_{z_0}^{z} [z - t]^{k-1} [dt] \psi(t) \tag{2.32}$$

does not depend on the integration path and defines a solution of (2.31). For an arbitrary D, function $\phi(z)$ does not depend on the integration path if and only if

$$\int_{\Gamma} [t]^i [dt] \psi(t) = 0, \quad i = 0, 1, \dots, k-1, \tag{2.33}$$

for any contour $\Gamma \subseteq D$.

Proof. We prove first the second part of the theorem. For $k = 1$, this statement is nearly evident. Indeed, according to (2.1), (2.3) for $t = \xi + i\eta$, expression

$$[dt]\, \phi'(t) = \frac{\partial \phi}{\partial \xi} \, d\xi + \frac{\partial \phi}{\partial \eta} \, d\eta$$

is a full differential of function ϕ so that

$$\int_{z_1}^{z_2} [dt]\, \phi'(t) = \phi(z_2) - \phi(z_1).$$

In particular, closing the arc $\overline{z_1 z_2}$ we get that (2.24) is valid for $\psi = \phi'$. Conversely, let ψ satisfy this condition. Let $\gamma_j,\ j = 1, 2$, be two arcs joining points z_0 and z in domain D. Choose a similar curve γ_0 so that the curves $\Gamma_j = \gamma_j \cup \gamma_0, j = 1, 2$, are contours, i.e. homeomorphic to the circle. Putting in (2.26) $\Gamma' = \Gamma_j$, we conclude that integral

$$\phi(z) = \int_{z_0}^{z} [dt]\, \psi(t)$$

does not depend on integration path. The fact that $\phi'(t) = \Psi(t)$ in the sense of definition (2.2) permits direct verification.

The general case $k \geq 2$ reduces to the preceding one. For $1 \leq j \leq k-1$, we have

$$[t]^j\, \phi^{(k)} = \left([t]^j\, \phi^{(k-1)}\right)' - j\, [t]^{j-1}\, \phi^{(k-1)}.$$

Repeating this procedure we see that, for $\psi = \phi^{(k)}$, functions $[t]^j \psi$ are full derivatives. Thus conditions (2.24) are necessary for solvability of equation (2.21) in the class of single-valued functions. Conversely, assume that these conditions are satisfied. Writing out expression $[z-t]^{k-1} = ([z]-[t])^{k-1}$ rewrite (2.23) in the form

$$\phi(z) = \sum_{j=0}^{k-1} \frac{1}{j!\,(k-1-j)!}\, [z]^{k-1-j} \int_{z_0}^{z} [t]^j\, [dt]\, \psi(t).$$

As was shown, here integrals do not depend on integration path. As above, direct verification shows that function (2.23) is differentiable in the sense of (2.1) and derivative ϕ' is defined by the same expression where k is substituted by $k-1$. Repeating this procedure we finish the proof.

As to the first part of the theorem, by Jordan's famous theorem a simple contour Γ decomposes the plane in two parts one of which is bounded and the second is a neighbourhood of ∞. Hence, if D is simply connected and $\Gamma \subseteq D$, then the domain inside Γ entirely lies in D so by Cauchy theorem condition (2.33) is automatically fulfilled.

If condition (2.33) is not fulfilled then integral (2.32) depends on the integration path and defines a multi-valued function. An example of such kind in domain $\mathbb{C} \setminus \{0\}$ gives function

$$\phi(z) = \ln[z]\; p(z), \quad p \in P_{k-1}, \tag{2.34}$$

first factor of which was already considered in subsection 2.1. Under a turn around $z = 0$ counterclockwise its element $\phi(z) = \phi_0(z)$ considered in a neighbourhood of $z = 1$, transforms into $\phi(z) = \phi_0(z) + 2\pi i p(z)$. On the other hand, the k-th derivative $\psi = \phi^{(k)}$ of this function is given by equality

$$\psi(z) = \sum_{s=1}^{k} \frac{(-1)^{k-1}}{(k-1)!} \binom{k}{s} [z]^{-s} p^{(k-s)}(z),$$

which defines a single-valued function.

Thus by Theorem 2.4 one of conditions (2.33) for a circle Γ around $z = 0$ should fail.

Functions of the form (2.34) enable one to describe the branching of integral (2.32) in a multiply-connected domain. Recall that domain $D \subseteq \mathbb{C}$ is called m-connected if its boundary on the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ consists of m connected components. For example, the boundary of $\mathbb{C}$ in this sense consists of one point $\{\infty\}$ while the boundary of $\mathbb{C} \setminus \{0\}$ consists of two points $z = 0$ and $z = \infty$.

Theorem 2.5. *Suppose D is m-connected, $m \geq 2$, and points z_j, $1 \leq j \leq m-1$, lie in different components of $\mathbb{C} \setminus D$. Then integral (2.32) is uniquely representable in the form*

$$\phi(z) = \phi_0(z) + \sum_{j=1}^{m-1} \ln[z - z_j] p_j(z), \tag{2.35}$$

where $\phi_0(z)$ is single-valued and $p_j \in P_{k-1}$.

Proof. Suppose $\Gamma_j \subseteq D$, $j = 1, \ldots, m-1$, surrounds z_j and leaves outside points z_k, $k \neq j$. Write (2.35) as $\phi = \phi_0 + \phi_1$ and let $\psi = \psi_0 + \psi_1$ correspond to kth derivatives of these functions.

By Cauchy theorem condition (2.33) of single-valuedness of (2.32) reduces to

$$\int_{G_j} [t]^i [dt] \psi(t) = 0, \quad i = 0, 1, \ldots, k-1, \quad j = 1, \ldots, m-1. \tag{2.36}$$

The space P_{k-1}^{m-1} of vectors $p = (p_1, \ldots, p_{m-1})$ with $p_j \in P_{k-1}$ has dimension $ks(m-1)$. To each $p \in P_{k-1}^{m-1}$ associate a vector $Lp = (L_{ij}p \in \mathbb{C}^s,\ 0 \leq i \leq k-1,\ 1 \leq j \leq m-1)$, where $L_{ij}p$ is defined by left hand side of (2.36) with respect to $\psi = \psi_1$. Thus we get a linear mapping $L: \ P_{k-1}^{m-1} \to \mathbb{C}^{sk(m-1)}$.

By Theorem 2.4, equality $Lp = 0$ means that function $\phi_1(z)$ is single-valued. As in the case of function (2.34) we verify that increment of element ϕ_1 along Γ_j equals $2\pi i p_j$. Hence function ϕ_1 is single-valued if and only if $p = 0$. Thus L is a one-to-one, hence invertible, mapping $P_{k-1}^{m-1} \to \mathbb{C}^{sk(m-1)}$.

Thus for a given right hand side ψ of (2.31) there exists unique $p \in P_{k-1}^{m-1}$ such that $L_{ij}p$ coincides with the left hand side of (2.36). By Theorem 2.4 function ϕ_0 from (2.35) is single-valued, which completes the proof.

As is clear from the proof, $p_j = 0$ in (2.35) if and only if $\psi = \phi^{(k)}$ satisfies (2.36) for considered j. In such case we say that function ϕ had no branching in the component of D' defined by z_j.

Let us consider the case when this component is $\{\infty\}$. Let D be a neighbourhood of ∞, so that D' consists of $m-1$ bounded components containing points z_j, and of ∞. Then, obviously, the following conditions are equivalent:

1) ϕ has no branching at ∞;

2) $\sum p_j = 0$ in (2.35);

3) in Laurent series of function $\psi = \phi^{(k)}$ do not appear members with degrees $[z]^j$,$1 \leq j \leq k$.

Hence if ψ has order $-k-1$ at ∞, then ϕ has order $k-1$ and in (2.32) one can take $z_0 = \infty$.

3. Elliptic systems of arbitrary order

3.1. **Representation of solutions.** Consider in domain D a system of partial differential equations of order n

$$\frac{\partial^n y}{\partial y^n} - \sum_{r=0}^{n-1} a_r \frac{\partial^n u}{\partial x^{r-n} \partial y^r} = 0 \tag{3.1}$$

with constant real coefficients $a_r \in \mathbb{R}^{l \times l}$. Under its solution is understood a real vector-function $u = (u_1, \ldots, u_l)$ which satisfies (3.1) everywhere.

Consider also matrix polynomial

$$P(z) = z^n - \sum_{r=0}^{n-1} a_r z^r \tag{3.2}$$

of degree n and scalar polynomial

$$\chi(z) = \det P(z) \tag{3.3}$$

of degree nl. By definition system (3.1) is elliptic if equation $\chi(z) = 0$ has no real roots. Since coefficients of χ are real its roots come in complex conjugate pairs so that the number

$$nl = 2s \tag{3.4}$$

is even and the upper half-plane contains exactly s roots.

Equation (3.1) is a canonical form of a more general equation

$$\sum_{r=0}^{n} a_{(r)} \frac{\partial^n u}{\partial x^{r-n} y^r} = 0.$$

Its ellipticity means that $\chi(z) = \det \sum a_{(r)} z^r$ has degree nl and its roots are non-real. In particular, matrices $a_{(0)}$ and $a_{(n)}$ are invertible and this equation can be reduced to the canonical form (3.1). The ellipticity condition is equivalent to

$$\det \left(\sum\nolimits_{r=0}^{n} \lambda_1^{n-r} \lambda_2^r a_{(r)} \right) \neq 0$$

for all non-zero $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$.

The general solution of elliptic system (3.1) can be expressed through hyperanalytic functions. This is especially visual for elliptic system

$$\frac{\partial u}{\partial y} - a \frac{\partial u}{\partial x} = 0 \tag{3.5}$$

of first order. Then polynomial (3.4) reduces to $P(z) = z - a$ so that ellipticity means that $a \in \mathbb{R}^{l \times l}$ has no real eigenvalues. In particular, $l = 2s$ is even. By Theorem 1.3 and lemmas 1.3, 1.4, matrix a can be transformed to a block-diagonal form

$$B^{-1} a B = \operatorname{diag}(J, \overline{J}), \ B = (b, \overline{b}); \quad J \in \mathbb{C}^{s \times s}, \quad b \in \mathbb{C}^{l \times s}, \tag{3.6}$$

where eigenvalues of J lie in the upper half-plane. Putting $u = bU$ transforms (3.5) into

$$\partial U / \partial y - \operatorname{diag}(J, \overline{J}) \partial U / \partial x = 0.$$

By (3.6) this substitution sends real vector u into a complex vector U of block structure $(\phi, \overline{\phi})$, where s-vector function ϕ satisfies (2.1) and $\overline{\phi}$ satisfies the complex conjugate equation. Changing now ϕ to $\phi/2$, we get the following result.

Theorem 3.1. *In notation* (3.6)*, each solution* $u = (u_1, \ldots, u_l)$ *of* (3.5) *is uniquely representable as*

$$u = \operatorname{Re} b\phi \tag{3.7}$$

with some J*-analytic function* ϕ*. This function is related to* u *by* $\phi = 2cu$*, where* $c \in \mathbb{C}^{s \times s}$*,* $(c, \overline{c}) = B^{-1}$*.*

By theorem 1.3 matrix B from (3.6) can be chosen so that (2.2) is fulfilled for matrix J. In accordance with theorem 1.4 J_k can be taken as Jordan blocks. If $J_k \in \mathbb{C}^{s_k \times s_k}$, $k = 1, \ldots, m$ and $b \in \mathbb{C}^{l \times s}$ is written as $b = (b_1, \ldots, b_m)$, $b_k \in \mathbb{C}^{l \times s_k}$, then (3.7) is transformed into representation

$$u = \sum\nolimits_{k=1}^{m} \operatorname{Re} b_k \phi_k \tag{3.8}$$

with J_k-analytic function s ϕ_k.

Consider now an elliptic system of order $n \geq 2$. By subsection 1.3, with polynomial (3.2) is associated the matrix $A \in \mathbb{R}^{nl \times nl}$ written as in (1.24):

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_0 & a_1 & a_2 & \ldots & a_{n-1} \end{pmatrix}. \tag{3.9}$$

By theorem 1.5 it can be brought to a block-diagonal form

$$\begin{aligned} &B^{-1}AB = \operatorname{diag}(J, \overline{J}), \quad B = (B_0, \overline{B}_0), \\ &B_0 = \downarrow (b, bJ, \ldots, bJ^{n-1}), \quad b \in \mathbb{C}^{l \times s}. \end{aligned} \tag{3.10}$$

As shows lemma 1.5, in notation (3.3), (3.4) eigenvalues of $J \in \mathbb{C}^{s \times s}$ are the roots of characteristic equation $\chi(\nu) = 0$ in upper half-plane and their multiplicities and orders coincide. From theorem 1.5 it follows also that matrices $b \in \mathbb{C}^{l \times s}$ and $J \in \mathbb{C}^{s \times s}$ can be considered as solutions of

$$bJ^n = \sum\nolimits_{r=0}^{n-1} a_r b J^r. \tag{3.11}$$

If s-vector-function $\phi(z)$ satisfies (2.1) then from (2.6) and (3.11) follows that l-vector-function $u = \operatorname{Re} b\phi$ is a solution of (3.1). As the following theorem shows, the converse is also true but, unlike to theorem 3.1, function ϕ is in general multi-valued.

Consider the class P_{n-2} of those polynomials $p(z) = c_0 + [z]c_1 + \ldots + [z]^{n-2}c_{n-2}$, $c_k \in \mathbb{C}^s$ which are J-analytic function s. The following decomposition is standard [5].

Lemma 3.1. (a) *The space P_{n-2} decomposes into direct sum of two subspaces*

$$\begin{aligned} &P^0 = \{p \in P_{n-2} \,|\, \operatorname{Re} bJ^r p^{(r)}(0) = 0, \quad 0 \leq k < r \leq n-1\}, \\ &P^1 = \{p \in P_{n-2} \,|\, \operatorname{Re} bJ^r p^{(r)}(0) = 0, \quad 0 \leq r \leq k < n-1\}, \end{aligned} \tag{3.12}$$

having the same dimension $(n-1)s$ (over $\mathbb{R}$).

(b) *For $p \in P_{n-2}$, function* $\operatorname{Re} bp \equiv 0$ *if and only if* $p \in P^0$. *In particular, each polynomial*

$$q(x, y) = \sum_{i+j \leq n-2} c_{ij} x^i y^j, \quad c_{ij} \in \mathbb{R}^l \tag{3.13}$$

is uniquely representable in the form $q = \operatorname{Re} bp$, $p \in P^1$.

Proof. (a) Put $L_{k,r}p = \operatorname{Re} bJ^r p^{(r)})(0)$ and consider linear mappings $L^0 = (L_{kr},\ 0 \leq k < r \leq n-1\}$ and $L^1 = (L_{kr},\ 0 \leq r \leq k < n-1\}$. As $2s = nl$ they both act as $P_{n-2} \to \mathbb{R}^{(n-1)s}$. Hence $L = (L^0, L^1)$ acts between spaces of dimension $2(n-1)$.

Let $Lp = 0$ for some $p \in P_{n-2}$. Then we have

$$\operatorname{Re} bJ^r p^{(k)}(0) = 0, \quad 0 \le r \le n-1,$$

for all $k = 0, 1, \dots, n-2$. Using notation (3.6) this can be rewritten in the form

$$Bx_k = 0, \quad x_k = (p^{(k)}(0), \overline{p^{(k)}(0)} \in \mathbb{C}^{nl}.$$

Since B is invertible $x_k = 0$, hence $p = 0$.

Thus L is an isomorphism between P_{n-2} and $\mathbb{R}^{(n-1)s} \times \mathbb{R}^{(n-1)s}$. In particular, $P^0 \cap P^1 = 0$ и $\dim P^j = (n-1)s$ so that $P_{n-2} = P^0 \oplus P^1$.

(b) Condition $\operatorname{Re} bp = 0$ is equivalent to

$$\frac{\partial^k}{\partial x^{k-r} \partial y^r} (\operatorname{Re} bp)(0) = 0, \quad 0 \le r \le k \le n-2.$$

By (2.6) and (3.12) they are in turn equivalent to $p \in P^0$. Since the space X of polynomials (3.14) has dimension $(n-1)s$ this implies that mapping $p \to \operatorname{Re} bp$ gives an isomorphism between P^1 and X.

Theorem 3.2. *(a) In notation* (3.10) *each solution $u(x, y)$ of* (3.1) *in simply connected domain D is representable in the form* (3.7) *with some J-analytic function ϕ and $u = 0$ implies $\phi \in P_{n-2}$. Function ϕ can be recovered from u by*

$$\phi^{(n-1)} = 2 \sum_{r=0}^{n-1} c_r \frac{\partial^{n-1} u}{\partial x^{n-1-r} \partial y^r}, \tag{3.14}$$

where for $B^{-1} = \downarrow (C_0, \overline{C}_0)$, matrices $c_r \in \mathbb{C}^{s \times l}$ are defined by $C_0 = (c_0, \dots, c_{n-1})$. This function is uniquely representable as $\phi_0 + p_0$, where $p_0 \in P^1$ and ϕ_0 satisfies conditions

$$\phi_0^{(r)}(z_0) = 0, \quad 0 \le r \le n-2 \tag{3.15}$$

at a fixed point $z_0 \in D$.

(b) Let D be m-connected, $m \ge 2$, and points z_j, $j = 1, \dots, m-1$ belong to different components of $\overline{\mathbb{C}} \setminus D$. Then in (3.7) *function ϕ is multi-valued and uniquely representable in the form*

$$\phi(z) = \phi_0(z) + p_0(z) + \sum_{j=1}^{m-1} \ln[z - z_j] p_j(z), \tag{3.16}$$

where function ϕ_0 is single-valued and satisfies (3.15)*, and*

$$p_0 \in P^1, \quad p_j \in P^0, \ j = 1, \dots, m-1. \tag{3.17}$$

Proof. (a) Let l-vector-function $u \in C^n(D)$ satisfy (3.1). Set

$$U = (U_0, \dots, U_{n-1}), \quad U_r = \frac{\partial^{n-1} u}{\partial^{n-1-r} x \partial^r y}. \tag{3.18}$$

In this notation (3.1) takes the form

$$\frac{\partial U_{n-1}}{\partial y} - \sum_{r=0}^{n-1} a_r \frac{\partial U_r}{\partial x} = 0.$$

Adding $n-1$ evident relations

$$\frac{\partial U_r}{\partial y}=\frac{\partial U_{r+1}}{\partial x},\quad r=0,1,\ldots,n-2,$$

in accordance with (3.9) for function U we get the system

$$\frac{\partial U}{\partial y}-A\frac{\partial U}{\partial x}=0.$$

Applying theorem 3.1 with notation (3.11) we arrive to $U=\operatorname{Re} B_0\psi$ with some J-analytic function ψ. Taking into account (3.10), (3.17) this representation can be rewritten as

$$\frac{\partial^{n-1}u}{\partial x^{n-1-r}\partial^r y}=\operatorname{Re} bJ^r\psi,\quad r=0,1,\ldots,n-1.$$

Let a J-analytic function ϕ be a solution of equation $\phi^{(n-1)}=\psi$, considered in subsection 2.4. Then by (2.6) we have

$$\frac{\partial^{n-1}u}{\partial x^{n-1-r}\partial y^r}(\operatorname{Re} b\phi)=\frac{\partial^{n-1}u}{\partial x^{n-1-r}\partial y^r}$$

hence $u-\operatorname{Re} b\phi$ is a polynomial of the form (3.14). Applying lemma 1(b) we get the following conclusion.

