To the theory of anisotropic plane elasticity

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Dedicated to Professor Heinrich Begehr on the occasion of his 70th birthday

Summary: The Lame system of general anisotropic plane elasticity is considered. A representation of a general solution or the system through a so-called Douglis analytic functions is given. The cases of orthotropic and isotropic media are also considered.

1 Lame system

The stress tensor

$$\sigma = \begin{pmatrix} \sigma_1 & \sigma_3 \\ \sigma_3 & \sigma_2 \end{pmatrix}$$

of plane elasticity medium is connected with a displacement vector $u = \downarrow (u_1, u_2)$ by the Hook law [1]

$$\begin{pmatrix} \sigma_1 \\ \sigma_3 \end{pmatrix} = a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}, \quad \begin{pmatrix} \sigma_3 \\ \sigma_2 \end{pmatrix} = a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}. \tag{1.1}$$

The coefficients $a_{ij} \in \mathbb{R}^{2\times 2}$ are defined by

$$\begin{aligned} a_{11} &= \begin{pmatrix} \alpha_1 & \alpha_6 \\ \alpha_6 & \alpha_3 \end{pmatrix}, \quad a_{12} = \begin{pmatrix} \alpha_6 & \alpha_4 \\ \alpha_3 & \alpha_5 \end{pmatrix}, \\ a_{21} &= \begin{pmatrix} \alpha_6 & \alpha_3 \\ \alpha_4 & \alpha_5 \end{pmatrix}, \quad a_{22} = \begin{pmatrix} \alpha_3 & \alpha_5 \\ \alpha_5 & \alpha_2 \end{pmatrix}, \end{aligned} \tag{1.2}$$

where modulus elasticity α_j form the positively defined matrix

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_4 & \alpha_6 \\ \alpha_4 & \alpha_2 & \alpha_5 \\ \alpha_6 & \alpha_5 & \alpha_3 \end{pmatrix}.$$

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By the Silvester criterium we have

$$\alpha_j > 0,\ j = 1, 2, 3, \quad \alpha_1\alpha_2 > \alpha_4^2, \tag{1.3a}$$

$$\alpha_1\alpha_2\alpha_3 + 2\alpha_4\alpha_5\alpha_6 > \alpha_1\alpha_5^2 + \alpha_2\alpha_6^2 + \alpha_3\alpha_4^2. \tag{1.3b}$$

The elastic medium is orthotropic if $\alpha_5 = \alpha_6 = 0$, and is isotropic if

$$\alpha_5 = \alpha_6 = 0, \quad \alpha_1 = \alpha_2 = 2\alpha_3 + \alpha_4. \tag{1.4}$$

The stress tensor satisfies the equilibrium equation

$$\frac{\partial}{\partial x}\begin{pmatrix}\sigma_1\\ \sigma_3\end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix}\sigma_3\\ \sigma_2\end{pmatrix} = 0.$$

Together with (1.1) it yields the Lame system

$$a_{11}\frac{\partial^2 u}{\partial x^2} + (a_{12} + a_{21})\frac{\partial^2 u}{\partial y^2} + a_{22}\frac{\partial^2 u}{\partial y^2} = 0 \tag{1.5}$$

for the displacement vector $u = (u_1, u_2)$. Besides there exists a so-called conjugate function $v(x, y)$ determined by the following relations:

$$\begin{pmatrix}\sigma_1\\ \sigma_3\end{pmatrix} = -\frac{\partial v}{\partial y}, \quad \begin{pmatrix}\sigma_3\\ \sigma_2\end{pmatrix} = \frac{\partial v}{\partial x}. \tag{1.6}$$

According to (1.1) this function is connected with u by relations

$$\frac{\partial v}{\partial x} = -\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right), \quad \frac{\partial v}{\partial y} = a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}. \tag{1.7}$$

From (1.2) it follows that the matrix

$$A = \begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} \in \mathbb{R}^{4\times 4}$$

is symmetric and nonnegatively defined i.e. $(A\xi, \xi) \geq 0$ for all $\xi \in \mathbb{R}^4$. Moreover

$$(A\xi, \xi) = 0 \Leftrightarrow A\xi = 0 \Leftrightarrow \xi = (0, t, -t, 0), t \in \mathbb{R}.$$

