GENERALIZED SOLUTIONS TO LINEARIZED EQUATIONS OF THERMOELASTIC SOLID AND VISCOUS THERMOFLUID

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ABSTRACT. Within the framework of continuum mechanics, the full description of joint motion of elastic bodies and compressible viscous fluids with taking into account thermal effects is given by the system consisting of the mass, momentum, and energy balance equations, the first and the second laws of thermodynamics, and an additional set of thermomechanical state laws. The present paper is devoted to the investigation of this system. Assuming that variations of the physical characteristics of the thermomechanical system of the fluid and the solid are small about some rest state, we derive the linearized non-stationary dynamical model, prove its well-posedness, establish additional refined global integral bounds for solutions, and further deduce the linearized incompressible models and models incorporating absolutely rigid skeleton, as asymptotic limits.

1. INTRODUCTION

We are interested in proposing a mathematical description for small perturbations in the thermomechanical system consisting of interacting elastic solid and viscous compressible fluid. In our study we aim to integrate purely mechanical, thermodynamical, and heat transfer effects altogether under one umbrella. Within such unified approach, a thermoconductive elastic body may be named a *thermoelastic solid*. Also we use the term *thermofluids*, which has been introduced quite recently for a subject that analyzes systems and processes involved in energy, various forms of energy, and transfer of energy in fluids [11, Sec. I.1].

The basic mathematical concept of reciprocal motion of thermoelastic solids and viscous thermofluids incorporates the classical conservation laws of continuum mechanics, the first and the second laws of thermodynamics, and a set of state laws specifying individual thermomechanical behavior of the components of media. In this article this concept is called *Model* O_1 and is stated in the beginning of Sec. 2. It is quite universal and spans a large variety of different phenomena in nature and technology. (A rather general relevant observation may be found, for example, in [11, Secs. I.5–I.7, VI].) At the same time this model is very complex and highly nonlinear. Therefore, some physically reasonable simplifications are necessary in view of further applications to natural problems and in engineering.

In investigation of small perturbations it is suitable to simplify *Model* O_1 by implementing the classical formalism of linearization about a rest state [3, Sec. V.7]. As the result, the linearized model arises, whose core consists of the heat equation coupled with the non-stationary compressible Stokes system in the fluid phase and the system of the wave equations (in the elasticity theory also called Lamé's equations) in the solid phase. The linearization procedure and the precise formulations of the resulting *Model* O_2 and its dimensionless version *Model A* are outlined in Secs. 2 and 3.

Also in Sec. 3 we introduce the notion of generalized solutions of initial-boundary value problem for *Model A* and formulate the first main result of this article – Theorem 3.4 on existence and uniqueness of solutions to *Model A*. The proof of this theorem relies on classical methods in the theory of generalized solutions of equations of mathematical physics and is fulfilled in Secs. 4 and 5.

After this, we are interested in studying of the limiting regimes in *Model A*, arising as some coefficients grow infinitely. To this end, in Secs. 6 and 7 additional global integral bounds are established for the pressure distributions (see Theorem 6.3) and for the deformation tensor in the solid phase (see Theorem 7.3). These bounds constitute the second main result of this article. With their help, in Secs. 8 and 9 *Models B1, B2*, and *B3* of incompressible media and *Models C1* and *C2* of viscous thermofluid contained in an absolutely rigid heat-conducting skeleton are established as respective incompressibility and solidification asymptotic limits of *Model A* (see Theorems 8.1, 9.1, and 9.4). These models are the third main result of this article.

2. Basic Nonlinear Formulation and Linearization

Let Ω be an open bounded set in $\mathbb{R}^3$ with a smooth boundary. From now on we assume that Ω is the cube with the side of a size L_0, i.e., $\Omega=(0,L_0)^3$. Next, assume that at time $t=0$ Ω is occupied by a solid component Ω_s and by a fluid component $\Omega_f=\Omega\setminus\overline{\Omega}_s$ such that the interface between the components $\Gamma:=\partial\Omega_f\cap\partial\Omega_s$ is a rather smooth surface or a finite union of such surfaces. The both solid and fluid phases obey the fundamental conservation laws, which have the following forms in Lagrangian variables $\mathbf{x}$ and t [3, Sec. V.5], [12, Sec. 10]. The balance of mass equation is

$$\rho_0=\rho J,\quad \mathbf{x}\text{ in }\Omega_f\text{ or }\Omega_s,\quad t>0,\tag{2.1}$$

the balance of momentum equation is

$$\rho_0\frac{\partial^2\mathbf{w}}{\partial t^2}=J\operatorname*{div}_x(\mathbf{T}\mathbf{P})-J\mathbf{T}\mathbf{P}\mathbf{T}^{-1}\operatorname*{div}_x(\mathbf{T}^t)^{-1}+\rho_0\mathbf{F},\quad \mathbf{x}\text{ in }\Omega_f\text{ or }\Omega_s,\quad t>0,\tag{2.2}$$

and the balance of energy equation is

$$\begin{gathered}\rho_0\frac{\partial U}{\partial t}=-J\operatorname*{div}_x(\mathbf{T}^{-1}\mathbf{q})+J\mathbf{q}\cdot\operatorname*{div}_x(\mathbf{T}^t)^{-1}+J\mathbf{P}:\frac{\partial\mathbb{E}}{\partial t}+\Psi,\\ \mathbf{q}=-\varkappa(\mathbf{T}^t)^{-1}\nabla_x\vartheta,\quad \mathbf{x}\text{ in }\Omega_f\text{ or }\Omega_s,\quad t>0,\end{gathered}\tag{2.3}$$

where

$$\mathbf{T}=\nabla_x\mathbf{r},\quad \mathbf{r}-\mathbf{x}=\mathbf{w},\quad J=\det\mathbf{T},\quad 2\mathbb{E}=\mathbf{T}^t\mathbf{T}-\mathbf{I}.\tag{2.4}$$

The thermodynamical state of the both phases is governed by the first law of thermodynamics

$$\vartheta ds=dU+pd\frac{1}{\rho}\tag{2.5}$$

and a thermodynamical state equation

$$U - \vartheta s = \mathcal{F}(\rho, \vartheta). \tag{2.6}$$

We postulate that thermomechanical behavior in the solid phase is described by the Duhamel–Neumann law of linear thermoelasticity, which is consistent with thermodynamical relations (2.5) and (2.6) [9, Sec. 1.6], [12, Sec. 10]:

$$\begin{gathered}\mathbf{P} := \mathbf{P}_s = \Big(- p_0 - \eta\gamma_s(\vartheta - \vartheta_*) + (\eta - \frac{2}{3}\lambda) \operatorname*{div}_x \mathbf{w}\Big)\mathbf{I} + 2\lambda\mathbf{D}(x, \mathbf{w}),\\ \mathbf{x} \in \Omega_s, \quad t > 0,\end{gathered} \tag{2.7}$$

and that thermomechanical behavior in the fluid phase is described by the Stokes state equation, which has the following form in Lagrangian variables [3, Sec. V.5], [12, Sec. 6]:

$$\begin{gathered}\mathbf{P} := \mathbf{P}_f = \Big(-p + \Big(\nu - \frac{2}{3}\mu\Big)\operatorname{tr}\Big((\mathbf{T}^t)^{-1}\frac{\partial \mathbb{E}}{\partial t}\mathbf{T}^{-1}\Big)\Big)\mathbf{I} + 2\mu\Big((\mathbf{T}^t)^{-1}\frac{\partial \mathbb{E}}{\partial t}\mathbf{T}^{-1}\Big),\\ \mathbf{x} \in \Omega_f, \quad t > 0.\end{gathered} \tag{2.8}$$

In (2.1)–(2.8), $\mathbf{w}$, ϑ, ρ, p, U, and s are unknown displacement, temperature, density, pressure, specific intrinsic energy, and entropy, respectively. The medium under consideration is a two-parameter thermomechanical system. Further we refer ϑ and ρ to as the independent thermodynamical parameters. The triple $\mathbf{r} = (r_1, r_2, r_3)$ is the set of spatial Eulerian coordinates of particles of solid or fluid. The vector $\mathbf{q}$ is the heat flux. The second equation in (2.3) is the Fourier law for the heat flux.

$\mathbf{T}$ is the distortion tensor. $\mathbf{T}^t$, $\mathbf{T}^{-1}$, and $(\mathbf{T}^t)^{-1}$ are its conjugate, inverse and inverse conjugate, respectively. $\mathbb{E}$ is the deformation tensor. It is connected with the displacement vector by the formula

$$2\mathbb{E} = (\nabla_x \mathbf{w})^t + \nabla_x \mathbf{w} + (\nabla_x \mathbf{w})^t \nabla_x \mathbf{w}.$$

$\mathbf{D}(x, \mathbf{w})$ is the symmetric part of the gradient $\nabla_x \mathbf{w}$, i.e.,

$$2\mathbf{D}(x, \mathbf{w}) = (\nabla_x \mathbf{w})^t + \nabla_x \mathbf{w}.$$

$\mathbf{P}$ is the stress tensor. Note that, due to the identity [3, Sec. V.4], [12, Sec. 6]

$$\frac{\partial \mathbb{E}}{\partial t} = \mathbf{T}^t \mathbf{D}(r, \mathbf{v})\mathbf{T},$$

where $\mathbf{D}(r, \mathbf{v}) = (1/2)((\nabla_r \mathbf{v})^t + \nabla_r \mathbf{v})$ is the deformation rate tensor in Eulerian variables $\mathbf{r}$ and t, and $\mathbf{v}$ is the velocity of particles of fluid or solid, (2.8) reduces in Eulerian variables to the well-known form

$$\mathbf{P}_f = \Big(-p + \Big(\nu - \frac{2}{3}\mu\Big) \operatorname*{div}_r \mathbf{v}\Big)\mathbf{I} + 2\mu\mathbf{D}(r, \mathbf{v}), \quad \mathbf{r} \in \Omega_f(t), \quad t > 0.$$

Coefficients $\varkappa$, ν, μ, η, λ, and γ_s are given. They are a thermal conductivity, bulk and shear viscosity coefficients of the fluid, bulk and shear elastic modules of the solid, and a thermal extension of the solid, respectively. In general, they may depend on the thermodynamical parameters ϑ and ρ. In line with the thermodynamics fundamentals, we have [8, Chap. 5], [9, Chap. 1]

$$\varkappa, \nu, \mu, \eta, \lambda, \gamma_s > 0, \quad \nu > \frac{2}{3}\mu, \quad \eta > \frac{2}{3}\lambda. \tag{2.9}$$

Constant positive coefficients ϑ_* and p_0 are given temperature of some rest state and atmosphere pressure, respectively.

Functions ρ_0, $\mathbf{F}$, Ψ, and $\mathcal{F}$ are given. They are an initial distribution of density, a density of mass distributed forces, a volumetric density of exterior heat application, and a specific free energy, respectively. Thermodynamical behavior in the solid and the fluid is different, in general. Therefore we are given, in fact, two distinct functions $\mathcal{F}_s$ and $\mathcal{F}_f$ such that

$$\mathcal{F}(\rho, \vartheta) = \begin{cases} \mathcal{F}_f(\rho, \vartheta) & \text{if } \mathbf{x} \in \Omega_f, \quad t \geq 0, \\ \mathcal{F}_s(\rho, \vartheta) & \text{if } \mathbf{x} \in \Omega_s, \quad t \geq 0 \end{cases}$$

and two distinct functions Ψ_s and Ψ_f such that

$$\Psi(\mathbf{x}, t) = \begin{cases} \Psi_f(\mathbf{x}, t) & \text{if } \mathbf{x} \in \Omega_f, \quad t \geq 0, \\ \Psi_s(\mathbf{x}, t) & \text{if } \mathbf{x} \in \Omega_s, \quad t \geq 0. \end{cases}$$

Similarly, we have two distinct coefficients of thermoconductivity, $\varkappa_s$ in the solid phase and $\varkappa_f$ in the liquid phase.

