# Problem of Bitsadze–Samarskii Type for Second-Order Elliptic Systems in the Plane[1]

**A. P. Soldatov**

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## FORMULATION OF THE PROBLEM

Let $D \subseteq \mathbb{C}$ be a bounded domain with a piecewise smooth boundary $\Gamma = \partial D$ without cuspidal points. Suppose that the set $F = \{\tau_1, \tau_2, \ldots, \tau_m\} \subseteq \Gamma$ contains all the angular points of the curve. A function $\varphi \in C(\Gamma \backslash F)$ is piecewise continuous on $\Gamma$ if there exist one-sided limits $\varphi(\tau \pm 0)$ at the points $\tau \in F$. A continuously differentiable mapping $\alpha: \Gamma \backslash F \to \overline{D}$ is called a shift if $\alpha'$ and its derivative $\alpha'$ (with respect to the arc length parameter measured in a fixed direction) are piecewise continuous on $\Gamma$, $\alpha(\tau \pm 0) \in F$, and $\alpha'(\tau \pm 0) \neq 0$ for all $\alpha(\Gamma \backslash F) \subseteq \Gamma$. In what follows, we consider only boundary shifts when $\alpha(\Gamma \backslash F) \subseteq \Gamma$ and inner shifts for which this curve lies in $D$ and is not tangent to $\Gamma$ at the points $\tau \in F$. For example, the identity mapping $e(t) = t$ is a boundary shift.

We consider the second-order elliptic system

$$\frac{\partial^2 u}{\partial y^2} - A_1 \frac{\partial^2 u}{\partial x\, \partial y} - A_0 \frac{\partial^2 u}{\partial x^2} = 0 \qquad (1)$$

with constant coefficients $A_0, A_1 \in \mathbb{R}^{l \times l}$ for an unknown vector-valued function $u = (u_1, u_2, \ldots, u_l) \in C^2(D)$.

The problem of the Bitsadze–Samarskii type is formulated as follows: Find a solution $u \in C(\overline{D})$ to system (1) that satisfies the boundary condition

$$\left. \left( u + \sum_{s=1}^{n} b_s u \circ \beta_s \right) \right|_{\Gamma} = f, \qquad (2)$$

where the $l \times l$ matrix-valued functions $b_s(t)$ and the right-hand-side $f(t)$ are piecewise continuous on $\Gamma$ and $\beta_s$ are inner shifts.

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*Belgorod State University, ul. Pobedy 85, Belgorod, 308015 Russia*
*e-mail: soldatov@bsu.edu.ru*

We can also consider the case when the coefficients $b_s$ are defined only on a part $\Gamma'$ of $\Gamma$. In this case, (2) is transformed into the following form:

$$u|_{\Gamma \backslash \Gamma'} = f_0, \quad \left. \left( u + \sum_{s=1}^{n} b_s u \circ \beta_s \right) \right|_{\Gamma'} = f_1. \qquad (2')$$

This problem can always be reduced to form (2) by extending $\beta_s$ to the whole $\Gamma$ and setting $b_s = 0$ on $\Gamma \backslash \Gamma'$.

In the case of a single shift, problem (1), (2') was first posed by Bitsadze and Samarskii [1] for the Laplace equation. For general elliptic systems and equations, it was investigated by many authors [2–7]. In this paper, a new approach to investigating this problem is developed. It is based on the reduction of the Bitsadze–Samarskii problem to a system of singular integral equations of the nonclassical type; the corresponding theory was developed in [8].

Note that, for the solvability of problem (1), (2), we need compatibility conditions imposed on the right side $f$ at the points $\tau \in F$. They can be described as follows. There exists a function $\tilde{u} \in C(\overline{D})$ such that the piecewise continuous function

$$\tilde{f} = \left. \left( \tilde{u} + \sum_{s=1}^{n} b_s \tilde{u} \circ \beta_s \right) \right|_{\Gamma}$$

coincides with $f$ at the points $\tau \in F$; i.e. $\tilde{f}(\tau_j \pm 0) = f(\tau_j \pm 0)$, $1 \leq j \leq m$. Obviously, the number of these linearly independent conditions is no less than $ml$. For example, for the Dirichlet problem, which corresponds to $b_s = 0$, these conditions are reduced to $f(\tau_i + 0) = f(\tau_i - 0)$ and their number is equal to $ml$. Let us consider a more general situation of this type.

