ON REPRESENTATION OF SOLUTIONS OF SECOND ORDER ELLIPTIC SYSTEMS ON THE PLANE

A.P. Soldatov

Belgorod State University
Pobeda str. 85, Belgorod, 308015, Russian Federation
E-mail: soldatov@bsu.edu.ru

The analogue of the Privalov theorem is established for solutions of elliptic systems of second order in Lipshitz domains. The corresponding result is also obtained with respect to the weighted Hölder spaces. The notion of conjugate functions to solutions of elliptic systems is introduced and representations formulas for them are received. For so called strengthen elliptic systems the class of degenerate solutions is described.

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1. Harmonic functions

Let $C^\mu(E) = C^{0,\mu}(E)$, $0 < \mu \le 1$, be the ordinary Hölder space of functions $\varphi(z), z \in E \subseteq \mathbb{C}$, which is Banach with respect to the norm

$$|\varphi|_\mu = |\varphi|_0 + [\varphi]_\mu; \quad |\varphi|_0 = \sup_{z\in E} |\varphi(z)|, \quad [\varphi]_\mu = \sup_{z,z'\in E,\, z\neq z'} \frac{|\varphi(z) - \varphi(z')|}{|z - z'|^\mu}. \tag{1.1}$$

In the case $\mu = 1$ it is called the Lipschitz space and denoted by $C^{0,1}(E)$. If the set E is not closed a function $\varphi \in C^\mu(E)$ is extended to the function $\tilde\varphi \in C^\mu(\overline{E})$ and $[\varphi]_{\mu,E} = [\tilde\varphi]_{\mu,\overline{E}}$. Besides if the set E is bounded then (1.1) is equivalent to the norm

$$|\varphi| = |\varphi(c)| + [\varphi]_\mu, \quad c \in E.$$

Let a function $u(z) = u(x, y)$ be harmonic in the unit disk $B = \{|z| < 1\}$ and belong to $C^\mu(\overline{B})$, $0 < \mu < 1$. The known Privalov theorem [1] asserts that the analytic function $\phi(z)$ for which

$$u = \operatorname{Re}\phi, \tag{1.2}$$

belongs to the same class and the estimate $[\phi]_\mu \le M|u|_\mu$ is valid where the constant $M > 0$ depends only on μ. We extend this result to the case when B is an arbitrary Lipschitz domain but u is a solution of a second order elliptic system with constant and only leading coefficient.

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Recall the definition of the mentioned domains. Following Stein [2] the set

$$\{z = x + iy \mid f(x) < y\}, \tag{1.3}$$

where $f \in C^{0,1}(\mathbb{R})$, is called by a special Lipschitz domain. We attribute the domains to the same type which result by motion from (1.3). In the general case a finite domain D is Lipschitz if there exist open sets $V_1, \dots, V_n$ and special Lipschitz domains $D_1, \dots, D_n$ such that

$$\partial D \subseteq V_1 \cup \dots \cup V_n, \quad D \cap V_i = D_i \cap V_i,\ i = 1, \dots, n. \tag{1.4}$$

To illustrate our approach we consider first the classic case of harmonic functions. Our approach is based on the following property of Hölder spaces.

Lemma 1. *Let a function $\varphi(x,y)$ be bounded and continuously differential in a Lipschitz domain D, and the following estimate*

$$\left|\frac{\partial\varphi}{\partial x}(z)\right| + \left|\frac{\partial\varphi}{\partial y}(z)\right| \le C[\rho(z, \partial D)]^{\mu-1}, \quad 0 < \mu \le 1, \tag{1.5}$$

hold where $\rho(z, \partial D)$ denotes the distance from a point $z \in D$ to the boundary ∂D. Then $\varphi \in C^{0,\mu}(\overline{D})$ and

$$[\varphi]_{\mu,D} \le MC, \tag{1.6}$$

where the constant $M > 0$ depends only on μ and D.

Apparently this result is known but for completeness we produce its proof below.

Proof. Let us first assume that D is the special Lipschitz domain D of the form (1.4). Let $\theta(z)$ be the angle between the vector $z \in \mathbb{C}$ and the $y-$axis and thus $\theta(z) = \arctan(|x|/y),\ z = x+iy$. Let $K(\alpha), 0 < \alpha < \pi/2$, be a sector $\{z \mid \theta(z) < \alpha\}$. Note that the distance from $z \in K$ to its boundary is defined by the equality

$$\rho[z, \partial K(\alpha)] = |z| \sin[\alpha - \theta(z)]. \tag{1.7}$$

Let the sector K_a be received by parallel translation of K to the vertex $a = a_1 + ia_2 \in \Gamma = \partial D$. Verify that $K_a(\alpha) \subseteq D$ under assumption

$$[f]_1 < \cot\alpha. \tag{1.8}$$

In fact let

$$\tan\theta(z-a) = \frac{|x - a_1|}{y - a_2} < \tan\alpha,$$

or $y > f(a_1) + |x - a_1| \cot\alpha$. As $|f(x) - f(a_1)| \le [f]_1 |x - a_1|$, we have $y > f(x) + (\cot\alpha - [f]_1)|x - a_1|$. Taking (1.8) into account we receive $y > f(x)$.