In terms of the indicator function of the fluid phase

$$\bar{\chi}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \Omega_f, \\ 0 & \text{if } \mathbf{x} \in \Omega_s, \end{cases} \tag{2.10}$$

we may write

$$\mathcal{F}(\rho, \vartheta) = \bar{\chi}\mathcal{F}_f(\rho, \vartheta) + (1 - \bar{\chi})\mathcal{F}_s(\rho, \vartheta) \quad \forall \mathbf{x} \in \Omega, \quad t \geq 0, \tag{2.11}$$

$$\Psi(\mathbf{x}, t) = \bar{\chi}\Psi_f(\mathbf{x}, t) + (1 - \bar{\chi})\Psi_s(\mathbf{x}, t) \quad \forall \mathbf{x} \in \Omega, \quad t \geq 0, \tag{2.12}$$

$$\bar{\varkappa}(\rho, \vartheta) = \bar{\chi}\varkappa_f(\rho, \vartheta) + (1 - \bar{\chi})\varkappa_s(\rho, \vartheta) \quad \forall \mathbf{x} \in \Omega, \quad t \geq 0. \tag{2.13}$$

Analogously,

$$\mathbf{P} = \bar{\chi}\mathbf{P}_f + (1 - \bar{\chi})\mathbf{P}_s \quad \forall \mathbf{x} \in \Omega, \quad t \geq 0. \tag{2.14}$$

Interactions between the fluid and the solid are governed by the classical conditions on discontinuity surfaces [3, Sec. II.3], [12, Sec. 12], [13, Sec. 2]. In order to state these conditions, we introduce the following notation. For any $\mathbf{x}_0 \in \Gamma$ and for any function $\varphi(\mathbf{x})$, continuous in the interior of Ω_s and in the interior of Ω_f, denote

$$\begin{gathered} \varphi_{(s)}(\mathbf{x}_0) = \lim_{\mathbf{x} \to \mathbf{x}_0,\, \mathbf{x} \in \Omega_s} \varphi(\mathbf{x}), \quad \varphi_{(f)}(\mathbf{x}_0) = \lim_{\mathbf{x} \to \mathbf{x}_0,\, \mathbf{x} \in \Omega_f} \varphi(\mathbf{x}), \\ [\varphi](\mathbf{x}_0) = \varphi_{(s)}(\mathbf{x}_0) - \varphi_{(f)}(\mathbf{x}_0). \end{gathered} \tag{2.15}$$

Remark 2.1. Clearly, Γ is immovable in Lagrangian coordinates in the sense that its any parametrization in Lagrangian variables does not depend on t.

Thus, in terms of notation (2.15) we write down the conditions on Γ as follows: the continuity of temperature

$$[\vartheta] = 0, \quad \mathbf{x}_0 \in \Gamma,\ t \geq 0, \tag{2.16}$$

the continuity of displacement

$$[\mathbf{w}] = 0, \quad \mathbf{x}_0 \in \Gamma,\ t \geq 0, \tag{2.17}$$

the continuity of normal stress

$$[\mathbf{P}\mathbf{n}] = 0, \quad \mathbf{x}_0 \in \Gamma,\ t \geq 0, \tag{2.18}$$

and the continuity of the normal heat flux

$$[\varkappa(\mathbf{T}^t)^{-1}\nabla_x \vartheta \cdot \mathbf{n}] = 0, \quad \mathbf{x}_0 \in \Gamma, \quad t \geq 0. \tag{2.19}$$

In (2.18)–(2.19) $\mathbf{n}(\mathbf{x}_0)$ is the unit normal to Γ at a point $\mathbf{x}_0 \in \Gamma$. We suppose that $\mathbf{n}$ is pointing into Ω_f.

The conditions of continuity of temperature and displacement on Γ reflect the local thermodynamical equilibrium and no-slip effect on the interface, respectively. Besides, the no-slip condition (2.17) includes the mass conservation law, applied to Γ, and manifests that Γ is a contact discontinuity, which means that the solid and the fluid do not exchange particles. The conditions of continuity of normal stress and normal heat flux are the respective momentum and energy conservation laws on Γ.

Equations (2.1)–(2.8) and (2.16)–(2.19) constitute the closed nonlinear model of joint motion of thermoelastic solid and viscous thermofluid. By the standard procedure [3, Sec 8.1], thermodynamical parameters s, p, and U may be expressed in terms of ρ and ϑ from (2.5) and (2.6) by virtue of the free energy $\mathcal{F}$:

$$s = -\mathcal{F}'_\vartheta(\rho,\vartheta), \quad p = \rho^2\mathcal{F}'_\rho(\rho,\vartheta), \quad U = -\vartheta\mathcal{F}'_\vartheta(\rho,\vartheta) + \mathcal{F}(\rho,\vartheta). \tag{2.20}$$

Insert (2.20) into (2.3) and (2.8), and discard (2.5) and (2.6) to get the equivalent system of equations and interface conditions, which we name *Model* O_1.

Now simplify *Model* O_1, applying the classical formal procedure of linearization [3, Sec. V.7], [16, Sec. 2.2–2.3] to it. Assume that ρ_s and ρ_f are mean constant densities of the solid and the liquid at rest. Suppose that the temperature in this rest state is equal to the constant temperature ϑ_*, which was introduced in (2.7). Denote

$$\begin{gathered}
c_s^* = \mathcal{F}_s(\rho_s,\vartheta_*), \quad c_{s\rho} = \mathcal{F}'_{s\rho}(\rho_s,\vartheta_*), \quad c_{s\vartheta} = \mathcal{F}'_{s\vartheta}(\rho_s,\vartheta_*), \\
c_{s\rho\rho} = \mathcal{F}''_{s\rho\rho}(\rho_s,\vartheta_*), \quad c_{s\rho\vartheta} = \mathcal{F}''_{s\rho\vartheta}(\rho_s,\vartheta_*), \quad c_{s\vartheta\vartheta} = \mathcal{F}''_{s\vartheta\vartheta}(\rho_s,\vartheta_*), \\
c_f^* = \mathcal{F}_f(\rho_f,\vartheta_*), \quad c_{f\rho} = \mathcal{F}'_{f\rho}(\rho_f,\vartheta_*), \quad c_{f\vartheta} = \mathcal{F}'_{f\vartheta}(\rho_f,\vartheta_*), \\
c_{f\rho\rho} = \mathcal{F}''_{f\rho\rho}(\rho_f,\vartheta_*), \quad c_{f\rho\vartheta} = \mathcal{F}''_{f\rho\vartheta}(\rho_f,\vartheta_*), \quad c_{f\vartheta\vartheta} = \mathcal{F}''_{f\vartheta\vartheta}(\rho_f,\vartheta_*).
\end{gathered} \tag{2.21}$$

Expand (2.20) for p and U in Taylor's series and skip terms of orders higher than one to get the linearized expressions

$$p(\rho,\vartheta) = c_{f\rho}\rho_f^2 + (2c_{f\rho}\rho_f + c_{f\rho\rho}\rho_f^2)(\rho - \rho_f) + c_{f\rho\vartheta}\rho_f^2(\vartheta - \vartheta_*), \tag{2.22}$$

$$U_s(\rho,\vartheta) = -c_{s\vartheta}\vartheta_* + c_s^* + (c_{s\rho} - c_{s\rho\vartheta}\vartheta_*)(\rho - \rho_s) - c_{s\vartheta\vartheta}\vartheta_*(\vartheta - \vartheta_*), \tag{2.23}$$

$$U_f(\rho,\vartheta) = -c_{f\vartheta}\vartheta_* + c_f^* + (c_{f\rho} - c_{f\rho\vartheta}\vartheta_*)(\rho - \rho_f) - c_{f\vartheta\vartheta}\vartheta_*(\vartheta - \vartheta_*). \tag{2.24}$$

Remark 2.2. In (2.22) and (2.24) notice that $p(\rho_f,\vartheta_*) = p_0$ due to interface condition (2.18). Also notice that from the physical point of view it is observed for the most of continuous media that $p(\rho,\vartheta)$ is increasing in ρ and ϑ and that $U_f(\rho,\vartheta)$ and $U_s(\rho,\vartheta)$ are increasing in ϑ and decreasing in ρ. Thus $c_{f\rho\vartheta} > 0$, $2c_{f\rho} + c_{f\rho\rho}\rho_f > 0$, $c_{s\vartheta\vartheta} < 0$, $c_{s\rho} - c_{s\rho\vartheta}\vartheta_* < 0$, $c_{f\vartheta\vartheta} < 0$, and $c_{f\rho} - c_{f\rho\vartheta}\vartheta < 0$.

Substituting (2.22)–(2.24) into *Model* O_1, expanding the terms of thus obtained equations in Taylor's series in ρ, ϑ, $\mathbf{w}$, and derivatives of $\mathbf{w}$ with respect to $\mathbf{x}$ and t about the rest state, and discarding the terms of orders higher than one, we arrive at the following linearized dynamical model of the system *thermoelastic body – compressible viscous thermofluid*, which we call further *Model* O_2.

Statement of Model $\boldsymbol{O_2}$**.** The motion in the solid phase is governed by the classical equations of the linear thermoelasticity theory:

$$\rho_s \frac{\partial^2 \mathbf{w}}{\partial t^2} = \operatorname*{div}_x \mathbf{P}_s + \rho_s\mathbf{F}, \quad \mathbf{x} \in \Omega_s,\ t > 0, \tag{2.25}$$

$$-\rho_s c_{s\vartheta\vartheta}\vartheta_* \frac{\partial \vartheta}{\partial t} = \varkappa_s \Delta_x \vartheta - \vartheta_* \gamma_s \eta \frac{\partial}{\partial t} \operatorname*{div}_x \mathbf{w} + \Psi_s, \quad \mathbf{x} \in \Omega_s, \ t > 0, \tag{2.26}$$

$$\mathbf{P}_s = \Big(-p_0 - \gamma_s \eta(\vartheta - \vartheta_*) + \Big(\eta - \frac{2}{3}\lambda\Big) \operatorname*{div}_x \mathbf{w}\Big)\mathbf{I} + 2\lambda \mathbf{D}(x, \mathbf{w}), \quad \mathbf{x} \in \Omega_s, \ t > 0, \tag{2.27}$$

(2.28)

$$\rho = \rho_s(1 - \operatorname*{div}_x \mathbf{w}), \quad \mathbf{x} \in \Omega_s, \ t > 0. \tag{2.29}$$

The motion of the liquid phase is described by the linearized classical model of liquid and gases:

$$\rho_f \frac{\partial^2 \mathbf{w}}{\partial t^2} = \operatorname*{div}_x \mathbf{P}_f + \rho_f \mathbf{F}, \quad \mathbf{x} \in \Omega_f, \ t > 0, \tag{2.30}$$

$$-\rho_f c_{f\vartheta\vartheta}\vartheta_* \frac{\partial \vartheta}{\partial t} = \varkappa_f \Delta_x \vartheta - \vartheta_* c_{f\rho\vartheta} \rho_f^2 \frac{\partial}{\partial t} \operatorname*{div}_x \mathbf{w} + \Psi_f, \quad \mathbf{x} \in \Omega_f, \ t > 0, \tag{2.31}$$

$$\begin{aligned} \mathbf{P}_f = \Big(&-p_0 - c_{f\rho\vartheta}\rho_f^2(\vartheta - \vartheta_*) + (2c_{f\rho} + c_{f\rho\rho}\rho_f)\rho_f^2 \operatorname*{div}_x \mathbf{w} \\ &+ \big(\nu - \frac{2}{3}\mu\big) \operatorname*{div}_x \frac{\partial \mathbf{w}}{\partial t}\Big)\mathbf{I} + 2\mu \mathbf{D}\Big(x, \frac{\partial \mathbf{w}}{\partial t}\Big), \quad \mathbf{x} \in \Omega_f, \ t > 0, \end{aligned} \tag{2.32}$$

$$\rho = \rho_f(1 - \operatorname*{div}_x \mathbf{w}), \quad \mathbf{x} \in \Omega_f, \ t > 0. \tag{2.33}$$

The linearized conditions on the interface between the solid and the fluid are

$$[\vartheta] = 0, \quad \mathbf{x}_0 \in \Gamma, \ t \geq 0, \tag{2.34}$$

$$[\mathbf{w}] = 0, \quad \mathbf{x}_0 \in \Gamma, \ t \geq 0, \tag{2.35}$$

$$\big(\mathbf{P}_s(\vartheta_{(s)}, \mathbf{w}_{(s)}) - \mathbf{P}_f(\vartheta_{(f)}, \mathbf{w}_{(f)})\big)\mathbf{n} = 0, \quad \mathbf{x}_0 \in \Gamma, \ t \geq 0, \tag{2.36}$$

$$\big(\varkappa_s \nabla_x \vartheta_{(s)} - \varkappa_f \nabla_x \vartheta_{(f)}\big) \cdot \mathbf{n} = 0, \quad \mathbf{x}_0 \in \Gamma, \ t \geq 0. \tag{2.37}$$

Remark 2.3. Transforming *Model* O_1 to Euler variables $(\mathbf{r}, t)$ and applying the same linearization formalism as above, one arrives exactly at *Model* O_2. Therefore in linearized setting the Lagrange and Euler descriptions of the thermomechanical system under consideration coincide.

