On a Boundary Value Problem for a Higher-Order Elliptic Equation

N. A. Malakhova and A. P. Soldatov

Belgorod State University, Belgorod, Russia
Institute for Applied Mathematics and Automation, Kabardino-Balkar Scientific Center, Russian Academy of Sciences, Nalchik, Russia

Received April 9, 2007

Abstract—For an elliptic $2l$th-order equation with constant (and only leading) real coefficients, we consider the boundary value problem in which the $(k_j - 1)$st normal derivatives, $j = 1, \dots, l$, are specified, where $1 \le k_1 < \dots < k_l$. If $k_j = j$, then it becomes the Dirichlet problem; and if $k_j = j + 1$, then it becomes the Neumann problem. We obtain a sufficient condition for this problem to be Fredholm and present a formula for the index of the problem.

In the generalized Neumann problem for an elliptic equation of even order $2l$, the successive normal derivatives $(\partial/\partial n)^j$, $j = 1, \dots, l$, are given on the boundary of the domain. This problem was studied in [1] for the polyharmonic equation with the use of the Almansi representation. Another version of the Neumann problem based on a variational principle was earlier suggested in [2].

In the present paper, for an elliptic equation with constant (and only leading) real coefficients, we consider the more general problem in which the $(k_j - 1)$st normal derivatives, $j = 1, \dots, l$, are specified, where $1 \le k_1 < \dots < k_l$. It becomes the Dirichlet problem for $k_j = j$ and the above-mentioned Neumann problem for $k_j = j + 1$. Therefore, it is natural to refer to this problem as a generalized Dirichlet–Neumann problem.

Consider the elliptic equation

$$\frac{\partial^{2l} u}{\partial y^{2l}} - \sum_{j=1}^{2l} a_j \frac{\partial^{2l} u}{\partial x^{2l+1-j} \partial y^{j-1}} = 0 \qquad (1)$$

in a domain D bounded by a simple smooth contour Γ on the plane. The ellipticity condition means that the roots of the characteristic polynomial

$$\chi(z) = z^{2l} - \sum_{j=1}^{2l} a_j z^{j-1}$$

do not lie on the real axis. For this polynomial, we also use the representation

$$\chi(z) = \prod_{i=1}^{m} (z - \nu_i)^{l_i} \prod_{i=1}^{m} (z - \bar{\nu}_i)^{l_i},$$

where the roots ν_i are pairwise distinct and lie in the upper half-plane. Clearly, their total multiplicity $l_1 + \dots + l_m$ is equal to l.

Consider the matrices

$$B = (B_1, \dots, B_m) \in \mathbb{C}^{1 \times l}, \qquad J = \operatorname{diag}(J_1, \dots, J_m) \in \mathbb{C}^{l \times l} \qquad (2)$$

with block entries $B_i \in \mathbb{C}^{1\times l_i}$ and $J_i \in \mathbb{C}^{l_i\times l_i}$ of the form

$$B_i = (1, 0, \dots, 0), \qquad J_i = \begin{pmatrix} \nu_i & 1 & 0 & \dots & 0 \\ 0 & \nu_i & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \nu_i \end{pmatrix}.$$

The matrix J defines the first-order canonical elliptic system

$$\frac{\partial\phi}{\partial y} - J\frac{\partial\phi}{\partial x} = 0, \tag{3}$$

and its solutions $\phi(z)$ treated as functions of the complex variable $z = x + iy$ are referred to as Douglis analytic functions [3]. These functions are real-analytic, and in a neighborhood of each point $\tau \in D$, they can be expanded in uniformly and absolutely convergent generalized Taylor series

$$\phi(z) = \sum_{k=0}^{\infty} \frac{(z-\tau)_J^k}{k!}\phi^{(k)}(\tau);$$

here and throughout the following, we use the notation

$$(x+iy)_J = x1 + yJ, \quad \phi^{(k)} = \frac{\partial^k\phi}{\partial x^k}. \tag{4}$$

By [4], the general solution of Eq. (1) can be represented in the form

$$u = \operatorname{Re} B\phi; \tag{5}$$

moreover, the relation $u = 0$ is possible only if ϕ is a polynomial of degree $\le l-2$. More precisely, the function ϕ is uniquely determined under the $l(2l-1)$ conditions

$$\operatorname{Re} BJ^r\phi^{(k)}(0) = 0, \qquad 0 \le k < r \le 2l-1, \tag{6}$$

where, to be definite, we assume that $0 \in D$.