Hence

$$(p(t)\xi_0, \xi_0) = (a_{11}\xi_0 + a_{12}t\xi_0, \xi_0) + (a_{21}\xi_0 + a_{22}t\xi_0, t\xi_0) > 0$$

for all $t \in \mathbb{R}$ and $\xi_0 \in \mathbb{R}^2$, where $p(z) = a_{11} + (a_{12} + a_{21})z + a_{22}z^2$. In particular the Lame system is strongly elliptic [2] and its characteristic equation $\chi(z) = \det p(z)$ has no real roots. Thus for the set σ_+ of these roots in the upper half- plane we have only the following two possibilities

$$(i)\ \sigma_+ = \{\nu_1, \nu_2\}, \nu_1 \neq \nu_2, \quad (ii)\ \sigma_+ = \{\nu\}. \tag{1.8}$$

In the explicit form we have

$$p = \begin{pmatrix} p_1 & p_3 \\ p_3 & p_2 \end{pmatrix}, \quad \begin{matrix} p_1(z) = \alpha_1 + 2\alpha_6 z + \alpha_3 z^2, \\ p_2(z) = \alpha_3 + 2\alpha_5 z + \alpha_2 z^2, \\ p_3(z) = \alpha_6 + (\alpha_3 + \alpha_4) z + \alpha_5 z^2, \end{matrix} \qquad \chi = p_1 p_2 - p_3^2. \tag{1.9}$$

The roots of the characteristic equation can be calculated explicitly in the orthotropic case. In this case

$$\chi(z) = \alpha_2\alpha_3(\rho^4 + 2m\rho^2 z^2 + z^4), \quad \rho = \sqrt[4]{\frac{\alpha_1}{\alpha_2}}, \quad m = \frac{\alpha_1\alpha_2 - \alpha_4^2 - 2\alpha_3\alpha_4}{2\alpha_3\sqrt{\alpha_1\alpha_2}}.$$

It is obvious

$$m + 1 = \frac{m_1}{2\alpha_3\sqrt{\alpha_1\alpha_2}}, \quad m - 1 = \frac{m_2}{2\alpha_3\sqrt{\alpha_1\alpha_2}}(\sqrt{\alpha_1\alpha_2} - 2\alpha_3 - \alpha_4),$$

where $m_1 = (\sqrt{\alpha_1\alpha_2} - \alpha_4)(\sqrt{\alpha_1\alpha_2} + \alpha_4 + 2\alpha_3)$ and $m_2 = \sqrt{\alpha_1\alpha_2} + \alpha_4$. By virtue of (1.3a) these numbers are positive. From this equation it follows that

$$\nu_1 = i\rho e^{i\theta},\ \nu_2 = i\rho e^{-i\theta}, \quad \sqrt{\alpha_1\alpha_2} < 2\alpha_3 + \alpha_4, \quad 2\theta = \arccos r, \tag{1.10a}$$

$$\nu_1 = i\rho e^{t},\ \nu_2 = i\rho e^{-t}, \quad \sqrt{\alpha_1\alpha_2} > 2\alpha_3 + \alpha_4, \quad 2t = \operatorname{arcch} r, \tag{1.10b}$$

$$\nu_1 = \nu_2 = i\rho, \quad \sqrt{\alpha_1\alpha_2} = 2\alpha_3 + \alpha_4, \tag{1.10c}$$

Very simple expressions we have in the case

$$\alpha_5 = \alpha_6 = 0, \quad \alpha_3 + \alpha_4 = 0. \tag{1.11}$$

Then Lame system is diagonal and

$$\nu_1 = i\sqrt{\frac{\alpha_1}{\alpha_3}}, \quad \nu_2 = i\sqrt{\frac{\alpha_3}{\alpha_2}}. \tag{1.12}$$

This corresponds to (1.10b) with

$$m = \frac{\alpha_1\alpha_2 + \alpha_3^2}{2\alpha_3\sqrt{\alpha_1\alpha_2}}.$$

The second possibility (ii) of multiple roots is corresponds to (1.10c). The equality $\rho = 1$ is valid if and only if the orthotropic medium is isotropic.

For general anisotropic Lame system let us consider a case when three elements of the matrix $p(\nu)$ are equal to zero.

Lemma 1.1 *(a) The equalities $p_2(\nu) = p_3(\nu) = 0$, $\nu \in \sigma_+$, hold if and only if*

$$\alpha_3^2 < \alpha_1\alpha_1, \quad |\alpha_5| < \alpha_2, \quad \alpha_3\alpha_5 = \alpha_2\alpha_6, \quad \alpha_2(\alpha_3 + \alpha_4) = 2\alpha_5^2. \tag{1.13a}$$

(b) The equalities $p_1(\nu) = p_3(\nu) = 0$, $\nu \in \sigma_+$, hold if and only if

$$\alpha_3^2 < \alpha_1\alpha_1, \quad |\alpha_6| < \alpha_3, \quad \alpha_1\alpha_5 = \alpha_3\alpha_6, \quad \alpha_1(\alpha_3 + \alpha_4) = 2\alpha_6^2. \tag{1.13b}$$

(c) The both conditions (1.13) *are equivalent to* (1.11).