Each solution u of system (3.1) is representable in the form (3.7) with some multi-valued J-analytic function ϕ and its $(n-1)$th derivative (18) is single-valued and related via (3.14) with the vector (3.18). In particular, if $u=0$ then $U=\psi=0$ hence $\phi=p\in P_{n-2}$. In fact, by lemma 1(b) one gets $p\in P^1_{n-2}$.

(b) By theorem 2.5 function ϕ can be uniquely represented in the form (2.35). It is convenient to decompose ϕ_0 in two summands one of which satisfies (3.15) and is again denoted ϕ_0 while the second belongs to P_{n-2}. Then for u we get representation (3.16) where ϕ_0 is single-valued and satisfies (3.15) and $p_j\in P_{n-2}$, $0\le j\le m-1$.

Let contours Γ_j, $j=1,\ldots,m-1$ be the same as in the proof of theorem 2.5. Along Γ_j element ϕ gets the increment $p_j(z)$. As function u is single-valued in multiply connected domain D we have relations $\operatorname{Re} bp_j\equiv 0$, $j=1,\ldots,m-1$. By lemma 1(b) they are equivalent to $p_j\in P^0$. As to polynomial $p_0\in P_{n-2}$ in (3.16), by lemma 2.1(b) it can be chosen to satisfy $p_0\in P^1$.

It remains to show that representation (3.16) with additional conditions (3.15), (3.17) is unique. If $u\equiv 0$ then as mentioned above we have $\phi=p\in P_{n-2}$. In particular, function

$$p-\phi_0-p_0=\sum_{j=1}^{m-1}\ln[z-z_j]p_j$$

is single-valued which is only possible if $p_1=\ldots=p_{m-1}=0$. Thus $p=\phi_0+p_0$ and by (3.15) we get $p=p_0$, $\phi_0=0$ so that $\operatorname{Re} bp_0=0$, $p_0\in P^1$. By lemma 3.1 this implies $p_0\in P^0\cap P^1$, i.e. $p_0=0$. Hence $u=0$ in (3.16) implies $\phi_0=0$ and $p_j=0$, $0\le j\le m-1$.

Suppose that D is a neighbourhood of ∞ and partial derivatives of $(n-1)$th order of solution $u(x,y)$ to (3.1) satisfy the following estimate in a neighbourhood ∞:

$$\left|\frac{\partial^{n-1}u}{\partial x^{n-1-r}\partial y^r}\right|\le C|z|^{-n},\quad 0\le r\le n-1 \tag{3.19}$$

with a constant $C > 0$. In virtue of (3.14) function ψ satisfies ana analogous estimate, i.e. is of order $-n$ at ∞. As was mentioned in subsection 2.4, in this case ϕ is of order $n-1$ at ∞ and has no branching at ∞. This is equivalent to $\sum p_j = 0$ in (3.16). Correspondingly, conditions (3.15) can be changed to $\phi_0(\infty) = 0$.

As in the case of first order systems, the choice of matrix B in (3.6) can be made so that (2.2) is satisfied. Then assuming $J_k \in \mathbb{C}^{s_k \times s_k}$, $b = (b_1, \ldots, b_m)$, $b_k \in \mathbb{C}^{l \times s_k}$, representation (3.16) can be written in the form (3.8) with J_k-analytic matrices ϕ_k. Expression for ϕ_k in (3.16) is written with respect to J_k and conditions (3.15), (3.17) are understood in this sense.

Substituting representation (2.10) instead of ϕ in (3.7), with s_k-vector-function s ψ_k, $k = 1, \ldots, m$, we get a representation

$$u(x, y) = \mathrm{Re} \sum_{k=1}^{m} \sum_{r=1}^{s-1} b_k \frac{y^r}{r!} (J_k - \nu_k)^r \psi_k^{(r)}(x + \nu_k y) \tag{3.20}$$

of the general solution to (3.1) in terms of s-vector $\psi = (\psi_1, \ldots, \psi_m)$, with the components ψ_k which are analytic in $D_k = \{x+\nu_k y \,|\, x+iy \in D\}$. This representation was obtained by A.Bitsadze [11].

3.2. **Complex systems.** Let coefficients of (3.1) be complex: $a_r \in \mathbb{C}^{l \times l}$. Then solution u is a complex l-vector-function . Ellipticity condition is defined analogously but the number nl is not necessarily even and the roots of characteristic equation can be arbitrarily distributed in upper half-plane and lower half-plane . The corresponding amounts of roots are denoted $s^{\pm}$. Instead of (3.4) here one has $s^+ + s^- = nl$. Cases when $s^+ = 0$ or $s^- = 0$ are not excluded. If $s^+ = s^-$ then equation (3.1) is called *correctly elliptic.*

For a first order elliptic system (3.50), ellipticity again means that $a \in \mathbb{C}^{s \times s}$ does not have real eigenvalues. In particular, equation (2.1) defining J-analytic function s is elliptic with $s^+ = s$, $s^- = 0$.

By theorem 1.3 matrix a can be reduced to a block-diagonal form analogous to (3.6)

$$\begin{gathered} B^{-1} a B = \mathrm{diag}(J^+, \overline{J^-}), \quad B = (b^+, \overline{b^-}), \\ J^{\pm} \in \mathbb{C}^{s^{\pm} \times s^{\pm}}, \quad b^{\pm} \in \mathbb{C}^{l \times s^{\pm}}. \end{gathered} \tag{3.20}$$

As a result we arrive at representation

$$u = b^+ \phi^+ + \overline{b^- \phi^-} \tag{3.21}$$

of the general solution to system (3.5) via a pair of $J^{\pm}$-analytic function s $\phi^{\pm}$.

Matrix B in (3.20) can be chosen so that $J^{\pm}$ satisfy condition

$$J^{\pm} = \mathrm{diag}(J_1^{\pm}, \ldots, J_{m^{\pm}}^{\pm}), \quad \sigma(J_k^{\pm}) = \nu_k^{\pm}. \tag{3.22}$$

Then (3.21) changes to

$$u = \sum\nolimits_k b_k^+ \phi_k^+ + \sum\nolimits_k \overline{b_k^- \phi_k^-}. \tag{3.23}$$

The same takes place for systems of order $n \geq 2$. In this case matrix (3.9) is complex and (3.10) is substituted by

$$\begin{gathered} B^{-1}AB = \operatorname{diag}(J^+, \overline{J^-}), \quad B = (B^+, \overline{B^-}), \\ B^{\pm} =\downarrow (b^{\pm}, b^{\pm}J^{\pm}, \ldots, b^{\pm}(J^{\pm})^{n-1}), \quad b^{\pm} \in \mathbb{C}^{l\times s^{\pm}}. \end{gathered} \tag{3.24}$$

Let $[z]^{\pm}$ be defined by $J^{\pm}$ as in (2.3) and P_{n-2} be the space of all pairs $p = (p^+, p^-)$, of $J^{\pm}$-analytic polynomials $p^{\pm}$ of degree $n-2$. Correspondingly, consider in P_{n-2} subspaces P^j, $j = 0, 1$, changing $\operatorname{Re} bJ^r p^{(r)}$ in (12) by expression

$$b^+J^+(p^+)^{(k)} \; + \; \overline{b^-J^-(p^-)^{(k)}}.$$

Then analogously to subsection 3.1 we get the following analog of theorem 3.2.

Theorem 3.3. *In conditions of theorem 3.2, each solution $u(x,y)$ of a complex equation (3.1) is uniquely representable in the form (3.21) with $J^{\pm}$-analytic function s*

$$\phi^{\pm}(z) = \phi_0^{\pm}(z) + p_0^{\pm}(z) + \sum_{j=1}^{m-1} \ln[z - z_j]^{\pm} p_j^{\pm}(z), \tag{3.25}$$

where $\phi_0^{\pm}$ and $p_j = (p_j^+, p_j^-)$ satisfy conditions (3.15), (3.16).

If matrix B in (3.24) is chosen so that $J^{\pm}$ are block-diagonal of the form (3.22) then representation (3.21) transforms into (3.230 with representation s (3.25) for $\phi_k^{\pm}$. For real systems we have $s^{\pm} = s$, $J^{\pm} = J$, $b^{\pm} = b$, so that up to the factor 2 representation (3.21) transforms into (3.16).

Each complex system can be reduced to a real one with respect to $2l$-vector $\hat{u} = (\operatorname{Re} u, \; \operatorname{Im} u)$. The easiest way to do that, is to add the complex conjugate of equation (3.1) and take into account the connection

$$\hat{u} \; = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \tilde{u}$$

between $\hat{u}$ and $\tilde{u} = (u, \overline{u})$. Then we come to a system

$$\frac{\partial^n \hat{u}}{\partial y^n} \; = \sum_{r=0}^{n-1} \hat{a}_r \, \frac{\partial^n \hat{u}}{\partial x^{n-r}\, \partial y^r} \tag{3.26}$$

with real coefficients

$$\hat{a}_r \; = \begin{pmatrix} \operatorname{Re} a_r & -\operatorname{Im} a_r \\ \operatorname{Re} a_r & \operatorname{Re} a_r \end{pmatrix}.$$

Let $\hat{P}$ and $\hat{A}$ are, respectively , the characteristic polynomial (3.2) and associated matrix (3.9) for this system. Since

$$\hat{a}_r \; = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}^{-1} \begin{pmatrix} a_r & 0 \\ 0 & \overline{a}_r \end{pmatrix} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix},$$

we have

$$\hat{P}(z) \; = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}^{-1} \begin{pmatrix} P(z) & 0 \\ 0 & \overline{P(\overline{z})} \end{pmatrix} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix},$$

$$\hat{A} \; = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}^{-1} \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}.$$

In particular, the characteristic equation of system (3.26) transforms into $\det P(z)\det P(\overline{z}) = 0$ and matrix $\hat{A}$ can be reduced to the form (10), where $J = \operatorname{diag}(J^+, J^-)$.

3.3. **The case of one equation.** For $l = 1$, system (3.1) transforms into a scalar equation with coefficients $a_r \in \mathbb{C}$. In this case it is completely determined by its characteristic equation $\det P(z) = 0$ which can be written as

$$\prod\nolimits_{k=1}^{m^+}(z-\nu_k^+)^{s_k^+}\prod\nolimits_{k=1}^{m^-}(z-\overline{\nu_k^-})^{s_k^-} = 0, \tag{3.27}$$

where $\operatorname{Im}\nu_k^\pm > 0$ and $s^+ + s^- = n$, $s^\pm = \sum s_k^\pm$. Denote by $J_k^\pm$ the Jordan block of order $s_k^\pm$ with eigenvalue $\nu_k^\pm$. According to subsection 1.3 in this case matrix B from (3.24) can be chosen so that $J^\pm$ have Jordan normal form (3.22). Then $b^\pm = (b_1^\pm, \ldots, b_{m^\pm}^\pm)$ with row-matrices $b_k^\pm = (1, 0, \ldots, 0) \in \mathbb{C}^{1\times s_k^\pm}$. Correspondingly, representation (3.23) for solution of scalar equation (3.1) in the formulation of theorem 3.3 transforms to

$$u = \sum\nolimits_{k=1}^{m^+}(\phi_k^+)_1 + \sum\nolimits_{k=1}^{m^-}\overline{(\phi_k^-)_1}, \tag{3.28}$$

where $(\phi_k^\pm)_1$ denotes the first component of $s_k^\pm$-vector $\phi_k^\pm$.

As an example consider equation

$$\left(\frac{\partial}{\partial\overline{z}}\right)^{s^+}\left(\frac{\partial}{\partial z}\right)^{s^-} u = 0, \quad s^+ + s^- = n, \tag{3.29}$$

(adopting complex notation (2.15) for derivatives). Solutions of this equation are called $(s^+, s^-)-$polyanalytic functions. They were thoroughly studied in [20]. For $s^\pm = s$, we get equation

$$\Delta^s u = 0, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Its solutions are called s-polyharmonic functions (harmonic for $s = 1$).

Let $J^\pm$ be Jordan cells of order $s^\pm$ with eigenvalues $\nu = i$. Then representation (3.28) for equation (3.29) transforms into

$$u = (\phi^+)_1 + \overline{(\phi^-)_1} \tag{3.30}$$

with $J^\pm-$analytic $s^\pm-$vector-functions $\phi^\pm$. In the case of polyharmonic equation matrices $J^\pm = J$ coincide and (3.30) reduces to

$$u = \operatorname{Re}(\phi)_1 \tag{3.31}$$

with a J-analytic function ϕ.

By a remark at the end of subsection 2.1 function ϕ can be expressed via an analytic vector-function ψ by the formula (2.17) with nilpotent matrix $Q = (1 - iJ)^{-1}(1 + iJ)$. Evidently, this matrix is triangular with zeroes on the diagonal. Consider the first component of vector equality (2.17):

$$(\phi)_1 = \sum_{r=0}^{s-1}\overline{z}^r\,\psi_r(z) \tag{3.32}$$

with scalar analytic functions $\psi_r = (1/r!)(Q^r\,\psi^{(r)})_1$. For $s = 2$, substitution of (3.32) into (3.31) gives the well known formulas of Goursat [10] for solutions of biharmonic equation. Analogously, combining (3.32) with (3.30) one gets a representation for (s^+, s^-)-polyanalytic functions

$$u(z) = \sum_{r=0}^{s^+-1} \overline{z}^r \, \psi_r^+(z) + \sum_{r=0}^{s^- -1} z^r \, \overline{\psi_r^-}(z)$$

using a family $(\psi_r^\pm)$ of analytic functions. This representation can be obtained by a direct integration of equation (3.29) if one considers z and $\overline{z}$ as independent variables, which is admissible in the case of real analytic functions.

3.4. **Application to boundary value problems.** Results of subsection 3.1 enable one to reduce the general boundary value problem for equation (3.1) to a boundary value problem for J-analytic functions. Using Cauchy integral from subsection 2.2 one can further reduce the latter problem to a system of singular integral equations. This program was realized in [21], [22] for domains with smooth and piecewise smooth boundaries, respectively. For this reason below we only outline the reduction scheme. Details can be found in [21].

Consider equation (3.1) in domain $D \subseteq \mathbb{C}$ with sufficiently smooth boundary $\Gamma = \partial D$. Define in Γ differential operators

$$B_i = \sum_{k+r \le n_i} B_{ikr} \frac{\partial^{k+r}}{\partial x^k \partial y^r}, \quad i = 1, \ldots, s$$

of orders $0 \le n_1 \le n_2, \ldots \le n_s = n_\Gamma$ ($n_\Gamma \ge n$ is not excluded). Here $B_{ikr}(t)$, $t \in \Gamma$ denote sufficiently smooth $1 \times l$-matrix-functions on Γ.

We seek for a solution $u \in C^n(\overline{D})$ to (3.1) satisfying boundary conditions

$$(B_i u)\big|_\Gamma = f_i, \quad i = 1, \ldots, s, \tag{3.33}$$

where f_i are given l-vector-functions on Γ.

Problem (3.1), (3.33) is called *Fredholm* (elliptic) if

(a) the homogeneous problem has a finite number k of linearly independent solutions (in the chosen class);

(в) there exist $k' < \infty$ linearly independent functionals on the given space of right hand sides such that their vanishing on $f = (f_1, \ldots, f_s)$ is necessary and sufficient for the solvability of the problem.

The difference $\Lambda = k - k'$ is as usual called the *index* of the problem.

Substituting the representation (3.7), (3.16) of the general solution to (3.1) into the boundary condition this problem can be equivalently reduced to the corresponding problem for pairs (ϕ, p) consisting of a J-analytic function $\phi \in C^n(\overline{D})$ and a family $p = (p_j,\ 0 \le j \le m-1)$ of polynomials $p_0 \in P^1$ and $p_j \in P^0$, $j = 1, \ldots, m-1$.

Summands with p_j in the boundary condition do not influence ellipticity of the problem. For this reason the problem (3.1), (3.33) is Fredholm equivalent to the problem

$$\mathrm{Re} \sum_{k+r \le n_i} [B_{ikr} b J^r \phi^{(k+r)}]\big|_\Gamma = f_i, \quad i = 1, \ldots, s$$

for J-analytic function $\phi \in C^{n_\Gamma}(\overline{D})$.

Let d/ds denote the differentiation operator on Γ with respect to arclength. Action of this operator does not influence ellipticity. Thus we can pass to a Fredholm equivalent problem

$$\operatorname{Re} \sum_{k+r\le n_i} \left(\frac{d}{ds}\right)^{n_\Gamma - n_i} \left[B_{ikr} b J^r \phi^{(k+r)}\right]\Big|_\Gamma = \tilde{f}_i, \;\; \tilde{f}_i = \left(\frac{d}{ds}\right)^{n_\Gamma - n_i} f_i \tag{3.34}$$

with homogenized orders of differentiations.

Let $s = (s_1, s_2)$ denote the unit tangent vector on Γ oriented positively with respect to D. It is a sufficiently smooth 2-vector-function . If the differential operator d/ds is considered as a boundary operator $s_1 \partial/\partial x + s_2 \partial/\partial y$ then

$$\left(\frac{d}{ds}\right)^k = \left(s_1 \frac{\partial}{\partial x} + s_2 \frac{\partial}{\partial y}\right)^k + \dots \;,$$

where dots denote lower order terms depending only on derivatives of functions s_1, s_2 with respect to arclength. For (3.34) this gives a relation

$$\operatorname{Re} \sum_{k+r\le n_i} [B_{ikr} b J^r (s_1 + s_2 J)^{n_\Gamma - n_i} \phi^{(n_\Gamma)} + \dots] = \tilde{f}_i.$$

Presence of the lower order terms in this boundary condition does not influence ellipticity of the problem. Hence the original problem (3.10), (3.33) is Fredholm equivalent to Riemann-Hilbert problem

$$\operatorname{Re} G\tilde{\phi}\big|_\Gamma = \tilde{f}, \quad \tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_s). \tag{3.35}$$

for J-analytic vector-function $\tilde{\phi} = \phi^{(n_\Gamma)}$. Here an $s \times s$-matrix-function G is defined by

$$G = \downarrow (G_1, \dots, G_s), \quad G_i = \sum_{k+r\le n_i} B_{ikr} b J^r (s_1 + s_2 J)^{n_\Gamma - n_i}. \tag{3.36}$$

For analytic vector-functions, there exists a well-known method of investigating boundary value problem (3.35) based on the use of Cauchy integrals and singular integral equations [23]. As was already mentioned, this method can be extended to J-analytic functions [21]. The condition

$$\det G(t) \ne 0, \quad t \in \Gamma, \tag{3.37}$$

is equivalent to the ellipticity of the problem. For matrix-function (3.36), this condition is just a different form of the famous Shapiro-Lopatinski condition for boundary value problem (3.1), (3.33). As a rule, problem (3.35) is considered in the class $C^{+0}(D)$ of functions satisfying Hölder condition in $\overline{D}$ (i.e. belonging to C^μ with some $0 < \mu < 1$). Correspondingly, the initial problem (3.1), (3.33) may be naturally considered in the class $C^{n_\Gamma,+0}(\overline{D})$.

If D coincides with the upper half-plane the affine transformation (3.7) preserves D so that in the conditions of theorem 2.1 analytic functions ψ_k are defined in the whole D.

Let us check that for $\phi \in C^{+0}(\overline{D})$ the analytic function ψ in representation (2.10) belongs to the same class and coincides with ϕ on the boundary $y = 0$ of D.