Let us set $0 < \alpha < \alpha^0$, where α^0 satisfies (1.8) and put for brevity $K_a = K_a(\alpha)$, $K_a^0 = K_a(\alpha^0)$. Then by virtue of (1.7) for $z \in K_a$ we have

$$\rho(z, \partial D) \ge \rho(z, \partial K_a^0) \ge |z - a| \sin(\alpha^0 - \alpha).$$

So the estimate (1.5) in K_a transforms into

$$\left|\frac{\partial\varphi}{\partial x}(z)\right| + \left|\frac{\partial\varphi}{\partial y}(z)\right| \le M_1 C |z - a|^{\mu-1}, \quad z \in K_a. \tag{1.9}$$

Let us consider the ray $L = \{a + te \mid t > 0\} \subseteq K_a$ with the end a and a unit vector e. For the function $\psi(t) = \varphi(a + te)$ on this ray (1.9) takes the form $|\psi'(t)| \le M_1 C t^{\mu-1}$, so

$$|\psi(t) - \psi(t')| \le M_1 C \int_t^{t'} s^{\mu-1} ds \le \mu^{-1} M_1 C (t' - t)^\mu, \quad t \le t'.$$

Thus we have the estimate

$$|\varphi(z) - \varphi(z')| \le M_2 C |z - z'|^\mu \tag{1.10}$$

for every $z, z' \in L \subseteq K_a$, which does not depend on $a \in \Gamma$ and the ray L.

If the points $z, z' \in D$ satisfy the condition

$$y \le y', \quad \theta(z' - z) < \alpha, \tag{1.11}$$

then these points lie on the ray $L \subseteq K_a$, where a is a point of intersection of the straight line through z and z' with ∂D. So (1.10) holds for these points.

In the opposite case $y \le y', \theta(z' - z) \ge \alpha$ let us consider the triangle with the vertices z, z' and $z'' = x' + it$, $t > f(x')$. Let θ, θ' and θ'' be the angles in these vertices. Let us choose the value t such that $\theta'' = \alpha$. Then the pairs of points z, z'' and z', z'' satisfy (1.11) and therefore

$$|\varphi(z) - \varphi(z')| \le |\varphi(z) - \varphi(z'')| + |\varphi(z') - \varphi(z'')| \le M_2 C (|z - z''|^\mu + |z' - z''|^\mu).$$

On the other hand from the sine theorem it follows that

$$\frac{|z - z''|}{|z - z'|} = \frac{\sin\theta'}{\sin\alpha} \le \frac{1}{\sin\alpha}, \quad \frac{|z' - z''|}{|z - z'|} = \frac{\sin\theta}{\sin\alpha} \le \frac{1}{\sin\alpha},$$

and hence $|z - z''|^\mu + |z' - z''|^\mu \le M_3 |z - z'|^\mu$. Combined with the previous inequality we receive (1.10) with a constant M_4, and (1.5) is proved in the considered case.

Suppose that the function φ satisfies (1.5) only in a disk $|z - z_0| < r$ with center $z_0 \in \partial D$. Then analogues reasonings show that there exists $0 < r_0 < r$ such that

$$[\varphi]_{\mu,G} \le M_0 C, \quad G = D \cap \{|z - z_0| < r_0\}. \tag{1.12}$$

Let us turn to a general Lipschitz domain D. By virtue of (1.4) there exists $r > 0$ such that for every point $z_0 \in \partial D$ the set $D \cap \{|z - z_0| < r\}$ is contained in V_i for some i. So we can suppose that (1.12) holds for every point $z_0 \in \partial D$. Let us consider the compact $K = \{z \in D,\ \rho(z, \partial D) \ge r_0\}$. The estimate (1.4) shows that the partial derivatives of φ do not exceed $C r_0^{\mu-1}$ by module on K and therefore $[\varphi]_{\mu,K} \le M_1 C$. Together with (1.12) this estimate proves (1.5).

With the help of the lemma it is easy to establish the following generalization of Privalov theorem.

Theorem 1. *Let a function $u \in C^\mu(\overline{D})$, $0 < \mu < 1$, be harmonic in a simply connected Lipschitz domain D. Then the analytic function ϕ in* (1.2) *belongs to the same class and*

$$[\phi]_\mu \le M [u]_\mu, \tag{1.13}$$

where the constant $M > 0$ depends only on μ and D.
Proof. If a disk $B = \{|z - z_0| < r\} \subseteq D$ then the estimate

$$\left|\frac{\partial u}{\partial x}(z_0)\right| + \left|\frac{\partial u}{\partial y}(z_0)\right| \leq M r^{\mu-1}[u]_{\mu,B} \tag{1.14}$$

is valid, where the constant $M > 0$ does not depend on B. In fact without loss of generality we can assume $u(z_0) = 0$. Putting $z_0 = 0$ and $\tilde{u}(z) = u(rz)$, we can also content ourself with the case $r = 1$. So B is the unit circle and $u(0) = 0$, and therefore $|u|_0 \leq [u]_\mu$. In this case (1.14) follows from Poisson formula.

Obviously we can set $r = \rho(z_0, \partial D)$ in (1.14). By virtue of the relation

$$\phi' = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}, \tag{1.15}$$

which is inverse to (1.2) we can substitute u for ϕ in the left side of this estimate. On the basis of Lemma 1 applying to ϕ we receive (1.13).

2. Elliptic systems

Let us consider the elliptic system

$$A_{11}\frac{\partial^2 u}{\partial x^2} + (A_{12} + A_{21})\frac{\partial^2 u}{\partial x \partial y} + A_{22}\frac{\partial^2 u}{\partial y^2} = 0 \tag{2.1}$$

with constant coefficients $A_{ij} \in \mathbb{R}^{l \times l}$ for an unknown vector-valued function $u = (u_1, \ldots, u_l) \in C^2$. The condition of ellipticity means that $\det A_{22} \neq 0$ and the characteristic polynomial

$$\chi(z) = \det P(z), \quad P(z) = A_{11} + (A_{12} + A_{21})z + A_{22}z^2, \tag{2.2}$$

has no real roots.