(d) The equalities $p_1(\nu) = p_2(\nu) = p_3(\nu) = 0$ *are impossible for all* ν. *The equalities* $p_2(\nu) = p_3(\nu) = 0$ *or* $p_1(\nu) = p_3(\nu) = 0$ *are only possible in the case (i).*

Proof:

(a) The equalities $p_2(\nu) = p_3(\nu) = 0$ are equivalent to the relation $p_3 = \lambda p_2$ for some $\lambda \in \mathbb{R}$, i.e.

$$\alpha_6 = \lambda\alpha_3, \quad \alpha_5 = \lambda\alpha_2 \quad \alpha_3 + \alpha_4 = 2\lambda\alpha_5 = 2\lambda^2\alpha_2. \tag{1.14}$$

By virtue of (1.3)

$$\alpha_3 - \sqrt{\alpha_1\alpha_2} < 2\lambda^2\alpha_2 < \alpha_3 + \sqrt{\alpha_1\alpha_2} \tag{1.15}$$

and

$$\alpha_1\alpha_2\alpha_3 + 2(2\lambda^2\alpha_2 - \alpha_3)\lambda^2\alpha_2\alpha_3 > (\alpha_1\alpha_2^2 + \alpha_2\alpha_3^2)\lambda^2 + \alpha_3(2\lambda^2\alpha_2 - \alpha_3)^2.$$

The last inequality can be written in the form $(\lambda^2\alpha_2 - \alpha_3)(\alpha_3^2 - \alpha_1\alpha_2) > 0$. The inequalities $\lambda^2\alpha_2 - \alpha_3 > 0$ and $\alpha_3^2 - \alpha_1\alpha_2 > 0$ contradict to (1.15), so $\lambda^2\alpha_2 - \alpha_3 < 0$ and $\alpha_3^2 - \alpha_1\alpha_2 < 0$. In this case (1.15) hold automatically and we receive (1.13a) after illuminating the parameter λ from (1.14).

(b) The proof is analogously to (a).

(c) Suppose that (1.13a) and (1.13b) hold but $\alpha_3 + \alpha_4 \neq 0$. Then $\alpha_5\alpha_6 \neq 0$ and from the system $\alpha_3\alpha_5 - \alpha_2\alpha_6 = 0$, $\alpha_1\alpha_5 - \alpha_3\alpha_6 = 0$ it follows that $\alpha_1\alpha_2 = \alpha_3^2$. But this equality contradicts to (1.13).

(d) The first assertion follows from (c). Suppose further that for example $p_2(\nu) = p_3(\nu) = 0$ for the multiple root ν. Then $\chi(\nu) = \chi'(\nu) = 0$. As $\chi' = p_1'p_2 + p_1p_2' - 2p_3p_3'$ and $p_i'(\nu) \neq 0$, $i = 1, 2$, for all ν, $\operatorname{Im}\nu \neq 0$. So we receive $p_2(\nu) = 0$ and $p_1(\nu) = p_2(\nu) = p_3(\nu) = 0$, that is impossible. □

2 Function theoretic approach

The classic approach to plane elasticity is based [3] on representation of general solution of the Lame system through two analytic functions. In the isotropic case this representation is known as Kolosov–Muskhelishvili formula [4]. Later there were developed various function theoretic methods [5, 6, 7], where the role of analytic functions play solutions of first order elliptic systems. Our approach to plane elasticity is based [8, 9] on the so called Douglis analytic functions which satisfy by definition the following system

$$\frac{\partial\phi}{\partial y} - J\frac{\partial\phi}{\partial x} = 0.$$

At this point the spectrum $\sigma(J)$ of the matrix $J \in \mathbb{C}^{2\times 2}$ here coincides with σ_+ and there exists the matrix $b \in \mathbb{C}^{2\times 2}$ such that

$$a_{11}b + (a_{12} + a_{21})bJ + a_{22}bJ^2 = 0, \quad \det \begin{pmatrix} b & \overline{b} \\ bJ & \overline{bJ} \end{pmatrix} \neq 0. \tag{2.1}$$

In this terms a general solution u of the Lame system and its conjugate function v can be represented by formulas

$$u = \mathrm{Re}\, b\phi, \quad v = \mathrm{Re}\, c\phi + \xi,$$

where $\xi \in \mathbb{R}^2$ and $c = -(a_{21}b + a_{22}bJ)$.