**Lemma 1.** *Suppose that there are sets*

$$F = F_0 \supset F_1 \supset \ldots \supset F_{n+1} = \emptyset,$$
$$F_p \backslash F_{p+1} \neq \emptyset, \quad 0 \leq p \leq n,$$

*such that* $\beta_s(\tau \pm 0) \in F_{p+1}$ *for* $\tau \in F_p$, $b_s(\tau \pm 0) \neq 0$, $0 \le p \le n$ (*in particular*, $b(\tau + 0) = b(\tau - 0) = 0$ *for* $\tau \in F_n$).

*Then the number of linearly independent compatibility conditions is equal to ml*

We may also consider problem (1), (2) in the class $C(\overline{D}\backslash F)$ with a right side $f \in C(\Gamma\backslash F)$. In this case, no compatibility conditions are imposed.

The condition of ellipticity for (1) means that the determinant of the characteristic polynomial $P(w) = w^2 - A_1 w - A_0$ has no real roots. This system is closely connected with the first-order elliptic system

$$\frac{\partial\phi}{\partial y} - J\frac{\partial\phi}{\partial x} = 0, \tag{3}$$

where $J \in \mathbb{C}^{l\times l}$ is a Jordan matrix and the eigenvalues of its blocks lie in the upper half-plane. This system was first considered by Douglis [9], so its solutions are called analytic functions in the Douglis sense or, briefly, $J$-analytic functions. If the solution of the original system (1) is sought in the form

$$u = \mathrm{Re}B\phi \tag{4}$$

with some matrix $B \in \mathbb{C}^{l\times l}$, then, by virtue of (3), we have the matrix relation $BJ^2 - A_1BJ - A_0B = 0$ for the pair $B$, $J$. It was shown in [10] that, on this pair, we can impose the supplementary condition

$$\det\begin{pmatrix} B & \overline{B} \\ BJ & \overline{BJ} \end{pmatrix} \neq 0.$$

In this case, formula (4) describes the general solution of (1). In the upper half-plane $\mathrm{Re}w > 0$, the zeroes of $\det P(w)$ coincide with the eigenvalues $\nu \in \sigma(J)$, including their multiplicities. The degrees of the poles of the matrix-valued functions $P^{-1}(w)$ and $(w - J)^{-1}$ coincide as well. The condition $\det B \neq 0$ defines so-called weakly connected elliptic systems (according to Bitsadze's nomenclature [11]). This condition is necessary for the Dirichlet problem to be Fredholm, so we assume that it is fulfilled for boundary condition (2).

The function $\phi$ in (4) is defined up to a constant vector $\eta \in \mathbb{C}^l$, $\mathrm{Re}B\eta = 0$. In general, it is a multivalued function in a multiply connected domain $D$. To be more precise, its derivative $\phi' = \dfrac{\partial\phi}{\partial x}$ is a univalent function in $D$. Thus, multivaluedness is of a logarithmic nature.

Representation (4) can be modified so that multivalued functions do not appear. Specifically, any solution to system (1) can be uniquely represented as

$$u = \mathrm{Re}B\phi + \sum_{j=1}^{k} u_j\xi_j, \quad \xi_j \in \mathbb{R}^l, \quad \phi(z_0) = 0, \tag{5}$$

where $u_j \in C^\infty(\overline{D})$ are completely defined $l \times l$ matrix-valued functions whose columns satisfy (1), $z_0$ is a fixed point of $D$, and $k$ is the number of connected components of $\Gamma = \partial D$

With the help of (5), problem (1), (2) reduces to the equivalent problem

$$\mathrm{Re}\left(B\phi + \sum_{s=1}^{n} b_sB\phi\circ\beta_s\right)\Bigg|_\Gamma + \sum_{j=1}^{k} c_j\xi_j = f, \tag{6}$$
$$\phi(z_0) = 0,$$

for a $J$-analytic function $\phi$ and a vector $\xi \in \mathbb{R}^{kl}$, where $c_j$ are piecewise continuous matrix-valued functions defined by $u_j$. It is obvious that (6) is a finite-dimensional perturbation of the problem

$$\mathrm{Re}\left(G\phi + \sum_{s=1}^{n} G_s\phi\circ\beta_s\right)\Bigg|_\Gamma = f \tag{7}$$

with piecewise continuous $l \times l$ matrix-valued functions $G = B$ and $G_s = b_sB$ on $\Gamma$.