The generalized Dirichlet–Neumann problem is the problem of finding the solution $u(x, y)$ of Eq. (1) in the domain D with the boundary conditions

$$\left.\frac{\partial^{k_j-1}u}{\partial n^{k_j-1}}\right|_\Gamma = g_j, \qquad j = 1, \dots, l, \tag{7}$$

where $n = n_1 + in_2$ is the unit outward normal. Here the kth normal derivative is treated as the boundary differential operator

$$\left(n_1\frac{\partial}{\partial x} + n_2\frac{\partial}{\partial y}\right)^k u = \sum_{r=0}^{k}\binom{k}{r} n_1^r n_2^{k-r}\frac{\partial^k u}{\partial x^r\partial y^{k-r}}. \tag{8}$$

Let $z = z(s) = x(s) + iy(s)$, $0 \le s \le s_\Gamma$, be a natural parametrization of the contour Γ. The parameter s is the arc length counted counterclockwise from a fixed point $z(0) \in \Gamma$. Accordingly, $e(t) = z'(s)$, $t = z(s)$, is the unit tangent vector related to the normal by the formula $e_1 + ie_2 = i\,(n_1 + in_2)$.

Throughout the following, we assume that Γ belongs to the class $C^{k_l,\mu+0}$, $0 < \mu < 1$; that is, the periodic function $z(s)$ belongs to the class $C^{k_l,\mu+\varepsilon}$ with some $\varepsilon > 0$. In particular, the functions n_1 and n_2 and hence the coefficients of the differential operator (8) belong to the class $C^{k_l-1,\mu+0}(\Gamma)$. We seek a solution of Eq. (1) in the class of functions $u \in C^{2l}(D)$ such that $\phi \in C^{k_l-1,\mu}\left(\bar{D}\right)$ in the representation (5). It is convenient to denote this function class by $\tilde{C}^{k_l-1,\mu}\left(\bar{D}\right)$; obviously, it is a Banach space with respect to the norm $|u| = |\phi|_{C^{k_l-1,\mu}}$. Obviously, for functions u in this class, the right-hand sides g_j in (7) should lie in $C^{k_l-k_j,\mu}(\Gamma)$.

Let $g^{(k)}$ be the kth derivative of the function $g \in C^k(\Gamma)$ with respect to the arc length parameter s; i.e.,

$$g^{(k)}[z(s)] = \frac{d^k}{ds^k} g[z(s)].$$

Lemma 1. *For each function $f \in C(\Gamma)$, the equation*

$$g^{(k)} + \lambda \int\limits_{\Gamma} g(t) ds_t = f,$$

where $\lambda > 0$, is uniquely solvable in the class $C^k(\Gamma)$.

Proof. Let us identify g with the function $g[z(s)]$. Then the considered equation can be reduced to the equation

$$T_{k,\lambda} g = f, \qquad \left(T_{k,\lambda}\right)(g) = g^{(k)}(s) + \lambda \int\limits_0^{s_\Gamma} g(s) ds$$

in the class C^k of s_Γ-periodic functions. (The periodicity condition also holds for the derivatives.) By virtue of the obvious relation

$$\left(T_{1,\alpha}\right)^k = T_{k,\lambda}, \qquad \lambda = \alpha^k s_\Gamma^{k-1},$$

we arrive at the case $k = 1$. A simple verification shows that the operator $T_{1,\alpha} : C^r \to C^{r-1}$ is invertible for any positive integer r and for any $\alpha > 0$.

The Dirichlet–Neumann problem (1), (7) belongs to the type of the Poincaré problem considered in [3]. All results in [3] remain valid for problem (1), (7). The novelty is that, in our case, the condition for the problem to be Fredholm can be written out in closed form in terms of generalized Vandermonde determinants. Let us briefly describe related constructions in [3] as applied to our problem.