The matrix b can be chosen in a Jordan form. In the case (ii) by virtue of Lemma 1.1(d) the matrix J doesn't have to be equal to scalar matrix ν. So according to (1.8) there are two possibilities

$$(i)\ J = \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}, \quad (ii)\ J = \begin{pmatrix} \nu & 1 \\ 0 & \nu \end{pmatrix}. \tag{2.2}$$

The matrix b is not uniquely defined by (2.1). If $\tilde{b}$ satisfies the same conditions and $\tilde{c} = -(a_{21}\tilde{b} + a_{22}\tilde{b}J)$, then we have [10]

$$\tilde{b} = bd, \quad \tilde{c} = cd, \tag{2.3}$$

where an invertible matrix d according two cases (i) and (ii) has a form

$$(i)\ d = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad (ii)\ d = \begin{pmatrix} d_1 & d_2 \\ 0 & d_1 \end{pmatrix}.$$

The matrixes b and c were described in [8, 9]. In this paper we give another more exact expressions for these matrixes. Let us introduce the matrixes

$$q = \begin{pmatrix} p_2 & -p_3 \\ -p_3 & p_1 \end{pmatrix}, \quad r(z) = -(a_{21} + a_{22}z)q(z). \tag{2.4}$$

In the explicit form

$$r(z) = \begin{pmatrix} -zq_3 & -q_1 \\ q_3 & q_2 - zq_3 \end{pmatrix}, \quad \begin{aligned} q_1(z) &= \beta_2 - \beta_5 z + \beta_4 z^2, \\ q_2(z) &= \beta_5 - \beta_3 z + \beta_6 z^2, \\ q_3(z) &= \beta_4 - \beta_6 z + \beta_1 z^2, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} &\beta_1 = \alpha_2\alpha_3 - \alpha_5^2, && \beta_2 = \alpha_1\alpha_3 - \alpha_6^2, && \beta_3 = \alpha_1\alpha_2 - \alpha_4^2, \\ &\beta_4 = \alpha_5\alpha_6 - \alpha_3\alpha_4, && \beta_5 = \alpha_4\alpha_6 - \alpha_1\alpha_5, && \beta_6 = \alpha_4\alpha_5 - \alpha_2\alpha_6. \end{aligned}$$

Note that β_j coincide with elements of 3×3-matrix β, which is adjoint to the matrix α, i.e.

$$\beta = (\det \alpha)\alpha^{-1} = \begin{pmatrix} \beta_1 & \beta_4 & \beta_6 \\ \beta_4 & \beta_2 & \beta_5 \\ \beta_6 & \beta_5 & \beta_3 \end{pmatrix}. \tag{2.6}$$

Theorem 2.1 *(i) Let $\sigma_+ = \{\nu_1, \nu_2\}$. If the condition* (1.13a) *doesn't valid then*

$$b = \begin{pmatrix} p_2(\nu_1) & p_2(\nu_2) \\ -p_3(\nu_1) & -p_3(\nu_2) \end{pmatrix}, \quad c = \begin{pmatrix} -\nu_1 q_3(\nu_1) & -\nu_2 q_3(\nu_2) \\ q_3(\nu_1) & q_3(\nu_2) \end{pmatrix}. \tag{2.7a}$$

If these conditions hold and $\alpha_3 + \alpha_4 \neq 0$, then

$$b = \begin{pmatrix} -p_3(\nu_1) & p_3(\nu_2) \\ p_1(\nu_1) & p_1(\nu_2) \end{pmatrix}, \quad c = \begin{pmatrix} -q_1(\nu_1) & -q_1(\nu_2) \\ q_2(\nu_1) - \nu_1 q_3(\nu_1) & q_2(\nu_2) - \nu_2 q_3(\nu_2) \end{pmatrix}. \tag{2.7b}$$

At last in the case (1.11) *we can put*

$$b = 1, \quad c = -(a_{21} + a_{22}J) = \begin{pmatrix} -\alpha_3\nu_1 & -\alpha_3 \\ \alpha_3 & -\alpha_2\nu_2 \end{pmatrix}. \tag{2.7c}$$

(ii) Let $\sigma_+ = \{\nu\}$. Then we can put

$$b = \begin{pmatrix} p_2(\nu) & p_2'(\nu) \\ -p_3(\nu) & -p_3'(\nu) \end{pmatrix}, \quad c = \begin{pmatrix} -\nu q_3(\nu) & -q_3(\nu) - \nu q_3'(\nu) \\ q_3(\nu) & q_3'(\nu) \end{pmatrix}. \tag{2.8}$$