3. Problem Formulation and Statement of the Existence and Uniqueness Theorem

In line with later asymptotic analysis in the present and forthcoming papers, we bring *Model* O_2 to a dimensionless form and absorb the interface conditions on Γ in the equations by introducing a uniform description of the both phases.

More precisely, choose the diameter L_0 of the domain Ω, a characteristic duration of physical processes τ_0, acceleration of gravity g, atmosphere pressure p_0, mean density of air ρ_0 at the temperature 273 K under atmosphere pressure, and the temperature difference ϑ_0 between the thawing and freezing points of water under atmosphere pressure as characteristic scales of length, time, density of mass distributed forces, pressure, density of matter, and temperature, respectively, and denote $\theta = \vartheta - \vartheta_*$.

Next, introduce the dimensionless variables (with primes) by the formulas

$$\begin{aligned} \mathbf{x} = L_0 \mathbf{x}', \quad t = \tau_0 t', \quad \mathbf{w} = L_0 \mathbf{w}', \\ p = p_0 p', \quad \rho = \rho_0 \rho', \quad \theta = \vartheta_0 \theta', \end{aligned} \tag{3.1}$$

the dimensionless vector of distributed mass forces, volumetric densities of exterior heat application, thermal conductivity coefficients, and mean densities of the solid and the fluid at rest (all with primes), respectively, by the formulas

$$\mathbf{F} = g\mathbf{F}', \quad \Psi_s = \frac{p_0\vartheta_*}{\tau_0\vartheta_0}\Psi'_s, \quad \Psi_f = \frac{p_0\vartheta_*}{\tau_0\vartheta_0}\Psi'_f, \tag{3.2}$$

$$\varkappa_s = \frac{L_0^2 p_0\vartheta_*}{\tau_0\vartheta_0^2}\varkappa'_s, \quad \varkappa_f = \frac{L_0^2 p_0\vartheta_*}{\tau_0\vartheta_0^2}\varkappa'_f, \quad \rho_s = \rho_0\rho'_s, \quad \rho_f = \rho_0\rho'_f, \tag{3.3}$$

and the dimensionless ratios by the formulas

$$\begin{gathered}
\alpha_\tau = \frac{\gamma_0 L_0^2}{c_0^2\tau_0^2}, \quad \alpha_F = \frac{\gamma_0 g L_0}{c_0^2}, \quad \alpha_\nu = \frac{1}{\tau_0 p_0}\big(\nu - \frac{2}{3}\mu\big),\\
\alpha_\eta = \frac{1}{p_0}\Big(\eta - \frac{2}{3}\lambda\Big), \quad \alpha_\lambda = \frac{2\lambda}{p_0}, \quad \alpha_p = \frac{\gamma_0 c^2}{c_0^2}\rho'_f,\\
c_0^2 = \frac{\gamma_0 p_0}{\rho_0}, \quad \alpha_{\theta s} = \frac{\gamma_s \eta\vartheta_0}{p_0}, \quad \alpha_{\theta f} = \frac{c_{f\rho\vartheta}\rho_f^2\vartheta_0}{p_0},\\
\alpha_\mu = \frac{2\mu}{\tau_0 p_0}, \quad c_{pf} = -\frac{c_{f\vartheta\vartheta}\rho_f\vartheta_0^2}{p_0}, \quad c_{ps} = -\frac{c_{s\vartheta\vartheta}\rho_s\vartheta_0^2}{p_0},
\end{gathered} \tag{3.4}$$

where $c = \sqrt{2c_{f\rho}\rho_f + c_{f\rho\rho}\rho_f^2}$ and $\gamma_0 = 7/5$. Quantity c_0 is the speed of sound in air at the temperature 273 K under atmosphere pressure. Quantity c is the speed of sound in the considered fluid at the temperature ϑ_* under atmosphere pressure. Dimensionless constant γ_0 is the ratio of specific heats (in other terms, the polytropic exponent) for air at the temperature 273 K under atmosphere pressure [11, Appendix 4], [16, Sec. 2.4].

Remark 3.1. On the strength of (2.9) and Remark 2.2, all dimensionless constants on the left-hand sides of relations (3.4) are positive.

Now, first shift the pressure scale on the constant value p_0 so that the stress tensors (2.28) and (2.32) become vanishing at the rest state. Next, multiply (2.25) and (2.30) by L/p_0, (2.26) and (2.31) by $\tau_0\vartheta_0/(p_0\vartheta_*)$, and divide (2.28) and (2.32) by p_0. After this, substitute expressions (3.1)–(3.4) into the resulting equations and then omit primes. Thus, *Model* O_2 is brought to a dimensionless form. Finally, use the notation (2.10)–(2.14) and additionally set

$$\begin{gathered}
\bar\rho = \bar\chi\rho_f + (1-\bar\chi)\rho_s, \quad \bar\alpha_\theta = \bar\chi\alpha_{\theta f} + (1-\bar\chi)\alpha_{\theta s},\\
\bar c_p = \bar\chi c_{pf} + (1-\bar\chi)c_{ps}, \quad \bar\varkappa = \bar\chi\varkappa_f + (1-\bar\chi)\varkappa_s
\end{gathered} \tag{3.5}$$

and introduce the dimensionless pressures p, q, and π in order to wrap the dimensionless *Model* O_2 into the following form.

Statement of Model A. In the space-time cylinder $Q = \Omega \times (0,T)$, where $\Omega = (0,1)^3$ and $T = \text{const} > 0$, it is necessary to find a displacement vector $\mathbf{w}$, a temperature distribution θ, and distributions of pressures p, q, and π, which satisfy the equations

$$\alpha_\tau\bar\rho\frac{\partial^2\mathbf{w}}{\partial t^2} = \operatorname*{div}_x \mathbf{P} + \alpha_F\bar\rho\mathbf{F}, \tag{3.6}$$

$$\mathbf{P} = \bar\chi\Big(-q\mathbf{I} + \alpha_\mu\mathbf{D}\Big(x, \frac{\partial\mathbf{w}}{\partial t}\Big)\Big) + (1-\bar\chi)\big(-\pi\mathbf{I} + \alpha_\lambda\mathbf{D}(x,\mathbf{w})\big) - \bar\alpha_\theta\theta\mathbf{I}, \tag{3.7}$$

$$\bar{c}_p\frac{\partial\theta}{\partial t}=\operatorname*{div}_x(\bar{\varkappa}\nabla_x\theta)-\bar{\alpha}_\theta\frac{\partial}{\partial t}\operatorname*{div}_x\mathbf{w}+\Psi, \tag{3.8}$$

$$p+\bar{\chi}\alpha_p\operatorname*{div}_x\mathbf{w}=0, \tag{3.9}$$

$$q=p+\frac{\alpha_\nu}{\alpha_p}\frac{\partial p}{\partial t}, \tag{3.10}$$

$$\pi+(1-\bar{\chi})\alpha_\eta\operatorname*{div}_x\mathbf{w}=0. \tag{3.11}$$

We endow *Model A* with initial data

$$\mathbf{w}|_{t=0}=\mathbf{w}_0,\quad \mathbf{w}_t|_{t=0}=\mathbf{v}_0,\quad \theta|_{t=0}=\theta_0,\quad \mathbf{x}\in\Omega \tag{3.12}$$

and homogeneous boundary conditions

$$\mathbf{w}=0,\quad \theta=0,\quad \mathbf{x}\in\partial\Omega,\quad t\geq 0. \tag{3.13}$$

Remark 3.2. On the strength of classical theory of conservation laws of continuum mechanics [3, Sec. II.3], (3.6)–(3.11) yield interface conditions (2.34)–(2.37) (in the dimensionless form). Thus (3.6)–(3.11) are equivalent to *Model O_2*.

Generalized solutions of *Model A* are understood in the following sense.

Definition 3.3. *Five functions $(\mathbf{w},\theta,p,q,\pi)$ are called a generalized solution of* Model A *if they satisfy the regularity conditions*

$$\mathbf{w},\ \frac{\partial\mathbf{w}}{\partial t},\ \mathbf{D}(x,\mathbf{w}),\ \bar{\chi}\mathbf{D}\Big(x,\frac{\partial\mathbf{w}}{\partial t}\Big),\ \operatorname*{div}_x\mathbf{w},\ \theta,\ \nabla_x\theta\in L^2(Q), \tag{3.14}$$

boundary conditions (3.13) *in the trace sense, equations* (3.9)–(3.11) *a.e. in Q, and the integral equalities*

$$\int_Q\Big(\alpha_\tau\bar{\rho}\frac{\partial\mathbf{w}}{\partial t}\cdot\frac{\partial\varphi}{\partial t}-\mathbf{P}:\nabla_x\varphi+\alpha_F\bar{\rho}\mathbf{F}\cdot\varphi\Big)d\mathbf{x}dt+\int_\Omega\alpha_\tau\bar{\rho}\mathbf{v}_0\cdot\varphi|_{t=0}d\mathbf{x}=0 \tag{3.15}$$

for all smooth $\varphi=\varphi(\mathbf{x},t)$ such that $\varphi|_{\partial\Omega}=\varphi|_{t=T}=0$ and

$$\begin{aligned}&\int_Q\Big(\bar{c}_p\theta\frac{\partial\psi}{\partial t}-\bar{\varkappa}\nabla_x\theta\cdot\nabla_x\psi+\bar{\alpha}_\theta(\operatorname*{div}_x\mathbf{w})\frac{\partial\psi}{\partial t}+\Psi\psi\Big)d\mathbf{x}dt\\&+\int_\Omega\big(\bar{c}_p\theta_0+\bar{\alpha}_\theta\operatorname*{div}_x\mathbf{w}_0\big)\psi|_{t=0}d\mathbf{x}=0\end{aligned} \tag{3.16}$$

for all smooth $\psi=\psi(\mathbf{x},t)$ such that $\psi|_{\partial\Omega}=\psi|_{t=T}=0$.

The first main result of the paper is the following theorem on existence and uniqueness of solutions to *Model A*.

Theorem 3.4. *Whenever $\mathbf{w}_0\in\overset{\circ}{W}{}^1_2(\Omega)$, $\mathbf{v}_0,\theta_0\in L^2(\Omega)$, and $\mathbf{F},\Psi\in L^2(Q)$,* Model A *has a unique generalized solution $(\mathbf{w},\ \theta,\ p,\ q,\ \pi)$ in the sense of Definition 3.3.*

4. The Energy Estimate and Uniqueness of Solution of Model A

Construction of the energy estimate is based on introducing the alternative equivalent definition of generalized solutions of *Model A* and on a special choice of test functions in the integral equalities in this definition. Namely, we state:

Definition 4.1. Five functions $(\mathbf{w},\theta,p,q,\pi)$ are called a generalized solution of *Model A* if they satisfy the regularity conditions

$$\mathbf{w},\ \frac{\partial\mathbf{w}}{\partial t},\ \mathbf{D}(x,\mathbf{w}),\ \bar{\chi}\mathbf{D}\Big(x,\frac{\partial\mathbf{w}}{\partial t}\Big),\ \operatorname*{div}_x\mathbf{w},\ \theta,\ \nabla_x\theta\in L^2(Q), \tag{4.1}$$

$$\text{functions } t \mapsto \tfrac{\partial \mathbf{w}}{\partial t}(t),\ t \mapsto \bar{c}_p \theta(t) + \bar{\alpha}_\theta \operatorname{div}_x \mathbf{w}(t) \text{ are weakly continuous mappings of the interval } [0,T] \text{ into } L^2(\Omega), \tag{4.2}$$

boundary conditions (3.13) in the trace sense, equations (3.9)–(3.11) a.e. in Q, and the integral equalities

$$\begin{aligned} &\int_0^\tau \int_\Omega \Big(\alpha_\tau \bar{\rho} \frac{\partial \mathbf{w}}{\partial t} \cdot \frac{\partial \varphi}{\partial t} - \mathbf{P} : \nabla_x \varphi + \alpha_F \bar{\rho} \mathbf{F} \cdot \varphi \Big) d\mathbf{x} dt \\ &= \int_\Omega \alpha_\tau \bar{\rho} \frac{\partial \mathbf{w}}{\partial t}(\mathbf{x}, \tau) \cdot \varphi(\mathbf{x}, \tau) d\mathbf{x} - \int_\Omega \alpha_\tau \bar{\rho} \mathbf{v}_0 \cdot \varphi|_{t=0} d\mathbf{x}, \quad \forall \tau \in [0,T] \end{aligned} \tag{4.3}$$

for all smooth $\varphi = \varphi(\mathbf{x}, t)$ such that $\varphi|_{\partial\Omega} = 0$ and

$$\begin{aligned} &\int_0^\tau \int_\Omega \Big[\Big(\bar{c}_p \theta + \bar{\alpha}_\theta \operatorname{div}_x \mathbf{w} \Big) \frac{\partial \psi}{\partial t} - \varkappa \nabla_x \theta \cdot \nabla_x \psi + \Psi \psi \Big] d\mathbf{x} dt \\ &= \int_\Omega \big(\bar{c}_p \theta(\mathbf{x}, \tau) + \bar{\alpha}_\theta \operatorname{div}_x \mathbf{w}(\mathbf{x}, \tau) \big) \psi(\mathbf{x}, \tau) d\mathbf{x} - \int_\Omega \big(\bar{c}_p \theta_0 + \bar{\alpha}_\theta \operatorname{div}_x \mathbf{w}_0 \big) \psi|_{t=0} d\mathbf{x}, \end{aligned} \tag{4.4}$$

for all $\tau \in [0,T]$ and all smooth $\psi = \psi(\mathbf{x}, t)$ such that $\psi|_{\partial\Omega} = 0$.