Let $h_j(z)$ be functions analytic in a neighborhood of each of the points ν_i, $i = 1, \ldots, m$, forming the spectrum of the matrix J in (2). Consider the upper-triangular matrices $h_j\left(J_i\right) = \left(h_{ps}\right)_{p,s=1}^{l_i}$ with entries $h_{ps} = h_j^{(s-p)}\left(\nu_i\right)/(s-p)!$, $p \le s$.

Given the vector function $h = (h_1, \ldots, h_l)$, by $W_h\left(\nu_i\right) \in \mathbb{C}^{l \times l_i}$, following [4], we denote the matrix whose jth row coincides with the $1 \times l_i$ matrix $B_i h_j\left(J_i\right)$. Its entries $w_{ps}\left(\nu_i\right)$ have the form $w_{ps}\left(\nu_i\right) = h_p^{(s-1)}\left(\nu_i\right)/(s-1)!$, $p = 1, \ldots, l$, $s = 1, \ldots, l_i$. By analogy with (2), we form the block matrix $W_h(\nu) = \left[W_h\left(\nu_1\right), \ldots, W_h\left(\nu_m\right)\right]$.

In this notation, we set

$$G(t) = W_h(\nu), \qquad h_j(z) = \left[e_1(t) + e_2(t) z\right]^{k_l-k_j} \left[e_2(t) - e_1(t) z\right]^{k_j-1}, \qquad t \in \Gamma, \tag{9}$$

where, recall, $e(t) = e_1(t) + i e_2(t)$ is the unit tangent vector to Γ at a point t.

Theorem 1. *Let D be a finite domain bounded by a smooth contour Γ of the class $C^{k_l,\mu+0}$, $0 < \mu < 1$, and let the condition*

$$\det G(t) \neq 0, \qquad t \in \Gamma, \tag{10}$$

be satisfied in notation (9). Then problem (1), (7) is Fredholm in the class $\tilde{C}^{k_l-1,\mu}\left(\bar{D}\right)$, and its index $\varkappa$ is given by the formula

$$\varkappa = -\frac{1}{\pi} \arg \det G \Big|_{\Gamma} + 2l\left(k_l - l\right). \tag{11}$$

Proof. If g is the boundary value of a function $v \in C^k\left(\bar{D}\right)$, $k \le k_l - 1$, then the derivative $g^{(k)}$ can be obtained by an application of the boundary operator

$$\left(v^+\right)^{(k)} = \left(e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y}\right)^k v + M_k v \tag{12}$$

to v, where M_k is a linear differential operator of order $k-1$ with coefficients that belong to the class $C^{\mu+0}(\Gamma)$ and can be expressed via the functions e_1 and e_2 and their derivatives. Therefore, by Lemma 1, we can rewrite the boundary condition (7) in the equivalent form

$$\left(\frac{\partial^{k_j-1}u}{\partial n^{k_j-1}}\right)^{(k_l-k_j)} + \int\limits_\Gamma \frac{\partial^{k_j-1}u}{\partial n^{k_j-1}}ds_t = f_j, \qquad j = 1,\dots,l, \tag{13}$$

where we have set

$$f_j = g_j^{(k_l-k_j)} + \int\limits_\Gamma g_j(t)ds_t \in C^\mu(\Gamma).$$

By setting $k = k_j$ in (8) and by applying the operator (12) to the partial derivatives of v on the right-hand side in (8), we can rewrite the boundary condition (13) in the form

$$\left(e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y}\right)^{k_l-k_j}\left(n_1\frac{\partial}{\partial x} + n_2\frac{\partial}{\partial y}\right)^{k_j-1} u + L_j u + \int\limits_\Gamma \frac{\partial^{k_j-1}u}{\partial n^{k_j-1}}ds = f_j, \tag{14}$$

where L_j is a linear differential operator of order $k_l - 2$ with coefficients in the class $C^{\mu+0}(\Gamma)$.