Proof:

(i) From (2.1) it follows that the columns $b_{(k)}$, $k = 1, 2$, satisfy the equation $p(\nu_k)b_{(k)} = 0$. Taking into account (2.4) we have $p(z)q(z) = \chi(z)$ and hence $p(\nu_k)q_{(i)}(\nu_k) = 0$, $i = 1, 2$. So we can put $b_{(k)} = d_k q_{(i)}(\nu_k)$, $d_k \neq 0$, under assumption $q_{(i)}(\nu_k) \neq 0$. If the conditions (1.13a) have no place then then according to Lemma 1.1 this assumption is fulfilled for $i = 1$.

Let the conditions (1.13a) hold. Then the unit matrix $b = 1$ satisfies (2.1) in the case (1.11). If $\alpha_3 + \alpha_4 \neq 0$, then by lemma 1 we have $b_{(k)} = d_k q_{(2)}(\nu_k),\ d_k \neq 0$ for all $k = 1, 2$. By virtue of (2.3) we can put here $d_1 = d_2 = 1$.

Let turn to the matrix $c = -(a_{21}b + a_{22}bJ)$. It is obviously that $c_{(k)} = -a_{21}b_{(k)} - \nu_k a_{22} b_{(k)}$ and therefore

$$c_{(k)} = -(a_{21} + a_{22}\nu_k)p_{(i)}(\nu_k), \quad b_{(k)} = p_{(i)}(\nu_k).$$

Taking into account (2.3) we complete the proof.

(ii) It follows from (2.1) that

$$p(\nu)b_{(1)} = 0, \quad p(\nu)b_{(2)} + p'(\nu)b_{(1)} = 0.$$

Since the root ν is multiple we have $p(\nu)q'(\nu)p'(\nu)q(\nu) = 0$. By virtue of Lemma 1.1 the column $q_{(1)}(\nu) \neq 0$ and therefore we can write

$$b_{(1)} = d_1 q_{(1)}(\nu), \quad b_{(2)} = d_1 q_{(1)}'(\nu) + d_2 q_{(1)}(\nu)$$

with $d_1 \neq 0$. Taking into account (2.3) we complete the proof for the matrix b.

As $(bJ)_{1)} = \nu b_{1)}$, $(bJ)_{2)} = b_{1)} + \nu b_{2)}$, we can write

$$c_{(1)} = -a_{21}b_{(1)} - \nu a_{22}b_{(1)}, \quad c_{(2)} = -a_{21}b_{(2)} - \nu a_{22}b_{(2)} - a_{22}b_{(1)}.$$

Putting $b_{(1)} = p_{(1)}(\nu)$, $b_{(2)} = p'_{(1)}(\nu)$ we receive

$$c_{(1)} = r_{(1)}(\nu), \quad c_{(2)} = -(a_{21} + \nu a_{22})p'_{(1)}(\nu) - a_{22}p_{(1)}(\nu) = r'_{(1)}(\nu),$$

that complete the proof. □

Due to [10] the matrix b is invertible for all strong elliptic system and in particular for Lame system. The matrix c has the same property.

Theorem 2.2 *Under assumptions of the Theorem 2.1 the matrix c is invertible.*

Proof: Within notations (2.6) the characteristic polynomial $\chi = p_1p_2 - p_3^2$ can be written in the form

$$\chi(z) = q_1(z) - zq_2(z) + z^2q_3(z). \tag{2.9}$$

The expressions (2.5) for q_j yield the relation $\xi = \beta\eta$ with respect to the vectors $\xi = (q_3, q_1, q_2)$ and $\eta = (z^2, 1, -z)$. Taking into account (2.6) we conclude that $(\det \alpha)\eta = \alpha\xi$ or

$$\begin{aligned} (\det \alpha)z^2 &= \alpha_4q_1 + \alpha_6q_2 + \alpha_1q_3, \\ \det \alpha &= \alpha_2q_1 + \alpha_5q_2 + \alpha_4q_3, \\ -(\det \alpha)z &= \alpha_5q_1 + \alpha_3q_2 + \alpha_6q_3. \end{aligned} \tag{2.10}$$

In particular the common equalities $q_1(\nu) = q_2(\nu) = q_3(\nu) = 0$ are impossible for all ν. From this and (2.9) it follows than only one of numbers $q_i(\nu)$, $i = 1, 2, 3$, where $\nu \in \sigma_+$, may be equal to zero.