Lemma 3.2. *Let function* $\psi(z)$, $z = x + iy$ *be analytic in* D *and satisfy Hölder condition*

$$\{\psi\}_\mu = \sup_{z_1 \neq z_2} \frac{|\psi_(z_1) - \psi(z_2)|}{|z_1 - z_2|^\mu} < \infty \tag{3.38}$$

with some $0 < \mu < 1$. *Then* J*-analytic function* ϕ *in representation* (2.10) *satisfies an analogous condition* $\{\phi\}_\mu < \infty$ *and coincides with* ψ *on the boundary* $y = 0$ *of the half-plane* D. *The converse is also true.*

Proof. Applying Cauchy formula for to the hemi-circle $\{|z| < R, \ \operatorname{Im} z > 0\}$, differentiating it, and taking the limit as $R \to \infty$ we get

$$\psi^{(r)}(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\psi(t) - \psi(t_0)}{(t - z)^{r+1}} dt, \quad r = 1, 2, \ldots, \tag{3.39}$$

for each $t_0 \in \mathbb{R}$. Hence

$$|\psi^{(r)}(t_0 + iy)| \leq \frac{\{\psi\}_\mu}{2\pi} \int_{\mathbb{R}} \frac{|t - t_0|^\mu \, dt}{|(t - t_0)^2 + y^2|^{(r+1)/2}} \leq C_r \{\psi\})_\mu \tag{3.40}$$

with some constant C_r, depending only on r and μ.

In particular, $|y^r \psi^{(r)}(z)| \leq C_r \{\psi\}_\mu y^\mu$, $r \geq 1$. Hence function ϕ from (2.10) is continuous in $\overline{D}$ and coincides with ψ on the boundary. Differentiating (2.10) and taking into account (3.40) we get the estimate $|\phi'(z)| \leq C y^{\mu - 1}$ which implies that $\{\phi\}_\mu < \infty$.

The converse is proved analogously. Let $\{\phi\}_\mu < \infty$ and analytic function ψ is given by (2.10), where y^r is changed by $(-y)^r$. By Cauchy theorem from subsection 2.2, for derivative s of J-analytic function s ϕ we have representation

$$\phi^{(r)}(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} [t - z]^{-r-1} \{\phi(t) - \phi(t_0)\} dt.$$

From this we get an estimate analogous to (3.40) for $\phi^{(r)}$. The rest of the argument remains unchanged.

We turn now to the Riemann-Hilbert problem (3.35) in half-plane D (the wave in notations is omitted). From lemma 3.2 follows that it is sufficient to solve this problem for analytic vector-function ψ and then return to ϕ using representation (2.10).

The situation is especially simple if G is constant. Let a real s-vector-function f be defined on $\mathbb{R}$ and $\{f\}_\mu < +\infty$. Then solution of problem $\operatorname{Re} G\psi = f(t)$, $t \in \mathbb{R}$ in the class of function s $\chi(z)$, $\{\psi\}_\mu < \infty$, analytic on D is given by Schwarz formula [23]

$$G\psi'(z) = \frac{1}{\pi i} \int_{\mathbb{R}} \frac{f(t) \, dt}{(t - z)^2}. \tag{3.41}$$

Strictly speaking, this formula is only applicable to function s f which are $O(|t|^{-1})$ at ∞. In general case it is sufficient to approximate f by functions of such type. From (3.41), in particular, follows that

$$\psi(z) = \frac{1}{\pi i} \int_{\mathbb{R}} \left[\frac{1}{t - z} - \frac{1}{t - z_0} \right] G^{-1} f(t) dt + \psi(z_0).$$

Substituting this expression in (2.10) we come to an explicit solution of problem (3.35) with constant matrix $G \in \mathbb{C}^{s \times s}$ for J-analytic function s ϕ from $\{\phi\}\mu < \infty$.

If $\det G = 0$ then there exists an infinite set of linearly independent analytic vector-functions $\psi \in C^\infty(\overline{D})$ for which $G\psi = 0$ on the boundary of half-plane D. These functions can be chosen to satisfy

$$|\psi^{(r)}(z)| \le C(1+|z|)^{-r-1}, \quad r = 0, 1, \ldots, s. \tag{3.42}$$

In this case formula (2.10) gives an infinity of linearly independent J-analytic functions ϕ satisfying (3.42) and boundary condition $G\phi = 0$ on the boundary of half-plane considered.

4. Elliptic systems of second order

4.1. **Strongly and weakly coupled systems.** For applications, especially important appear second order real elliptic systems

$$\frac{\partial^2 u}{\partial y^2} = a_0 \frac{\partial^2 u}{\partial x^2} + a_1 \frac{\partial^2 u}{\partial y^2}. \tag{4.1}$$

For such a system, the characteristic polynomial (3.2) is a matrix quadratic trinomial $P(z) = z^2 - a_1 z - a_0$, quantity $s = nl/2$ from (3.4) coincides with l and expressions (3.9) – (3.11) take the form

$$\begin{gathered} A = \begin{pmatrix} 0 & 1 \\ a_0 & a_1 \end{pmatrix}, \quad B = \begin{pmatrix} b & \overline{b} \\ bJ & \overline{bJ} \end{pmatrix}, \\ B^{-1}AB = \operatorname{diag}(J, \overline{J}), \quad b, J \in \mathbb{C}^{l\times l}. \end{gathered} \tag{4.2}$$

Matrices b and J can be considered as solutions to equation

$$J^2 = a_0 b + a_1 bJ, \tag{4.3}$$

where

$$\det \begin{pmatrix} b & \overline{b} \\ bJ & \overline{bJ} \end{pmatrix} \ne 0. \tag{4.4}$$

For $n = 2$, the space P_{n-2} coincides with $\mathbb{C}^l$ so that (3.12) reduces to $P^0 = \{\xi \in \mathbb{C}^l \mid \operatorname{Re} b\xi = 0\}$ and $P^1 = \{\xi \in \mathbb{C}^l \mid \operatorname{Re} bJ\xi = 0\}$.

One of the basic boundary value problems for such system is the Dirichlet problem which consists in finding a solution u to (3.33) in domain D satisfying boundary condition

$$u\big|_\Gamma = f \tag{4.5}$$

on its boundary $\Gamma = \partial D$. Unlike to the case $l = 1$ for systems this problem can appear not well-posed. This fact was discovered by A.Bitsadze. In [24] was indicated an elliptic $2\times 2-$system with coefficients

$$a_0 = 1, \quad a_1 = \pm \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \tag{4.6}$$

for which the Dirichlet problem in the unit circle has an infinity of linearly independent solutions. Later on A.Bitsadze [24] introduced a class of elliptic systems for which Dirichlet problem is Fredholm. Such systems are called *weakly coupled.* In notation (2.34) this class is defined by condition

$$\det b \ne 0. \tag{4.7}$$

Correspondingly, the systems for which this condition is not fulfilled are called *strongly coupled.*

Using (3.7) Dirichlet problem can be reduced to Riemann-Hilbert problem (3.35) with constant coefficient $G = b$. For this reason condition (4.7) determining the normal solvability of the latter problem is simultaneously an ellipticity criterion for Dirichlet problem . Acting within the scheme of subsection 3.4 it is not difficult to show that the index is equal to zero [21].

As shows a remark at the end of subsection 3.4, for strongly coupled systems the homogeneous Dirichlet problem in the upper half-plane has an infinity of linearly independent solutions. The following lemma shows that the concept of weakly coupled system is well-defined, i.e. does not depend of the choice of block matrix B in (4.2). This lemma gives an answer to a question posed by A.Bitsadze in [11].

Lemma 4.1. *System (4.1) is weakly coupled if and only if the real matrix*

$$\Delta = \int_{\mathbb{R}} (\lambda^2 - a_1\lambda - a_0)^{-1} d\lambda \tag{4.8}$$

is invertible.

Notice that due to ellipticity, matrix-function $P(\lambda) = \lambda^2 - a_1\lambda - a_0$ is invertible for $\lambda \in \mathbb{R}$ and elements of its inverse have order -2 at ∞. Thus the integral (4.8) makes sense.

Proof. For $n = 2$, identity (1.25) takes the form

$$(z - A)\begin{pmatrix} 1 & 1 \\ 0 & z \end{pmatrix} = \begin{pmatrix} z & 0 \\ -a_0 & P(z) \end{pmatrix}, \quad P(z) = z^2 - a_1 z - a_0.$$

Passing to inverse matrices and applying (3.34) we get

$$\begin{pmatrix} z^{-1} & 0 \\ z^{-1}P^{-1}a_0P^{-1} & p^{-1} \end{pmatrix} = \begin{pmatrix} 1 & -z^{-1} \\ 0 & -z^{-1} \end{pmatrix} (z - A)^{-1} =$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} B \begin{pmatrix} (z - J)^{-1} & 0 \\ 0 & (z - J)^{-1} \end{pmatrix} B^{-1} + \tag{4.9}$$

$$\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} B \begin{pmatrix} z^{-1}(z - J)^{-1} & 0 \\ 0 & z^{-1}(z - J)^{-1} \end{pmatrix} B^{-1}.$$

Consider a contour Γ in upper half-plane $\operatorname{Im} z > 0$ which embraces all eigenvalues of A. Then relations

$$\frac{1}{2\pi i}\int_\Gamma z^k(z - J)^{-1}dz = J^k, \quad \int_\Gamma z^k(z - \overline{J})^{-1}dz = 0$$

hold for each $k = 0, \pm 1, \ldots$. The first one follows from (1.11) while the second follows from Cauchy theorem since matrix-function $(z - \overline{J})^{-1}$ is analytic inside Γ.

Integrating (4.9) along Γ and using the above relations we get

$$\frac{1}{2\pi i}\int_\Gamma \begin{pmatrix} z^{-1} & 0 \\ z^{-1}P^{-1}(z)a_1 & P^{-1}(z) \end{pmatrix} dz = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} B \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} B^{-1} +$$

$$+\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} B \begin{pmatrix} J^{-1} & 0 \\ 0 & 0 \end{pmatrix} B^{-1} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} B^{-1},$$

where the block structure of B from (4.2) is taken into account. Hence

$$\int_\Gamma P_0^{-1}(z)dz = 2\pi i b c_{12}, \quad B^{-1} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

By Cauchy theorem contour Γ can be replaced by the real axis, which gives the expression

$$\Delta = 2\pi i b c_{12} \tag{4.10}$$

for integral (4.8).

The last relation shows that $\det \Delta \neq 0$ implies $\det b \neq 0$. Conversely, let $\det b \neq 0$. According to (2.34) equation $B\xi = \eta$ can be rewritten as the system $b\xi_1 + \overline{b}\xi_2 = \eta_1$, $bJ\xi_1 + \overline{bJ}\xi_2 = \eta_2$. From this we get $\xi_2 = \overline{b}^{-1}(\eta_1 - b\xi_1)$ hence $q\xi_1 = -bJ\overline{b}^{-1}\eta_1 + \eta_2$, $q = bJ - \overline{bJ}\overline{b}^{-1}b$. Since B is invertible the factor q is also invertible so that the block c_{12} of B^{-1} coincides with q^{-1}. Thus all matrices in the right hand side of (4.10) are invertible hence $\det \Delta \neq 0$.

For Bitsadze system with coefficients (4.6), characteristic polynomial $P(z) = z^2 - a_1 z - a_0$ and rational function $P^{-1}(z)$ are given by

$$P(z) = \begin{pmatrix} z^1 - 1 & \pm 2z \\ \mp 2z & z^2 - 1 \end{pmatrix}, \quad P^{-1}(z) = \frac{1}{z^2 - 1}\begin{pmatrix} z^1 - 1 & \mp 2z \\ \pm 2z & z^2 - 1 \end{pmatrix}.$$

Thus integral (4.8) is equal to the zero-matrix hence this system is strongly coupled.

As was mentioned, (4.3) together with (4.4) are decisive for (4.2). Namely, if (b, J) satisfies (4.3), (4.4) then (4.2) is automatically fulfilled. This can be also checked by a direct verification.

Pair (b, J) is not uniquely defined by equation (4.8). For example, the same property has a pair (b_1, J_1), where $b_1 = bd$, $J_1 = d^{-1}Jd$, and matrix $d \in \mathbb{C}^{l\times l}$ is invertible. The corresponding matrices B and B_1 are related by $B = B_1 \operatorname{diag}(d, \overline{d})$ so that condition (4.9) is fulfilled.

In particular, if (4.7) is fulfilled one can put $d = b^{-1}$ and then (4.8) transforms into equation $J_1^2 = a_0 + a_1 J_1$ with respect to J_1. Thus for weakly coupled systems in (4.2) one can always put $b = 1$. In this case $\det B \neq 0$ is equivalent to $\det(\operatorname{Im} J) = 0$ and the following analog of theorem 1.1 holds (cf. [16]): if $J^2 = a_0 + a_1 J$ then matrix J satisfies equation $\chi(J) = 0$, $\chi(z) = \det(z^2 - a_1 z - a_0)$. This result shows, in particular, that the order of Jordan ν-block in the Jordan normal form of J does not exceed the multiplicity of ν as the root of characteristic equation $\chi(z) = 0$.

In general case, bringing matrix A in (4.2) to Jordan form we can choose matrix J in (4.3) to be block-diagonal consisting of Jordan blocks. Then by theorem 1.5 columns of matrix b can be distributed into groups consisting of chains of eigenvectors and associated vectors $x_0, \ldots, x_r \in \mathbb{C}^l$ of polynomial $P(z) = z^2 - a_1 z - a_0$. In the case considered, relations (1.26) defining this chain take the form

$$\begin{gathered} P(\nu)x_0 = 0, \; P(\nu)x_1 + P'(\nu)x_0 = 0, \\ P(\nu)x_2 + P'(\nu)x_1 + 2x_0 = 0, \ldots, \\ P(\nu)x_r + P'(\nu)x_{r-1} + 2x_{r-2} = 0. \end{gathered} \tag{4.11}$$

Thus if $J = \operatorname{diag}(J_1, \ldots, J_m)$, where J_k is a Jordan ν_k-block of order r_k (numbers ν_k, as well as numbers r_k, may coincide for different k), then matrix b has block structure $(b_1, \ldots, b_m)$, where b_k for $\nu_k = \nu$ and $r_k = r + 1$ is given by matrix $\downarrow (x_0, \ldots, x_r)$ constituted by the chain of vectors $x_0, \ldots, x_r \in \mathbb{C}^l$.

Precisely this structure of matrix b was described in [11] with respect to representation (3.20) for system (4.1). Condition of weak coupling (4.7) means that the totality of above chains gives a basis in $\mathbb{C}^l$. In this sense for the correspoding polynomial holds an analogue of Jordan theorem.

If $\det b = 0$, i.e. system (4.1) is strongly coupled, then the necessary condition (4.4) imposes a certain restriction on the rank of matrix b.

Lemma 4.2. *For a strongly coupled system, rank of matrix $b \in \mathbb{C}^{l\times l}$ is not less than $l/2$.*

Proof. As was mentioned, the class of solutions (b, J) to equation (4.3), (4.4) is invariant under transformation $b \to bd$, $J \to d^{-1}Jd$. Hence by a proper choice of d one can achieve that first r columns of b are linearly independent and the last $l - r$ columns are equal to zero. Evidently, multiplication of coefficients of (4.1) by a non-invertible real matrix e gives an equivalent system which is obtained by a substitution $u = e\tilde{u}$ of the sought solution u. Hence matrix b can be multiplied from the left by non-invertible real matrices. Thus if $\det b = 0$, matrixy b in (4.3) can be always brought to the form, where the last $l - r$ columns are zero and in the first l rows and columns stays the identity matrix. According to (4.4) this is only possible if $\operatorname{rang} b \geq l/2$.

4.2. **Strongly and perfectly elliptic systems.** Elliptic system (4.1) is often written in the form

$$\sum_{i,j=1}^{2} a_{ij}\frac{\partial^2 u}{\partial x_i \partial y_j} = 0, \quad x_1 = x,\ x_2 = y. \tag{4.12}$$

Then the ellipticity condition is

$$\det\left(\sum\nolimits_{i,j=1}^{2} \lambda_i\lambda_j a_{ij}\right) \neq 0 \tag{4.13}$$

for $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$, $\lambda \neq 0$. In particular, matrices a_{ii} are invertible and (4.12) may be transformed to a canonical form (4.1) with coefficients

$$a_0 = -a_{22}^{-1}a_{11}, \quad a_1 = -a_{22}^{-1}(a_{12} + a_{21}). \tag{4.14}$$

Correspondingly, up to a factor a_{22} the matrix polynomial (3.2) transforms into

$$P(z) = a_{11} + (a_{12} + a_{21})z + a_{22}z^2. \tag{4.15}$$

Obviously, eigenvectors and adjoint vectors which constitute columns of matrix b in (4.2) (supposing that J is in Jordan form) can be determined with respect to this polynomial.

Work of A.Bitsadze [24] stimulated introduction of various classes of elliptic system s for which Dirichlet problem is Fredholm. The most useful appeared introduced by M.Vishik [25] concept of *strong ellipticity*. It means the positive definiteness, for all non-vanishing $(\lambda_1,\ \lambda_2)$, of the matrix under the sign of determinant in (4.13). Let us write $d > 0$ ($d \geq 0$) for a positively (negatively) defined matrix d (this notation tacitly assumes that d is symmetric). Then the strong ellipticity of (4.12) is expressed by condition

$$\sum_{i,j=1}^{2} a_{ij}\lambda_i\lambda_j > 0. \tag{4.16}$$

Obviously, matrices $P^{\pm 1}(\lambda)$, $\lambda \in \mathbb{R}$ in (4.15) are also positive definite so that Δ from (4.8) also has this property. According to lemma 4.1 this implies that strongly elliptic systems are weakly coupled, i.e. Dirichlet problem for them is really Fredholm.

An example of strongly elliptic system is given by the self-adjoint (in the sense of Lagrange) system (4.12) with coefficients

$$a_{11} = a_{22} = 1, \quad a_{12} = a_{21}^{\mathrm{T}} = p, \tag{4.17}$$

where p is an orthogonal matrix without real eigenvalues. In particular, the order l of the system should be even. For this system $P(z) = (z + p)(z + p)^{\mathrm{T}}$ so that the condition $P(\lambda) > 0$ of strong ellipticity (4.16) is fulfilled.

Still more narrow class of elliptic systems is defined by the notion of *perfect ellipticity* introduced in [26] which means that the $(2l \times 2l)$-matrix constituted by the coefficients of the system is non-negatively determined:

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \geq 0, \tag{4.18}$$

and the homogeneous system $a\xi = 0$ has no non-zero solutions $\xi = (\xi_1, \xi_2)$ with linearly independent vectors $\xi_1, \xi_2 \in \mathbb{R}^l$. Obviously, (4.18) is equivalent to

$$a_{ji}^{\mathrm{T}} = a_{ij}, \quad \sum_{i,j=1}^{2} (a_{ij}\xi_j)\xi_i \geq 0 \tag{4.19}$$

for all $\xi_i, \xi_j \in \mathbb{R}^l$. In particular, this system is self-adjoint. For $\xi_i = \lambda_i \xi$, $\lambda_i \in \mathbb{R}$, inequality (4.19) is strict by the second requirement so that for such systems the strong ellipticity condition is fulfilled.

Notice that system (4.12) with coefficients (4.17) is perfectly elliptic as

$$\sum (a_{ij}\xi_j)\xi_i = \xi_1^2 + 2(p\xi_1)\xi_2 + \xi_2^2 \geq \xi_1^2 - 2|\xi_1|\,|\xi_2| + \xi_2^2 \geq 0,$$

where is used that $|p\xi_1| = |\xi_1|$ by orthogonality of matrix p.

If for $\xi \neq 0$ inequality (4.19) is strict, i.e. matrix (4.18) is positively defined, then system (4.12) is called elliptic in the sense of Somilliano [11]. These notions are especially useful for dealing with Dirichlet problem [26].