Recall the representation formula [3] of a general solution of the system (2.1) in a simply connected domain D. Let us introduce the block- matrix

$$A_* = \begin{pmatrix} 0 & 1 \\ -A_{22}^{-1}A_{11} & -A_{22}^{-1}(A_{12} + A_{21}) \end{pmatrix},$$

its spectrum $\sigma(A_*)$ consists of roots of the characteristic polynomial (2.2). Then (2.1) can be written in the form

$$\frac{\partial \nabla u}{\partial y} = A_* \frac{\partial \nabla u}{\partial x}, \quad \nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right).$$

Let the matrix B_* reduce A_* to the Jordan form

$$B_*^{-1} A_* B_* = J_*, \quad J_* = \operatorname{diag}(J, \overline{J}), \tag{2.3}$$

where the spectrum of the matrix $J \in \mathbb{C}^{l \times l}$ lies in the upper half plane. As the matrix A_* is real the matrix B_* can be chosen in block form

$$B_* = \begin{pmatrix} B & \overline{B} \\ BJ & \overline{BJ} \end{pmatrix}. \tag{2.4}$$

So the $l-$vector-valued function ψ from the equality

$$B_*\psi_* = 2\nabla u, \quad \psi_* = (\psi, \overline{\psi}), \tag{2.5}$$

satisfies the system

$$\frac{\partial \psi}{\partial y} - J\frac{\partial \psi}{\partial x} = 0. \tag{2.6}$$

Solutions of this system are said to be Douglis analytic functions or shortly $J-$analytic functions. For the case of a Toeplitz matrix J the system was firstly investigated by Douglis [4] in the frame of so called hypercomplex numbers. Later first order elliptic systems studied by many authors ([5]-[10]). It is easy to show [3] that a Douglis analytic function $\psi(z)$ in the neighborhood of each point $z_0 \in D$ expands in a generalized power series

$$\psi(z) = \sum_{k=0}^{\infty} \frac{1}{k!}(z - z_0)_J^k \psi^{(k)}(z_0), \quad \psi^{(k)} = \frac{\partial^k \psi}{\partial x^k},$$

where z_J, $z = x + iy$, denotes the matrix $x1 + yJ$. Besides, if this function is continuous up to the boundary ∂D, then the generalized Cauchy formula

$$\psi(z) = \frac{1}{2\pi i}\int_{\partial D} dt_J (t - z)_J^{-1}\psi(t), \quad z \in D, \tag{2.7}$$

holds, where the matrix differential dt_J has the same sense and the contour ∂D is oriented positively with respect to D. The function ψ can be also expressed through usual analytic functions. Let the matrix J in (2.3) be written in the form $J = \operatorname{diag}(J_1, \ldots, J_n)$, $J_k \in \mathcal{C}^{l_k \times l_k}$, where $\sigma(J_k) = \nu_k$, $\operatorname{Im}\nu_k > 0$, and therefore $(J_k - \nu_k)^{l_k} = 0$. Then the system (2.6) decompose into

$$\frac{\partial \psi_k}{\partial y} - J_k\frac{\partial \psi_k}{\partial x} = 0, \quad \psi = (\psi_1, \ldots, \psi_n).$$

In these notations $J-$analytic vector- valued functions $\psi = (\psi_1, \ldots, \psi_n)$ can be uniquely represented by the formula

$$\psi = \Lambda\tilde{\psi}, \quad (\Lambda\tilde{\psi})_k(x + iy) = \sum_{r=0}^{l-1} \frac{y^r}{r!}(J_k - \nu_k)^r \tilde{\psi}_k^{(r)}(x + \nu_k y), \tag{2.8}$$

where the l_k-vector $\tilde{\psi}_k$ is analytic in the domain $D_k = \{x + \nu_k y \mid z = x + iy \in D\}$. The inverse formula is

$$\tilde{\psi}_k(x + \nu_k y) = \sum_{r=0}^{l-1} \frac{(-y)^r}{r!}(J_k - \nu_k)^r \psi_k^{(r)}(x + iy), \quad k = 1, \ldots, n.$$

Let us turn to the relation (2.5). After integrating it takes the form

$$u = \operatorname{Re} B\phi, \tag{2.9}$$

where ϕ is the Douglis analytic function

$$\phi(z) = \int_{z_0}^{z} dt_J \psi(t) + \phi(z_0).$$

Accordingly the inverse relation is

$$\phi' = B_1 \frac{\partial u}{\partial x} + B_2 \frac{\partial u}{\partial y}, \quad 2B_*^{-1} = \begin{pmatrix} B_1 & B_2 \\ \overline{B_1} & \overline{B_2} \end{pmatrix}. \tag{2.10}$$

The representation (2.9) of a general solution of the elliptic system (2.1) was firstly received by Soldatov [11] and Yeh [12]. Note that (2.9), (2.10) are analogous to (1.2), (1.15) with respect to solutions of (2.1). Substitution the formula (2.8) for corresponding functions ϕ and $\tilde{\phi}$ into (2.9) gives the known Bitsadze representation [13] through the analytic function $\tilde{\phi}$.

Recall that the elliptic system (2.1) is called weakly connected [13] if the matrix B in (2.4) is invertible. This condition is equivalent [14] to invertibility of the matrix

$$\int_{\mathbb{R}} P^{-1}(t)dt \in \mathbb{R}^{l\times l}, \tag{2.11}$$

where P is defined by (2.2).

Theorem 2. *Let the elliptic system* (2.1) *be weakly connected and a function* $u \in C^\mu(\overline{D})$, $0 < \mu < 1$, *be its solution in a simply connected Lipschitz domain* D. *Then the function* ϕ *in* (2.9) *belongs to the same class and* (1.13) *holds with a constant* M *depending only on* μ *and* D.

Proof. The Proof is identical with the one of Theorem 1 under the assumption that the following fact is true. If $u \in C^\mu(\overline{B})$ is a solution of (2.1) in the unit disk $B = \{|z| < 1\}$, then the inequality

$$\left|\frac{\partial u}{\partial x}(0)\right| + \left|\frac{\partial u}{\partial y}(0)\right| \le M[u]_\mu \tag{2.12}$$

is valid. Substituting B for the circle $\{|z| < 1/2\}$, we can assume $u \in C^1(\overline{B})$. Then taking (2.10) into account the function ϕ in (2.9) belongs to the same class and (2.12) reduces to

$$|\phi'(0)| \le M[\operatorname{Re} B\phi]_\mu, \tag{2.13}$$

where the constant $M > 0$ does not depend on the Douglis analytic function $\phi \in C^\mu(\overline{B})$.