Remark 4.2. Definitions 3.3 and 4.1 are equivalent. The fact that a generalized solution in the sense of Definition 4.1 is a generalized solution in the sense of Definition 3.3 is quite obvious. The inverse proposition is true thanks to the simple standard considerations. Its justification can be fulfilled similarly to, for example, [1, Sec. III.1].

Now let us follow the track of considerations of [10, Chap. 2, Sec. 5.2].

Fix arbitrary $\tau_*, \tau_{**} \in (0, \tau)$, $\tau_* < \tau_{**}$. Take a continuous piece-wise linear function on $[0, \tau]$ such that $\phi_m(t) = 1$ if $\tau_* + (2/m) < t < \tau_{**} - (2/m)$ and $\phi_m(t) = 0$ if $t > \tau_{**} - (1/m)$ and $t < \tau_* + (1/m)$. Take a regularizing sequence $\omega_n \in C_0^\infty(\mathbf{R})$ such that

$$\omega_n(t) = \omega_n(-t), \quad \omega_n(t) \geq 0, \quad \int_{-\infty}^{\infty} \omega_n(t) dt = 1, \quad \operatorname{supp} \omega_n \subset \Big[-\frac{1}{n}, \frac{1}{n} \Big].$$

For $n > 2m$, set

$$\varphi_{mn} = \Big(\Big(\phi_m \frac{\partial \mathbf{w}}{\partial t} \Big) * \omega_n * \omega_n \Big) \phi_m, \quad \psi_{mn} = ((\phi_m \theta) * \omega_n * \omega_n) \phi_m, \tag{4.5}$$

where the asterisk$*$ means the integral convolution in $\mathbf{R}$, and substitute for φ and ψ into (4.3) and (4.4), respectively. Clearly, this choice of test functions is valid due to regularity properties (4.1) and (4.2).

Insert (3.9)–(3.11) into (4.3) (with $\varphi = \varphi_{mn}$) and then sum the result with (4.4) (with $\psi = \psi_{mn}$). In thus obtained equality represent

$$\begin{aligned} &\int_0^\tau \int_\Omega \bar{\alpha}_\theta \theta \operatorname{div}_x \varphi_{mn} d\mathbf{x} dt \\ &= \int_0^\tau \int_\Omega \bar{\alpha}_\theta \big((\phi_m \theta) * \omega_n \big) \frac{\partial}{\partial t} \operatorname{div}_x \big((\phi_m \mathbf{w}) * \omega_n \big) d\mathbf{x} dt \\ &\quad - \int_0^\tau \int_\Omega \bar{\alpha}_\theta \big((\phi_m \theta) * \omega_n \big) \operatorname{div}_x \Big(\Big(\frac{\partial \phi_m}{\partial t} \mathbf{w} \Big) * \omega_n \Big) d\mathbf{x} dt \end{aligned} \tag{4.6}$$

and

$$\int_0^\tau \int_\Omega \bar{\alpha}_\theta \operatorname{div}_x \mathbf{w} \frac{\partial \psi_{mn}}{\partial t} d\mathbf{x} dt = \int_0^\tau \int_\Omega \bar{\alpha}_\theta \operatorname{div}_x \mathbf{w} \big((\phi_m \theta) * \omega_n * \omega_n \big) \frac{\partial \phi_m}{\partial t} d\mathbf{x} dt$$

$$+\int_0^\tau \int_\Omega \bar{\alpha}_\theta \operatorname*{div}_x \big((\phi_m \mathbf{w}) * \omega_n\big) \frac{\partial}{\partial t}\big((\phi_m \theta) * \omega_n\big) d\mathbf{x}dt.$$

Then we integrate by parts with respect to t in the last summand of the above expression and combine similar terms.

Applying the arguments of [10, Chap. 2, Sec. 5.2], after some technical transformations and passage to the limit as $n \nearrow \infty$, $m \nearrow \infty$, $\tau_* \searrow 0$, and $\tau_{**} \nearrow \tau$, successively, we finally arrive at the *energy identity*, as follows:

$$\begin{aligned}
&\frac{1}{2}\alpha_\tau \Big\|\sqrt{\bar{\rho}}\frac{\partial \mathbf{w}}{\partial t}(\tau)\Big\|^2_{2,\Omega} + \frac{1}{2}\alpha_\eta \|(1-\bar{\chi}) \operatorname*{div}_x \mathbf{w}(\tau)\|^2_{2,\Omega} + \frac{1}{2}\alpha_p \|\bar{\chi} \operatorname*{div}_x \mathbf{w}(\tau)\|^2_{2,\Omega} \\
&+ \frac{1}{2}\alpha_\lambda \|(1-\bar{\chi})\mathbf{D}(x, \mathbf{w}(\tau))\|^2_{2,\Omega} + \frac{1}{2}\|\sqrt{\bar{c}_p}\theta(\tau)\|^2_{2,\Omega} \\
&+ \alpha_\nu \Big\|\bar{\chi} \operatorname*{div}_x \frac{\partial \mathbf{w}}{\partial t}\Big\|^2_{2,\Omega\times(0,\tau)} + \alpha_\mu \Big\|\bar{\chi}\mathbf{D}\Big(x, \frac{\partial \mathbf{w}}{\partial t}\Big)\Big\|^2_{2,\Omega\times(0,\tau)} + \|\sqrt{\bar{\varkappa}}\nabla_x \theta\|^2_{2,\Omega\times(0,\tau)} \\
&= \frac{1}{2}\alpha_\tau \|\sqrt{\bar{\rho}}\mathbf{v}_0\|^2_{2,\Omega} + \frac{1}{2}\alpha_\eta \|(1-\bar{\chi}) \operatorname*{div}_x \mathbf{w}_0\|^2_{2,\Omega} + \frac{1}{2}\alpha_p \|\bar{\chi} \operatorname*{div}_x \mathbf{w}_0\|^2_{2,\Omega} \\
&\quad + \frac{1}{2}\alpha_\lambda \|(1-\bar{\chi})\mathbf{D}(x, \mathbf{w}_0)\|^2_{2,\Omega} + \frac{1}{2}\|\sqrt{\bar{c}_p}\theta_0\|^2_{2,\Omega} \\
&\quad - \int_0^\tau \int_\Omega \Big(\alpha_F \bar{\rho}\mathbf{F} \cdot \frac{\partial \mathbf{w}}{\partial t} + \Psi\theta\Big) d\mathbf{x}dt, \quad \forall\, \tau \in [0, T].
\end{aligned} \tag{4.7}$$

Discarding all the terms on the left-hand side except for the first and the fifth ones and applying the Cauchy–Schwartz inequality on the right-hand side, we get

$$\begin{aligned}
&\frac{1}{2}\alpha_\tau \Big\|\sqrt{\bar{\rho}}\frac{\partial \mathbf{w}}{\partial t}(\tau)\Big\|^2_{2,\Omega} + \frac{1}{2}\|\sqrt{\bar{c}_p}\theta(\tau)\|^2_{2,\Omega} \\
&\le \frac{1}{2}\alpha_\tau \|\sqrt{\bar{\rho}}\mathbf{v}_0\|^2_{2,\Omega} + \frac{1}{2}\|\sqrt{\bar{c}_p}\theta_0\|^2_{2,\Omega} + \int_0^\tau \frac{1}{2}\alpha_\tau \Big\|\sqrt{\bar{\rho}}\frac{\partial \mathbf{w}}{\partial t}(t)\Big\|^2_{2,\Omega} dt \\
&\quad + \int_0^\tau \frac{1}{2}\|\sqrt{\bar{c}_p}\theta(t)\|^2_{2,\Omega} dt + \frac{\alpha_F^2}{2\alpha_\tau}\|\sqrt{\bar{\rho}}\mathbf{F}\|^2_{2,\Omega\times(0,\tau)} + \frac{1}{2\min\{c_{pf}, c_{ps}\}}\|\Psi\|^2_{2,\Omega\times(0,\tau)} \\
&\quad + \frac{1}{2}\alpha_\eta \|(1-\bar{\chi}) \operatorname*{div}_x \mathbf{w}_0\|^2_{2,\Omega} + \frac{1}{2}\alpha_p \|\bar{\chi} \operatorname*{div}_x \mathbf{w}_0\|^2_{2,\Omega} \\
&\quad + \frac{1}{2}\alpha_\lambda \|(1-\bar{\chi})\mathbf{D}(x, \mathbf{w}_0)\|^2_{2,\Omega}, \quad \forall\, \tau \in [0, T].
\end{aligned} \tag{4.8}$$

Applying Grownwall's lemma to this inequality, we conclude that

$$\begin{aligned}
&\frac{1}{2}\alpha_\tau \Big\|\sqrt{\bar{\rho}}\frac{\partial \mathbf{w}}{\partial t}\Big\|^2_{2,\Omega\times(0,\tau)} + \frac{1}{2}\|\sqrt{\bar{c}_p}\theta\|^2_{2,\Omega\times(0,\tau)} \\
&\le \int_0^\tau \Big[\frac{\alpha_F^2}{2\alpha_\tau}\|\sqrt{\bar{\rho}}\mathbf{F}\|^2_{2,\Omega\times(0,t)} + \frac{1}{2\min\{c_{pf}, c_{ps}\}}\|\Psi\|^2_{2,\Omega\times(0,t)}\Big] e^{\tau - t} dt \\
&\quad + e^\tau \Big[\frac{1}{2}\alpha_\tau \|\sqrt{\bar{\rho}}\mathbf{v}_0\|^2_{2,\Omega} + \frac{1}{2}\|\sqrt{\bar{c}_p}\theta_0\|^2_{2,\Omega}\Big] \\
&\quad + (e^\tau - 1)\Big[\frac{1}{2}\alpha_\eta \|(1-\bar{\chi}) \operatorname*{div}_x \mathbf{w}_0\|^2_{2,\Omega} + \frac{1}{2}\alpha_p \|\bar{\chi} \operatorname*{div}_x \mathbf{w}_0\|^2_{2,\Omega} \\
&\quad + \frac{1}{2}\alpha_\lambda \|(1-\bar{\chi})\mathbf{D}(x, \mathbf{w}_0)\|^2_{2,\Omega}\Big], \quad \forall\, \tau \in [0, T].
\end{aligned}$$

Combining energy identity (4.7) with the above estimate and (4.8), we finally establish the following result.