The Fredholm solvability of problem (3), (7) was studied in [12] for the case of a single shift. Let us consider its particular case

$$\mathrm{Re}(G\phi)|_{\Gamma\backslash\Gamma'} = f_0, \quad \mathrm{Re}(G\phi + G^0\phi\circ\beta)|_{\Gamma'} = f_1,$$

where $\Gamma'$ is a smooth arc and $\beta(\Gamma')$ divides $D$ into two subdomains. As was indicated in [13], this problem can be reduced to the so-called generalized Riemann–Hilbert problem for $J$-analytic functions. In this context, we can also note [14, 15], where this problem was considered for solutions of the linearized Stokes system and for usual analytic functions, respectively.

## MAIN RESULTS

We consider the problem in the weighted Hölder space $C^\mu_\lambda(\overline{D}; F)$ where $0 < \mu < 1$ and $\lambda < 0$, and in the modified weighted class $C^\mu_{(\lambda)}(\overline{D}; F)$, where $0 < \lambda < 1$. Let us recall their definitions [8]. Let $C^\mu_\lambda$ with $\lambda \in \mathbb{R}$ be the space of all functions $\varphi \in C(\overline{D}\backslash F)$ that belong to $C^\mu(K)$ for every compact subset $K \subseteq \overline{D}\backslash F$ and $O(1)|z - \tau|^\lambda$ as $z \to \tau \in F$. To be more precise, in the curvilinear sectors $D_i = D \cap \{|z - \tau_i| < \delta\}$, $i = 1, 2, \ldots, m$, where $\delta > 0$ is sufficiently small, we have $\varphi_i(z) = \varphi(z)|z - \tau_i|^{\mu-\lambda} \in C^\mu(\overline{D}_i)$, $\varphi_i(\tau_i) = 0$. The space $C^\mu_{(\lambda)}$ with $0 < \lambda < 1$ is a finite-dimensional expansion of $C^\mu_\lambda$ by smooth functions that are constant in a neighborhood on $D_j$. This space is embedded in $C^{\min(\mu,\lambda)}(\overline{D})$, and the embedding becomes an exact equality when $\lambda = \mu$.

We will also use these spaces for piecewise continuous functions defined on $\Gamma\backslash F$. The boundary of the sector $D_i$ consists of two smooth arcs $\Gamma_{ik}$ ($k = 1, 2$) with a common endpoint $\tau_i$, which are called its lateral sides, and of an arc of the circle $|z - \tau_i| = \delta$. It is convenient to denote the one-sided limits $\varphi(\tau \pm 0)$ at the point $\tau_i$ by $\varphi(\tau_{ik}) = \lim\varphi(t)$ as $t \to \tau_i$, $t \in \Gamma_{ik}$. For definiteness, we numerate the lateral sides $\Gamma_{ik}$ so that $\tau_{i1} = \tau_i + 0$ and $\tau_{i2} = \tau_i - 0$. Now the space $C^{\mu}_{(\lambda)}(\Gamma, F)$ can be defined as above by replacing $\overline{D}_i$ with $\Gamma_{ik}$.

We consider problem (1), (2) in the classes $C^{\mu}_{\lambda}$, $\lambda < 0$ and $C^{\mu}_{(\lambda)}$, $0 < l < 1$, with respect to solutions $\phi$ to (3), (5). It is assumed that the data $e'$, $\beta'$, $G$, and $b$ of the problem belong to the class $C^{\mu+0}_{(+0)}(\Gamma, F) = \bigcup_{\epsilon > 0} C^{\mu+\epsilon}_{(\epsilon)}$ in the former case and to the class $C^{\mu+0}_{(\lambda+0)}(\Gamma, F)$ in the latter case. Here, the derivatives on $\Gamma\backslash F$ are meant with respect to the parameter of length arc measured in the positive direction (so that $D$ is on the left). Under these assumptions, the operators of the problems are bounded from the space $C^{\mu}_{(\lambda)}(\overline{D}, F)$ of solutions to system (1) to the space $C^{\mu}_{(\lambda)}(\overline{\Gamma}, F)$, $0 < \lambda < 1$, and the same is true for $C^{\mu}_{\lambda}$, $\lambda < 0$. The Fredholm solvability and the index of the problems are meant with respect to these operators.