By using relation (3) and the notation accepted in (4), for the differentiation of the right-hand side of (5), we have the formula

$$\left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y}\right)^k (\operatorname{Re} B\phi) = \operatorname{Re} B a_J^k \phi^{(k)}, \qquad a = a_1 + ia_2.$$

Consequently, for J-analytic functions ϕ [solutions of the Douglis system (3)], the set of l boundary conditions (14) can be represented in the matrix form

$$\operatorname{Re} G\phi^{(k_l-1)} + \operatorname{Re}\sum_{r=0}^{k_l-2}\left[G_r^1\phi^{(r)} + \int\limits_\Gamma G_r^0\phi^{(r)}ds\right] = f, \tag{15}$$

where $f = (f_1,\dots,f_l)$, the jth row of the matrix G coincides with the $1\times l$ matrix $Be_J^{k_l-k_j}n_J^{k_j-1}$, $j = 1,\dots,l$, and the matrix coefficients G_r^0 and G_r^1, whose specific form is inessential for our considerations, belong to the class $C^{\mu+0}(\Gamma)$.

Therefore, problem (1), (7) is equivalent to problem (6), (15) considered in the class $C^{k_l-1,\mu}\left(\bar{D}\right)$ of Douglis analytic vector functions.

Set $\phi^{(k_l-1)} = \psi$; then

$$\phi = c_0 + z_J c_1 + \dots + z_J^{k_l-2}c_{k_l-2} + \psi^{(1-k_l)}(z), \qquad c_r \in \mathbb{C}^l, \tag{16}$$

where, for a positive integer n, we have set

$$\psi^{(-n)}(z) = \frac{1}{(n-1)!}\int\limits_0^z (t-z)_J^{n-1}dt_J\psi(t).$$

In notation (4), we introduce the generalized Cauchy integral

$$(I\varphi)(z) = \frac{1}{2\pi i}\int_\Gamma (t-z)_J^{-1} dt_J \varphi(t), \qquad z \notin \Gamma.$$

If $\varphi \in C^\mu(\Gamma)$, then $I\varphi \in C^\mu(\bar{D})$, and the boundary values of the latter function satisfy the Sokhotskii–Plemelj formula [3]

$$2(I\varphi)^+ = \varphi + K\varphi, \tag{17}$$

where K is the singular Cauchy integral

$$(K\varphi)(t_0) = \frac{1}{\pi i}\int_\Gamma (t-t_0)_J^{-1} dt_J \varphi(t), \qquad t_0 \in \Gamma. \tag{18}$$

As was shown in [3], if J is a triangular matrix, then every J-analytic function $\psi \in C^\mu(\bar{D})$ can be uniquely represented in the form

$$\psi = I\varphi + i\xi, \qquad \xi \in \mathbb{R}^l, \tag{19}$$

with a real function $\varphi \in C^\mu(\Gamma)$.

If $J = i$, then the integral (18) becomes the classical singular operator

$$(S\varphi)(t_0) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)dt}{t-t_0}, \qquad t_0 \in \Gamma.$$

For each linear operator N, we introduce the operation of complex conjugation by the formula $\bar{N}\varphi = \overline{N\bar{\varphi}}$. In particular, it follows from (18) that

$$\left(\bar{K}\varphi\right)(t_0) = -\frac{1}{\pi i}\int_\Gamma (t-t_0)_{\bar{J}}^{-1} dt_{\bar{J}} \varphi(t), \qquad t_0 \in \Gamma.$$

If $k(t_0, t) \in C^{\mu+0}(\Gamma \times \Gamma)$ and $k(t,t) \equiv 0$, then one can readily show that the integral operator

$$(M\varphi)(t_0) = \int_\Gamma \frac{k(t_0,t)}{|t-t_0|}\varphi(t)dt, \qquad t_0 \in \Gamma,$$

is compact in the space $C^\mu(\Gamma)$. The class of such operators will be denoted by $\mathcal{T}_0(\Gamma)$.

Let us use the following assertion (see [3]).

Let $\Gamma \in C^{1,\mu+0}$. Then each of the operators

$$(M_1\varphi)(t_0) = (I\varphi)^{(-n)}(t_0), \qquad t_0 \in \Gamma, \qquad M_2 = K - S, \qquad M_3 = K + \bar{K}$$

belongs to the class $\mathcal{T}_0(\Gamma)$.