Let X be the class of these functions with the additional condition $\phi(0) = 0$. It follows from the Cauchy formula (2.7) that

$$|\phi'(0)| \le M[\phi]_\mu$$

and (2.13) reduces to the estimate

$$[\phi]_\mu \le M[\operatorname{Re} B\phi]_\mu \quad \phi \in X. \tag{2.14}$$

As it was shown in [15] the Riemann- Hilbert problem

$$\operatorname{Re} B\phi\,|_{\partial B} = f$$

is Fredholm in the class X, i.e. the operator $(R\phi)(t) = \operatorname{Re} B\phi(t)$, $t \in \partial B$, is Fredholm $X \to C^\mu(\partial B)$. So its image $Y = R(X)$ is a closed subspace of $C^\mu(\partial B)$ of finite co-dimension but its kernel $X_0 = \ker R$ is a finite dimensional subspace of X. Thus there exists a bounded operator $R^{(-1)} : Y \to X$ such that $RR^{(-1)}f = f, f \in Y$, and hence

$$X = X_0 \oplus X_1,\ X_1 = R^{(-1)}(Y). \tag{2.15}$$

By virtue of the boundedness of $R^{(-1)}$ the norms $[\phi]_\mu$ and $[\mathrm{Re}\, B\phi]_\mu$ are equivalent on X_1. Obviously the same is true on the finite dimensional space X_0. Taking (2.15) into account (2.14) follows. That completes the proof.

3. Estimates in weighted Hölder spaces

Let D be a simply connected domain with piecewise smooth boundary ∂D and one sided tangents at its angular points are different. In particular D is a Lipschitz domain. Let $0 \in \partial D$ and a solution u of the weakly connected system (2.1) belongs to $C^\mu_{loc}(\overline{D}\backslash 0)$. The last implies that u satisfies a Hölder condition in $D_\varepsilon = D \cap \{|z| > \varepsilon\}$ for each $\varepsilon > 0$. Then on basis of Theorem 2 the Douglis analytic function in (2.9) belongs to the same class. We are interested in the question whether if $u(z) = O(|z|^\lambda)$ as $z \to 0$ it is true for ϕ. More exactly we describe the power behavior of functions with the help of weighted Hölder spaces. Recall [15] that the Banach space $C^\mu_\lambda = C^\mu_\lambda(D, 0), \lambda \in \mathbb{R}$, consists of all functions $\varphi(z)$ on $D \setminus 0$ with finite norm

$$|\varphi| = ||z|^{-\lambda}\varphi(z)|_0 + [|z|^{\mu-\lambda}\varphi(z)]_\mu. \tag{3.1}$$

In notations (1.1) we can give another definition of this space.

Lemma 2. *Let the integer m be such that $D_k = \{z' \mid 2^{-k}z' \in D,\ 1/2 < |z'| < 2\} \neq \emptyset,\ k \geq m$ and $D_k = \emptyset,\ k < m - 1$. Then* (3.1) *is equivalent to the norm*

$$|\varphi|' = \sup_{k \geq m} 2^{k\lambda} |\varphi(2^{-k}z')|_{\mu, D_k}. \tag{3.2}$$

Proof. Obviously

$$||z|^{-\lambda}\varphi|_0 \leq |2|^{|\lambda|}|\varphi|'.$$

Let $|z_1| \leq |z_2|$ and $z_j = 2^{-k}z'_j$, where k is defined by condition $1 \leq |z'_1| < 2$. Then there are two cases, when $|z'_2| \leq 1/2$ and $|z'_2| > 1/2$. In the first case

$$A = \frac{||z_1|^{\mu-\lambda}\varphi(z_1) - |z_2|^{\mu-\lambda}\varphi(z_2)|}{|z_1 - z_2|^\mu} \leq \left\{ \frac{|z'_1|^{\mu-\lambda} + |z'_2|^{\mu-\lambda}}{|z'_1 - z'_2|^\mu} \right\} |\varphi|'$$

and in the second case

$$A \leq \left\{ |z'_1|^{\mu-\lambda} + \frac{||z'_1|^{\mu-\lambda} - |z'_2|^{\mu-\lambda}|}{|z'_1 - z'_2|^\mu} \right\} |\varphi|'.$$

In both cases the figured expressions are bounded by a constant depending only on μ and λ. According to (3.1), (3.2) these estimates yield $|\varphi| \leq M|\varphi|'$.

To receive the inverse inequality let us denote $\psi(z) = |z|^{\mu-\lambda}\varphi(z)$. Then $|\varphi(z)| \leq |z|^\lambda[\psi]_\mu \leq |z|^\lambda|\varphi|$ and hence

$$2^{k\lambda}|\varphi(2^{-k}z')| \leq |z'|^\lambda|\varphi|, \quad 1/2 < |z'| < 2.$$

Further we have

$$2^{k\lambda}\frac{|\varphi(2^{-k}z_1')-\varphi(2^{-k}z_2')|}{|z_1'-z_2'|^\mu}=\frac{|z_1'|^{\mu-\lambda}\psi(z_1)-|z_2'|^{\mu-\lambda}\psi(z_2)}{|z_1-z_2|^\mu}$$

$$\le\left\{|z_1'|^{\mu-\lambda}+\frac{||z_1'|^{\lambda-\mu}-|z_2'|^{\lambda-\mu}|}{|z_1'-z_2'|^\mu}|z_2'|^\mu\right\}|\varphi|.$$

Together with the previous inequality we receive the required estimate $|\varphi|'\le M|\varphi|$.