The following implications

$$q_3(\nu) = 0 \quad \Leftrightarrow \quad p_2(\nu) = p_3(\nu) = 0, \tag{2.11a}$$

$$q_1(\nu) = 0 \quad \Rightarrow \quad p_3(\nu) = 0. \tag{2.11b}$$

for every $\nu \in \sigma_+$ hold.

In fact let $\chi(\nu) = q_3(\nu) = 0$. Then by virtue of (2.9) we can write $q_1(\nu) = \lambda\nu$, $q_2(\nu) = \lambda \neq 0$, $q_3(\nu) = 0$. Putting $z = \nu$ in (2.10) we conclude that $\alpha_3 + 2\alpha_5\nu + \alpha_2\nu^2 = 0$. Accordingly (1.9) this expression coincides with $p_2(\nu) = 0$. Since $p_1(\nu)p_2(\nu) - p_3^2(\nu) = 0$ we have also $p_3(\nu) = 0$. Conversely if $p_2(\nu) = p_3(\nu) = 0$, then by virtue of (2.4), (2.5) $q_3(\nu) = 0$.

The second implication (2.11b) is proved analogously. If $\chi(\nu) = q_1(\nu) = 0$, then $q_1 = 0$, $q_2 = \lambda\nu$, $q_3 = \lambda$ and we derive from (2.10) that $p_3(\nu) = 0$.

Let the conditions (1.13a) be broken. Then by virtue of (2.11a) we have $q_2(\nu) \neq 0$ for $\nu \in \sigma_+$ and it is easily verified that $\det c \neq 0$ in the cases (2.7a) and (2.8). Let the conditions (1.13a) hold and therefor $p_2(\nu) = p_3(\nu) = 0$ for some $\nu \in \sigma_+$. For definiteness let $\nu = \nu_1$. Then by virtue of (2.11a) $q_3(\nu_1) = 0$ and therefor $q_1(\nu_1) \neq 0$.

Let us prove that also $q_1(\nu_2) \neq 0$ out of the exceptional case (1.11). Really if $q_1(\nu_1) = 0$ then according to (2.11b) we will have $p_3(\nu_2) = 0$. As $p_2(\nu_2) \neq 0$ it follows from the equality $p_1(\nu)p_2(\nu) - p_3^2(\nu) = 0$ that $p_1(\nu_2) = 0$. So $p_2(\nu) = p_3(\nu) = 0$, $\nu = \nu_1$ and $p_1(\nu) = p_3(\nu) = 0$, $\nu = \nu_2$ and by virtue of Lemma 1.1(a), (b) the both conditions (1.13) hold which is equivalent to (1.11).

Thus the numbers $q_1(\nu_j)$ in (2.7b) are not equal to zero. It follows from (2.9) that $q_2(\nu_j) - \nu_j q_3(\nu_j) = \nu_j^{-1} q_1(\nu_j)$ and so $\det c \neq 0$.

According to (22c) in the exceptional case (1.11) we have $\det c = \alpha_3^2 + \alpha_2\alpha_3\nu_1\nu_2$. Taking into account (1.12) we also receive $\det c \neq 0$.

The expressions of the Theorem 2.1 are simplified in the orthotropic case. In this case (1.9) and (2.5) have the form

$$\begin{aligned} p_1(z) &= \alpha_1 + \alpha_3 z^2, & p_2(z) &= \alpha_3 + \alpha_2 z^2, \\ p_3(z) &= (\alpha_3 + \alpha_4) z, & q_3(z) &= -\alpha_3(\alpha_4 - \alpha_2 z^2). \end{aligned}$$

If $\alpha_3 + \alpha_4 \neq 0$ then we can use the formulas (2.7a) and (2.8). So we have the expressions

$$b = \begin{pmatrix} \alpha_3 + \alpha_2\nu_1^2 & \alpha_3 + \alpha_2\nu_2^2 \\ -(\alpha_3 + \alpha_4)\nu_1 & -(\alpha_3 + \alpha_4)\nu_2 \end{pmatrix}, \quad c = \alpha_3 \begin{pmatrix} \nu_1(\alpha_4 - \alpha_2\nu_1^2) & \nu_2(\alpha_4 - \alpha_2\nu_2^2) \\ -(\alpha_4 - \alpha_2\nu_1^2) & -(\alpha_4 - \alpha_2\nu_2^2) \end{pmatrix},$$

where ν_j are defined by (1.10a) or (1.10b), and

$$b = \begin{pmatrix} \alpha_3 - \alpha_2\rho^2 & 2i\alpha_2\rho \\ -i(\alpha_3 + \alpha_4)\rho & -(\alpha_3 + \alpha_4) \end{pmatrix}, \quad c = \alpha_3 \begin{pmatrix} i\rho(\alpha_4 + \alpha_2\rho^2) & \alpha_4 + 3\alpha_2\rho^2 \\ -(\alpha_4 + \alpha_2\rho^2) & 2i\alpha_2\rho \end{pmatrix}$$

in the case (ii).