Recall that the eigenvalues of the matrix $J$ of system (3) lie in the upper half-plane and coincide with the roots of the characteristic equation of the original system (1). For every nonzero complex number $q$, we introduce the invertible matrix $q_J = \mathrm{Re}\,q + J\,\mathrm{Im}\,q$. It will be used below for the derivatives $q = \alpha'(\tau_{ik})$ of shifts at the points $\tau_i \pm 0$. These derivatives are meant with respect to the length arc parameter on $\Gamma_{ik}$ measured from the point $\tau_i$. In particular, for the identity shift $e(t) = t$, the number $e'(\tau_{ik})$ is a unique tangent vector on $\Gamma_{ik}$ at the point $\tau_i$. By assumption, the arcs $\Gamma_{i1}$ and $\Gamma_{i2}$ are not tangent to each other at the point $\tau_i$, so $e'(\tau_{i1}) \neq e'(\tau_{i2})$. From the same considerations, the limit values $\beta(\tau_{ik})$ of an inner shift $\beta = \beta_s$ belong to $F$, and if $\beta(\tau_{ik}) = \tau_j$, then the arc $\beta(\Gamma_{ik})$ is not tangent to the lateral sides of $D_j$ at the point $\tau_j$. Hence, the vector $\beta'(\tau_{ik})$ lies between the tangent vectors on $\Gamma_{jk}$ at the point $\tau_j$.

In this notation, for each shift $\alpha\colon \Gamma\backslash F \to \overline{D}$, we can introduce the matrices $Q_{ik}(\alpha)$

$$Q_{ik}(\alpha) = [\alpha'(\tau_{ik})]_J[e'(\tau_{j1})]_J^{-1}, \quad \alpha(\tau_{ik}) = \tau_j,$$

whose eigenvalues do not lie on the positive half-axis. Therefore, we can define the matrices $\ln Q$ as the values of the analytic functions $\ln w$ ($0 < \arg\ \arg w < 2\pi$) of $Q$, and we can define the complex degrees $Q^{\zeta}_{ik}(a) = \exp[\zeta \ln Q_{ik}(\alpha)]$ and $\overline{Q}^{\zeta}_{ik}(\alpha) = \exp[\zeta\,\overline{\ln Q_{ik}(\alpha)}\,]$.

On the basis of these degrees and a piecewise continuous matrix-valued function $G$ on $\Gamma$, we introduce the following $m \times m$ block matrices:

$$(G;\alpha)^{\hat{}}_{ij} = \{(G;\alpha)^{\hat{}}_{ijkr}\}_1^2,$$

$$(G;\alpha)^{\hat{}}_{ijkr} = \begin{cases} G(\tau_{ik})Q^{\zeta}_{ik}(\alpha), & \alpha(\tau_{ik}) = \tau_j, \quad r = 1, \\ \overline{G}(\tau_{ik})\overline{Q}^{\zeta}_{ik}(\alpha), & \alpha(\tau_{ik}) = \tau_j, \quad r = 2, \\ 0, \quad \alpha(\tau_{ik}) \neq \tau_j. \end{cases} \tag{8}$$

In the accepted notation, problem (1), (2) is associated with two matrices:

$$X = (B;e)^{\hat{}} + \sum_{s=1}^{n}(b_sB;\beta_s)^{\hat{}}, \quad Y = (1;e)^{\hat{}},$$

The former is called the end symbol of this problem.

The determinant of $Y$ can be explicitly calculated. It expression shows that the function $\det Y(\zeta)$ has the unique zero $\zeta = 0$ in the strip $|\mathrm{Re}\,\zeta| < \frac{1}{2}$ and its degree is equal to $ml$. For fixed $\mathrm{Re}\,\zeta = \lambda$, the function $\det(XY^{-1})(\zeta)$ has a finite limit as $\mathrm{Im}\,\zeta \to \infty$, which, by virtue of the assumption $\det B \neq 0$ is not equal to zero. Then, in the strips $\lambda < \mathrm{Re}\,\zeta < 0$ and $0 \le \mathrm{Re}\,\zeta < \lambda$, the function $\det X(\zeta)$ has a finite number of zeroes. We denote this number, counting multiplicities, by $-\Delta(\lambda)$ and $\Delta(\lambda)$, respectively. Thus, the piecewise constant function $\Delta(\lambda)$ is monotone nondecreasing and, for $\lambda_1 < \lambda_2$, the difference $\Delta(\lambda_2) - \Delta(\lambda_1)$ is equal to the number of zeroes of $\det X(\zeta)$ counting multiplicities in the strip $\lambda_1 \le \mathrm{Re}\,\zeta < \lambda_2$.