By definition the space $C^\mu_{(\lambda)}=C^\mu_{(\lambda)}(D,0), 0<\lambda<1$, consists of functions $\varphi\in C(\overline{D})$ such that $\varphi(z)-\varphi(0)\in C^\mu_\lambda$. It is convenient to set $C^\mu_{(\lambda)}=C^\mu_\lambda$ for $\lambda\le 0$.

Lemma 3. *The space $C^\mu_{(\lambda)},\ \lambda\ne 0$, can be defined by the equivalent norm*

$$|\varphi|'=|\varphi(c)|+[\varphi]_{\mu,(\lambda)},\quad [\varphi]_{\mu,(\lambda)}=\sup_{k\ge m}2^{k\lambda}[\varphi(2^{-k}z')]_{\mu,D_k},\tag{3.3}$$

where $c\in D$ is a fixed point.

Proof. Let us first consider the case $\lambda<0$. Let us choose points $z_i'\in D_i\cap D_{i+1}$, $i=m,m+1,\dots$. By virtue of Lemma 1 the equality

$$|\varphi|=\sup_{k\ge m}2^{k\lambda}\{|\varphi(2^{-k}z_k')|+[\varphi(2^{-k}z')]_{\mu,D_k}\}$$

defines an equivalent norm in C^μ_λ. Without loss of generality we can set $c=2^{-m}z_m'$ in (3.3). Then

$$|\varphi(2^{-k}z_k')|\le|\varphi(c)|+\sum_{i=m}^{k-1}|\varphi(2^{-k}z_{i+1}')-\varphi(2^{-k}z_i')|.$$

As

$$|\varphi(2^{-k}z_{i+1}')-\varphi(2^{-k}z_i')|\le 2^{-i\lambda}[\varphi]_{\mu,(\lambda)}|z_{i+1}'-z_i'|^\mu\le 4\,2^{-i\lambda}[\varphi]_{\mu,(\lambda)},$$

it follows that

$$|\varphi(2^{-k}z_k')|\le|\varphi(c)|+4(2^{-\lambda}-1)^{-1}2^{-k\lambda}[\varphi]_{\mu,(\lambda)}$$

and thus $|\varphi|\le M|\varphi|'$. The inverse inequality is obvious.

The case $0<\lambda<1$ can be considered analogously. Without loss of generality we can set $c=0$ in (3.3). On basis of Lemma 2 the equality

$$|\varphi|=|\varphi(0)|+\sup_{k\ge m}2^{k\lambda}\{|\tilde\varphi(2^{-k}z_k')|+[\tilde\varphi(2^{-k}z')]_{\mu,D_k}\},\quad\tilde\varphi(z)=\varphi(z)-\varphi(0),$$

defines an equivalent norm in $C^\mu_{(\lambda)}$. So

$$|\tilde\varphi(2^{-k}z_k')|\le\sum_{i=k}^{\infty}|\varphi(2^{-k}z_{i+1}')-\varphi(2^{-k}z_i')|\le 4(1-2^{-\lambda})^{-1}2^{-k\lambda}[\varphi]_{\mu,(\lambda)},$$

and therefore $|\varphi|\le M|\varphi|'$.

Theorem 3. *Let D be a simply connected domain with piecewise smooth boundary ∂D and with different one-sided tangents at its angular points. Let $0\in\partial D$ and a function $u\in C^\mu_{(\lambda)}(\overline{D},0)$, where $0<\mu<1,\lambda<1,\lambda\ne 0$, satisfy the weakly connected* *elliptic system* (2.1) *in* D. *Then the function* ϕ *in* (2.9) *belongs to the same class and the estimate*

$$[\phi]_{\mu,(\lambda)} \leq M[u]_{\mu,(\lambda)}, \tag{3.4}$$

holds with a constant M *depending only on* μ, λ *and* D.

Proof. Let us set

$$D_0 = D \cap \{|z| < \varepsilon\}, \quad D_1 = D \cap \{|z| > \varepsilon/2\},$$

where $\varepsilon > 0$ is sufficiently small. Then D_1 is a Lipschitz domain and on basis of Theorem 2 the function $\phi \in C^\mu(\overline{D}_1)$. So without loss of generality we can assume $D = D_0$. Then ∂D is formed by an arc L of the circumference $|z| = \varepsilon$ and two smooth arcs Γ_1, Γ_2 with the common end $z = 0$. In polar coordinates $r = |z|, \theta = \arg z$ these arcs are described by equations $\theta = h_j(r)$, where

$$h_j(r) \in C[0, \varepsilon] \cap C^1(0, \varepsilon], \quad \lim_{r \to 0} r h_j'(r) = 0, \quad j = 1, 2. \tag{3.5}$$

Assuming $h_1(0) \neq h_2(0)$ modulo 2π, we can suppose that $h_1(r) < h_2(r)$ for all $0 \leq r \leq \varepsilon$ and D is defined by the inequalities $h_1(r) < \theta < h_2(r)$, $0 < r < \varepsilon$. In particular the domains $D_k = \{z' \mid 2^{-k}z' \in D,\, 1/2 < |z'| < 2\}$ in (3.1) are described by the inequalities $h_1(2^{-k}r) < \theta < h_2(2^{-k}r)$, $1/2 < r < 2$. By virtue of (3.5) the functions $h_j(2^{-k}r) \to h_j(0)$ as $k \to \infty$ in $C^1[1/2, 2]$. Hence the estimate (1.13) in Theorem 2 with respect to D_k holds uniformly by $k \geq m$. In particular

$$[\phi(2^{-k}z')]_{\mu, D_k} \leq M[u(2^{-k}z')]_{\mu, D_k},$$

where the constant $M > 0$ does not depend on k. On basis of Lemma 3 it follows that the function $\phi \in C^\mu_{(\lambda)}(\overline{D}, 0)$ and the estimate (3.4) is valid.