Proposition 4.3. *Let* $\mathbf{F},\Psi\in L^2(Q)$, $\mathbf{w}_0\in \overset{\circ}{W}{}^1_2(\Omega)$, *and* $\mathbf{v}_0,\theta_0\in L^2(\Omega)$. *Assume that five functions* $(\mathbf{w},\theta,p,q,\pi)$ *are a generalized solution of Model A. Then* $\mathbf{w}$ *and* θ *satisfy the energy estimate*

$$
\begin{aligned}
&\frac{1}{2}\operatorname*{ess\,sup}_{t\in[0,\tau]}\alpha_\tau\Big\|\sqrt{\bar\rho}\frac{\partial\mathbf{w}}{\partial t}(t)\Big\|^2_{2,\Omega}+\frac{1}{2}\operatorname{ess\,sup}t\in[0,\tau]\|\sqrt{\bar c_p}\theta(t)\|^2_{2,\Omega}\\
&\quad+\frac{1}{2}\operatorname*{ess\,sup}_{t\in[0,\tau]}\alpha_\eta\|(1-\bar\chi)\operatorname*{div}_x\mathbf{w}(t)\|^2_{2,\Omega}+\frac{1}{2}\operatorname*{ess\,sup}_{t\in[0,\tau]}\alpha_p\|\bar\chi\operatorname*{div}_x\mathbf{w}(t)\|^2_{2,\Omega}\\
&\quad+\frac{1}{2}\operatorname*{ess\,sup}_{t\in[0,\tau]}\alpha_\lambda\|(1-\bar\chi)\mathbf{D}(x,\mathbf{w}(t))\|^2_{2,\Omega}+\alpha_\nu\Big\|\bar\chi\operatorname*{div}_x\frac{\partial\mathbf{w}}{\partial t}\Big\|^2_{2,\Omega\times(0,\tau)}\\
&\quad+\alpha_\mu\Big\|\bar\chi\mathbf{D}\Big(x,\frac{\partial\mathbf{w}}{\partial t}\Big)\Big\|^2_{2,\Omega\times(0,\tau)}+\|\sqrt{\bar\varkappa}\nabla_x\theta\|^2_{2,\Omega\times(0,\tau)}\\
&\le\frac{\alpha_F^2}{2\alpha_\tau}\|\sqrt{\bar\rho}\mathbf{F}\|^2_{2,\Omega\times(0,\tau)}+\frac{1}{2\min\{c_{pf},c_{ps}\}}\|\Psi\|^2_{2,\Omega\times(0,\tau)}\\
&\quad+\int_0^\tau\Big[\frac{\alpha_F^2}{2\alpha_\tau}\|\sqrt{\bar\rho}\mathbf{F}\|^2_{2,\Omega\times(0,t)}+\frac{1}{2\min\{c_{pf},c_{ps}\}}\|\Psi\|^2_{2,\Omega\times(0,t)}\Big]e^{\tau-t}dt\\
&\quad+\frac{e^\tau+1}{2}\Big[\alpha_\tau\|\sqrt{\bar\rho}\mathbf{v}_0\|^2_{2,\Omega}+\|\sqrt{\bar c_p}\theta_0\|^2_{2,\Omega}\Big]\\
&\quad+\frac{e^\tau}{2}\Big[\alpha_\eta\|(1-\bar\chi)\operatorname*{div}_x\mathbf{w}_0\|^2_{2,\Omega}+\alpha_p\|\bar\chi\operatorname*{div}_x\mathbf{w}_0\|^2_{2,\Omega}\\
&\quad+\alpha_\lambda\|(1-\bar\chi)\mathbf{D}(x,\mathbf{w}_0)\|^2_{2,\Omega}\Big]:=C_{\mathrm{en}}(\tau),\quad\forall\,\tau\in[0,T].
\end{aligned}\tag{4.9}
$$

Corollary 4.4. *The displacement* $\mathbf{w}$ *belongs to the space* $L^\infty(0,T;\overset{\circ}{W}{}^1_2(\Omega))$ *and admits the bound*

$$
\operatorname*{ess\,sup}_{t\in[0,T]}\|\mathbf{w}(t)\|_{W^1_2(\Omega)}\le C\big(C_{\mathrm{en}}(T),C_k(\Omega)\big),\tag{4.10}
$$

where $C_k(\Omega)$ *depends only on geometry of* $\partial\Omega$.

Proof. On the strength of the Newton-Leibnitz formula,

$$
\begin{aligned}
&\int_\Omega\bar\chi(\mathbf{x})\mathbf{D}(x,\mathbf{w}(\tau)):\mathbb{G}(\mathbf{x})d\mathbf{x}\\
&=\int_0^\tau\int_\Omega\bar\chi(\mathbf{x})\mathbf{D}\Big(x,\frac{\partial\mathbf{w}}{\partial t}(t)\Big):\mathbb{G}(\mathbf{x})d\mathbf{x}dt\\
&\quad+\int_\Omega\bar\chi(\mathbf{x})\mathbf{D}(x,\mathbf{w}_0):\mathbb{G}(\mathbf{x})d\mathbf{x}\quad\forall\,\mathbb{G}\in(L^2(\Omega))^{3\times3},\quad\forall\,\tau\in[0,T].
\end{aligned}\tag{4.11}
$$

Substitute $\mathbb{G}(\mathbf{x})=\mathbf{D}(x,\mathbf{w}(\tau))$ and use inequality (4.8) and the Cauchy-Schwartz inequality with some positive ε_0 and ε_1 to get

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{t\in[0,\tau]}\int_\Omega\bar\chi|\mathbf{D}(x,\mathbf{w}(t))\|^2d\mathbf{x}\\
&\le\int_0^\tau\int_\Omega\bar\chi\Big[\frac{1}{2\varepsilon_0}\Big|\mathbf{D}\Big(x,\frac{\partial\mathbf{w}}{\partial t}(t)\Big)\Big|^2+\frac{\varepsilon_0}{2}|\mathbf{D}(x,\mathbf{w}(\tau))|^2\Big]d\mathbf{x}dt\\
&\times\int_\Omega\bar\chi\Big[\frac{1}{2\varepsilon_1}|\mathbf{D}(x,\mathbf{w}_0)|^2+\frac{\varepsilon_1}{2}|\mathbf{D}(x,\mathbf{w}(\tau))|^2\Big]d\mathbf{x}
\end{aligned}
$$

$$\leq \int_0^\tau \int_\Omega \bar{\chi}\frac{1}{2\varepsilon_0}\Big|\mathbf{D}\Big(x, \frac{\partial \mathbf{w}}{\partial t}(t)\Big)\Big|^2 d\mathbf{x}dt + \Big(\frac{\tau\varepsilon_0}{2} + \frac{\varepsilon_1}{2}\Big) \operatorname*{ess\,sup}_{t\in[0,\tau]} \int_\Omega \bar{\chi}|\mathbf{D}(x, \mathbf{w}(t))|^2 d\mathbf{x}$$
$$+ \int_\Omega \bar{\chi}\frac{1}{2\varepsilon_1}|\mathbf{D}(x, \mathbf{w}_0)|^2 d\mathbf{x}.$$

Choosing here $\varepsilon_0 = 1/(2\tau)$ and $\varepsilon_1 = 1/2$ we derive

$$\frac{1}{2}\operatorname*{ess\,sup}_{t\in[0,\tau]} \|\bar{\chi}\mathbf{D}(x, \mathbf{w}(t))\|^2_{2,\Omega} \leq \tau\Big\|\bar{\chi}\mathbf{D}\big(x, \frac{\partial \mathbf{w}}{\partial t}\big)\Big\|^2_{2,\Omega\times(0,\tau)} + \|\bar{\chi}\mathbf{D}(x, \mathbf{w}_0)\|^2_{2,\Omega}, \quad (4.12)$$

for all $\tau \in [0, T]$. We end up with estimate (4.10), by combining inequality (4.12) with energy estimate (4.9) and Korn's inequality

$$\|\phi\|_{\overset{\circ}{W}{}^1_2(\Omega)} \leq C_k(\Omega)\|\mathbf{D}(x, \phi)\|_{2,\Omega}, \quad (4.13)$$

which is valid for all functions $\phi \in L^2(\Omega)$ such that $\mathbf{D}(x, \phi) \in L^2(\Omega)$ and ϕ vanishes in the trace sense on some open subset of $\partial\Omega$ [14, Chap. III, Sec. 3.2]. □

Corollary 4.5. *Let* $\mathbf{F}, \Psi \in L^2(Q)$, $\mathbf{v}_0, \theta_0 \in L^2(\Omega)$, *and* $\mathbf{w}_0 \in \overset{\circ}{W}{}^1_2(\Omega)$. *If Model A is solvable, then there exists exactly one generalized solution.*

Proof. Since *Model A* is linear, the uniqueness assertion amounts to the proposition that, if $\mathbf{F}, \mathbf{v}_0, \mathbf{w}_0 = 0$ and $\Psi, \theta_0 = 0$, then there is only trivial solution. The latter proposition is obvious due to energy estimate (4.9). □

5. Galerkin's Approximations and Existence of Solutions to Model A

Let $\{\phi_l\} \subset C_0^\infty(\Omega)^3$ and $\{\psi_l\} \subset C_0^\infty(\Omega)$ be total systems in $\overset{\circ}{W}{}^1_2(\Omega)^3$ and $\overset{\circ}{W}{}^1_2(\Omega)$, respectively.

We construct Galerkin's approximations of the displacement vector and of the temperature distribution in the forms

$$\mathbf{w}_n(\mathbf{x}, t) = \sum_{l=1}^n a_l(t)\phi_l(\mathbf{x}), \quad \theta_n(\mathbf{x}, t) = \sum_{l=1}^n b_l(t)\psi_l(\mathbf{x}), \quad (5.1)$$

where unknown functions $a_l(t)$ and $b_l(t)$ $(l = 1, \ldots, n)$ are found from Galerkin's system

$$\sum_{l=1}^n \frac{d^2 a_l(t)}{dt^2}\int_\Omega \alpha_\tau\bar{\rho}(\mathbf{x})\phi_l(\mathbf{x})\cdot\phi_j(\mathbf{x})d\mathbf{x}$$
$$= -\sum_{l=1}^n \frac{da_l(t)}{dt}\int_\Omega \bar{\chi}(\mathbf{x})\big(\alpha_\nu \operatorname*{div}_x \phi_l(\mathbf{x})\cdot\operatorname*{div}_x \phi_j(\mathbf{x}) + \alpha_\mu\mathbf{D}(x, \phi_l(\mathbf{x})) : \mathbf{D}(x, \phi_j(\mathbf{x}))\big)d\mathbf{x}$$
$$- \sum_{l=1}^n a_l(t)\int_\Omega \big[\bar{\chi}(\mathbf{x})\alpha_p \operatorname*{div}_x \phi_l(\mathbf{x})\cdot\operatorname*{div}_x \phi_j(\mathbf{x}) + (1-\bar{\chi}(\mathbf{x}))\alpha_\eta \operatorname*{div}_x \phi_l(\mathbf{x})\cdot\operatorname*{div}_x \phi_j(\mathbf{x})$$
$$+ (1-\bar{\chi}(\mathbf{x}))\alpha_\lambda\mathbf{D}(x, \phi_l(\mathbf{x})) : \mathbf{D}(x, \phi_j(\mathbf{x}))\big]d\mathbf{x}$$
$$+ \sum_{l=1}^n b_l(t)\int_\Omega \bar{\alpha}_\theta(\mathbf{x})\psi_l(\mathbf{x})\operatorname*{div}_x \phi_j(\mathbf{x})d\mathbf{x} + \int_\Omega \alpha_F\bar{\rho}(\mathbf{x})\mathbf{F}_n(\mathbf{x}, t)\cdot\phi_j(\mathbf{x})d\mathbf{x}, \quad (5.2)$$

$j = 1, \ldots, n$, where $\mathbf{F}_n$ is a given approximation of $\mathbf{F}$ such that

$$\mathbf{F}_n \in C^\infty(Q), \quad \mathbf{F}_n \to \mathbf{F} \text{ in } L^2(Q) \quad \text{as } n \nearrow \infty, \quad (5.3)$$

$$\sum_{l=1}^{n} \frac{db_l(t)}{dt} \int_\Omega \bar{c}_p(\mathbf{x})\psi_l(\mathbf{x})\psi_j(\mathbf{x})d\mathbf{x}$$
$$= -\sum_{l=1}^{n} b_l(t) \int_\Omega \bar{\varkappa}(\mathbf{x})\nabla_x\psi_l(\mathbf{x}) \cdot \nabla_x\psi_j(\mathbf{x})d\mathbf{x} \tag{5.4}$$
$$+ \sum_{l=1}^{n} \frac{da_l(t)}{dt} \int_\Omega \bar{\alpha}_\theta(\mathbf{x})(\operatorname*{div}_x \phi_l(\mathbf{x}))\psi_j(\mathbf{x})d\mathbf{x} + \int_\Omega \Psi_n(\mathbf{x},t)\psi_j(\mathbf{x})d\mathbf{x},$$

$j = 1, \ldots, n$, where Ψ_n is a given approximation of Ψ such that

$$\Psi_n \in C^\infty(Q), \quad \Psi_n \to \Psi \text{ in } L^2(Q) \quad \text{as } n \nearrow \infty. \tag{5.5}$$