Nevertheless, if

$$\det X(\zeta) \neq 0, \quad \mathrm{Re}\,\zeta = \lambda, \tag{9}$$

then we can introduce the increment $\arg\det(XY^{-1})(\lambda + i\infty) - \arg\det(XY^{-1})(\lambda - i\infty)$ of a continuous branch of the argument, which is divisible by $2\pi$. Regarded as a function of $\lambda$, this increment is piecewise constant and, by Rouche's theorem, $\arg\det(XY^{-1})|_{\lambda_2} - \arg\det(XY^{-1})|_{\lambda_1} = 2\pi[\Delta(\lambda_2) - \Delta(\lambda_1)]$. In particular, we can set $\lim\arg\det(XY^{-1})|_{-\varepsilon}$ as $\varepsilon \to 0$, $\varepsilon > 0$.

**Theorem 1.** *Problem* (1), (2) *is Fredholm in the classes* $C^{\mu}_{\lambda}$, $\lambda < 0$, *and* $C^{\mu}_{(\lambda)}$, $0 < \lambda < 1$, *if and only if it is of the normal type and condition* (9) *holds. In this case, its index* $\kappa$ *is given by the formula* $\kappa = -\frac{1}{2\pi}\arg\det(XY^{-1})|_{-0} - \Delta(\lambda)$.

Let us consider the problem in the classes $C^{\mu}_{-0} = \bigcap_{\varepsilon > 0} C^{\mu}_{-\varepsilon}$ and $C^{\mu}_{+0} = \bigcup_{\varepsilon > 0} C^{\mu}_{\varepsilon}$. It may happen that the function $f$ in (2) belongs to $C^{\mu}_{+0}(\Gamma, F)$ for some solution of

(1) in the class $C^{\mu}_{-0}(\bar{D}, F)$. The question arises about the asymptotics of this solution at the vertex $\tau_i$ of the sector $D_i$. To formulate the corresponding result, we introduce the analytic functions $\ln(z - \tau_i)$ in these sectors. As above, they define the matrix-valued functions $\ln(z - \tau_i)_J$ and $(z - \tau_i)^{\zeta}_J$. Note that $z - \tau_i)^{\zeta}_J [\ln(z - \tau_i)_J]^k \in C^{\mu}_{-0}(\bar{D}_i, \tau_i)$, $\mathrm{Re}\zeta = 0$, $k = 0, 1, \ldots$.

Let us introduce nonnegative integer-valued functions $k(\zeta)$ and $r(\zeta)$ that characterize the degrees of zeroes and poles of the functions $\det X(\zeta)$ and $X^{-1}(\zeta)$, respectively. If $\det X(\zeta) \neq 0$, we set $k(\zeta) = r(\zeta) = 0$. Obviously, $r(\zeta) \le k(\zeta)$ for all $\zeta$ and, in the above notations, $\Delta(0)$ is equal to the sum of $k(\zeta)$ over $\mathrm{Re}\zeta = 0$.

**Theorem 2.** *Suppose that the function $f$ in* (2) *belongs to $C^{\mu}_{+0}(\Gamma, F)$ for some solution $u(z) \in C^{\mu}_{-0}(\bar{D}, F)$ of* (1).

*Then, for any sector $D_i$, there exist $c_k(\zeta) \in \mathbb{C}^l$, $0 \le k \le r(\zeta) - 1$ such that*

$$u(z) - \mathrm{Re}B\phi_i(z) \in C^{\mu}_{+0}(\bar{D}_i, \tau_i),$$

$$\phi_i(z) = \sum_{\mathrm{Re}\zeta = 0} \sum_{k=0}^{r(\zeta)-1} (z - \tau_i)^{\zeta}_J [\ln(z - \tau_i)_J]^k c_k(\zeta).$$

Of course, the inner sum in the expression for $\phi_i$ is equal to zero for $r(\zeta) = 0$; therefore, the outer sum is finite.

**Corollary.** *If $r(0) \le 1$, $r(\zeta) = 0$ for $\zeta \neq 0$, and* $\mathrm{Re}\zeta = 0$, *then, under the assumptions of Theorem* 2, $u \in C^{\mu}_{(+0)}(\bar{D}_i, \tau_i)$.

This corollary shows that if $u \in C^{\mu}_{(-0)}(\bar{D}_i, F)$ and the function $f$ does not satisfy the compatibility conditions, then the solution $u$ permits logarithmic singularities at the points $\tau \in F$.