Theorem 4.1. *For a perfectly elliptic system, Dirichlet problem is uniquely solvable.*

Proof. According to subsection 4.1, for this system Dirichlet problem is Fredholm and has index zero so that it suffices to establish uniqueness of its solution. Let $u \in C^1(\overline{D})$ be a solution of homogeneous Dirichlet problem. Then applying to the left hand side of (4.12) Green's formula one gets

$$0 = -\int_D \sum_{i,j=1}^{2} \left(a_{ij}\frac{\partial u}{\partial x_j}\right)\frac{\partial u}{\partial x_i}dx + \int_\Gamma u \sum_{i=1}^{2}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) n_i\, ds. \tag{4.20}$$

Hence

$$\int_D (a\nabla u)\nabla u dx = 0$$

for the gradient $\nabla u = (u_x, u_y)$.

Since a is non-negatively determined, $(a\nabla u)\nabla u = 0$ hence $a\nabla u = 0$ or, in more detailed form.

$$a_{i1}\frac{\partial u}{\partial x} + a_{i2}\frac{\partial u}{\partial y} = 0, \quad i = 1, 2. \tag{4.21}$$

Since $u = 0$ on Γ vectors u_x и u_y are linearly dependent at each $t \in \Gamma$. By definition of perfect ellipticity we see that $u_x = u_y = 0$ on the boundary Γ of D.

Consider $J-$analytic function ϕ participating in representation (3.7) of solution u. By theorem 2.2 its derivative ϕ' is related to u by (3.14) so that it also vanishes on Γ. Applying Cauchy theorem from subsection 2.2 we derive that $\phi' = 0$ everywhere in D hence u is constant. Taking into account the boundary condition (4.5) this is only possible if $u = 0$.

4.3. **Conjugate and degenerate solutions.** With each solution $u(x, y)$ of system (4.12) one can associate function $v(x, y)$ defined by

$$\frac{\partial v}{\partial x} = -\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right), \quad \frac{\partial v}{\partial y} = a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}. \tag{4.22}$$

The necessary condition for existence of this function follows from equation (4.12) written in the form

$$\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right) = 0.$$

Function v is called conjugate to solution u of system (4.12). It is determined up to additive constant and in the case of multiply connected domain it is in general multi-valued .

Using matrix

$$c = -(a_{21}b + a_{22}bJ) \tag{4.23}$$

the conjugate function v can be expressed via J-analytic function ϕ analogously to (3.7).

Theorem 4.2. *The function conjugate to* $u = \operatorname{Re} b\phi$ *is given by the formula*

$$v = \operatorname{Re} c\phi + \xi, \quad \xi \in \mathbb{R}^l. \tag{4.24}$$

Proof. Substituting (3.7) into (4.22) we get

$$v_x = -\operatorname{Re}(a_{21}b + a_{22}bJ)\phi', \quad v_y = \operatorname{Re}(a_{11}b + a_{12}bJ)\phi'.$$

Writing equation (4.3) for the system (4.12), (4.14) in the form $a_{11}b + a_{12}bJ = -(a_{21}b + a_{22}bJ)J$ (in notation (4.23)), we obtain:

$$v_x = \operatorname{Re} c\phi', \quad v_y = \operatorname{Re} cJ\phi'.$$

Hence partial derivative s of function $v - \operatorname{Re} c\phi$ vanish, which gives representation (4.23).

Conjugate function v may appear constant, i.e. the right hand side of (4.22) is identically zero. In such case solution u of system (4.12) is called *degenerate*. Thus it is determined by an over-determined first order system(4.21). Evident examples of degenerate solutions give polynomials of first degree

$$u_0(x) = \eta + \xi_1 x + \xi_2 x_2, \quad (a_{i1}\xi_1 + a_{i2}\xi_2) = 0, \quad i = 1, 2. \tag{4.25}$$

Let us describe systems (4.12) whose degenerate soltuions coincide with these polynomials. Put for brevity

$$\tilde{a}_1 = a_{11}^{-1} a_{12}, \quad \tilde{a}_2 = a_{22}^{-1} a_{21},$$
$$X = \{x \in \mathbb{R}^l \,|\, x = \tilde{a}_1 \tilde{a}_2 x = \tilde{a}_2 \tilde{a}_1 x\}. \tag{4.26}$$

Theorem 4.3. *If $X = \{0\}$ then each degenerate solution of elliptic system* (4.12) *has the form* (4.25). *Otherwise the class of degenerate solutions is infinite dimensional.*

Proof. Adopting notation (4.2), (4.18) let us show that $x \in X$ is equivalent to the system of equations

$$a\xi = aA\xi = 0 \tag{4.27}$$

for vector $\xi = (-\tilde{a}_1 x, x) \in \mathbb{R}^{2l}$.

Obviously, a can be changed by

$$\tilde{a} = \begin{pmatrix} 1 & \tilde{a}_1 \\ \tilde{a}_2 & 1 \end{pmatrix}. \tag{4.28}$$

It is also clear that $\tilde{a}\xi = 0$ is equivalent to $\xi = (-\tilde{a}_1 x, x)$, $x - \tilde{a}_1 \tilde{a}_2 x = 0$, and also to $\xi = (y, -\tilde{a}_2 y)$, $y - \tilde{a}_2 \tilde{a}_1 y = 0$. Hence the statement follows from the equality

$$A \begin{pmatrix} -\tilde{a}_1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -\tilde{a}_2 \end{pmatrix}, \tag{4.29}$$

which can be checked using (4.14), (4.26).

From (4.29) also follows that (4.27) implies

$$aA^r\xi = 0, \quad r = 0, 1, 2, \ldots\,. \tag{4.30}$$

Indeed, in the space $X \subseteq \mathbb{R}^l$ matrices $\tilde{a}_j$ define mutually inverse transformations so that $(x, \tilde{a}_2 x) = (-\tilde{a}_1 y, y)$, $y = -\tilde{a}_2 x$. Thus (4.29) gives relation

$$A^r \begin{pmatrix} -\tilde{a}_1 x \\ x \end{pmatrix} = \begin{pmatrix} -\tilde{a}_1 y \\ y_r \end{pmatrix}, \quad y_r = (-\tilde{a}_2)^r x, \; r \geq 1,$$

which implies (4.30).

We pass to the statements of the theorem. Let $u = 2\,\mathrm{Re}\, b\phi$ be a solution to (4.12). Then $\nabla u = (u_x, u_y)$ and $\tilde{\psi} = (\phi', \overline{\phi'})$ are related by $\nabla u = B\tilde{\psi}$. Hence solution u is degenerate, i.e. satisfies system (4.24) if and only if $aB\tilde{\psi} = 0$. Differentiating this by x and y and using (2.6) we get

$$aB\tilde{J}^r \tilde{\Psi}^{(k)} = 0, \quad 0 \leq r \leq k, \tag{4.31}$$

where is put $\tilde{J} = \mathrm{diag}(J, \overline{J})$. By (4.2), $B\tilde{J}^r = A^r B$ and previous relations are equivalent to

$$aA^r \tilde{\phi}^{(k)} = 0, \quad 0 \leq r \leq k.$$

Hence if (4.28) has only zero solution then $\psi' = 0$ and $\psi = \phi' \in \mathbb{C}^l$. In other words, degenerate solution u has the form (4.25).

Conversely, let the space $Y \subseteq \mathbb{R}^{2l}$ of all solutions to (4.27) have non-zero dimension. Then for $\xi \in Y$ are fulfilled all relations (4.30). Choose a bounded sequence $\xi_k \in Y$, $k = 0, 1, \ldots$ and put $\tilde{\eta}_k = (\eta_k, \overline{\eta}_k) = B^{-1}\xi_k$, $\eta_k \in \mathbb{C}^l$. Consider J-analytic function

$$\psi(z) = \sum_{k=0}^{\infty} \frac{1}{k!}[z]^k \eta_k = \sum_{0 \le r \le k} \frac{x^{k-r}y^y}{(k-r)!r!} J^r \eta_k.$$

An analogous decomposition is valid for $\tilde{\psi}$ with respect to $\tilde{J}$ and $\tilde{\eta}_k$. Thus

$$aB\tilde{\psi} = \sum_{0 \le r \le k} \frac{x^{k-r}y^r}{(k-r)!r!} aA^r \xi_k, \quad \xi_k = B\tilde{\eta}_k,$$

and by (4.31) we get $aB\tilde{\psi} \equiv 0$. Hence $u = \operatorname{Re} b\phi$, where $\phi' = \psi$, is a degenerate solution. It's clear that the class of such solutions is infinite dimensional.

Lemma 4.3. (a) *The dimension of X is even.*

(b) *The rank of matrix $a = (a_{ij})_1^2$ lies between l and $2l$. For $\mathtt{rk}a = l$, the space X coincides with $\mathbb{R}^l$ (so that l is even). For $\mathtt{rk}a \ge 2l - 1$, one has $X = 0$.*

Proof. (a) Matrices $\tilde{a}_j$ define on X mutually inverse transformations denoted by $\tilde{a}_{(j)}$. We claim that $\tilde{a}_{(j)}$ have no real eigenvalues.

Suppose the converse, i.e. that for some $\mu \in \mathbb{R}$ and non-zero $x \in X$ we have relations $\tilde{a}_1 x = \mu x$, $\tilde{a}_2 x = \mu^{-1} x$. Then

$$\sum\nolimits_{i,j=1}^{2} (a_{ij}\lambda_i\lambda_j)x = \{a_{11}(\lambda_1 + a_1'\lambda_2)\lambda_1 + a_{22}(a_2'\lambda_1 + \lambda_2)\lambda_1\}x$$

and for $\lambda_1 + \mu\lambda_2 = 0$ the left hand side of this expression vanishes, which contradicts ellipticity of system (4.12).

(b) Obviously, rank of $\tilde{a}$ in (4.29) is not less than l. If it equals l then matrices $\tilde{a}_j$, $j = 1, 2$, are mutually inverse and $X = \mathbb{R}^l$. If $\mathtt{rk}a \ge 2l - 1$ then $\dim X \le 1$ and by (a) this is only possible if $X = 0$.

First statement in (b) is satisfied by system (4.12), (4.17) with orthogonal matrix p having no real eigenvalues.

Let us slightly extend the notion of conjugate solution. Consider matrices $d \in \mathbb{R}^{2l\times 2l}$ and $dB \in \mathbb{C}^{2l\times 2l}$ written in 2×2-block form

$$d = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}, \quad dB = \begin{pmatrix} c_0 & \overline{c}_0 \\ c_1 & \overline{c}_1 \end{pmatrix}. \tag{4.32}$$

Let matrix d be related to coefficients of equation (4.1) by

$$d_{21} = d_{12}a_0, \quad d_{22} - d_{11} = d_{12}a_1. \tag{4.33}$$

Then using an evident identity

$$\frac{\partial}{\partial y}\left(d_{11}\frac{\partial u}{\partial x} + d_{12}\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial x}\left(d_{21}\frac{\partial u}{\partial x} + d_{22}\frac{\partial u}{\partial y}\right) =$$

$$= d_{12}\left(\frac{\partial^2 u}{\partial y^2} - a_0\frac{\partial^2 u}{\partial x^2} - a_1\frac{\partial^2 u}{\partial x \partial y}\right)$$

with each solution u of equation (4.1) one can associate v for which

$$\frac{\partial v}{\partial x} = d_{11}\frac{\partial u}{\partial x} + d_{12}\frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial y} = d_{21}\frac{\partial u}{\partial x} + d_{22}\frac{\partial u}{\partial y}. \tag{4.34}$$

It is called d-conjugate to u and determined up to an additve constant. In general it may be multi-valued (if domain D is multiply connected).

Analogously to theorem 4.2 one checks that there exists representation (4.23) with matrix $c = c_0$ from (4.32). Indeed, by (4.2), (4.32)

$$c_0 = d_{11}b + d_{12}bJ, \quad c_1 = d_{12}b + d_{22}bJ.$$

Taking into account (4.3), (4.33) we get

$$c_0J - c_1 = d_{12}(bJ^2 - a_0b - a_1bJ) = 0.$$

Thus differentiating the equality $v = \operatorname{Re} c_0\phi$ one obtains

$$v_x = \operatorname{Re} c_0\phi' = d_{11} \operatorname{Re} b\phi' + d_{12} \operatorname{Re} bJ\phi',$$

$$v_y = \operatorname{Re} c_0J\phi' = d_{21} \operatorname{Re} b\phi' + d_{22} \operatorname{Re} bJ\phi',$$

which proves (4.34).

Under some additional assumptions function v is solution to a certain second order elliptic system .

Lemma 4.4. *If* $\det d \neq 0$ *then function* v *is solution to elliptic system*

$$\frac{\partial^2 v}{\partial y^2} = \tilde{a}_0\frac{\partial^2 v}{\partial x^2} + \tilde{a}_1\frac{\partial^2 v}{\partial x \partial y}, \quad \begin{pmatrix} 0 & 1 \\ \tilde{a}_0 & \tilde{a}_1 \end{pmatrix} = dAd^{-1}. \tag{4.35}$$

In particular, if $da = ad$ *then* (4.32) *is fulfilled and function* v *is solution of the same equation* (4.1) *as* u.

Proof. Obvously, gradient $\nabla u = (u_x, u_y)$ satisfies the system

$$(\nabla u)_y - A(\nabla u)_x \;=\; 0.$$

Writing (4.34) in the form $v = du$ we arrive to system

$$(\nabla v)_y\tilde{A}(\nabla v)_x \;=\; 0, \quad \tilde{A} \;=\; d\,A\,d^{-1}. \tag{4.36}$$

Taking into account special form (4.2) of matrix A, relations (4.32) can be rewritten in the form

$$(dA)_{11} \;=\; d_{21}, \quad (dA)_{12} \;=\; d_{22}, \tag{4.37}$$

hence matrix $\tilde{A}$ has a block structure similar to (4.2). Combining this with (4.36) we arrive to (4.35).

Finally, if $Ad = dA$ then due to $(Ad)_{11} = d_{21}$, $(Ad)_{12} = d_{22}$, relations (4.37) are fulfilled hence (4.32) is also fulfilled.

With A commutes, for example, matrix

$$d \;=\; B\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}B^{-1}.$$

Here $c_0 = -ib$ hence $v = \operatorname{Im} b\phi$.

4.4. **Neumann problem.** Along with Dirichlet problem (4.5) for elliptic system (4.12) an important role plays boundary value problem

$$\sum_{i,j=1}^{2}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)n_i\bigg|_{\Gamma} = g, \tag{4.38}$$

where $n = (n_1, n_2)$ is the unit outer normal to the boundary $\Gamma = \partial D$ of D. In the scalar case ($l = 1$) the left hand side of this expression defines differential along vector with components $(a_{11}n_1 + a_{21}n_2,\, a_{12}n_1 + a_{22}n_2)$ called conormal. In this case the problem is called Neumann problem. We preserve this name also for $l > 1$.

From (4.21) is clear that degenerate solutions satisfy homogeneous boundary condition (4.38) in any domain D. This and theorem 4.3 implies that for $\dim X > 0$ the homogeneous Neumann problem has an infinity of linearly independent solutions.

Degenerate solutions $\tilde{u}$ corresponding to elliptic system

$$\sum_{i,j=1}^{2} a^{\mathrm{T}}_{ji} \frac{\partial^2 \tilde{u}}{\partial x_i \partial x_j} = 0, \tag{4.12$^\sim$}$$

which is Lagrange conjugate to (4.12), define the natural solvability conditions of the homogeneous problem (4.38).

Indeed, denote by Lu and $\tilde{L}\tilde{u}$ the left hand sides of (4.12) and и (4.12$^\sim$), respectively. Then applying Green's formula to the identity

$$(Lu)\tilde{u} - u(\tilde{L}\tilde{u}) = \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left[\left(a_{ij}\frac{\partial u}{\partial x_j}\right)\tilde{u} - u\left(a^{\mathrm{T}}_{ji}\frac{\partial \tilde{u}}{\partial x_j}\right)\right]$$

the above solvability conditions can be rewritten as the conditions of orthogonality

$$\int_\Gamma g u_0 ds = 0 \tag{4.39}$$

to all degenerate solutions u_0 of system (4.12$^\sim$). For $u_0 = \xi \in \mathbb{R}^l$, they include the necessary condition of solvability

$$\int_\Gamma g ds = 0.$$

In particular, on a smooth contour one can introduce function f whose derivative with respect to arclength coincides with g. In terms of conjugate function v the left hand side (4.38) coincides with tangential derivative $v' = v_x s_1 + v_y s_2 = -v_x n_2 + v_y n_1$. Hence the Neumann problem can be rewritten in the form

$$v\big|_\Gamma = f$$

of Dirichlet problem for v. Taking into account theorem 4.2, this is Fredholm equivalent to Riemann-Hilbert problem (3.35) with a constant coefficient $G = c$ determined by matrix (4.23). Correspondingly, condition

$$\det c \neq 0 \tag{4.40}$$

is necessary and sufficient for Neumann problem to be Fredholm. As in the case of Dirichlet problem one checks that the index of this problem is equal to zero.

If (4.40) is fulfilled then the class of degenerate solutions should be finite-dimensional, which by theorem 4.3 is only possible if $X = 0$. For perfectly elliptic systems the converse is also true.

Lemma 4.5. *Let system* (4.12) *be perfectly elliptic. Then each solution of homogeneous Neumann problem is degenerate and condition* (4.40) *(adopting notation* (4.26)*) is equivalent to* $X = 0$.

Proof. Let $u \in C^1(\overline{D})$ be a solution of homogeneous Neumann problem. Then the second summand in (4.20) vanishes. As in theorem 4.2 one deduces that $a\nabla u = 0$, i.e. that u is a degenerate solution.

If $\det c \neq 0$ then the space of solutions of the homogeneous problem is finite-dimensional and theorem 4.3 implies that $X = 0$.

Let $\det c = 0$. Consider Neumann problem in the half-plane $\operatorname{Im} z > 0$. By a remark at the end of subsection 3.4 there exist infinitely many linearly independent J-analytic functions ϕ satisfying the estimate

$$|\phi'(z)| \leq C(1 + |z|^2)^{-1}$$

and for which $c\phi(z) = 0$ on the real axis $\operatorname{Im} z = 0$. In virtue of this estimate, relation (4.20) is valid for $u = \operatorname{Re} b\phi$ in the whole half-plane hence all such solutions u are degenerate. By theorem 4.3 we conclude that $\dim X > 0$.

As in the case of Dirichlet problem, for perfectly elliptic systems the nature of solvability of Neumann problem is much simpler.

Theorem 4.4. *Let system* (4.12) *be perfectly elliptic and satisfy* (4.40). *Then solutions of the homogeneous Neumann problem reduce to polynomials* (4.25) *and the non-homogeneous problem is solvable if and only if the orthogonality conditions* (4.39) *are fulfilled with respect to this polynomials.*

If (4.40) is violated then the homogeneous Neumann problem has an infinity of linearly independent solutions.

Proof. Let (4.40) be fulfilled. Then since the problem is Fredholm lemma 4.5 and theorem 4.4 imply that the kernel of the homogeneous problem consists only of polynomials (4.25). As the conjugate system to (4.12) coincides with ($4.12^{\sim}$), orthogonality conditions (4.39) with respect to these polynomials are necessary for the solvability of non-homogenous problem. As was mentioned, the index vanishes so these conditions are also sufficient.

Let now $\det c = 0$. Then according to lemma 4.5 and theorem 4.3, the space of degenerate solutions of system (4.12), i.e. the kernel of the homogeneous problem, is infinite dimensional.