Since $\alpha_\tau\bar{\rho}(\mathbf{x}) > 0$, $\bar{c}_p(\mathbf{x}) > 0$ and the sets $(\phi_1, \ldots, \phi_n)$ and $(\psi_1, \ldots, \psi_n)$ are linearly independent in $L^2(\Omega)^3$ and $L^2(\Omega)$, respectively, then the matrices

$$\mathbb{A}_n = \Big(\int_\Omega \alpha_\tau\bar{\rho}(\mathbf{x})\phi_l(\mathbf{x}) \cdot \phi_j(\mathbf{x})d\mathbf{x}\Big)_{l,j=1}^n, \quad \mathbb{B}_n = \Big(\int_\Omega \bar{c}_p(\mathbf{x})\psi_l(\mathbf{x})\psi_j(\mathbf{x})d\mathbf{x}\Big)_{l,j=1}^n$$

are invertible, due to the classical theory of Hilbert spaces. Hence, setting

$$c_l(t) = \frac{da_l(t)}{dt}, \quad \mathbf{a}_n(t) = (a_1(t), \ldots, a_n(t)),$$
$$\mathbf{b}_n(t) = (b_1(t), \ldots, b_n(t)), \quad \mathbf{c}_n(t) = (c_1(t), \ldots, c_n(t)),$$
$$\mathbb{A}_n^{(1)} = \Big(\int_\Omega \bar{\chi}(\mathbf{x})\big(\alpha_\nu \operatorname*{div}_x \phi_l(\mathbf{x}) \cdot \operatorname*{div}_x \phi_j(\mathbf{x}) + \alpha_\mu \mathbf{D}(x, \phi_l(\mathbf{x})) : \mathbf{D}(x, \phi_j(\mathbf{x}))\big)d\mathbf{x}\Big)_{l,j=1}^n,$$
$$\mathbb{A}_n^{(2)} = \Big(\int_\Omega \big[\bar{\chi}(\mathbf{x})\alpha_p \operatorname*{div}_x \phi_l(\mathbf{x}) \cdot \operatorname*{div}_x \phi_j(\mathbf{x}) + (1-\bar{\chi}(\mathbf{x}))\alpha_\eta \operatorname*{div}_x \phi_l(\mathbf{x}) \cdot \operatorname*{div}_x \phi_j(\mathbf{x})$$
$$+ (1-\bar{\chi}(\mathbf{x}))\alpha_\lambda \mathbf{D}(x, \phi_l(\mathbf{x})) : \mathbf{D}(x, \phi_j(\mathbf{x}))\big]d\mathbf{x}\Big)_{l,j=1}^n,$$
$$\mathbb{A}_n^{(3)} = \Big(\int_\Omega \bar{\alpha}_\theta(\mathbf{x})\psi_l(\mathbf{x}) \operatorname*{div}_x \phi_j(\mathbf{x})d\mathbf{x}\Big)_{l,j=1}^n,$$
$$\mathbb{B}_n^{(1)} = \Big(\int_\Omega \bar{\varkappa}(\mathbf{x})\nabla_x\psi_l(\mathbf{x}) \cdot \nabla_x\psi_j(\mathbf{x})d\mathbf{x}\Big)_{l,j=1}^n,$$
$$\mathbb{B}_n^{(2)} = \Big(\int_\Omega \bar{\alpha}_\theta(\mathbf{x})(\operatorname*{div}_x \phi_l(\mathbf{x}))\psi_j(\mathbf{x})d\mathbf{x}\Big)_{l,j=1}^n,$$
$$\tilde{\mathbf{F}}_n(t) = \Big(\int_\Omega \alpha_F\bar{\rho}(\mathbf{x})\mathbf{F}_n(\mathbf{x},t) \cdot \phi_j(\mathbf{x})d\mathbf{x}\Big)_{j=1}^n,$$
$$\tilde{\Psi}_n(t) = \Big(\int_\Omega \Psi_n(\mathbf{x},t)\psi_j(\mathbf{x})d\mathbf{x}\Big)_{j=1}^n,$$

we see that system (5.2)–(5.4) is equivalent to the system of the first-order linear differential equations with constant coefficients in the normal form:

$$\frac{d\mathbf{c}_n(t)}{dt} = -\mathbb{A}_n^{-1}\mathbb{A}_n^{(1)}\mathbf{c}_n(t) - \mathbb{A}_n^{-1}\mathbb{A}_n^{(2)}\mathbf{a}_n(t) + \mathbb{A}_n^{-1}(\mathbb{A}_n^{(3)})^t\mathbf{b}_n(t) + \mathbb{A}_n^{-1}\tilde{\mathbf{F}}_n(t), \tag{5.6}$$
$$\frac{d\mathbf{b}_n(t)}{dt} = -\mathbb{B}_n^{-1}\mathbb{B}_n^{(1)}\mathbf{b}_n(t) + \mathbb{B}_n^{-1}(\mathbb{B}_n^{(2)})^t\mathbf{c}_n(t) + \mathbb{B}_n^{-1}\tilde{\Psi}_n(t), \tag{5.7}$$
$$\frac{d\mathbf{a}_n(t)}{dt} = \mathbf{c}_n(t). \tag{5.8}$$

System (5.2)–(5.4) or, equivalently, system (5.6)–(5.8), is supplemented with initial data

$$a_l(t)|_{t=0} = a_l^0 := \int_\Omega \mathbf{w}_0(\mathbf{x}) \cdot \phi_l(\mathbf{x}) d\mathbf{x}, \tag{5.9}$$

$$b_l(t)|_{t=0} = b_l^0 := \int_\Omega \theta_0(\mathbf{x}) \psi_l(\mathbf{x}) d\mathbf{x}, \tag{5.10}$$

$$c_l(t)|_{t=0} = c_l^0 := \int_\Omega \mathbf{v}_0(\mathbf{x}) \cdot \phi_l(\mathbf{x}) d\mathbf{x}. \tag{5.11}$$

On the strength of the classical theory of systems of first-order ordinary linear differential equations, the Cauchy problem (5.6)–(5.11) has a unique infinitely smooth solution $(\mathbf{a}_n(t), \mathbf{b}_n(t), \mathbf{c}_n(t))$ for any $n \in \mathbb{N}$. This amounts to the following result.

Proposition 5.1. *Galerkin's system* (5.2)–(5.4)*, supplemented with initial data* (5.9)–(5.11)*, has a unique smooth solution* $(a_1(t), \ldots, a_n(t), b_1(t), \ldots, b_n(t))$ *on the interval* $[0, T]$ *for any* $n \in \mathbb{N}$*.*

The approximate distributions of pressures now can be found from the equations

$$p_n(\mathbf{x}, t) = -\bar{\chi}(\mathbf{x}) \alpha_p \operatorname*{div}_x \mathbf{w}_n(\mathbf{x}, t), \tag{5.12}$$

$$q_n(\mathbf{x}, t) = p_n(\mathbf{x}, t) + \frac{\alpha_\nu}{\alpha_p} \frac{\partial p_n(\mathbf{x}, t)}{\partial t}, \tag{5.13}$$

$$\pi_n(\mathbf{x}, t) = -(1 - \bar{\chi}(\mathbf{x})) \alpha_\eta \operatorname*{div}_x \mathbf{w}_n(\mathbf{x}, t). \tag{5.14}$$

Remark 5.2. Using the standard technics [4, Sec. III.3] we easily conclude that five approximate functions ($\mathbf{w}_n$, θ_n, p_n, q_n, π_n), which are obtained by virtue of (5.1)–(5.4) and (5.9)–(5.14), are a generalized solution of *Model A*, provided with the given approximate functions $\mathbf{F}_n(\mathbf{x}, t)$ and $\Psi_n(\mathbf{x}, t)$, and with the initial data

$$\mathbf{w}_n(\mathbf{x}, t)|_{t=0} = \sum_{l=1}^n a_l^0 \phi_l(\mathbf{x}), \tag{5.15}$$

$$\frac{\partial \mathbf{w}_n(\mathbf{x}, t)}{\partial t}|_{t=0} = \sum_{l=1}^n c_l^0 \phi_l(\mathbf{x}), \tag{5.16}$$

$$\theta_n(\mathbf{x}, t)|_{t=0} = \sum_{l=1}^n b_l^0 \psi_l(\mathbf{x}), \tag{5.17}$$

where a_l^0, b_l^0, and c_l^0 are given by (5.9)–(5.11), i.e., they are Fourier coefficients of initial data $\mathbf{w}_0$, $\mathbf{v}_0$, and θ_0, and hence initial data (5.15)–(5.17) are the partial Fourier sums of $\mathbf{w}_0$, $\mathbf{v}_0$, and θ_0.

Due to this remark and energy estimate (4.9), the sequence ($\mathbf{w}_n$, θ_n, p_n, q_n, π_n) has a weak limiting point ($\mathbf{w}$, θ, p, q, π) as $n \nearrow \infty$, and, due to linearity of *Model A*, the functions $\mathbf{w}$, θ, p, q, and π are a generalized solution of *Model A*, provided with initial data $\mathbf{w}_0$, $\mathbf{v}_0$, and θ_0, which completes the proof of Theorem 3.4.

6. Additional Estimates for the Pressures

Energy inequality (4.9) includes some estimates for the pressures p, q, and π, due to (3.9)–(3.11). However, these estimates are not applicable for analysis of incompressible limiting regimes, because they are not uniform in α_p and α_η, as one or both of these coefficients grow infinitely. Hence, in order to do such analysis, it is necessary to obtain additional bounds on the pressures. We start with justification of the following technical result.

Lemma 6.1. *Let* $\mathbf{F}$, $\partial\mathbf{F}/\partial t$, Ψ, $\partial\Psi/\partial t \in L^2(Q)$, *and initial data be homogeneous, i.e.,*

$$\mathbf{w}_0 = 0, \quad \mathbf{v}_0 = 0, \quad \theta_0 = 0. \tag{6.1}$$

Then the following bound is valid:

$$\begin{aligned}
&\frac{1}{2}\operatorname*{ess\,sup}_{t\in[0,\tau]} \alpha_\tau \Big\|\sqrt{\bar\rho}\frac{\partial^2\mathbf{w}}{\partial t^2}(t)\Big\|^2_{2,\Omega} + \frac{1}{2}\operatorname*{ess\,sup}_{t\in[0,\tau]} \Big\|\sqrt{\bar c_p}\frac{\partial\theta}{\partial t}(t)\Big\|^2_{2,\Omega} \\
&+ \frac{1}{2}\operatorname*{ess\,sup}_{t\in[0,\tau]} \alpha_\eta \Big\|(1-\bar\chi)\operatorname*{div}_x \frac{\partial\mathbf{w}}{\partial t}(t)\Big\|^2_{2,\Omega} + \frac{1}{2}\operatorname*{ess\,sup}_{t\in[0,\tau]} \alpha_p \Big\|\bar\chi \operatorname*{div}_x \frac{\partial\mathbf{w}}{\partial t}(t)\Big\|^2_{2,\Omega} \\
&+ \frac{1}{2}\operatorname*{ess\,sup}_{t\in[0,\tau]} \alpha_\lambda \Big\|(1-\bar\chi)\mathbf{D}\Big(x, \frac{\partial\mathbf{w}}{\partial t}(t)\Big)\Big\|^2_{2,\Omega} + \alpha_\nu \Big\|\bar\chi \operatorname*{div}_x \frac{\partial^2\mathbf{w}}{\partial t^2}\Big\|^2_{2,\Omega\times(0,\tau)} \\
&+ \alpha_\mu \Big\|\bar\chi\mathbf{D}\Big(x, \frac{\partial^2\mathbf{w}}{\partial t^2}\Big)\Big\|^2_{2,\Omega\times(0,\tau)} + \Big\|\sqrt{\bar\varkappa}\frac{\partial}{\partial t}\nabla_x\theta\Big\|^2_{2,\Omega\times(0,\tau)} \\
&\le \frac{\alpha_F^2}{2\alpha_\tau}\Big\|\sqrt{\bar\rho}\frac{\partial\mathbf{F}}{\partial t}\Big\|^2_{2,\Omega\times(0,\tau)} + \frac{1}{2\min\{c_{pf}, c_{ps}\}}\Big\|\frac{\partial\Psi}{\partial t}\Big\|^2_{2,\Omega\times(0,\tau)} \\
&\quad + \int_0^\tau \Big[\frac{\alpha_F^2}{2\alpha_\tau}\Big\|\sqrt{\bar\rho}\frac{\partial\mathbf{F}}{\partial t}\Big\|^2_{2,\Omega\times(0,t)} + \frac{1}{2\min\{c_{pf}, c_{ps}\}}\Big\|\frac{\partial\Psi}{\partial t}\Big\|^2_{2,\Omega\times(0,t)}\Big] e^{\tau-t}dt \\
&\quad + \frac{e^\tau+1}{2}\Big[\frac{\alpha_F^2}{\alpha_\tau}\|\sqrt{\bar\rho}\mathbf{F}|_{t=0}\|^2_{2,\Omega} + \frac{1}{\min\{c_{pf}, c_{ps}\}}\|\Psi|_{t=0}\|^2_{2,\Omega}\Big] := C^{(2)}_{\mathrm{en}}(\tau),
\end{aligned} \tag{6.2}$$

for all $\tau \in [0, T]$.