4. Conjugate functions

Let $u(z)$ be a solution of the elliptic system (2.1). Consider the function $v(z)$ which is defined by the relation

$$\frac{\partial v}{\partial x} = -\left(A_{21}\frac{\partial u}{\partial x} + A_{22}\frac{\partial u}{\partial y}\right), \quad \frac{\partial v}{\partial y} = A_{11}\frac{\partial u}{\partial x} + A_{12}\frac{\partial u}{\partial y}. \tag{4.1}$$

The existence of this function follows from (2.1) written in the form

$$\frac{\partial}{\partial x}\left(A_{11}\frac{\partial u}{\partial x} + A_{12}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(A_{21}\frac{\partial u}{\partial x} + A_{22}\frac{\partial u}{\partial y}\right) = 0.$$

The function v is said to be conjugate to the solution u. It is defined with accuracy of a constant vector $\xi \in \mathbb{R}^l$. This function is closely connected with the second boundary value problem

$$\sum_{i,j=1}^{2} A_{ij} n_i \frac{\partial u}{\partial x_j}\bigg|_{\partial D} = g, \tag{4.2}$$

where $x_1 = x, x_2 = y$ and $n = (n_1, n_2)$ denotes the external normal to the boundary. In fact by virtue of (4.1) this boundary condition we can write in the form

$$v'_s = g, \tag{4.3}$$

where $(\ )'_s$ means the tangent differentiation.

As it is seen from (4.1) theorems 2 and 3 are valid with respect to the conjugate function v. This fact also follows from the representation of this function

$$v = \xi + \operatorname{Re} C\phi, \quad C = -(A_{21}B + A_{22}BJ), \tag{4.4}$$

where the Douglis analytic function ϕ is given in (2.9).

The proof of this representation is not complicated. On the basis of (2.6), (2.9) the equation (2.1) gives the identity

$$\operatorname{Re}\,[A_{11}B + (A_{12} + A_{21})BJ + A_{22}BJ^2]\phi'' = 0.$$

It follows that $A_{11}B + (A_{12} + A_{21})BJ + A_{22}BJ^2 = 0$ or

$$C = -(A_{21}B + A_{22}BJ), \quad CJ = A_{11}B + A_{12}BJ. \tag{4.5}$$

Thus we can rewrite the relations (4.1) in the form

$$\frac{\partial v}{\partial x} = \operatorname{Re} C\phi', \quad \frac{\partial v}{\partial y} = \operatorname{Re} CJ\phi'.$$

So the partial derivatives of $v - \operatorname{Re} C\phi$ are equal to 0 that proves (4.4).

It follows from (4.3), (4.4) that the second boundary value problem (4.2) reduces to the Riemann-Hilbert problem

$$\operatorname{Re} C\phi|_{\partial D} = f, \quad f'_s = g,$$

with a constant matrix coefficient C. Hence the condition $\det C \neq 0$ is necessary [15] for this problem to be Fredholm. The equality $\det C = 0$ is closely connected with the case of a constant conjugate function v, when the right-hand side of (4.1) is identically equal to 0. In this case the solution u of the system (2.1) is called degenerate.

In such a way the degenerate solutions are defined by the over-determined first order system

$$A_{i1}\frac{\partial u}{\partial x} + A_{i2}\frac{\partial u}{\partial y} = 0, \quad i = 1, 2. \tag{4.6}$$

It is obvious that the polynomials of first degree $u(x) = \xi_0 + \xi_1 x + \xi_2 y$, where $A_{i1}\xi_1 + A_{i2}\xi_2 = 0$, give the simplest example of degenerate solutions.

It is convenient below to consider numerical $l \times l-$matrices as linear operators in $\mathbb{C}^l$. Let us put

$$P = A_{11}^{-1}A_{12},\ Q = A_{22}^{-1}A_{21}, \quad X = \operatorname{Ker}(1 - PQ) \cap \operatorname{Ker}(1 - QP), \tag{4.7}$$

and introduce the subspace Y of vectors $\eta \in \mathbb{C}^l$ such that

$$\operatorname{Re} CJ^k\eta = 0, \quad k = 0, 1, 2. \tag{4.8}$$

The class of all degenerate solutions can be described with the help of this space. Namely the solution u is degenerate if and only if all partial derivatives of the second order of the function (4.4) are equal to 0, that is equivalent to the condition $\eta = \phi''(z) \in Y$ for all z.

Lemma 4. *The space Y is invariant with respect to the operator J and described by the equivalent conditions*

$$\mathrm{Re}\, B\eta \in X, \ \mathrm{Re}\, C\eta = 0. \tag{4.9}$$

Proof. Make sure first that the space defined by (4.9) is invariant with respect to J. Let η satisfy (4.9). It is obviously that the operators P and Q are invariant and mutually inverse on X. Besides we can rewrite (4.5) in the form

$$C = -A_{22}(QB + BJ), \quad CJ = -A_{11}(B + PBJ). \tag{4.10}$$

So (4.8) becomes

$$\mathrm{Re}\, BJ\eta = -Q\mathrm{Re}\, B\eta \in X,$$

$$\mathrm{Re}\, CJ\eta = A_{11}P\mathrm{Re}\,(B\eta + PBJ)\eta = A_{11}P\mathrm{Re}\,(QB\eta + BJ\eta) = 0$$

and (4.9) is really invariant with respect to J.

In particular (4.9) implies (4.8). Conversely let η satisfy (4.9). It follows from (4.10) that

$$CJ^2 = A_{11}BJ - A_{11}P(A_{22}^{-1}C + QB)J = A_{11}(1 - PQ)BJ - A_{11}PA22^{-1}CJ.$$

Let us set $x = \mathrm{Re}\, B\eta, y = \mathrm{Re}\, BJ\eta$ for brevity and substitute (4.10) and the last expression into (4.8). Then we receive the relations $Qx + y = 0$, $x + Py = 0$, $(1 - PQ)y = 0$, that imply $x = \mathrm{Re}\, B\eta \in X$.