By substituting the representation (19) into the expression (16), we obtain

$$\phi(z) = p(z) + (I\varphi)^{(1-k_l)}(z), \qquad p(z) = \sum_{k=0}^{k_l-1} z_J^k c_k,$$

where $c_r \in \mathbb{C}^l$, $r = 0, \ldots, k_l - 2$, and $ic_{k_l-1} \in \mathbb{R}^l$.

By using the Sokhotskii–Plemelj formula (17) and the last relation, from the boundary value problem (15), we arrive at the equivalent singular equation

$$\operatorname{Re}[G(1+K)\varphi] + R\varphi + \sum_{j=1}^{2k_l-1} a_j\xi_j = 2f, \tag{20}$$

where $R \in \mathcal{T}_0(\Gamma)$ and the $l \times l$ matrix functions a_j belong to the class $C^{\mu+0}(\Gamma)$. The unknowns in this equation are a real l-vector function $\varphi \in C^{\mu}(\Gamma)$ and a set (ξ_j), $\xi_j \in \mathbb{R}^l$.

From condition (6), we obtain the relations

$$\int\limits_{\Gamma} c(t)\varphi(t)ds_t + \sum_{j=1}^{2k_l-1} b_j\xi_j = 0, \qquad c(t) \in \mathbb{R}^{l(2l-1)\times l}, \tag{21}$$

with some $(l(2l-1) \times l)$ matrix functions b_j of the class $C^{\mu+0}(\Gamma)$.

By the preceding, Eq. (20) can be rewritten in the form

$$G(1+S)\varphi + \bar{G}(1-S)\varphi + R_1\varphi + 2\sum_{j=1}^{2k_l-1} a_j\xi_j = 4f \tag{22}$$

with an operator R_1 in the class $\mathcal{T}_0(\Gamma)$ and the matrix function G given by relation (9). Here we have used the fact that the tangent vector $e = e_1 + ie_2$ and the normal vector $n = n_1 + in_2$ are related by the formula $e = in$.

The classical theory of singular integral equations [6, p. 315] can be applied to system (21), (22). This theory implies that, under assumption (10), the system belongs to the normal type and its index $\varkappa$ is given by the formula

$$\varkappa = \operatorname{Ind}\left(G^{-1}\bar{G}\right) + l\,(2k_l - 1) - l(2l-1).$$

By virtue of the equivalence, the same result is valid for the original problem. Elementary transformations reduce this formula to (11), which completes the proof of the theorem.

In some cases, condition (10) and formula (11) can be described in closed form.

Theorem 2. *Let one of the following two conditions be satisfied*:
(a) *the characteristic polynomial of Eq.* (1) *has a unique root in the upper half-plane*;
(b) $k_{j+1} - k_j = 1$, $1 \le j \le l$.
Then problem (1), (7) *has the Fredholm property, and its index* $\varkappa$ *is zero.*

Proof. We write $h_j(z) = (e_1 + e_2 z)^{k_l - 1} g_j[w]$, where

$$w = \omega(z), \qquad \omega(z) = \frac{e_2 - e_1 z}{e_1 + e_2 z}, \qquad g_j(w) = w^{k_j - 1}, \qquad j = 1, \dots, l.$$

As was shown in [4], the matrix W_h has the following properties.

1. If $a \in \mathbb{C}^{l\times l}$, then $aW_h = W_{ah}$.

2. If $\varphi(z)$ is a scalar analytic function, then $W_{\varphi h} = W_h A$, where $A \in \mathbb{C}^{l\times l}$ is a block diagonal matrix and the $A_r \in \mathbb{C}^{l_r\times l_r}$, $r = 1, \dots, m$, are the triangular matrices with entries $[A_r]_{ij} = \varphi^{(j-i)}/(j-i)!$ for all $i \le j$.