Proof. We take advantage of the fact that the generalized solution is unique and may be constructed, using Galerkin's method.

First of all, we notice that the values of all the constants a_l^0, b_l^0, and c_l^0, defined in (5.9)–(5.11), are equal to zero. Passing to the limit in the right hand sides of (5.2)–(5.4) (or, equivalently, of (5.6)–(5.8)), we conclude that

$$\alpha_\tau\bar\rho\frac{\partial^2\mathbf{w}_n}{\partial t^2}\Big|_{t=0} = \alpha_F\bar\rho\mathbf{F}_n|_{t=0}, \quad \bar c_p\frac{\partial\theta_n}{\partial t}\Big|_{t=0} = \Psi_n|_{t=0}. \tag{6.3}$$

Next, differentiating Galerkin's system (5.2)–(5.4) and equations (5.12)–(5.14) with respect to t, on the strength of Remark 5.2 and (6.3) we conclude that the derivatives

$$\Big(\frac{\partial\mathbf{w}_n}{\partial t}, \frac{\partial\theta_n}{\partial t}, \frac{\partial p_n}{\partial t}, \frac{\partial q_n}{\partial t}, \frac{\partial\pi_n}{\partial t}\Big)$$

are the generalized solutions of *Model A*, provided with the dimensionless approximate density of distributed mass forces $\partial\mathbf{F}_n/\partial t$ and volumetric density of heat application $\partial\Psi_n/\partial t$, and with the initial data (6.3) and (5.16). Now passing to the limit as $n \nearrow \infty$ in energy estimate (4.9), we immediately derive bound (6.2). □

Next, we introduce the normalized pressures $\tilde p$, $\tilde q$, and $\tilde\pi$ as follows:

$$\tilde p(\mathbf{x}, t) = -\bar\chi(\mathbf{x})\alpha_p \operatorname*{div}_x \mathbf{w}(\mathbf{x}, t) + \frac{\bar\chi(\mathbf{x})}{\operatorname{meas}\Omega_f}\int_\Omega \bar\chi(\mathbf{x})\alpha_p \operatorname*{div}_x \mathbf{w}(\mathbf{x}, t)d\mathbf{x}, \tag{6.4}$$

$$
\begin{aligned}
\tilde{\pi}(\mathbf{x},t) = &-(1-\bar{\chi}(\mathbf{x}))\alpha_\eta \operatorname*{div}_x \mathbf{w}(\mathbf{x},t) \\
&+ \frac{1-\bar{\chi}(\mathbf{x})}{\operatorname{meas}\Omega_s}\int_\Omega (1-\bar{\chi}(\mathbf{x}))\alpha_\eta \operatorname*{div}_x \mathbf{w}(\mathbf{x},t)d\mathbf{x},
\end{aligned} \tag{6.5}
$$

$$
\tilde{q}(\mathbf{x},t) = \tilde{p}(\mathbf{x},t) + \frac{\alpha_\nu}{\alpha_p}\frac{\partial \tilde{p}(\mathbf{x},t)}{\partial t}. \tag{6.6}
$$

Remark 6.2. By the straightforward calculation, using Green's formula, we conclude that integral equalities (3.15) and (4.3) with (3.7) being inserted into them, are valid with $\tilde{q}$ and $\tilde{\pi}$ on the places of q and π, respectively, and that

$$
\int_\Omega \tilde{p}d\mathbf{x} = \int_\Omega \tilde{\pi}d\mathbf{x} = \int_\Omega \tilde{q}d\mathbf{x} = 0. \tag{6.7}
$$

This means that all the results previously obtained in this article, in particular, Theorem 3.4 and energy estimate (4.9), remain true for the modified model, which appears if we substitute equations (3.9)–(3.11) in the statement of *Model A* by equations (6.4)–(6.6).

For the normalized pressures $\tilde{p}$, $\tilde{q}$, and $\tilde{\pi}$ we prove the following result.

Theorem 6.3. *Let $\mathbf{F}$, $\partial\mathbf{F}/\partial t$, Ψ, $\partial\Psi/\partial t \in L^2(Q)$ and initial data be homogeneous, i.e., satisfy* (6.1)*. Then the normalized pressures $\tilde{p}$, $\tilde{q}$, and $\tilde{\pi}$ satisfy the estimates*

$$
\begin{aligned}
\|\tilde{q}\|^2_{2,Q} + \|\tilde{\pi}\|^2_{2,Q} \le C_{\mathrm{inc}} \cdot \Big(&\|\mathbf{F}\|^2_{2,Q} + \|\Psi\|^2_{2,Q} + \Big\|\frac{\partial \mathbf{F}}{\partial t}\Big\|^2_{2,Q} \\
+ \Big\|\frac{\partial \Psi}{\partial t}\Big\|^2_{2,Q} &+ \|\mathbf{F}|_{t=0}\|^2_{2,\Omega} + \|\Psi|_{t=0}\|^2_{2,\Omega}\Big) := C^*_{\mathrm{inc}},
\end{aligned} \tag{6.8}
$$

$$
\|\tilde{p}\|^2_{2,Q} \le C^*_{\mathrm{inc}} + C_{\mathrm{en}}(T), \tag{6.9}
$$

where $C_{\mathrm{inc}} = C_{\mathrm{inc}}(T,\Omega,\alpha_\tau,\alpha_F,\alpha_\mu,\alpha_\lambda,\alpha_{\theta f},\alpha_{\theta s},\rho_f,\rho_s,c_{pf},c_{ps})$.

Remark 6.4. We emphasize that the constants $C_{\mathrm{en}}(T)$ and C_{inc} do not depend on α_p and α_η, which implies that the obtained bounds are applicable for studying asymptotic as α_p and α_η grow infinitely. Moreover, notice that the constants $C_{\mathrm{en}}(T)$ and C_{inc} also do not depend on geometry of Ω_f and Ω_s. This fact seems to be very useful in view of possible studies of homogenization topics in periodic structures (like in [2, 15]) for *Model A*.

Proof. From Lemma 6.1, we have the bound

$$
\begin{aligned}
&\frac{1}{2}\alpha_\tau\Big\|\sqrt{\bar{\rho}}\frac{\partial^2\mathbf{w}}{\partial t^2}\Big\|^2_{2,\Omega\times(0,\tau)} + \frac{1}{2}\Big\|\sqrt{\bar{c}_p}\frac{\partial\theta}{\partial t}\Big\|^2_{2,\Omega\times(0,\tau)} \\
&\le \int_0^\tau \Big[\frac{\alpha_F^2}{2\alpha_\tau}\Big\|\sqrt{\bar{\rho}}\frac{\partial\mathbf{F}}{\partial t}\Big\|^2_{2,\Omega\times(0,t)} + \frac{1}{2\min\{c_{pf},c_{ps}\}}\Big\|\frac{\partial\Psi}{\partial t}\Big\|^2_{2,\Omega\times(0,t)}\Big]e^{\tau-t}dt \\
&\quad + \frac{\alpha_F^2}{2\alpha_\tau}\Big\|\sqrt{\bar{\rho}}\frac{\partial\mathbf{F}}{\partial t}\Big\|^2_{2,\Omega\times(0,\tau)} + \frac{1}{2\min\{c_{pf},c_{ps}\}}\Big\|\frac{\partial\Psi}{\partial t}\Big\|^2_{2,\Omega\times(0,\tau)} \\
&\quad + \frac{e^\tau+1}{2}\Big[\frac{\alpha_F^2}{\alpha_\tau}\big\|\sqrt{\bar{\rho}}\mathbf{F}|_{t=0}\big\|^2_{2,\Omega} + \frac{1}{\min\{c_{pf},c_{ps}\}}\big\|\Psi|_{t=0}\big\|^2_{2,\Omega}\Big] \\
&\le \frac{e^\tau+1}{2}\Big[\frac{\alpha_F^2}{\alpha_\tau}\Big\|\sqrt{\bar{\rho}}\frac{\partial\mathbf{F}}{\partial t}\Big\|^2_{2,\Omega\times(0,\tau)} + \frac{1}{\min\{c_{pf},c_{ps}\}}\Big\|\frac{\partial\Psi}{\partial t}\Big\|^2_{2,\Omega\times(0,\tau)} \\
&\quad + \frac{\alpha_F^2}{\alpha_\tau}\big\|\sqrt{\bar{\rho}}\mathbf{F}|_{t=0}\big\|^2_{2,\Omega} + \frac{1}{\min\{c_{pf},c_{ps}\}}\big\|\Psi|_{t=0}\big\|^2_{2,\Omega}\Big], \quad \forall\,\tau\in[0,T].
\end{aligned} \tag{6.10}
$$

Justification of estimates (6.9)–(6.8) is based on use of this bound, on a special choice of test functions in integral equality (4.3), and on application of energy inequality (4.9) and Remark 6.2.

We integrate the first term in (4.3) by parts, substitute (3.7) and (3.10), and replace p and π by $\tilde{p}$ and $\tilde{\pi}$, respectively, which is legal due to Remark 6.2. Thus we get

$$\begin{aligned}&\int_{\Omega\times(0,\tau)}\Big(\alpha_\tau\bar{\rho}\frac{\partial^2\mathbf{w}}{\partial t^2}\cdot\varphi-\tilde{z}\operatorname*{div}_x\varphi+\bar{\chi}\alpha_\mu\mathbf{D}\Big(x,\frac{\partial\mathbf{w}}{\partial t}\Big):\mathbf{D}(x,\varphi)\\&+(1-\bar{\chi})\alpha_\lambda\mathbf{D}(x,\mathbf{w}):\mathbf{D}(x,\varphi)-\bar{\alpha}_\theta\theta\operatorname*{div}_x\varphi-\alpha_F\bar{\rho}\mathbf{F}\cdot\varphi\Big)d\mathbf{x}dt=0,\end{aligned}\tag{6.11}$$

for all $\tau\in[0,T]$. Here we denote $\tilde{z}:=\tilde{q}+\tilde{\pi}$ for briefness.

Now our aim is to choose a test function φ in this equality such that $\operatorname{div}_x\varphi=\tilde{z}$ and all integrals make perfect sense and admit estimates in terms of $\|\tilde{z}\|_{2,\Omega\times(0,\tau)}$ independently of α_p and α_η. Evidently, if this choice of φ is possible then it leads directly to a bound on $\|\tilde{z}\|_{2,\Omega\times(0,\tau)}$, uniform in α_p and α_η. We succeed to pick up such a test function as follows. Introduce successively $\varphi_1=\varphi_1(\mathbf{x},t)$ as the solution of the Dirichlet problem for Poisson's equation on Ω for $t\in[0,T]$:

$$\Delta_x\varphi_1=\tilde{z},\quad \varphi_1\big|_{\partial\Omega}=0\tag{6.12}$$

and $\varphi_2=\varphi_2(\mathbf{x},t)$ such that

$$\operatorname*{div}_x\varphi_2=0,\quad \varphi_2|_{\partial\Omega}=\nabla_x\varphi_1|_{\partial\Omega}.\tag{6.13}$$

Note that due to the classical theory of elliptic equations, a solution of problem (6.12) exists, is uniquely defined by $\tilde{z}$, and admits the bound [4, Sec. II.7]

$$\|\varphi_1\|_{\overset{\circ}{W}{}_2^2(\Omega)}\le C_1(\Omega)\|\tilde{z}\|_{2,\Omega}\quad\forall t\in[0,T],\tag{6.14}$$

and that, on the strength of the construction in [5, Chap. I, Sec. 2.1] and property (6.7), a vector-function φ_2 satisfying (6.13) may be found such that

$$\|\varphi_2\|_{W_2^1(\Omega)}\le C_0(\Omega)\|\nabla_x\varphi_1\|_{2,\partial\Omega}\quad\forall t\in[0,T],\tag{6.15}$$

which immediately implies the bound

$$\|\varphi_2\|_{W_2^1(\Omega)}\le C_2(\Omega)\|\varphi_1\|_{\overset{\circ}{W}{}_2^2(\Omega)}\quad\forall t\in[0,T],\tag{6.16}$$

thanks to the well-known trace theorem [4, Sec. I.6]. In inequalities (6.14)–(6.16) the constants $C_0(\Omega)$, $C_1(\Omega)$, and $C_2(\Omega)$ depend merely on regularity of $\partial\Omega$ and make sense, for example, for C^2-piecewise smooth surfaces $\partial\Omega$.