Theorem 4. *If $X = 0$, then every degenerate solution of the elliptic system is a polynomial of first order. Otherwise the class of degenerate solutions is infinitely dimensional. The case $X = 0$ is provided by* $\det\, C \neq 0$. *In general case the dimension of X is even.*

Proof. By virtue of Lemma 4 the dimension of Y coincides with $\dim X$. So if $X = 0$, then $\phi''(z) \in Y$ is equivalent $\phi'' = 0$.

If $X \neq 0$, then the class of all Douglis analytic functions ϕ, such that $\phi''(z) \in Y$ is invariant with respect to J and therefore it contains all functions of the form

$$\phi(z) = (z - z_0)_J^{-k}\eta, \quad \eta \in Y, \, k = 1, 2 \ldots, \tag{4.11}$$

where the fixed point $z_0 = x_0 + iy_0$ lies outside D. So this class is infinitely dimensional.

Suppose further by contradiction that $\det C \neq 0$ but $\dim\, X > 0$. Let us consider the system (2.1) in the upper half-plane $\mathrm{Im}\, z > 0$. Then the functions (4.11) for $\mathrm{Im}\, z_0 < 0$ define degenerate solutions $u = \mathrm{Re}\, B\phi$ of this system. In particular, $\mathrm{Re}\, C\phi'(x) = 0, x \in \mathbb{R}$, where $\mathbb{R}$ is the real axis of the complex plane $\mathbb{C}$. According to (2.8) we can write $\phi = \Lambda\tilde{\phi}, \ \tilde{\phi} = (\tilde{\phi}_1, \ldots, \tilde{\phi}_n)$, where the l_k-vector valued function $\tilde{\phi}_k$

is analytic on $\mathbb{C}\setminus\{x_0+\nu_k y_0\}$ and has the same degree $-k$ at infinity as ϕ. Formula (2.8) also shows that $\phi(x)=\tilde{\phi}(x)$, $x\in\mathbb{R}$. Hence $\operatorname{Re} C\tilde{\phi}'(x)=0, x\in\mathbb{R}$, and in particular the function $C\tilde{\phi}'(z)$ is analytically extended to the lower half-plane. As $\tilde{\phi}'(z)\to 0$ at ∞ and under the assumption $\det C\neq 0$, it follows $\tilde{\phi}'=0$. Hence $\phi'=0$, that contradicts the choice of ϕ.

Recall that P and Q from Lemma 4 act on X as mutually inverse operators. We assert that these operators have no real eigenvalues. Really assuming the contrary let $P\xi=\mu\xi$ and $Q\xi=\mu^{-1}\xi$ for $\mu\in\mathbb{R}$ and $\xi\in X$ not equal to 0. Then

$$\sum_{i,j=1}^{2}(A_{ij}t_it_j)\xi=[A_{11}(t_1+Pt_2)t_1+A_{22}(Qt_1+t_2)t_2]\xi=0,\quad t_1+\mu t_2=0.$$

That contradicts to the ellipticity condition

$$\det\left(\sum\nolimits_{i,j=1}^{2}A_{ij}t_it_j\right)\neq 0,\quad |t_1|+|t_2|>0, \tag{4.12}$$

of the system (2.1). Thus the operators P and Q acting on X have no real eigenvalues and hence $\dim X$ is even.

5. Strengthen elliptic systems

Due to Vishik[16] the system (2.1) is strongly elliptic if

$$\left(\left(\sum\nolimits_{i,j=1}^{2}A_{ij}t_it_j\right)\xi,\xi\right)>0 \tag{5.1}$$

for all $t_j\in\mathbb{R}$, $|t_1|+|t_2|\neq 0$ and nonzero vectors $\xi\in\mathbb{R}^l$. Here and below $(\ ,\ )$ denotes the inner product in $\mathbb{R}^l$. In particular the condition (2.11) for the system to be weakly connected is fulfilled. A narrower class of elliptic systems is defined [14] by the conditions

$$A_{ji}^{\top}=A_{ij},\quad \sum\nolimits_{i,j=1}^{2}(A_{ij}\xi_j,\xi_i)\geq 0,\quad \xi_i\in\mathbb{R}^l, \tag{5.2}$$

i.e the block matrix

$$A=\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix} \tag{5.3}$$

is positive semidefinite. These systems are said to be strengthened elliptic. Note that the ellipticity condition (4.12) for these systems is equivalent to the following property. If $A\xi=0, \xi=(\xi_1,\xi_2)$, where $t_1\xi_1+t_2\xi_2=0, |t_1|+|t_2|\neq 0$, then $\xi_1=\xi_2=0$.

Theorem 5. *Let the system* (2.1) *be strengthened elliptic. Then* $X=0$ *is equivalent to* $\det C\neq 0$ *and is provided by*

$$\operatorname{rang} A\geq 2l-1. \tag{5.4}$$

Proof. On basis of Theorem 4 $\det C \neq 0$ implies $X \neq 0$. To prove that $\det C = 0$ implies $\dim X > 0$ we suppose by contradiction that $\det C = 0$ but $X = 0$. Let $C\eta = 0$ for a nonzero $\eta \in \mathbb{C}^l$ and set

$$\tilde{\phi}(z) = (z - z_0)^{-1}\eta, \quad \operatorname{Im} z_0 < 0, \tag{5.5}$$

Let the Douglis analytic function ϕ be connected with $\tilde{\phi}$ by (2.8) i.e $\phi = \Lambda\tilde{\phi}$. As $\phi(x) = \tilde{\phi}(x)$, $x \in \mathbb{R}$, we have the equality

$$C\phi(x) = 0, \quad x \in \mathbb{R}. \tag{5.6}$$

In particular the conjugate function $v = \operatorname{Re} C\phi$, to the solution $u = \operatorname{Re} B\phi$ of (2.1) is equal to 0 on the boundary $\mathbb{R} = \partial D$ of the half-plane $D = \{\operatorname{Im} z > 0\}$.