Note that the linear independent solvability conditions in the definition of the index of the problem include the compatibility conditions for $0 < \lambda < 1$. Let us denote by $\kappa^+$ and $\kappa^-$ the indices of problem (1), (2) in the classes $C^{\mu}_{-0}$ and $C^{\mu}_{(+0)}$, respectively. According to Theorem 1, they are connected by the relation $\kappa^- - \kappa^+ = \Delta(0)$. If the conditions of the corollary are fulfilled, then $\Delta(0) = k(0)$ and the number of compatibility conditions is equal to $k(0)$.

In the scalar case $l = 1$, Eq. (1) reduces to the Laplace equation by a change of variables. In particular, the maximum principle holds for this equation. This fact allows us to completely study the solvability of the problem.

**Theorem 3.** *Suppose that $l = 1$, the inequality $\sum_{s=1}^{n} |b_s| \le 1$ holds, and the conditions of Lemma* 1 *are fulfilled. Then, under the compatibility conditions, problem* (1), (2) *is uniquely solved in the class $C^{\mu}_{\lambda}$, $-\frac{1}{2} < \lambda < 1/2$.*

Note that, in this theorem, the condition on $r(\zeta)$ imposed in the corollary is satisfied and $k(0) = m$.

We can complete Theorems 1 and 2 by adding the corresponding result on the smoothness of the solution.

**Theorem 4.** *Under the assumptions of Theorem* 1, *let the solution $u$ be such that $f$ in* (2) *is continuously differentiable on $\Gamma \backslash F$ and $f' \in C^{\mu}_{\lambda-1}(\Gamma, F)$. Then the partial derivatives $u_x$ and $u_y$ of the solution $u$ belong to $C^{\mu}_{\lambda-1}(\bar{D}, F)$. Analogously, if $f' \in C^{\mu}_{-1+0}(\Gamma, F)$ in Theorem* 2, *then the partial derivatives of the difference $u(z) - \mathrm{Re}B\phi_i(z)$ belong to $C^{\mu}_{-1+0}(\bar{D}_i, \tau_i)$ in the sector $D_i$.*

If a shift $\alpha$ satisfies the condition $\alpha(\tau_i + 0) = \alpha(\tau_i - 0) = \tau_i$, $1 \le i \le m$, then matrix (8) has the block diagonal structure $(G;\ \alpha)^{\hat{}}_{ij} = (G;\ \alpha)^{\hat{}}_i \delta_{ij}$, where the diagonals blocks

$$(G;\ \alpha)^{\hat{}}_i = \begin{pmatrix} G(\tau_{i1}) Q^{\zeta}_{i1}(\alpha) & \bar{G}(\tau_{i1}) \bar{Q}^{\zeta}_{i1}(\alpha) \\ G(\tau_{i2}) Q^{\zeta}_{i2}(\alpha) & \bar{G}(\tau_{i2}) \bar{Q}^{\zeta}_{i2}(\alpha) \end{pmatrix},$$

$$Q_{ik} = [\alpha'(\tau_{ik})]_J [e'(\tau_{i1})]^{-1}_J$$

are associated with the corresponding sectors $D_i$.

Let all the shifts $\beta_s$ satisfy this condition. Then the end symbol $X$ of the problem has the same block diagonal structure $(X_i \delta_{ij})^1_m$ with diagonals blocks $X_i = (B;\ e)^{\hat{}}_i = \sum_s (b_s B;\ \beta_s)^{\hat{}}_i$. Moreover, $Y$ has the same structure with $Y_i = (1;\ e)^{\hat{}}_i$. In this case, we can regard the weighted order $\lambda$ as a vector whose coordinates $\lambda_i$ are associated with the corresponding space $C^{\mu}_{\lambda_i}(\bar{D}_i, \tau_i)$.

Theorem 1 also holds in this case. It is only necessary to replace (9) by the condition $\det X_i(\zeta) \neq 0$, $\mathrm{Re}\zeta = \lambda_i$, $1 \le i \le m$. The index formula in this case has the form

$$\kappa = \sum \kappa_i,$$

$$\kappa_i = -\frac{1}{2\pi} \sum_1^m \arg\det(X_i Y_i^{-1})\big|_{-0} - \sum_{i=1}^m \Delta_i(l_i),$$

where $\Delta_i$ is defined with respect to $X_i$ as above. In the same way, Theorem 2 is valid when the characteristic $r(\zeta)$ of poles is meant with respect to $X_i(\zeta)$.


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