5. System of two equations of second order

5.1. **Classification of systems.** For $l = 2$ elliptic systems (4.12) admit complete description. In this case the characteristic polynomial is given by a 2×2-matrix

$$P(z) = a_1 + (a_{12} + a_{21})z + a_{22}z^2 = (P_{ij})_1^2, \tag{5.1}$$

and the scalar characteristic polynomial $\chi(z) = \det P(z)$ has degree 4. Thus in the upper half-plane it has two roots which may coincide. More precisely three cases are possible: (i) roots ν_1 and ν_2 differ; (ii) there is one multiple root ν and $P(\nu) \neq 0$; (iii) $P(\nu) = 0$.

Let us suppose that J in (4.2) is Jordan so that to these three cases correspond

$$\text{i)} \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}, \quad \text{ii)} \begin{pmatrix} \nu_1 & 1 \\ 0 & \nu \end{pmatrix}, \quad \text{iii)} \begin{pmatrix} \nu & 0 \\ 0 & \nu \end{pmatrix}. \tag{5.2}$$

As was explained in subsection 4.1 columns $x \in \mathbb{R}^2$ of matrix b are solutions of the chain of equations (4.11). In the three cases (5.2) these equations take the form:

$$\begin{array}{ll} \text{(i)} & P(\nu_1)x = P(\nu_2)y = 0; \\ \text{(ii)} & P(\nu)x = 0,\ P(\nu)y + P'(\nu)x = 0; \\ \text{(iii)} & P(\nu)x = P(\nu)y = 0. \end{array} \tag{5.3}$$

In each of these cases let us find the connection between $b \in \mathbb{C}^{2\times 2}$ and $B \in \mathbb{C}^{4\times 4}$ в (4.2).

Lemma 5.1. *Adopting notation*

$$b = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}, \qquad \begin{matrix} p = \operatorname{Im}(x_1\overline{x}_2), \\ q = \operatorname{Im}(y_1\overline{y}_2), \end{matrix} \tag{5.4}$$

we have equality

$$\frac{1}{4}\det B = \begin{cases} |\nu_1 - \nu_2|^2 pq - (\operatorname{Im}\nu_1)(\operatorname{Im}\nu_2)|\det b|^2, & \text{(i)} \\ p^2 - (\operatorname{Im}\nu)^2|\det B|^2, & \text{(ii)} \\ -(\operatorname{Im}\nu)^2|\det b|^2, & \text{(iii)}. \end{cases} \tag{5.5}$$

Proof. Introduce an operation over 2×2-matrices

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}^* = \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix}. \tag{5.6}$$

Obviously, $(xy)^* = y^*x^*$ и $xx^* = x^*x = \det x$. In particular, if $\det x \neq 0$ one has $x^{-1} = (\det x)^{-1}x^*$. Consider the equality (4.2). If $\det b \neq 0$ then

$$\det B = \det b \det(bJ) \det \begin{pmatrix} 1 & b^{-1}\overline{b} \\ 1 & J^{-1}b^{-1}\overline{bJ} \end{pmatrix},$$

hence

$$\det B = \det \Delta, \quad \Delta = b^*\overline{bJ} - Jb^*\overline{b}. \tag{5.7}$$

Since Δ continuously depends on b, by a density argument (5.6) remains valid when $\det b = 0$.

Adopting notation (5.4), matrix $b^*\overline{b}$ in the right hand side of (5.7) can be written in the form

$$b^*\overline{b} = \begin{pmatrix} r & -2iq \\ 2ip & r \end{pmatrix}, \quad r = y_2\overline{x}_1 - y_1\overline{x}_2. \tag{5.8}$$

In particular,

$$|r|^2 = |\det b|^2 + 4pq. \tag{5.9}$$

Calculating elements of Δ in (5.7) in each of the three cases (i)) – (iii) and using (5.9) we conclude that (5.5) holds.

From (5.5) , in particular, follows that in case (iii) matrix b is invertible. Thus its columns x, y are linearly independent so by (5.3) we have $P(\nu) = 0$. Then polynomial $P(z)$ is divisible by $z - \nu$ and $z - \overline{\nu}$ so it is given by

$$P(z) = a_{22}[z^2 - 2(\operatorname{Re}\nu)z + |\nu|^2]. \tag{5.10}$$

Notice that equality $P(\nu) = 0$ also follows from equation (4.3) satisfied by scalar matrix $J = \nu$.

Now (5.10) shows that in the case (iii) system (4.12) can be reduced to one scalar equation by multiplying it on the left by a_{22}^{-1}. For this reason in the sequel the main attention is given to the cases (i) and (ii).

Columns x, y of matrix b in equation s (5.3) are determined up to linear transformations

$$\begin{matrix} \text{(i)} & x' = \lambda_1 x, & y' = \lambda_2 y, & \lambda_j \neq 0; \\ \text{(ii)} & x' = \lambda_1 x, & y' = \lambda_1 x + \lambda_2 y, & \lambda_1 \neq 0. \end{matrix} \tag{5.11}$$

In other words, performing these transformations over the columns does not change matrix A in (4.2).

According to subsection 4.1 strongly coupled systems are characterized by the linear dependence of their columns x and y. For such systems, applying an appropriate transformation (5.11) one can always achieve that $x = y$ in the case (i) and $y = 0$ in the case (ii). In other words, the strongly coupled systems are characterized by the condition

$$\begin{array}{ll} \text{(i)} & P(\nu_1)x = P(\nu_2)x = 0; \\ \text{(ii)} & P(\nu)x = P'(\nu)x = 0 \end{array} \tag{5.12}$$

for some non-zero $x \in \mathbb{C}^2$.

Vectors x and y in (5.3) can be defined in terms of the elements of the matrix $P(z)$ itself. Along with b it is convenient to describe matrix c in (4.23) which palys an important role for the Neumann problem. It suffices to consider the cases (i) and (ii) since in the case (iii) we can put $b = 1$ and $c = -(a_{11} + a_{22}\nu)$.

Adopting notation (5.1), (5.6) let us associate with matrix $P(z)$ matrix polynomial s

$$Q(z) = P^*(z), \quad R(z) = -(a_{21} + a_{22}z)Q(z). \tag{5.13}$$

Lemma 5.2. (i) *Let numbers $1 \le i, j \le 2$ be chosen so that ith and jth column of, respectively, matrices $Q(\nu_1)$ and $Q(\nu_2)$ are non-zero. Then we can put*

$$\begin{array}{ll} b_{(1)} = Q_{(i)}(\nu_1), & b_{(2)} = Q_{(j)}(\nu_2), \\ c_{(1)} = R_{(i)}(\nu_1), & c_{(2)} = R_{(j)}(\nu_2), \end{array} \tag{5.14}$$

where $x_{(k)}$ denotes the kth column of matrix x.

(ii) *Let $i = 1, 2$ be chosen so that ith column of matrix $Q(\nu)$ is non-zero. Then we can put*

$$\begin{array}{ll} b_{(1)} = Q_{(i)}(\nu), & b_{(2)} = Q'_{(j)}(\nu), \\ c_{(1)} = R_{(i)}(\nu), & c_{(2)} = R'_{(j)}(\nu). \end{array} \tag{5.15}$$

Proof. (i) From definitions (5.6), (5.13) is clear that

$$P(z)Q(z) = \chi(z), \tag{5.16}$$

where $\chi = \det P$. Since $\chi(\nu_1) = \chi(\nu_2) = 0$ columns x of matrix $Q(\nu_k)$ are solutions of equation $P(\nu_k)x = 0$, which implies relations (5.14) for b. According to (1.17) and (5.2), for the columns of matrix c in (4.23) we have $c_{(k)} = -(a_{21}b_{(k)} + a_{22}b_{(k)}\nu_k) = -[(a_{21} + a_{22}\nu_k)n]_{(k)}$. According to (5.13) this leads to relations (5.14) for c.

(ii) Since ν is a multiple root of polynomial $\chi(z)$ we have $\chi(\nu) = \chi'(\nu) = 0$. By differentiating (5.16) we derive equality $P(\nu)Q'(\nu) + P'(\nu)Q(\nu) = 0$. Comparing it with equation (5.3) for this case and acting as above we get (5.15) for b.

As to c, as above we have: $c_{(k)} = -[a_{21}b_{(k)} + a_{22}(bJ)_{(k)}]$, $k = 1, 2$. In the case of Jordan ν-block J, one has $(bJ)_{(1)} = b_{(1)}\nu$, $(bJ)_{(2)} = b_{(1)} + b_{(2)}\nu$, so that $c_{(1)} = -(a_{21} + a_{22}\nu)b_{(1)}$, $c_{(2)} = -(a_{21} + a_{22}\nu)b_{(2)} - a_{22}b_{(1)}$. Taking into account (5.13) and relations (5.15) for b we get

$$c_{(1)} = -(a_{21} + a_{22}\nu)Q_{(i)}(\nu) = Q_{(i)}(\nu),$$

$$c_{(2)} = -(a_{21} + a_{22}\nu)Q'_{(i)}(\nu) - a_{22}Q_{(i)}(\nu) = R'_{(i)}(\nu),$$

which completes the proof of (5.15) and lemma.

Let us illustrate preceding considerations in the case of system (4.1) obtained by separating real and imaginary parts of one complex equation

$$\left(\frac{\partial}{\partial y}-\alpha\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial y}-\beta\frac{\partial}{\partial x}\right)(u_1+iu_2)=0.$$

Following subsection 3.2 we get system (4.1) with matrix coefficients

$$a_0=-(\alpha\beta)^\wedge,\quad a_1=-(\alpha+\beta)^\wedge, \tag{5.17}$$

where we use notation

$$(x+iy)^\wedge=\begin{pmatrix} x & -y \\ y & x \end{pmatrix},\quad x,y\in\mathbb{R}.$$

For appropriate α,β one can get a system of any of the three types (i)-(iii). More precisely, for $\alpha=\beta=\pm i$ one gets Bitsadze systems with coefficients (4.6). For $\alpha=\nu$, $\beta=\overline{\nu}$, one gets the case (iii) because then polynomial $P(z)$ with coefficients (5.17) coincides with (5.10). Cases (i) and (ii) are obtained, respectively, for $\alpha\neq\beta,\overline{\beta}$ and $\alpha=\beta$. Moreover, $\nu_1=\operatorname{Re}\alpha+i|\operatorname{Im}\alpha|$, $\nu_2=\operatorname{Re}\beta+i|\operatorname{Im}\beta|$.

For $\operatorname{Re}\alpha=\operatorname{Re}\beta=0$, coefficients (5.17) have the form

$$a_0=\begin{pmatrix} x_0 & 0 \\ 0 & s_0 \end{pmatrix},\quad a_1=\begin{pmatrix} 0 & r_1 \\ s_1 & 0 \end{pmatrix} \tag{5.18}$$

with some $r_j,s_j\in\mathbb{R}$. Consider system (4.1) with such coefficients for which polynomial $\chi(z)$ has $z=i$ as a multiple root. Since

$$P(z)=\begin{pmatrix} z^2-r_0 & -r_1z \\ -s_1z & z^2-s_0 \end{pmatrix}$$

and $\chi(z)=\det P(z)=z^4-(r_0+s_0+r_1s_1)z^2+r_0s_0$, the above requirement is equivalent to conditions

$$r_0+s_0+r_1s_0=-2,\quad r_0s_0=1. \tag{5.19}$$

For $r_1=s_1=0$, equation (4.1), (5.18) transforms into Laplace equation so this case can be excluded. In remaining cases, for polynomial $P(z)$, matrix Q and its derivative Q' at $z=i$ are equal to

$$Q(i)=\begin{pmatrix} -s_0-1 & ir_1 \\ is_1 & -r_0-1 \end{pmatrix},\quad Q'(i)=\begin{pmatrix} 2i & r_1 \\ s_1 & 2i \end{pmatrix}.$$

Hence by lemma 5.2 we can put

$$b=\begin{pmatrix} -s_0-1 & 2i \\ is_1 & s_1 \end{pmatrix},\quad b=\begin{pmatrix} ir_1 & r_1 \\ -r_0-1 & 2i \end{pmatrix}$$

in case $s_1\neq 0$ and $r_1\neq 0$, respectively. In particular, taking into account (5.19) we see that the system is strongly coupled if $s_0=r_0=1$ and weakly coupled in other cases.

5.2. **Parametric description of systems.** Elliptic systems (4.12) are invariant under linear transformations of eq s and of the vector of unknowns u, and also under linear transformations of independent variables x, y. In terms of characteristic polynomial these transformations act as multiplication of $P(z)$ from the left and from the right by invertible matrices $d \in \mathbb{R}^{2\times 2}$ and application of fractional-linear substitution

$$z = \frac{\alpha_{11}u + \alpha_{12}}{\alpha_{21}u + \alpha_{22}}, \quad \det\alpha \neq 0.$$

If one works with systems (4.1) then the first two transformations can be reduced to similarity transformations $P(z) \to dP(z)d^{-1}$. Obviously, the three case (i)-(iii) are invariant for these transformations.

Systems of two equation s of second order were systematically investigated in the monograph [5]. ipp it was shown there that using these transformations system (4.1) can be brought to a canonical form of the type (5.19).

Consider another approach based on the relation (4.2) where matrix J belonging to one the three types (5.2) is fixed and matrix $b \in \mathbb{C}^{2\times 2}$ is considered as parameter. The only requirement is (4.4), i.e. invertibility of matrix B in (4.2). Lemma 5.1 describes it in an explicit form. If it is fulfilled, to each parameter $b \in \mathbb{C}^{2\times 2}$ corresponds an elliptic system (4.1) c with coefficients a_0, a_1 determined from

$$\begin{pmatrix} 0 & 1 \\ a_0 & a_1 \end{pmatrix} = B \begin{pmatrix} J & 0 \\ 0 & \overline{J} \end{pmatrix} B^{-1}, \quad B = \begin{pmatrix} b & \overline{b} \\ bJ & \overline{bJ} \end{pmatrix}. \tag{5.20}$$

The whole set of elliptic system s is decomposed into three subsets corresponding to the three types (5.2) of matrix J. Using lemma 5.1 it is not difficult to give homotopy description of each of these subsets.

Theorem 5.1. *The set E of elliptic system s corresponding to the case (iii) is connected. In the cases (i) and (ii) there are three connected components $E^{\pm}$ and E^0 with parametric descriptions (5.20) as follows*

$$\det B > 0, \quad \pm p > 0 \tag{5.21$^{\pm}$}$$

$$\det B < 0, \tag{5.21^{0}}$$

where p is taken from (5.4).

Proof. For a fixed J denote by G the set of all matrices $b \in \mathbb{C}^{2\times 2}$ for which $\det B \neq 0$ in formula (5.5) of lemma 5.1. Dependence of coefficients a_0, a_1 in (5.20) on b will be denoted as $(a_0, a_1) = h(b)$. As a result we get a continuous mapping h of $G \subseteq \mathbb{C}^{2\times 2} \cong \mathbb{R}^8$ onto $E \subseteq \mathbb{R}^2 \times \mathbb{R}^{2\times 2} \cong \mathbb{R}^8$. As (5.5) shows, set G is open. From lemma 5.2 follows that h has a continuous right inverse $h^{(-1)} : E \to G$, i.e. one has $h\, h^{(-1)}(a_0, a_1) = (a_0, a_1)$. Hence E is open and h sends connected components of G to those of G. The induced mapping of connected components is one-to-one. Indeed, if to pair $(a_0, a_1) \in E$ correspond two matrices b and $\overline{b}$ then these two matrices are related by a transformation (5.11). Hence b и $\overline{b}$ belong to the same connected component of G.

Thus it suffices to prove the statement for G. In the case (iii) according to (5.5) the set G coincides with $\{b \in \mathbb{C}^{2\times 2},\ \det b \neq 0\}$ and is thus connected. Consider now the cases (i) and (ii). Then matrix b by elementary transformations (5.11) can be brought to one of the following two forms in each of the cases (i), (ii):

$$b = \begin{pmatrix} 1 & 1 \\ x_2 & y_2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ x_2 & 1 \end{pmatrix}; \tag{5.22i}$$

$$b = \begin{pmatrix} 1 & 0 \\ x_2 & y_2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & y_1 \\ 1 & 0 \end{pmatrix}. \tag{5.22ii}$$

Here is used that in the case (i) inequality $x_1 \neq 0$ can be always achieved by rearranging columns and renumbering ν_1, ν_1.

The rest of the argument can be performed for these matrices. Consider matrix of the first type in (5.22). Setting $x_2 - y_2 = x - iy$, $x_1 + y_2 = u + iv$, equality (5.5) for (i) can be written in the form

$$\det B = -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{v^2}{c^2} \tag{5.23}$$

with positive constants a, b, c, depending only on ν_1, ν_2. Hence equation $\det B = 0$ defines an elliptic cone in the space $\mathbb{R}^3$ of variables x, y, v. Its complement consists of three components K^0, $K^{\pm}$ determined by the signs of $\det B$ and v as in (5.21). In the four-dimensional space of variables x, y, u, v, defining matrix b we get the corresponding components $K^0 \times \mathbb{R}$ и $K^{\pm} \times \mathbb{R}$. It remains to notice that $\det B > 0$ implies that $pq > 0$ in (5.4), (5.5) hence the signs of p, q and $p+q$ coincide.

An analogous argument works for matrix b of the first type in в (5.22ii). Here one should put $y_2 = x - iy$, $x_2 = u - iv$ and then (5.5) again takes the form (5.23).

For matrix b of the second type in (5.22), one has $\det B < 0$. It is clear that such matrices constitute a connected component G^0. This completes the proof of theorem 5.1.

It would be interesting to derive theorem 5.1 from the general approach to homotopy classification of elliptic systems and boundary value problems suggested in [27]. The same refers to ellipticity criteria for systems with constant coefficients presented in subsection 3.3.

As was noticed in subsection 5.1, strongly coupled systems can only appear in cases (i) and (ii). From (5.5) and theorem 5.1 it follows that they correspond to matrices $b \in G^{\pm}$. For example, for Bitsadze system with coefficients (4.6) one has $b \in G^{\pm}$. To strongly coupled systems correspond matrices $b \in G^0$. This follows from the fact that matrix polynomials $P_t(\lambda) = tP(\lambda) + (1-t)(1+\lambda^2)$ of variable λ, depending on parameter $0 \leq t \leq 1$, are positively determined for all t.

Using the mapping $h: \ b \to (a_1, a_2)$ one can construct elliptic systems with a prescribed matrix b. For this reason it is useful to describe this mapping explicitly. Let operation $*$ be the same as in (5.6).

Lemma 5.3. *For coefficients* $a_0, a_1 \in \mathbb{R}^{2\times 2}$ *of system* (4.1) *one has*

$$\begin{aligned} a_0 &= 2(\det B)^{-1}\,\mathrm{Re}[(\det \overline{bJ})d_2 - (\det J)d_1\overline{d}_1], \\ a_1 &= 2(\det B)^{-1}\,\mathrm{Re}[(\det \overline{b}J)d_1 - d_2\overline{d}_1^*], \end{aligned} \tag{5.24}$$

where $d_k = bJ^k b^*$, $k = 1, 2$.

Proof. We use explicit expressions for the block elements of B^{-1}. They are obtained by inverting the system $B\xi = \eta$ or, in more detail, $b\xi_1 + \overline{b}\xi_2 = \eta_1$, $bJ\xi_1 + \overline{bJ}\xi_2 = \eta_2$.