The Green formula applied to the scalar product of (2.1) with u gives the equality

$$\int_D \sum_{i,j=1}^{2} \left(A_{ij}\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j} \right) dx_1 dx_2 = -\int_{\mathbb{R}} \left(A_{21}\frac{\partial u}{\partial x} + A_{22}\frac{\partial u}{\partial y}, u \right) dx,$$

where $x_1 = x, x_2 = y$. By virtue of (5.6)

$$A_{21}\frac{\partial u}{\partial x} + A_{22}\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$$

on the boundary $\mathbb{R} = \partial D$, so we have

$$\int_D \sum_{i,j=1}^{2} \left(A_{ij}\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j} \right) dx_1 dx_2 = 0.$$

Together with (5.2) the relations (4.4) follow. Thus the solution u is degenerate and by virtue of Theorem 4 it is a polynomial of the first degree. In particular $\operatorname{Re} B\phi'' = 0$ on the half-plane D and therefore the function ϕ' is constant. But this fact contradicts to (5.5).

Let us turn to the second assertion of the theorem. It follows from (4.7), (5.3) that

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} 1 & P \\ Q & 1 \end{pmatrix}$$

and therefore $\xi \in X$ implies $A\tilde{\xi} = 0, \quad \tilde{\xi} = (\xi, Q\xi)$. Hence $\dim X \leq \dim(\operatorname{Ker} A) = 2l - \operatorname{rang} A$. Taking (5.4) into account we have the inequality $\dim X \leq 1$. But on basis of Theorem 4 the dimension of X is even, and this dimension has to be equal to 0.

The assertion of Theorem 4 on the evenness of $\dim X$ we can complete in the following way.

Lemma 5. *For every even number s between* 0 *and l there exists a strengthened elliptic system such that* $\dim X = s$.

Proof. Let the matrices $P_0, Q_0 = P_0^{-1} \in \mathbb{R}^{s\times s}$ be orthogonal and have no real eigenvalues. Let us set $\mathbb{R}^l = \mathbb{R}^s \times \mathbb{R}^{l-s}$ and according to this representation introduce the matrices $P = \mathrm{diag}\,(P_0, 0)$ and $Q = \mathrm{diag}\,(Q_0, 0)$. Then $P^\top = O,\ PQ = \mathrm{diag}\,(1, 0)$ and therefore

$$|P(\xi_1, \xi_2)| = |(\xi_1, Q\xi_2)| \le |\xi_1||\xi_2|, \quad \xi_j \in \mathbb{R}^l,$$
$$|P(\xi, \xi)| = |(\xi, Q\xi)| < |\xi|^2, \quad \xi \ne 0. \tag{5.7}$$

It is also clear that the space X in (4.7) coincides with $\mathbb{R}^s \times 0$ for these matrices.

Let us consider the system (2.1) with the coefficients $A_{11} = A_{22} = 1, \quad A_{12} = P,\ A_{21} = Q$ and make sure that it is strengthened elliptic. By virtue of (5.7) we have for this system

$$(\xi_1 + P\xi_2, \xi_1) + (Q\xi_1 + \xi_2, \xi_2) = |\xi_1|^2 + |\xi_2|^2 + 2(Q\xi_1, \xi_2) \ge 0,$$

i.e. condition (3.2) is fulfilled. Besides the left hand side of this expression is positive for $\xi_j = t_j\xi$ and therefore the considered system is strengthened elliptic.

References

1. Mushelishvili, N.I. *Singular integral equations.* Noordhoff, Groningen, 1953.
2. Stein, E.M. *Singular integrals and differentiability properties of functions.* Princeton Univ. Press, Princeton,N. J., 1970.
3. Soldatov, A.P., *Hyperanalytic functions and their applications.* J. Math. Sciences, 2004, V. 15, P.142–199.
4. Douglis, A. *A function theoretic approach to elliptic systems of equations in two variables.* Comm. Pure Appl. Math. 1953, 6, 259–289.
5. Gilbert, R.P., Buchanan, J.L. *First order elliptic systems.* New York, Acad. Press, 1983.
6. Begehr, H., Wen, G.-C. *Boundary value problem for elliptic equations and systems.* Pitman Monographs and Surveys in Pure and Appl. Math., 46, Pitman, Harlow, 1990.
7. Wendland, W. *Elliptic systems in the plane.* Pitman, London etc., 1979.
8. Gilbert R.P., Hile G.N. *Generalized hypercomplex function theory.* Trans. Amer. Math. Soc. 195 (1974), 1–29.
9. Hile, G.N. *Elliptic systems in the plane with lower order terms and constant coefficients.* Comm. Pure Appl. Math. 3(10), (1978), 949–977.
10. Begehr H., Gilbert, R.P., *Pseudohyperanalytic functions,* Complex Variables, Theory Appl. 9 (1988), 343–357.
11. Soldatov, A.P., *Higher order elliptic systems,* Differentcial'nye Uravneniya, 25 (1989), 136–144; Differential Equations, 25 (1989) (English transl).
12. Yeh, R.Z. *Hyperholomorphic functions and higher order partial differentials equations in the plane.* Pacific Journ. of Mathem, 142, 2, (1990), 379–399.
13. Bitsadze, A.V. *Boundary value problems for elliptic equations of second order.* North Holland Publ. Co., Amsterdam, 1968.
14. Soldatov, A.P., *On the first and second boundary value problems for elliptic systems on the plane*, Differentcial'nye Uravneniya, 39, (2003), 674-686;. Differential Equations, 39, (2003) (English transl.).
15. Soldatov, A.P., *On Dirichlet problem for elliptic systems on the plane*, 2005, to appear.
16. Vishik, M.I. *On strong elliptic systems of differencial equations*, Matematicheskiy Sbornik, 29, (1951), No 3, 615–676 (Russian).