3. If $w(z)$ is a scalar analytic function and $w'(\nu_i) \ne 0$, $i = 1, \dots, m$, then

$$W_{g\circ w}(\nu) = W_g[w(\nu)]H,$$

where $w(\nu)$ stands for the set $(w(\nu_1), \dots, w(\nu_m))$, $H \in \mathbb{C}^{l\times l}$ is a block diagonal matrix, and

$$H_r = \operatorname{diag}\left(1, w'(\nu_r), \dots, [w'(\nu_r)]^{l_r - 1}\right) \in \mathbb{C}^{l_r\times l_r}, \qquad r = 1, \dots, m.$$

Then, by virtue of properties 2 and 3, we have

$$\det G(t) = \prod_{j=1}^{m} [e_1(t) + e_2(t)\nu_j]^{l_j(k_l - l_j)} \det G_0(t),$$

where we have set

$$G_0(t) = W_g(w), \quad w_j = \frac{e_2(t) - e_1(t)\nu_j}{e_1(t) + e_2(t)\nu_j}.$$

Since $\operatorname{Ind}(e_1 + e_2\nu_j)|_\Gamma = 1$ for $\operatorname{Im}\nu_j > 0$, it follows that

$$\operatorname{Ind} G = \sum_{j=1}^{m} l_j (k_l - l_j) + \operatorname{Ind} G_0. \tag{23}$$

As t goes around the contour Γ, the point $e(t)$ of the unit circle makes a complete revolution; therefore, the condition $\det G_0(t) \neq 0$ is equivalent to the condition

$$\det W_g[w(e)] \neq 0, \qquad |e| = 1, \tag{24}$$

where

$$w = (w_1, \dots, w_m), \qquad w_j(e) = (e_2 - e_1\nu_j)/(e_1 + e_2\nu_j).$$

In view of (23), this condition is equivalent to (10).

It was shown in [4] that if one of assumptions (a) or (b) of the theorem is satisfied, then condition (24) always holds. Therefore, it only remains to show that the index of problem (1), (7) is zero.

Let assumption (a) of the theorem be true; i.e., let $m = 1$. Then, by [4], the determinant of the matrix W_g has the form

$$\det W_g (w_1, \dots, w_m) = \prod_{i>j} \left(\frac{k_i - k_j}{i - j}\right) w_1^{\sum (k_j - j)}.$$

Therefore, the index of the matrix function G_0 is given by the relation

$$\operatorname{Ind} G_0 = \sum_{i=1}^{m} (k_j - j) \operatorname{Ind} w_1(t).$$

The Cauchy index of the functions $w_i(t)$ is the difference of the indices of the functions $e_2(t) - e_1(t)\nu_i$ and $e_1(t) + e_2(t)\nu_i$ and hence is zero. Consequently, $\operatorname{Ind} G_0 = 0$, and this, together with (23), implies that $\varkappa = 0$.

Let assumption (b) of the theorem be true. Then the determinant W_g is called a generalized Vandermonde determinant and, as was shown in [4], can be represented in the closed form

$$\det W_g (w_1, \dots, w_m) = \prod_i w_i^{l_i(k_1 - 1)} \prod_{i>j} (w_i - w_j)^{l_i l_j}.$$

From this, for the index of the matrix function G_0, we obtain

$$\operatorname{Ind} G_0 = \sum_{i=1}^{m} l_i (k_1 - 1) \operatorname{Ind} w_i(t) + \sum_{i>j} l_i l_j \operatorname{Ind} (w_i - w_j)(t).$$

If $i \neq j$, then for the difference $(w_i - w_j)(t)$ occurring in the second sum, we obtain the expression

$$(w_i - w_j)(t) = \frac{\nu_j - \nu_i}{(e_1(t) + e_2(t)\nu_i)(e_1(t) + e_2(t)\nu_j)},$$

and consequently, the Cauchy index of this function is equal to -2. Hence we obtain

$$\operatorname{Ind} G_0 = -2 \sum_{i>j} l_i l_j = \sum_i l_i^2 - l^2,$$

which, together with (23), implies that $\varkappa = 0$.

ACKNOWLEDGMENTS

The research was financially supported by the Russian Foundation for Basic Research (project no. 0701002999).

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