Suppose first that $\det b \neq 0$. Then eliminating ξ из from the first equation and substituting in the second one we get:

$$
\begin{aligned}
(bJ - \overline{bJb}^{-1} b)\xi_1 &= -\overline{bJb}^{-1}\eta_1 + \eta_2,\\
(\overline{bJ} - bJb^{-1}\overline{b})\xi_2 &= -bJ\overline{b}^{-1}\eta_1 + \eta_2.
\end{aligned}
$$

Using notation (5.7) for matrix $bJ - \overline{bJb}^{-1} b = \overline{b}(\overline{b}^{-1} bJ - \overline{Jb}^{-1} b)$ we have $\overline{b}(\det \overline{b})^{-1}\overline{\Delta} = (\overline{b}^*)^{-1}\overline{\Delta}$. From (5.7) and reality of $\det B$ it follows that $\overline{\Delta}^{-1} = (\det B)^{-1}\overline{\Delta}^*$. Hence the formulae for ξ_1, ξ_2 can be rewritten in the form $\xi_1 = c_0\eta_1 + c_1\eta_2$, $\xi_2 = \overline{c}_0\eta_1 + \overline{c}_1\eta_2$ with matrices

$$
c_0 = -(\det B)^{-1}\overline{\Delta}^*\overline{b}^*\overline{bJb}^{-1}, \quad c_1 = (\det B)^{-1}\overline{\Delta}^*\overline{b}^*.
$$

Since $\overline{\Delta}^* = J^*b^*\overline{b} - b^*\overline{bJ}^*$ we have $\overline{\Delta}^*\overline{b}^* = (\det \overline{b})J^*b^*\overline{b}^* - \overline{bJ}^*\overline{b}^*$ hence $\overline{\delta}^*\overline{b}^*\overline{bJb}^{-1} = J^*b^*\overline{bJb}^* - (\overline{\det bJ})b^*$. Thus

$$
B^{-1} = \begin{pmatrix} c_0 & c_1 \\ \overline{c}_0 & \overline{c}_1 \end{pmatrix}, \qquad
\begin{aligned}
c_0 &= (\det B)^{-1}[(\det \overline{bJ})b^* - J^*b^*\overline{bJb}],\\
c_1 &= (\det B)^{-1}[(\overline{\det b})J^*b^* - b^*\overline{bJ}^*\overline{b}^*].
\end{aligned}
$$

By a density argument these formulae remain valid for $\det b = 0$.

Now by (5.20) we have:

$$
\begin{pmatrix} 0 & 1 \\ a_1 & a_2 \end{pmatrix} = \begin{pmatrix} bJ & \overline{bJ} \\ bJ^2 & \overline{bJ}^2 \end{pmatrix}\begin{pmatrix} c_1 & c_2 \\ \overline{c}_1 & \overline{c}_2 \end{pmatrix},
$$

which implies $a_j = 2\operatorname{Re} bJ^2 c_j$, $j = 1, 2$. Substituting here explicit expressions for matrices c_j we arrive to formulae (5.24).

Notice that in the case (iii) matrices $d_k = \nu^k \det b$ are scalar and formulae (5.24) transform into the corresponding coefficients of the polynomial entering in the right hand side of (5.10).

As an illustration consider a strongly coupled system corresponding to matrix

$$
b = \begin{pmatrix} 1 & 1 \\ i & i \end{pmatrix}. \tag{5.25}
$$

This is clearly possible only in the cases (i), (ii).

By (5.4), (5.5) we have $\det B = 4|\nu_1 - \nu_2|^2$ for (i) and $\det B = 4$ for (ii). Hence substitution of (5.25) into (5.24) gives

$$
a_j = \operatorname{Re}\alpha_j(1 + ie), \quad j = 0, 1, \tag{5.26}
$$

where

$$
\alpha_0 = \begin{cases} -\nu_1\nu_2, \\ -\nu^2, \end{cases}, \quad \alpha_1 = \begin{cases} \nu_1 + \nu_2, \\ 2\nu, \end{cases}, \quad e = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
$$

In particular, for $\nu = i$ formulae (5.26) agree with (5.18).

Putting $a_{22} = 1$ write system (4.1), (5.26) in the form (4.12). Then (4.14) and (4.23) transform into

$$
a_0 = -a_{11}, \quad a_1 = -a_{12} - a_{21}, \quad c = -a_{21}b - bJ.
$$

Here a_{21} can be chosen to satisfy $\det c \neq 0$. Indeed, by (5.25) we can write

$$
a_{21}b = \begin{pmatrix} x & x \\ iy & iy \end{pmatrix}
$$

with arbitrarily prescribed x, y.

As a simple verification shows, $\det c \neq 0$ for $x \neq y$ in both cases (i) and (ii). Thus Neumann problem may appear Fredholm for a strongly coupled system.

On the other hand, condition $\det c \neq 0$ of ellipticity of Neumann problem may not be fulfilled even for strongly elliptic systems of the type (iii). Indeed, by (5.10) in this case we have $a_{21} + a_{12} = -2(\operatorname{Re}\nu)a_{22}$, where a_{22} is positively determined and $a_{21}^{\mathrm{T}} = a_{12}$. Hence we can put $a_{21} = -(\operatorname{Re}\nu)a_{22} - \Delta$ with a skew-symmetric matrix Δ. Then $c = -a_{21}b - a_{22}\nu = [\Delta - i(\operatorname{Im}\nu)a_{22}]b$. Thus it suffices to show that, for an appropriate choice of Δ, the matrix in square brackets has zero determinant. Write

$$a_{22} = \begin{pmatrix} p & r \\ r & q \end{pmatrix}, \quad \Delta = \begin{pmatrix} 0 & s\beta \\ -s\beta & 0 \end{pmatrix}, \ \beta = \operatorname{Im}\nu,$$

where p, q are positive and $pq > r^2$. Then we have $\det[\Delta - i(\operatorname{Im}\nu)a_{22}] = 0$ for $s^2 = pq - r^2$.

5.3. **Lamé system of anisotropic plane elasticity theory.** In anisotropic plane elasticity theory [28], [29] a medium is characterized by stress tensor σ and deformation tensor ε which can be expressed as symmetric 2×2-matrix-function s with the elements written as $\sigma_{ii} = \sigma_i$, $\varepsilon_{ii} = \varepsilon_i$, $i = 1, 2$ и $\sigma_{12} = \sigma_{21} = \sigma_3$, $\varepsilon_{21} = \varepsilon_{12} = \varepsilon_3$. Here ε_j are expressed through displacement vector $u = (u_1, u_2)$ by the formulae

$$\varepsilon_1 = \frac{\partial u_1}{\partial x}, \ \varepsilon_2 = \frac{\partial u_2}{\partial y}, \ 2\varepsilon_3 = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}. \tag{5.27}$$

If one considers a cut in the medium along an arc with normal $n = (n_1, n_2)$ then on a unit length of this arc acts force σn called the normal component of stress tensor. Write columns of 2×2-matrix σ as $\sigma_{(1)}$ and $\sigma_{(2)}$ then vector σn is a linear combination $\sigma_{(1)}n_1 + \sigma_{(2)}n_2$. In absence of mass forces matrix σ satisfies the equilibrium equation s

$$\frac{\partial \sigma_{(1)}}{\partial x} + \frac{\partial \sigma_{(2)}}{\partial y} = 0 \tag{5.28}$$

and is related to deformation tensor ε by Hooke's law. In the linear theory this relation is expressed by

$$\begin{array}{l} \sigma_1 = \alpha_1\varepsilon_1 + \alpha_4\varepsilon_2 + 2\alpha_6\varepsilon_3, \\ \sigma_2 = \alpha_4\varepsilon_1 + \alpha_2\varepsilon_2 + 2\alpha_5\varepsilon_3, \\ \sigma_3 = \alpha_6\varepsilon_1 + \alpha_5\varepsilon_2 + 2\alpha_3\varepsilon_3, \end{array} \quad \alpha = \begin{pmatrix} \alpha_1 & \alpha_4 & \alpha_6 \\ \alpha_4 & \alpha_2 & \alpha_5 \\ \alpha_6 & \alpha_5 & \alpha_3 \end{pmatrix} > 0 \tag{5.29}$$

with constant coefficients α_j, called elasticity modules. Here, as above, inequality symbol after a matrix means that it is positively determined. In particular, all principal minors of α including its determinant

$$\det \alpha = \alpha_1\alpha_2\alpha_3 + 2\alpha_4\alpha_5\alpha_5 - \alpha_1\alpha_5^2 - \alpha_2\alpha_6^2 - \alpha_3\alpha_4^2$$

are positive. Thus $\alpha_j > 0$, $j = 1, 2, 3$, and $\alpha_1\alpha_2 > \alpha_4^2$, $\alpha_1\alpha_3 > \alpha_6^2$, $\alpha_2\alpha_3 > \alpha_5^2$.

Using (5.27) formulae (5.29) can be written as

$$\sigma_{(i)} = a_{i1}\frac{\partial u}{\partial x} + a_{i2}\frac{\partial u}{\partial y}, \quad i = 1, 2 \tag{5.30}$$

with matrix coefficients

$$a_{11} = \begin{pmatrix} \alpha_1 & \alpha_6 \\ \alpha_6 & \alpha_3 \end{pmatrix}, \qquad a_{12} = \begin{pmatrix} \alpha_6 & \alpha_4 \\ \alpha_3 & \alpha_5 \end{pmatrix},$$
$$a_{21} = \begin{pmatrix} \alpha_6 & \alpha_3 \\ \alpha_4 & \alpha_5 \end{pmatrix}, \qquad a_{22} = \begin{pmatrix} \alpha_3 & \alpha_5 \\ \alpha_5 & \alpha_2 \end{pmatrix}. \tag{5.31}$$

Substituting expressions (5.30) into (5.28) we get a system of equations (4.12) for displacement vector $u = (u_1, u_2)$ which is called *Lamé system.*

Consider block matrix $a = (a_{ij})_1^2 \in \mathbb{R}^{4\times 4}$ composed from matrices (5.31). Rearranging its columns and rows with numbers 2 and 4 one gets a symmetric matrix with equal two last columns (and rows) and with the third principal minor coinciding with matrix α entering in (5.29). Hence matrix a is non-negatively defined and its rank equals 3. Moreover, the solution space of system $a_{i1}\xi_1 + a_{i2}\xi_2 = 0$, $i = 1, 2$, is spanned by vector $e = (e_1, e_2) \in \mathbb{R}^4$ with block components

$$e_1 = (0, 1), \;\; e_2 = (-1, 0) \in \mathbb{R}^2. \tag{5.32}$$

Remembering definition 4.2 we see that Lamé system is perfectly elliptic.

As is known typical boundary conditions for Lamé system in a plane domain D assume that prescribed is either the displacement vector u on its boundary $\Gamma = \partial D$ or the normal component $\sigma_{(1)}n_1 + \sigma_{(2)}n_2$ of stress tensor σ, where $n = (n_1, n_2)$ is the unit outer normal on Γ, or else given is a combination of these two conditions. Taking into account (5.30) this corresponds to Dirichlet problem (4.5) or Neumann problem (4.38) which are often called the first and the second boundary value problem, respectively.

In the theory of boundary value problems of anisotropic plane elasticity one can distinguish two classical directions. The first one is based on using analytic functions in the spirit of Kolosov-Muskhelishvili formulae in isotropic case [30], [31]. The second direction uses the methods of potential theory [29].

Results of preceding sections enable us to develop a functional theoretic approach based on the use of hyperanalytic functions. In isotropic case it was described in [32]. In a general anisotropic case this method was used by S.Mitin [33]. Some other possible approaches were discussed in [34], [35], [36].

For Lamé system (4.12), (5.31) consider matrices b and c appearing in (4.2) and (4.23). By the strong ellipticity of Lamé system we have $\det b \neq 0$. According to lemmas 4.3(a) and 4.5 an analogous condition is fulfilled for c. Thus theorems 4.1 and 4.4 lead to the following result.

Theorem 5.2. *For Lamé system, Dirichlet problem is uniquely solvable, while solutions of homogeneous Neumann problem reduce to polynomials* $u_0(x, y) = c + \lambda_1 x e_1 + \lambda_2 y e_2$, $c \in \mathbb{R}^2$, *where* $\lambda_j \in \mathbb{R}$ *and vectors* e_j *are as in* (5.32). *Correspondingly, the non-homogeneous problem is solvable if and only if orthogonality conditions* (4.39) *are fulfilled with respect to these polynomials.*

Matrices b and c discussed earlier may be explicitly described applying lemma 5.2 to Lamé system (4.12), (5.31). Let us make this more concrete. The characteristic polynomial (4.25) and associated polynomial Q in (5.13) can be written as

$$P = \begin{pmatrix} g_1 & g_3 \\ g_3 & g_2 \end{pmatrix}, \quad Q = \begin{pmatrix} g_2 & -g_3 \\ -g_3 & g_1 \end{pmatrix}$$

with polynomials

$$\begin{aligned} g_1(z) &= \alpha_1 + 2\alpha_6 z + \alpha_3 z^2, \\ g_2(z) &= \alpha_3 + 2\alpha_5 z + \alpha_2 z^2, \\ g_3(z) &= \alpha_6 + (\alpha_3 + \alpha_4) z + \alpha_5 z^2. \end{aligned} \tag{5.33}$$

Along with 3×3-matrix α in (5.29) consider matrix

$$\beta = (\det \alpha)\alpha^{-1} = \begin{pmatrix} \beta_1 & \beta_4 & \beta_6 \\ \beta_4 & \beta_2 & \beta_5 \\ \beta_6 & \beta_5 & \beta_3 \end{pmatrix},$$

which is also positively determined. In more detail:

$$\begin{aligned} &\beta_1 = \alpha_2\alpha_3 - \alpha_5^2, \qquad &&\beta_2 = \alpha_1\alpha_3 - \alpha_6^2, \qquad &&\beta_3 = \alpha_1\alpha_2 - \alpha_4^2, \\ &\beta_4 = \alpha_5\alpha_6 - \alpha_3\alpha_4, \qquad &&\beta_5 = \alpha_4\alpha_6 - \alpha_1\alpha_5, \qquad &&\beta_6 = \alpha_4\alpha_5 - \alpha_2\alpha_6. \end{aligned} \tag{5.34}$$

In this notation the scalar characteristic polynomial $\chi(z) = g_1 g_2 - g_3^2$ can be written as

$$\chi(z) = h_1(z) - z h_2(z) + z^2 h_3(z), \quad \begin{aligned} h_1(z) &= \beta_2 - \beta_5 z + \beta_4 z^2, \\ h_2(z) &= \beta_5 - \beta_3 z + \beta_6 z^2, \\ h_3(z) &= \beta_4 - \beta_6 z + \beta_1 z^2. \end{aligned} \tag{5.35}$$

Notice that polynomial s $h_j(z)$, $j = 1, 2, 3$, never vanish simultaneously. Indeed, if $h_1(z) = h_2(z) = h_3(z) = 0$ then $(1, -z, z^2)$ is a solution of the homogeneous system with determinant equal to $\det \beta$. However this contradicts the inequality $\det \beta > 0$.

Simple computations show that for matrix

$$R(z) = -(a_{21} + a_{22} z) Q(z) = \begin{pmatrix} \alpha_6 + \alpha_3 z & \alpha_3 + \alpha_5 z \\ \alpha_4 + \alpha_5 z & \alpha_5 + \alpha_2 z \end{pmatrix} \begin{pmatrix} -g_2 & g_3 \\ g_3 & -g_1 \end{pmatrix}$$

in (5.13) we get the following expression:

$$R = \begin{pmatrix} -z h_3 & -h_1 \\ h_3 & h_2 - z h_3 \end{pmatrix}. \tag{5.36}$$

In particular, pol s $g_j(z)$, $j = 1, 2, 3$, also do not vanish simultaneously. Thus for Lamé system only first two cases in (5.20) are actually possible. Notice that in the case (i) one of the numbers $h_3(\nu_1)$, $h_3(\nu_2)$ is certainly non-zero as only one of the roots of h_3 may lie in the upper half-plane . In both cases (i), (ii) matrices b and c can be described as follows.

Theorem 5.3. *Let for definiteness $h_3(\nu_2) \neq 0$. Then*

$$b = \begin{pmatrix} g_2(\nu_1) & g_2(\nu_2) \\ -g_3(\nu_1) & -g_3(\nu_2) \end{pmatrix}, \quad c = \begin{pmatrix} -\nu_1 h_3(\nu_1) & -\nu_2 h_3(\nu_2) \\ h_3(\nu_1) & h_3(\nu_2) \end{pmatrix}, \quad h_3(\nu_1) \neq 0,$$

$$b = \begin{pmatrix} -g_3(\nu_1) & g_2(\nu_2) \\ g_1(\nu_1) & -g_3(\nu_2) \end{pmatrix}, \quad c = \begin{pmatrix} -\nu_1 h_2(\nu_1) & -\nu_2 h_3(\nu_2) \\ h_2(\nu_1) & h_3(\nu_2) \end{pmatrix}, \quad h_3(\nu_1) = 0.$$

(ii) *Moreover,*

$$b = \begin{pmatrix} g_2(\nu) & g_2'(\nu) \\ -g_3(\nu) & -g_3'(\nu) \end{pmatrix}, \qquad c = \begin{pmatrix} -\nu h_3(\nu) & -h_3(\nu) - \nu h_3'(\nu) \\ h_3(\nu) & h_3'(\nu) \end{pmatrix}.$$

In particular, we see that here determinants of matrices b and c are always non-zero, which was earlier derived from general reasons.

Proof. (i) If $h_3(\nu) \neq 0$ then by (5.36) the first column of $Q(\nu)$ is non-zero. If $h_3(\nu) = 0$ then by (5.35) each of the numbers $h_2(\nu)$, $h_3(\nu)$ is non-zero, hence the second column of Q is non-zero. Thus the desired conclusion follows from lemma 5.2.

(ii) Acting as in (i) it is sufficient to show that $h_3(\nu) \neq 0$. Suppose the contrary: $\chi(\nu) = \chi'(\nu) = h_3(\nu) = 0$. Then by (5.35) we have $h_j(\nu) \neq 0$, $j = 1, 2$. Putting $p = \alpha_6 + \alpha_3\nu$, $q = \alpha_3 + \alpha_5\nu$ from (5.36) we deduce that $pg_2(\nu) - qg_3(\nu) = 0$, $-pg_3(\nu)+qg_1(\nu) = h_1(\nu)$. Since $g_1(\nu)g_2(\nu) = g_3^2(\nu)$ and $pq \neq 0$, this is only possible if $g_2(\nu) = g_3(\nu) = 0$. Using equalities $\chi = g_1g_2 - g_3^2$, $\chi' = g_1'g_2 + g_1g_2' - 2g_3g_3'$ we see that $g_1(\nu) = 0$, which gives a contradiction.