The last formulas permit further simplification in the isotropic case. In this case $\rho = 1$ and $\alpha_1 > \alpha_3$ by virtue of (1.3a), (1.4). So

$$b = \begin{pmatrix} \alpha_3 - \alpha_1 & 2\alpha_1 i \\ (\alpha_3 - \alpha_1)i & \alpha_3 - \alpha_1 \end{pmatrix}, \quad c = 2\alpha_3 \begin{pmatrix} (\alpha_1 - \alpha_3)i & 2\alpha_1 - \alpha_3 \\ \alpha_3 - \alpha_1 & \alpha_1 i \end{pmatrix}.$$

According to (2.3) we can multiply these matrices by

$$d = (\alpha_3 - \alpha_1)^{-1} \begin{pmatrix} 1 & 2\alpha_1(\alpha_1 - \alpha_3)^{-1} i \\ 0 & 1 \end{pmatrix}.$$

As a result we have

$$b = \begin{pmatrix} 1 & 0 \\ i & -æ \end{pmatrix}, \quad c = \alpha_3 \begin{pmatrix} 2i & æ - 1 \\ 2 & i(æ + 1) \end{pmatrix},$$

where $æ = (\alpha_1 + \alpha_3)/(\alpha_1 - \alpha_3)$. □

3 Conjugate function

Let us consider a second order elliptic system

$$a_{11}\frac{\partial^2 u}{\partial x^2} + a_{(12)}\frac{\partial^2 u}{\partial y^2} + a_{22}\frac{\partial^2 u}{\partial y^2} = 0, \quad a_{(12)} = a_{12} + a_{21}, \tag{3.1}$$

with coefficients $a_{ij} \in \mathbb{R}^{l\times l}$. We can introduce the notion of conjugate function v to solution $u = (u_1, \dots, u_l)$ of this equation as above by (1.7). Of course this definition depends on the partition $a_{(12)} = a_{12} + a_{21}$. There is a question which second order system defines the function v? Let us put

$$a_1 = a_{11}^{-1} a_{12}, \quad a_2 = a_{22}^{-1} a_{21} \tag{3.2}$$

and define matrixes $d_1, d_2 \in \mathbb{R}^{l\times l}$ such that

$$d_1 a_{22}(1 - a_2 a_1) = d_2 a_{11}(1 - a_1 a_2). \tag{3.3}$$

Lemma 3.1 *The conjugate function v satisfies the system*

$$d_1 \frac{\partial^2 v}{\partial x^2} + (d_1 a_{21} a_{11}^{-1} + d_2 a_{12} a_{22}^{-1}) \frac{\partial^2 v}{\partial x \partial y} + d_2 \frac{\partial^2 v}{\partial x^2} = 0. \tag{3.4}$$

Proof: With respect to the vectors

$$U = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right), \quad V = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)$$

(1.7) takes a form

$$V = \begin{pmatrix} -a_{21} & -a_{22} \\ a_{11} & a_{12} \end{pmatrix} U. \tag{3.5}$$

Hence (3.1) and the analogous equation

$$d_{11} \frac{\partial^2 u}{\partial x^2} + d_{(12)} \frac{\partial^2 u}{\partial y^2} + d_{22} \frac{\partial^2 u}{\partial y^2} = 0 \tag{3.6}$$

for the function v can be rewritten as

$$\frac{\partial U}{\partial y} = \begin{pmatrix} 0 & 1 \\ -a_{22}^{-1} a_{11} & -a_{22}^{-1} a_{(12)} \end{pmatrix} \frac{\partial U}{\partial x}, \quad \begin{pmatrix} 1 & 0 \\ 0 & d_{22} \end{pmatrix} \frac{\partial V}{\partial y} = \begin{pmatrix} 0 & 1 \\ -d_{11} & -d_{(12)} \end{pmatrix} \frac{\partial V}{\partial x}.$$