5.4. **Orthotropic and anisotropic cases.** An elastic medium is called *orthotropic* if coordinate axes are the axes of symmetry. This case corresponds to

$$\alpha_5 = \alpha_6 = 0 \tag{5.37}$$

for modules of elasticity of matrix α appearing in Hooke's law (5.29). In particular, matrix α is block-diagonal. Then expressions (5.33) and (5.35) are simplified:

$$\begin{array}{l} g_1 = \alpha_1 + \alpha_3 z^2,\ g_2 = \alpha_3 + \alpha_2 z^2,\ g_3 = (\alpha_3 + \alpha_4)z, \\ h_1 = \beta_2 + \beta_4 z^2,\ h_2 = -\beta_3 z,\ h_3 = \beta_4 + \beta_1 z^2, \\ \chi = \beta_2 + (\beta_3 + 2\beta_4)z^2 + \beta_1 z^4. \end{array} \tag{5.38}$$

In particular, the roots ν_j of biquadratic equation $\chi(z) = 0$ are defined from

$$2\beta_1\nu^2 = -\beta_3 - 2\beta_4 \pm \sqrt{\gamma}, \tag{5.39}$$

where $\gamma = (\beta_3+2\beta_4)^2 - 4\beta_1\beta_2$. In particular, for $\gamma \geq 0$ the roots lie on the imaginary axis. The case (ii) of multiple roots corresponds to $\gamma = 0$, i.e. $\beta_3 + 2\beta_4 = 2\sqrt{\beta_1\beta_2}$. By (5.34) this is equivalent to

$$2\alpha_3 + \alpha_4 = \sqrt{\alpha_1\alpha_2}. \tag{5.40}$$

From (5.38), (5.39) follows that $2h_3(\nu) = -\beta_3 \pm \sqrt{\gamma}$. Hence $h_3(\nu_j) = 0$ is equivalent to $\beta_3\beta_4 = \beta_1\beta_2 - \beta_4^2$, or, in terms of α, to $\alpha_3 + \alpha_4 = 0$. Thus (5.40) corresponds to theorem 5.3(ii), while the first and second cases of theorem 5.3(i) are obtained, respectively , for $\alpha_3 + \alpha_4 \neq 0$ and $\alpha_3 + \alpha_4 = 0$.

As to the formulae for matrices b и c given in the theorem, one should substitute there expressions (5.38).

Suppose that in addition to (5.37) are fulfilled relations $\alpha_1 = \alpha_2 = \alpha_4 + 2\alpha_3$ or, equivalently,

$$\alpha_1 = \alpha_2 = \lambda + 2\mu, \quad \alpha_3 = \mu, \quad \alpha_4 = \lambda \tag{5.41}$$

with some positive λ and μ. Then we get the isotropic case when each straight line is an axis of symmetry of medium. In such case linear relations (5.27), (5.29) transform into

$$\begin{array}{l} \sigma_1 = \lambda\left(\dfrac{\partial u_1}{\partial x} + \dfrac{\partial u_2}{\partial y}\right) + 2\mu\dfrac{\partial u_1}{\partial x}, \\ \sigma_2 = \lambda\left(\dfrac{\partial u_1}{\partial x} + \dfrac{\partial u_2}{\partial y}\right) + 2\mu\dfrac{\partial u_2}{\partial y}, \end{array} \quad \sigma_3 = \mu\left(\dfrac{\partial u_1}{\partial y} + \dfrac{\partial u_2}{\partial x}\right), \tag{5.42}$$

and Lamé system can be written in the form

$$\begin{aligned}
&\mu\Delta u_1 + (\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right) = 0,\\
&\mu\Delta u_2 + (\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right) = 0.
\end{aligned} \tag{5.43}$$

In this case (5.40) is obviously fulfilled so that ν is multiple and equal to i. Thus we are in the conditions of case (ii) with Jordan block J corresponding to eigenvalue $\nu = i$.

From (5.34), (5.41) we have $\beta_1 = \beta_2 = \mu(\lambda+2\mu)$, $\beta_3 = (\lambda+2\mu)^2$, $\beta_4 = -\mu\lambda$, $\beta_5 = \beta_6 = 0$. Thus by theorem 5.3(ii)

$$b = \begin{pmatrix} -(\lambda+\mu) & 2i(\lambda+2\mu) \\ -i(\lambda+\mu) & -(\lambda+\mu) \end{pmatrix}, \quad c = 2\mu \begin{pmatrix} i(\lambda+\mu) & 2\lambda+3\mu \\ -(\lambda+\mu) & i(\lambda+2\mu) \end{pmatrix}.$$

Adding to the second column the first multiplied by $2i(\lambda+2\mu)/(\lambda+\mu)$ and reducing the common scalar multiple $-(\lambda+\mu)$ these matrices can be also brought to the form [37]

$$b = \begin{pmatrix} 1 & 0 \\ i & -\Lambda \end{pmatrix}, \quad c = \mu \begin{pmatrix} 2i & \Lambda-1 \\ 2 & i(\Lambda+1) \end{pmatrix}, \quad \Lambda = \frac{\lambda+3\mu}{\lambda+\mu}. \tag{5.44}$$

By theorem 3.2 the general solution of Lamé system (5.43) is expressed through a $J-$analytic function by formula

$$u = \operatorname{Re} b\phi. \tag{5.45}$$

On the other hand, we have the classical formulae due to Kolosov-Muskhelishvili [30] which give representation of σ and u by a pair of analytic functions:

$$\begin{aligned}
\sigma_1 + \sigma_2 &= 4\operatorname{Re}\chi_1'(z),\\
\sigma_2 - \sigma_1 + 2i\sigma_3 &= 2[\overline{z}\chi_1''(z) + \chi_2'(z)],\\
2\mu(u_1 + iu_2) &= \Lambda\chi_1(z) - z\overline{\chi_1'(z)} - \overline{\chi_2(z)}.
\end{aligned} \tag{5.46}$$

Connection of these formulae with (5.45) is established using theorem 2.1 in the form (2.17). In our case formula (2.17) takes the form

$$\phi(z) \;=\; \psi(z) + \frac{\overline{z}}{2}\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}\psi'(z),$$

Substituting this into (5.45) we get for the components of vector

$$u \;=\; \operatorname{Re}\, b\phi \;=\; \operatorname{Re}\begin{pmatrix} 1 & 0 \\ i & -\Lambda \end{pmatrix}\psi + \frac{1}{2}\operatorname{Re}\overline{z}\begin{pmatrix} 0 & i \\ 0 & -1 \end{pmatrix}\psi'$$

expressions

$$\begin{aligned}
2u_1 &= \psi_1 + \overline{\psi}_1 + (i\overline{z}\psi_2' - iz\overline{\psi}_2')/2,\\
2u_2 &= i\psi_1 - \Lambda\psi_2 - i\overline{\psi}_1 - \Lambda\overline{\psi}_2 - (\overline{z}\psi_2' + z\overline{\psi}_2')/2,
\end{aligned}$$

and, respectively, $2(u_1 + iu_2) \;=\; 2\overline{\psi}_1 - i\Lambda(\psi_2 + \overline{\psi}_2) - iz\overline{\psi}_2'$. Putting here $\chi_1 = -\mu i\psi_2$, $\chi_2 = -2\mu\psi_1 - i\Lambda\mu\psi_2$, we come to the last equality in (5.46).

As to the first two equalities in (5.46), they can be obtained from the last one using (5.42). It is also possible to use vector-function v conjugate to u as in subsection 4.3. By theorem 4.2 it is expressed through a J-analytic function ϕ by formula (4.24) with matrix c from (5.42). From (5.30) the tangential derivative of function v along a smooth arc coincides with the normal component of stress tensor

$$\frac{dv}{ds} = \sigma n, \tag{5.47}$$

which gives the desired expression for elements σ_j of matrix σ.

6. Douglis-Nirenberg systems

6.1. **Analog of Jordan's theorem.** Let a natural number l be decomposed into sums of non-negative integers $l = l_1 + l_2 = m_1 + m_2$. Given matrices $a_k^{ij} \in \mathbb{R}^{m_i \times l_j}$ consider a system of equation s

$$\begin{aligned} &a_0^{11}\frac{\partial^2 u_1}{\partial x_2} + a_1^{11}\frac{\partial^2 u_1}{\partial x \partial y} + a_2^{11}\frac{\partial^2 u_1}{\partial y_2} + a_0^{12}\frac{\partial u_2}{\partial x} + a_1^{12}\frac{\partial u_2}{\partial y} = 0,\\ &a_0^{21}\frac{\partial u_1}{\partial x} + a_1^{21}\frac{\partial u_1}{\partial y} = 0, \end{aligned} \tag{6.1}$$

for real l_j-vector-function s u_j,$j = 1, 2$. As usual under a regular solution are meant function s $u_1 \in C^2$,$u_2 \in C^1$ satisfying (6.1). As in (3.2), (3.3), with system (6.1) one can associate its matrix characteristic polynomial

$$P(z) = \begin{pmatrix} a_0^{11} + a_1^{11}z + a_2^{11}z^2 & a_0^{12} + a_1^{12}z \\ a_0^{21} + a_1^{21}z & 0 \end{pmatrix} \tag{6.2}$$

and scalar polynomial

$$\chi(z) = \det P(z). \tag{6.3}$$

System (6.1) is called *elliptic in the sense of Douglis-Nirenberg* (in short *DN-elliptic*) if the degree of polynomial χ is equal to l_1+m_1 and characterisitc equation $\chi(z) = 0$ does not have real roots. Such systems were introduced by M.Douglis and L.Nirenberg in 1995 in a slightly different form [38]. For $l_2 = 0$, such systems were earlier considered by I.Petrovsky [39].

Lemma 6.1. *If system* (6.1) *is ND-elliptic then number* $l_1 + m_1$ *is even and* $m_2 \le l_1$, $l_2 \le m_1$. *In particular,*

$$2s = l_1 + m_1 = m_1 + m_2 + s_0 = l_1 + l_2 + s_0 \tag{6.4}$$

for some $s_0 \ge 0$.

Proof. According to (6.2) product

$$\mathrm{diag}(1 \in \mathbb{C}^{m_1\times m_1},\ z \in \mathbb{C}^{m_2\times m_2})\ P(z)\,\mathrm{diag}(1 \in \mathbb{C}^{l_1\times l_1},\ z \in \mathbb{C}^{l_2\times l_2})$$

is a matrix polynomial $\tilde P(z) = \tilde P_0 + \tilde P_1 z + \tilde P_2 z^2$ with highest coefficient

$$\tilde P_2 = \begin{pmatrix} a_2^{11} & a_1^{12} \\ a_1^{21} & 0 \end{pmatrix}.$$

By definition, $\det \tilde P(z)$ is a polynomial of degree $2l = m_2 + l_2 + m_1 + l_1$ hence $\det \tilde P_2 \ne 0$. This implies that rows of a_1^{21} are linearly independent so that $m_2 \le l_1$. Analogously, considering columns of a_1^{12} we get inequality $l_2 \le m_1$. Finally, the fact that $m_1 + l_1$ is even follows from the reality of coefficients of polynomial $\chi(z)$.

As in subsection 1.3 roots ν of characteristic equation $\chi(z) = 0$ are called the eigenvalues of polynomial $P(z)$ of corresponding multiplicity. The order of ν is the order of pole of matrix-function $P^{-1}(z)$ at $z = \nu$.

The main aim of this subsection is to establish an analog of theorem 1.5 for polynomial (6.2). Write it in the form

$$P(z) \; = \; P_0 \; + \; P_1 z \; + \; P_2 z^2 \tag{6.5}$$

Unlike to subsection 1.3 determinant of P_2 vanishes. Let $J \in \mathbb{C}^{s\times s}$ be a Jordan matrix with eigenvalues in the upper half-plane. By analogy with subsection 1.3 we say that matrix $b \in \mathbb{C}^{(l_1+l_2)\times s}$ transforms $P(z)$ to Jordan form J if

$$P_0 b \; + \; P_1 bJ \; + \; P_2 bJ^2 \; = \; 0. \tag{6.6}$$

As in theorem 1.5 it is easy to see that columns of b are given by the chains of eigenvectors and adjoint vectors of polynomial $P(z)$, i.e. by vectors $x_0, x_1, \ldots, x_r \in \mathbb{C}^s$ satisfying equalities (1.26):

$$\begin{aligned} &P(\nu)x_0 \; = \; 0, \;\; P(\nu)x_1 \; + \; P'(\nu)x_0 \; = \; 0, \\ &P(\nu)x_i \; + \; P'(\nu)x_{i-1} \; + \; (1/2)P''(\nu)x_{i-2} \; = \; 0, \quad i = 2, ..., r, \end{aligned}$$

where is used that $P^{(k)} \; = 0$ for $k > 2$.

Here number ν corresponds to the eigenvalue of the corresponding Jordan block. By the first equality it is also an eigenvalue of polynomial $P(z)$.

Theorem 6.1. *There exists matrix $b =\downarrow (b_1, b_2)$, $b_j \in \mathbb{C}^{l_j\times s}$, transforming polynomial $P(z)$ to Jordan normal form J. Matrix J has the same eigenvalues that polynomial $P(z)$ in the upper half-plane $\{\operatorname{Im} z \; > \; 0\}$ and their multiplicities and orders coincide. Columns of $(2l_1 + l_2) \times 2s$-matrix B defined as*

$$B = (B_0, \overline{B_0}), \quad B_0 =\downarrow (b_1, b_1 J, b_2), \tag{6.7}$$

are linearly independent.

Proof. Comparing (6.5) with the block form (6.2) equality (6.6) can be written as:

$$\sum\nolimits_{k,j} a_k^{ij} b_j J^k \; = \; 0, \quad i = 1, 2, \tag{6.8}$$

where $b_j \in \mathbb{C}^{l_j\times s}$ are determined by (6.7). With polynomial (6.2) are associated two block matrices

$$a_0 \; = \; \begin{pmatrix} 0 & -1 & 0 \\ a_0^{11} & a_1^{11} & a_0^{12} \\ a_0^{21} & a_1^{21} & 0 \end{pmatrix}, \quad a_1 \; = \; \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_2^{11} & a_1^{12} \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.9}$$

which according to (6.4) are square matrices of order $2l_1 + l_2 = 2s + m_2$. By a direct verification one gets that

$$(a_0 \; + \; a_1 z) \begin{array}{c} \begin{matrix} l_1 & l_1 & l_2 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & z & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \; = \; \left(\begin{array}{c|c} -1 & 0 \\ \hline \begin{matrix} a_1^{11} + a_2^{11} z \\ a_1^{21} \end{matrix} & \mathrm{P(z)} \end{array} \right). \tag{6.10}$$

This identity which is an analog of (1.25) shows that polynomials $P(z)$ and $a_0 + a_1 z$ have the same eigenvalues with the same multiplicities and orders. It is also easy to see that adopting notation (6.7) equalities (6.8) and

$$a_0 B_0 + a_1 B_0 J = 0 \tag{6.11}$$

are equivalent. Thus it suffices to prove the theorem for affine polynomial $a_0 + a_1 z$.

According to (6.9) matrices a_j have block structure

$$a_0 =\downarrow (a_0^1, a_0^2),\ a_1 =\downarrow (a_1^1, 0),\quad a_j^1 \in \mathbb{R}^{2s\times(2s+m_2)}. \tag{6.12}$$

By (6.5) rows of a_0^2 are linearly independent , hence there exist m_2 linearly independent columns. Choose an invertible matrix d such that $a_0\,d$ is block-diagonal with identity $(m_2 \times m_2)$-block in the right low corner and $a_1\,d$ is a column with vanishing last m_2 elements.

Denoting $a_j d$ and $d^{-1}B_0$ again by a_j and B_0 we may count that $a_0^2 = (0, 1)$, $1 \in \mathbb{R}^{m_2\times m_2}$. Then by putting $B_0 =\downarrow (B_0^1, 0)$, $B_0^1 \in \mathbb{C}^{2s\times s}$ (6.12) is reduced to the case

$$a_0^1 + a_1^1 B_0^1 J = 0 \tag{6.13}$$

of $2s \times 2s$-matrices a_j^1. Since equation $\det(a_0^1 + a_1^1 z) = 0$ has exactly s roots in the upper half-plane and $\det a_1^1 \neq 0$, equality (6.13) expresses the statement of theorem 1.4.

6.2. **Representation of solutions.** For the general solution u of equation (6.1) we have an analog of theorem 3.2. As in §4 we notice that for $n = 2$ relations (3.12) define the corresponding subspaces in $\mathbb{C}^s$.

Theorem 6.2. (a) *Adopting notation* (6.7) *each solution* $u = (u_1, u_2)$ *of equation* (6.1) *in simply connected domain* D *is representable in the form*

$$u_1 \;=\; \operatorname{Re} b_1\phi, \quad u_2 \;=\; \operatorname{Re} b_2\phi' \tag{6.14}$$

with some J*-analytic function* ϕ *and* $u = 0$ *implies* $\phi \in \mathbb{C}^s$. *Function* ϕ *is uniquely representable as a sum* $\phi_0 + c$, *where* $c \in \mathbb{C}^s$ *and* ϕ_0 *satisfy conditions*

$$\phi_0(z_0) = 0,\ \operatorname{Re} b_1 Jc = \operatorname{Re} b_2 c = 0, \tag{6.15}$$

at a fixed point $z_0 \in D$.

(b) *Let* D *be a* m*-connected domain,* $m \geq 2$, *and points* z_j, $j = 1, \dots, m-1$, *belong to different components of the complement* $\overline{\mathbb{C}}\backslash D$. *Then in representation* (6.14) *function* ϕ *is multi-valued and uniquely representable as a sum*

$$\phi(z) = \phi_0(z) + c + \sum_{j=1}^{m-1} \ln[z - z_j]c_j,\ \ c, c_j \in \mathbb{C}^s, \tag{6.16}$$

where ϕ_0 *is single-valued and, in addition to* (6.15), *conditions* $\operatorname{Re} b_1 c_j = \operatorname{Re} b_1 c_j = 0$, $j = 1, \dots, m-1$, *are fulfilled.*

Proof. Let (u_1, u_2) be a solution of (6.1) and

$$U \;=\; \left(\frac{\partial u_1}{\partial x}, \frac{\partial u_1}{\partial y}, u_2\right). \tag{6.17}$$

Comparing (6.1) with (6.9),(6.12) one sees that vector-function U satisfies system of equation s

$$a_0^1\frac{\partial U}{\partial x}+a_1^1\frac{\partial U}{\partial y}=0,\quad a_0^2U=0. \tag{6.18}$$

Analogously, system (6.11) can be written as:

$$a_0^1B_0+a_1^1B_0J=0,\quad a_0^2B_0=0. \tag{6.19}$$

Consider in $\mathbb{R}^{2l_1+l_2}$ subspace X of vectors $\xi,\quad a_0^2\xi=0$. Since rows of a_0^2 are linearly independent its dimension equals $2l_1+l_2-m_2=2s$. By (6.18) transformation $\xi\to\operatorname{Re}B_0\xi$ acts as $\mathbb{C}^s\to X$. By theorem 6.1 columns of matrix $B=(B_0,\overline{B_0})$ are linearly independent hence this transformation defines an isomorphism of $\mathbb{C}^s$ onto X. Consequently, equality

$$U=\operatorname{Re}B_0\psi \tag{6.20}$$

establishes a one-to-one correspondence between $(2l_1+l_2)-$ vectors U satisfying the second equation in (6.18) and complex s-vector-function s ψ. Substituting (6.20) in the first equation we get

$$0=a_1^1\operatorname{Re}B_0\frac{\partial\psi}{\partial y}-J\frac{\partial\psi}{\partial x}.$$

Comparing (6.5) with (6.9), (6.12) one sees that equalities $a_1^1\xi=0$, $a_0^2\xi=0$ take place only if $\xi=0$. In particular, the figure bracket in the latter expression vanishes, which means $J-$analyticity of function ψ. The remaining part of the argument is the same in in the proof of theorem 3.2.

Similarly, the remark at the end of subsection 3.1 remains valid. Namely, let D be a neighbourhood of ∞ and solution u_1,u_2 satisfies in some neighbourhood of this point estimate

$$\left|\frac{\partial u_1}{\partial x}\right|+\left|\frac{\partial u_1}{\partial y}\right|+|u_2|\le C|z|^{-2}.$$

Then function ϕ in (6.14) does not have branching at ∞, i.e. $\sum_j=0$ in (6.16) and its order at ∞ equals 0 so that it is bounded in a neighbourhood of ∞.

A complex version of (6.1) can be studied in the same way as in subsection 3.2. In this case number l_1+m_1 need not be even and (6.4) is changed by equalities

$$s^++s^-=l_1+m_1=m_1+m_2+s_0=l_1+l_2+s_0,$$

where $s^+(s^-)$ is the number of roots of characteristic equation $\chi(z)=0$ in the upper (lower) half-plane . Some changes should also be done in theorem 6.1. Equality (6.7) should be changed to

$$B=(B^+,\overline{B^-}),\quad B^\pm=\downarrow(b_1^\pm,b_1^\pm J^\pm,b_2^\pm),\ b_j^\pm\in\mathbb{C}^{l_j\times s^\pm}.$$

Matrices $J^\pm$ correspond to the roots of polynomial (6.3) in the upper and lower half-plane s. Matrix relation (6.6) should hold for $b^\pm=\downarrow(b_1^\pm,b_2^\pm)$ and $J^\pm$:

$$P_0b^\pm+P_1b^\pm J^\pm+P_2b^\pm(J^\pm)^2=0.$$

In this notation an analog of theorem 6.2 holds in the complex case. One only needs to change (6.14) by

$$u_1=b_1^+\phi^++\overline{b_1^-\phi^-},\quad u_2=b_2^+(\phi^+)'+\overline{b_2^-(\phi^-)'},$$

and make corresponding changes in conditions (6.15) (as it was done in subsection 3.2).

The case when J is a direct sum of blocks can be treated analogously. The scalar case $l_1 = l_2 = 1$ corresponds to situation in subsection 3.3.

Consider now systems elliptic in the sense of Petrovsky which are obtained from (6.1) for $l_2 = 0$:

$$
\begin{aligned}
&a_0^1 \frac{\partial^2 u}{\partial x^2} + a_1^1 \frac{\partial^2 u}{\partial x \partial y} + a_2^1 \frac{\partial^2 u}{\partial y^2} = 0, \\
&a_0^2 \frac{\partial u}{\partial x} + a_1^2 \frac{\partial u}{\partial y} = 0,
\end{aligned}
\tag{6.21}
$$

where $a_k^i \in \mathbb{R}^{m_i \times l}$, $m_1 + m_2 = l$. For such systems, matrix b_2 in (6.7) is absent and representation (6.14) for their solutions coincides with representation (3.7) for systems of classical type (3.1).

In the complex case system (6.21) can be approached differently. Namely, applying Cauchy-Riemann operator $\partial/\partial\overline{z}$ to the second equation one gets a system

$$
\begin{aligned}
a_0^1 \frac{\partial^2 u}{\partial x^2} + a_1^1 \frac{\partial^2 u}{\partial x \partial y} + a_2^1 \frac{\partial^2 u}{\partial y^2} = 0, \\
\left(\frac{\partial}{\partial y} - i \frac{\partial}{\partial x}\right) \left(a_0^2 \frac{\partial}{\partial x} + a_1^2 \frac{\partial}{\partial y}\right) \;=\; 0,
\end{aligned}
$$

which is elliptic in the classical sense.

Thus theorem 6.2 for system (6.21) can be deduced from theorem 3.2. In principle, analogous considerations are applicable to general system (6.1) if, in addition to differentiation, one makes a substitution

$$
u_2 = \left(\frac{\partial}{\partial y} - i \frac{\partial}{\partial x}\right) \tilde{u_2}
$$

in the first equation of the system. Finding then representation $u_1, \tilde{u_2}$ of the resulting system as in §3 one can get representation for function u_2 by differentiating $\tilde{u_2}$. In a slightly different form theorem 6.2 was established by N.Zhura [15].

6.3. **Conjugate functions.** Concepts from subsection 4.3 can be naturally extended to systems (6.1). Suppose we are given l_i-vector-function s $u_i, v_i, i = 1, 2$, subject to linear relations

$$
\begin{aligned}
\frac{\partial v_1}{\partial x} \;&=\; d_{11} \frac{\partial u_1}{\partial x} \;+\; d_{12} \frac{\partial u_1}{\partial y} \;+\; d_{13} u_2, \\
\frac{\partial v_1}{\partial y} \;&=\; d_{21} \frac{\partial u_1}{\partial x} \;+\; d_{22} \frac{\partial u_1}{\partial y} \;+\; d_{23} u_2, \\
v_2 \;&=\; d_{31} \frac{\partial u_1}{\partial x} \;+\; d_{32} \frac{\partial u_1}{\partial y} \;+\; d_{33} u_2,
\end{aligned}
\tag{6.22}
$$

where d_{ij} are constant matrices of corresponding orders. The vector-function $v = (v_1, v_2)$ is uniquely determined from u by these (up to an additive constant vector $(\xi, 0)$) and it is called conjugate to u.

Square $(2l_1 + l_2) \times (2l_1 + l_2)$-matrix $d = (d_{ij})$ is called *admissible* for system (6.1) if each of solutions $u = (u_1, u_2)$ of the latter admits conjugate function . Description of such matrices can be given as in lemma 4.4

Lemma 6.2. *Matrix d is admissible for* (6.1) *if and only if*

$$
(d_{11} b_1 \;+\; d_{12} b_1 J \;+\; d_{13} b_2)\, J \;=\; d_{21} b_1 \;+\; d_{22} b_1 J \;+\; d_{23} b_2. \tag{6.23}
$$

Granted this condition function v *conjugate to* (6.14) *is given by*

$$v_1 = \mathrm{Re}\, c_1\phi + \xi,\ v_2 = \mathrm{Re}\, c_2\phi' \tag{6.24}$$

with matrices $c_1 = d_{11}b_1 + d_{12}b_1J + d_{13}b_2,\ c_2 = d_{31}b_1 + d_{32}b_1J + d_{33}b_2$.

Condition (6.23) *is fulfilled if there exist matrices* $\tilde{a}_j$ *such that in the block structure* (6.9) *their first rows coincide with the first row of* a_j *and*

$$d\, a_j = \tilde{a}_j\, d, \quad j = 0, 1. \tag{6.25}$$

In particular, if d *commutes with* a_j *then function* v *is solution of the same system* (6.1).

Proof. Existence of function v in (6.22) is equivalent to

$$\frac{\partial}{\partial y}\left(d_{11}\frac{\partial u_1}{\partial x} + d_{12}\frac{\partial u_1}{\partial y} + d_{13}u_2\right) -$$
$$-\frac{\partial}{\partial x}\left(d_{21}\frac{\partial u_1}{\partial x} + d_{22}\frac{\partial u_1}{\partial y} + d_{23}u_2\right) \equiv 0.$$

By theorem 6.2 each solution u of (6.1) is representable in the form (6.14). Substituting it in the previous identity rewrite it as

$$\mathrm{Re}\, Q\,\phi'' = 0,$$

where Q is the difference between the left hand side and right hand side of (6.23). Since J-analytic function ϕ is arbitrary, this implies $Q = 0$.

For the second part of the lemma suppose that (6.25) holds with indicated $\tilde{a}_j$. Then multiplying (6.11) from the left by d we get a similar equality

$$\tilde{a}_0\tilde{B}_0 + \tilde{a}_1\tilde{B}_0 J = 0 \tag{6.26}$$

for $\tilde{B}_0 = dB_0$. As in the proof of theorem 6.1, we conclude that $\tilde{B}_0$ has a block structure analogous to (6.7). This , in particular, implies (6.23).

Let $\tilde{b}_j$ be determined by $\tilde{B}_0$ as in (6.7). Then for $\tilde{a}_j = a_j$ equality (6.26) ca be rewritten in block form (6.8), where b_j should be marked by wave. Putting $c_1 = \tilde{b}_1,\ c_2 = \tilde{b}_2$ in (6.24) we come to equation (6.1) for v.

If matrix d is invertible, then analogously to (4.32) condition (6.25) can be rewritten in terms of relations between elements of matrices a and d. According to block structure (6.9) represent vectors $\xi \in \mathbb{R}^{2l_1+l_2}$ in the form (ξ_1, ξ_2, ξ_3), where $\xi_1, \xi_2 \in \mathbb{R}^{l_1},\ \xi_3 \in \mathbb{R}^{l_2}$. Then element of the first row of a_j in (6.9) can be described by conditions

$$(a_1\xi)_1 = -\xi_2, \quad (a_2\xi)_1 = \xi_1.$$

The same conditions should be satisfied by matrices $\tilde{a}_j$ appearing in (6.25). Putting $\xi = d\eta$ assumption (6.25) can be changed to conditions $(d\, a_1\, \eta)_1 = -(d\,\eta)_2, \quad (d\, a_2\, \eta)_1 = (d\,\eta)_1$ or, in matrix notation,

$$(d\, a_1)_{1j} = -d_{2j}, \quad (d\, a_2)_{1j} = d_{1j}, \quad j = 1, 2, 3.$$

In order to calculate here matrix products write $l_1 \times (l_1 + l_2)-$matrix (d_{12}, d_{13}) in block form

$$(d_{(12)}, d_{(13)}), \quad d_{(12)} \in \mathbb{R}^{l_1\times m_1}, d_{(13)} \in \mathbb{R}^{l_1\times m_2}.$$

Then taking into account (6.9) preceding equalities take the form

$$
\begin{gathered}
d_{(12)}a_0^{11} + d_{(13)}a_0^{21} = -d_{21},\ d_{(12)}a_1^{11} + d_{(13)}a_1^{21} = d_{11} - d_{22}, \\
d_{(12)}a_0^{12} = -d_{23},\ d_{(12)}a_2^{11} = d_{12}, d_{(13)}a_2^{12} = d_{13}.
\end{gathered} \tag{6.28}
$$

Hence if matrix d is invertible and, in notation (6.27), is related to $a_0,\ a_1$ by relations (6.28) then this matrix is admissible for system (6.1).

It is easy to write down a system of equation for conjugate function υ. By assumption matrices $\tilde{a}_j$ have block structure

$$
\tilde{a}_0 = \begin{array}{c} \\ l_1 \\ m_1 \\ m_2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} l_1 & l_1 & l_2 \end{array} \\ \begin{pmatrix} 0 & -1 & 0 \\ p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \end{pmatrix} \end{array}, \quad \tilde{a}_1 = \begin{pmatrix} 1 & 0 & 0 \\ q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \end{pmatrix}.
$$

Differentiating the second equation in (6.18) we get system

$$
a_0 \frac{\partial U}{\partial x} + a_1 \frac{\partial U}{\partial y} = 0.
$$

Since $\tilde{a}_j = da_jd^{-1}$ we have

$$
\tilde{a}_0 \frac{\partial V}{\partial x} + \tilde{a}_1 \frac{\partial V}{\partial y} = 0, \quad V = dU.
$$

By (6.22) vector V is constructed from functions υ_1, υ_2 as in (6.17), hence taking into account expressions for $\tilde{a}_j$, for these functions we get the following equations:

$$
\begin{gathered}
p_{i1} \frac{\partial^2 \upsilon_1}{\partial x^2} + (p_{i2} + q_{i1}) \frac{\partial^2 \upsilon_1}{\partial x \partial y} + q_{i2} \frac{\partial^2 \upsilon_1}{\partial y^2} + \\
+ p_{i3} \frac{\partial \upsilon_2}{\partial x} + q_{i3} \frac{\partial \upsilon_2}{\partial y} = 0,\ i = 1, 2.
\end{gathered}
$$

Each of them has the same form as the first equation of system (6.1).

Despite relations (6.28) express conditions of compatibility of d with (6.1), conditions (6.23) are practically more convenient.

6.4. **Stokes system.** Linearized two-dimensional stationary Navier-Stokes system describing viscous incompressible fluid [40] is called *Stokes system.* In dimensionless variables it has the form

$$
\begin{gathered}
\frac{\partial^2 u^1}{\partial x^2} + \frac{\partial^2 u^1}{\partial y^2} - \frac{\partial p}{\partial x} = 0,\ \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} - \frac{\partial p}{\partial y} = 0, \\
\frac{\partial u^1}{\partial x} + \frac{\partial u^2}{\partial y} = 0,
\end{gathered} \tag{6.29}
$$

where $u = (u^1, u^2)$ is velocity vector and p is the pressure. With respect to $u_1 = u$,$u_2 = p$ this system is elliptic in the sense of Douglis-Nirenberg of the form (6.1) with $m_1 = l_1 = 2,\ m_2 = l_2 = 1$.

Its solutions can be represented using analytic functions [40]. Put $\omega = u^1 + iu^2$ and add the second equation in (6.29) multiplied by i to the first one. Then system (6.29) can be written in the form

$$2\frac{\partial^2\omega}{\partial z\partial\overline{z}} - \frac{\partial p}{\partial\overline{z}} = 0, \quad \frac{\partial\overline{\omega}}{\partial\overline{z}} + \frac{\partial\omega}{\partial z} = 0, \tag{6.30}$$

which shows analyticity of function

$$4\varphi' = 2\frac{\partial\omega}{\partial z} - p.$$

Since

$$4\overline{\varphi}' = 2\frac{\partial\overline{\omega}}{\partial\overline{z}} - p,$$

for p we get expression

$$p = -4\mathrm{Re}\,\varphi'. \tag{6.31}$$

Eliminating p from preceding equalities and using the second equation in (6.30) we obtain

$$\frac{\partial\omega}{\partial z} = \varphi' - \overline{\varphi}',$$

hence

$$\omega = \varphi - z\overline{\varphi}' + \overline{\psi} \tag{6.32}$$

with some analytic function ψ. Representation (6.31) , in particular, shows that function p is harmonic.

On the other hand, Stokes system (6.30) can be treated using results of subsection 6.2. Putting $u_1 = u,\ u_2 = p$, write characteristic polynomial (6.2) of this system as

$$P(z) = \begin{pmatrix} 1+z^2 & 0 & -1 \\ 0 & 1+z^2 & -z \\ 1 & z & 0 \end{pmatrix}.$$

Since $\det P(z) = (1+z^2)^2$, characteristic polynomial has in the upper half-plane one root $\nu = i$ of multiplicity 2. By theorem 6.1 matrix $b \in \mathbb{C}^{3\times 2}$ transforming $P(z)$ to Jordan form is constructed from eigenvector x_0 and adjoint vector x_1:

$$P(i)x_0 = 0, \quad P(i)x_1 + P'(i)x_0 = 0.$$

Thus

$$b = \begin{pmatrix} 1 & 0 \\ i & -1 \\ 0 & 2 \end{pmatrix}, \quad J = \begin{pmatrix} i & 1 \\ 0 & i \end{pmatrix},$$

and representation (6.14) from theorem 6.2 takes the form

$$u = \mathrm{Re}\begin{pmatrix} 1 & 0 \\ i & -1 \end{pmatrix}\phi, \quad p = \mathrm{Re}\,(2i\phi_2'), \tag{6.33}$$

where ϕ_2 is the second component of 2-vector ϕ.

By theorem 2.1 function ϕ can be expressed via an analytic vector-function ψ by formula (2.17)

$$\phi = \psi + \frac{i\overline{z}}{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\psi'.$$

Substituting this into (6.33) we get

$$2u_1 = \mathrm{Re}(2\psi_1 + i\overline{z}\psi_2'), \quad 2u_2 = \mathrm{Re}(2i\psi_1 - 2\psi_2 - \overline{z}\psi_2'),$$
$$p = \mathrm{Re}(2i\psi_2'),$$

or putting $\omega = u_1 + iu_2$,

$$2\omega = \overline{\psi}_1 - 4i(\psi_2 + \overline{\psi}_2) + \overline{zi\psi_2'}, \ p = \mathrm{Re}(2i\psi_2').$$

For $2\varphi = -i\psi_2'$, $2\psi = \psi_1 + 4i\psi_2$, these formulae transform into (6.31), (6.32). By analogy with subsection 5.3 introduce stress tensor

$$T = \begin{pmatrix} T_1 & T_3 \\ T_3 & T_2 \end{pmatrix} \tag{6.34}$$

with elements

$$T_1 = \frac{\partial u_1}{\partial x} - \frac{p}{2}, \ T_2 = \frac{\partial u_2}{\partial y} - \frac{p}{2}, \ T_3 = \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right).$$

With a solution (u, p) of Stokes system one can associate pair of function s (v, q) by formulae

$$\frac{\partial v}{\partial s} = Tn, \quad q = \frac{\partial u_1}{\partial y} - \frac{\partial u_2}{\partial x}, \tag{6.35}$$

first of which has the same sense as equality (5.47). These formulae can be rewritten in the form (6.22). Namely, denoting by $T_{(j)}$, $j = 1, 2$, columns of matrix T, its definition (6.34) can be rewritten in the form

$$T_{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \frac{\partial u}{\partial x} + \begin{pmatrix} 0 & 0 \\ 1/2 & 0 \end{pmatrix} \frac{\partial u}{\partial y} + \begin{pmatrix} -1/2 \\ 0 \end{pmatrix} p,$$

$$T_{(2)} = \begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix} \frac{\partial u}{\partial x} + \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial u}{\partial y} + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} p.$$

Writing out the first equation (6.35)

$$T_{(1)}\, n_1 + T_{(2)}\, n_2 = \frac{\partial v}{\partial x}(-n_2) + \frac{\partial v}{\partial y} n_1,$$

we arrive to relations (6.22)

$$\frac{\partial v}{\partial x} = \begin{pmatrix} 0 & -1/2 \\ 0 & 0 \end{pmatrix} \frac{\partial u}{\partial x} + \begin{pmatrix} -1/2 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial u}{\partial y} + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} p,$$

$$\frac{\partial v}{\partial y} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \frac{\partial u}{\partial x} + \begin{pmatrix} 0 & 0 \\ 1/2 & 0 \end{pmatrix} \frac{\partial u}{\partial y} + \begin{pmatrix} -1/2 \\ 0 \end{pmatrix} p, \tag{6.36}$$

$$q = (0, -1)\frac{\partial u}{\partial x} + (1, 0)\frac{\partial u}{\partial y}$$

for pair (v, q). Direct verification shows that the main condition (6.23) of lemma 6.2 holds for given matrices d_{ij} and matrices

$$b_1 = \begin{pmatrix} 1 & 0 \\ i & -1 \end{pmatrix}, \quad b_2 = (0, 2i), \quad J = \begin{pmatrix} i & 1 \\ 0 & i \end{pmatrix},$$

appearing in (6.33). Adopting notation (6.24) we also have

$$c_1 = \begin{pmatrix} -i & 0 \\ 1 & i \end{pmatrix} = -i\, b_1, \quad c_2 = (0,2) = i\, b_2.$$

Thus pair (v, q) admits a representation analogous to (6.33):

$$v = \mathrm{Im} \begin{pmatrix} 1 & 0 \\ i & -1 \end{pmatrix} \phi, \quad q = \mathrm{Im}(2i\phi_2'), \tag{6.37}$$

and , in particular, it is a solution of the same system (6.29). It is not difficult to show that matrix d in (6.36) commutes with matrices (6.9) corresponding to system (6.29). This agrees with the last statement of lemma 6.2.

Relations (6.35) are natural analogs of Cauchy-Riemann conditions for the real and imaginary parts of an analytic function. Along with representations (6.33), (6.37) they were obtained by N.Zhura and used for investigation of new non-local problems of hydromechanics [41]. Notice that changing in (6.33), (6.37) function ϕ to $i\phi$ Cauchy-Riemann conditions (6.35) can be reversed. In particular, we have

$$p = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}, \quad q = \frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial x}.$$

From the physical point of view this enables one to interpret pressure as the curl of conjugate velocity vector v while the conjugate pressure is equal to the minus curl of the original velocity vector.

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