Together with (3.5) it follows that

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & d_{22} \end{pmatrix} \begin{pmatrix} -a_{21} & -a_{22} \\ a_{11} & a_{12} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -a_{22}^{-1} a_{11} & -a_{22}^{-1} a_{(12)} \end{pmatrix} \right.$$
$$\left. - \begin{pmatrix} 0 & 1 \\ -d_{11} & -d_{(12)} \end{pmatrix} \begin{pmatrix} -a_{21} & -a_{22} \\ a_{11} & a_{12} \end{pmatrix} \right] \frac{\partial U}{\partial x} = 0$$

for all U. This is equivalent to system

$$d_{11} a_{21} - d_{(12)} a_{11} = -d_{22}(a_{12} a_{22}^{-1} a_{11}), \quad d_{11} a_{22} - d_{(12)} a_{12} = d_{22}(a_{11} - a_{12} a_{22}^{-1} a_{(12)})$$

with respect to unknown coefficients d_{11}, d_{22} è $d_{(12)}$. This system we can rewrite as following:

$$d_{22}(a_{11} - a_{12} a_{22}^{-1} a_{21}) = d_{11}(a_{22} - a_{21} a_{11}^{-1} a_{12}), \quad d_{(12)} = d_{11} a_{21} a_{11}^{-1} + d_{22} a_{12} a_{22}^{-1}.$$

The first equation coincides with (3.3) with respect to $d_i = d_{ii}$, but a substitution of second one to (3.6) gives (3.4).

Let us apply this result to Lame system. According to (2) the matrices (3.2) have the form

$$a_1 = \frac{1}{\alpha_1\alpha_3 - \alpha_6^2}\begin{pmatrix} 0 & \alpha_3\alpha_4 - \alpha_5\alpha_6 \\ \alpha_1\alpha_3 - \alpha_6^2 & \alpha_1\alpha_5 - \alpha_4\alpha_6 \end{pmatrix} = \begin{pmatrix} 0 & -\beta_4/\beta_2 \\ 1 & -\beta_5/\beta_2 \end{pmatrix},$$

$$a_2 = \frac{1}{\alpha_2\alpha_3 - \alpha_5^2}\begin{pmatrix} \alpha_2\alpha_6 - \alpha_4\alpha_5 & \alpha_2\alpha_3 - \alpha_5^2 \\ \alpha_3\alpha_4 - \alpha_5\alpha_6 & 0 \end{pmatrix} = \begin{pmatrix} -\beta_6/\beta_1 & 1 \\ -\beta_4/\beta_1 & 0 \end{pmatrix},$$

where β_j figure in (2.5), (2.6). Analogously

$$a_{21}a_{11}^{-1} = \begin{pmatrix} 0 & 1 \\ -\beta_4/\beta_2 & -\beta_5/\beta_2 \end{pmatrix}, \quad a_{12}a_{22}^{-1} = \begin{pmatrix} -\beta_6/\beta_1 & -\beta_4/\beta_1 \\ 1 & 0 \end{pmatrix}.$$

Hence

$$1 - a_1a_2 = \frac{1}{\beta_1\beta_2}\begin{pmatrix} \beta_1\beta_2 - \beta_4^2 & 0 \\ \beta_2\beta_6 - \beta_4\beta_5 & 0 \end{pmatrix}, \quad 1 - a_2a_1 = \frac{1}{\beta_1\beta_2}\begin{pmatrix} 0 & \beta_1\beta_5 - \beta_4\beta_6 \\ 0 & \beta_1\beta_2 - \beta_4^2 \end{pmatrix}$$

and (3.3) reduces to conditions $(d_1)_{(2)} = (d_2)_{(1)}$ with respect to columns of the matrixes d_j. So we can take

$$d_1 = \begin{pmatrix} s_1 & 0 \\ t_1 & 0 \end{pmatrix}, \quad d_2 = \begin{pmatrix} 0 & s_2 \\ 0 & t_2 \end{pmatrix}$$

with some $s_j, t_j \in \mathbb{R}$ and (3.4) reduces to

$$\begin{pmatrix} s_1 & 0 \\ t_1 & 0 \end{pmatrix}\frac{\partial^2 v}{\partial x^2} + \begin{pmatrix} s_2 & s_1 \\ t_2 & t_1 \end{pmatrix}\frac{\partial^2 v}{\partial x \partial y} + \begin{pmatrix} 0 & s_2 \\ 0 & t_2 \end{pmatrix}\frac{\partial^2 v}{\partial x^2} = 0.$$

This system is equivalent to equations

$$\frac{\partial}{\partial x}\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}\right) = 0, \quad \frac{\partial}{\partial y}\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}\right) = 0.$$

But they are consequence of the equation (1.6) from which it follows

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0.$$

Therefor the result of Lemma 3.1 for the Lame system is reduced to the last equation. □

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