The sum $\varphi=\nabla_x\varphi_1-\varphi_2$ is a valid test function for integral equality (6.11). Inserting it and rearranging terms, we obtain the equality

$$\begin{aligned}&\int_{\Omega\times(0,\tau)}\tilde{z}^2d\mathbf{x}dt\\&=\int_{\Omega\times(0,\tau)}\Big(\alpha_\tau\bar{\rho}\frac{\partial^2\mathbf{w}}{\partial t^2}\cdot(\nabla_x\varphi_1-\varphi_2)+\bar{\chi}\alpha_\mu\mathbf{D}\Big(x,\frac{\partial\mathbf{w}}{\partial t}\Big):\mathbf{D}(x,\nabla_x\varphi_1)\\&-\bar{\chi}\alpha_\mu\mathbf{D}\Big(x,\frac{\partial\mathbf{w}}{\partial t}\Big):\mathbf{D}(x,\varphi_2)+(1-\bar{\chi})\alpha_\lambda\mathbf{D}(x,\mathbf{w}):\mathbf{D}(x,\nabla_x\varphi_1)\\&-(1-\bar{\chi})\alpha_\lambda\mathbf{D}(x,\mathbf{w}):\mathbf{D}(x,\varphi_2)-\bar{\alpha}_\theta\theta\tilde{z}-\alpha_F\bar{\rho}\mathbf{F}\cdot(\nabla_x\varphi_1-\varphi_2)\Big)d\mathbf{x}dt,\end{aligned}\tag{6.17}$$

for all $\tau\in[0,T]$, due to (6.12) and (6.13).

Applying Hölder's and the Cauchy-Schwartz inequalities, bounds (6.10) and (6.14)–(6.16), and energy inequality (4.9) to the right hand side of equality (6.17), we estimate

$$
\begin{aligned}
&\|\tilde z\|^2_{2,\Omega\times(0,\tau)}\\
&\leq \frac{1}{2}\Big(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\Big)\Big[\alpha_\tau\Big\|\sqrt{\bar\rho}\frac{\partial^2\mathbf{w}}{\partial t^2}\Big\|^2_{2,\Omega\times(0,\tau)}+\alpha_F\|\sqrt{\bar\rho}\mathbf{F}\|^2_{2,\Omega\times(0,\tau)}\Big]\\
&\quad+\frac{1}{2}\Big(\frac{1}{\varepsilon_3}+\frac{1}{\varepsilon_4}\Big)\Big[\alpha_\mu\Big\|\bar\chi\mathbf{D}\Big(x,\frac{\partial\mathbf{w}}{\partial t}\Big)\Big\|^2_{2,\Omega\times(0,\tau)}+\alpha_\lambda\|(1-\bar\chi)\mathbf{D}(x,\mathbf{w})\|^2_{2,\Omega\times(0,\tau)}\Big]\\
&\quad+\frac{1}{2\varepsilon_5}\|\sqrt{\bar\alpha_\theta}\theta\|^2_{2,\Omega\times(0,\tau)}+\frac{(\alpha_\tau+\alpha_F)\max\{\rho_f,\rho_s\}\varepsilon_1}{2}\|\nabla_x\varphi_1\|^2_{2,\Omega\times(0,\tau)}\\
&\quad+\frac{\max\{\alpha_\mu,\alpha_\lambda\}\varepsilon_3}{2}\|\mathbf{D}(x,\nabla_x\varphi_1)\|^2_{2,\Omega\times(0,\tau)}+\frac{\max\{\alpha_\mu,\alpha_\lambda\}\varepsilon_4}{2}\|\mathbf{D}(x,\varphi_2)\|^2_{2,\Omega\times(0,\tau)}\\
&\quad+\frac{(\alpha_\tau+\alpha_F)\max\{\rho_f,\rho_s\}\varepsilon_2}{2}\|\varphi_2\|^2_{2,\Omega\times(0,\tau)}+\frac{\max\{\alpha_{\theta f},\alpha_{\theta s}\}\varepsilon_5}{2}\|\tilde z\|^2_{2,\Omega\times(0,\tau)}\\
&\leq\frac{e^\tau+1}{2}\Big(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\Big)\Big[\frac{\alpha_F^2}{\alpha_\tau}\Big\|\sqrt{\bar\rho}\frac{\partial\mathbf{F}}{\partial t}\Big\|^2_{2,\Omega\times(0,\tau)}+\frac{1}{\min\{c_{pf},c_{ps}\}}\Big\|\frac{\partial\Psi}{\partial t}\Big\|^2_{2,\Omega\times(0,\tau)}\\
&\quad+\frac{\alpha_F^2}{\alpha_\tau}\big\|\sqrt{\bar\rho}\mathbf{F}|_{t=0}\big\|^2_{2,\Omega}+\frac{1}{\min\{c_{pf},c_{ps}\}}\big\|\Psi|_{t=0}\big\|^2_{2,\Omega}+\alpha_F\|\sqrt{\bar\rho}\mathbf{F}\|^2_{2,\Omega\times(0,\tau)}\Big]\\
&\quad+\frac{1}{2}\Big(\frac{1}{\varepsilon_3}+\frac{1}{\varepsilon_4}\Big)\Bigg[\frac{\alpha_F^2}{\alpha_\tau}\|\sqrt{\bar\rho}\mathbf{F}\|^2_{2,\Omega\times(0,\tau)}+\frac{1}{\min\{c_{pf},c_{ps}\}}\|\Psi\|^2_{2,\Omega\times(0,\tau)}\\
&\quad+\int_0^\tau\Big(\frac{\alpha_F^2}{\alpha_\tau}\|\sqrt{\bar\rho}\mathbf{F}\|^2_{2,\Omega\times(0,t)}+\frac{1}{\min\{c_{pf},c_{ps}\}}\|\Psi\|^2_{2,\Omega\times(0,t)}\Big)e^{\tau-t}dt\Bigg]\\
&\quad+\frac{1}{2\varepsilon_5}\frac{\max\{\alpha_{\theta f},\alpha_{\theta s}\}}{\min\{c_{pf},c_{ps}\}}\Bigg[\frac{\alpha_F^2}{\alpha_\tau}\|\sqrt{\bar\rho}\mathbf{F}\|^2_{2,\Omega\times(0,\tau)}+\frac{1}{\min\{c_{pf},c_{ps}\}}\|\Psi\|^2_{2,\Omega\times(0,\tau)}\\
&\quad+\int_0^\tau\Big(\frac{\alpha_F^2}{\alpha_\tau}\|\sqrt{\bar\rho}\mathbf{F}\|^2_{2,\Omega\times(0,t)}+\frac{1}{\min\{c_{pf},c_{ps}\}}\|\Psi\|^2_{2,\Omega\times(0,t)}\Big)e^{\tau-t}dt\Bigg]\\
&\quad+\frac{C_1^2(\Omega)}{2}\Big[(\alpha_\tau+\alpha_F)\max\{\rho_f,\rho_s\}(\varepsilon_1+C_2^2(\Omega)\varepsilon_2)\\
&\quad+\max\{\alpha_\mu,\alpha_\lambda\}(\varepsilon_3+C_2^2(\Omega)\varepsilon_4)+\max\{\alpha_{\theta f},\alpha_{\theta s}\}\varepsilon_5\Big]\|\tilde z\|^2_{2,\Omega\times(0,\tau)}.
\end{aligned}
\tag{6.18}
$$

Choosing

$$
\begin{gathered}
\varepsilon_1=2/(5C_1^2(\Omega)(\alpha_\tau+\alpha_F)\max\{\rho_f,\rho_s\}),\\
\varepsilon_2=2/(5C_1^2(\Omega)C_2^2(\Omega)(\alpha_\tau+\alpha_F)\max\{\rho_f,\rho_s\}),\\
\varepsilon_3=2/(5C_1^2(\Omega)\max\{\alpha_\mu,\alpha_\lambda\}),\\
\varepsilon_4=2/(5C_1^2(\Omega)C_2^2(\Omega)\max\{\alpha_\mu,\alpha_\lambda\}),\\
\varepsilon_5=2/(5C_1^2(\Omega)\max\{\alpha_{\theta f},\alpha_{\theta s}\}),
\end{gathered}
$$

we deduce the estimate

$$
\begin{aligned}
&\frac{1}{2}\|\tilde{z}\|^2_{2,\Omega\times(0,\tau)} \\
&\leq C_*^{(1)}\Big[\frac{\alpha_F^2}{\alpha_\tau}\|\sqrt{\rho}\mathbf{F}\|^2_{2,\Omega\times(0,\tau)} + \frac{1}{\min\{c_{pf}, c_{ps}\}}\|\Psi\|^2_{2,\Omega\times(0,\tau)}\Big] \\
&\quad + C_*^{(2)}\Big[\frac{\alpha_F^2}{\alpha_\tau}\Big\|\sqrt{\rho}\frac{\partial\mathbf{F}}{\partial t}\Big\|^2_{2,\Omega\times(0,\tau)} + \frac{1}{\min\{c_{pf}, c_{ps}\}}\Big\|\frac{\partial\Psi}{\partial t}\Big\|^2_{2,\Omega\times(0,\tau)} \\
&\quad + \frac{\alpha_F^2}{\alpha_\tau}\|\sqrt{\rho}\mathbf{F}|_{t=0}\|^2_{2,\Omega} + \frac{1}{\min\{c_{pf}, c_{ps}\}}\|\Psi|_{t=0}\|^2_{2,\Omega} + \alpha_F\|\sqrt{\rho}\mathbf{F}\|^2_{2,\Omega\times(0,\tau)}\Big],
\end{aligned}
\tag{6.19}
$$

where

$$
\begin{aligned}
C_*^{(1)} &= (5/4)(e^\tau+1)C_1^2(\Omega)(1+C_2^2(\Omega))\max\{\alpha_\mu, \alpha_\lambda\} \\
&\quad + (5/4)(e^\tau+1)C_1^2(\Omega)(\max\{\alpha_{\theta f}, \alpha_{\theta s}\})^2/\min\{c_{pf}, c_{ps}\}, \\
C_*^{(2)} &= (5/4)(e^\tau+1)C_1^2(\Omega)(1+C_2^2(\Omega))(\alpha_\tau+\alpha_F)\max\{\rho_f, \rho_s\}.
\end{aligned}
$$

Since the supports of $\tilde{q}$ and $\tilde{\pi}$ do not intersect, we have

$$\|\tilde{z}\|^2_{2,\Omega\times(0,\tau)} = \|\tilde{q}\|^2_{2,\Omega\times(0,\tau)} + \|\tilde{\pi}\|^2_{2,\Omega\times(0,\tau)}.$$

Thus bound (6.8) immediately follows from inequality (6.19). Finally, estimate (6.9) follows from bound (6.8), energy estimate (4.9), and equation (6.6). □

7. Additional Estimates for the Deformation Tensor

Also we are interested in investigating limiting regimes arising as α_λ grows infinitely. In order to fulfill this study, it is necessary to establish additional bounds on solution of *Model A*, which should be in certain sense uniform in α_λ.

In this line, the fundamental estimate immediately follows from energy inequality (4.9):

$$\operatorname*{ess\,sup}_{t\in[0,\tau]}\|(1-\bar{\chi})\mathbf{D}(x,\mathbf{w}(t))\|^2_{2,\Omega} \leq C_{\mathrm{en}}(\tau)/\alpha_\lambda, \quad \forall\,\tau\in[0,T]. \tag{7.1}$$

Remark 7.1. Assume that $(1-\bar{\chi})\mathbf{D}(x,\mathbf{w}_0) = 0$, which implies that $C_{\mathrm{en}}(\tau)$ is independent of α_λ. Then from bound (7.1) it obviously follows that

$$\|(1-\bar{\chi})\mathbf{D}(x,\mathbf{w}(t))\|_{2,\Omega} \longrightarrow 0 \quad \text{as } \alpha_\lambda \nearrow \infty.$$

Since the kernel of the operator $\phi \mapsto \mathbf{D}(x,\phi)$ is the set of absolutely rigid body motions, i.e., translations and rotations $\phi(\mathbf{x}) = \mathbf{x}_0 + \omega\times\mathbf{x}$ $(\mathbf{x}_0, \omega = const)$ [14, Chap. III, §2.1], we see that the infinite growth of α_λ leads to *solidification* limiting regimes.

The following assertion provides one more useful bound in the case of potential (e.g., gravitational) mass forces under additional assumption on geometry of Ω_s:

Assumption 7.2. Let the Lebesgue measure of $\partial\Omega\cap\partial\Omega_s$ be strictly positive and Ω_s be connected.

Theorem 7.3. *Let Assumption 7.2 hold. Suppose that initial data satisfy* (6.1)*, the derivative $\partial\Psi/\partial t$ belongs to $L^2(Q)$, and the given density of distributed mass forces has the form*

$$\mathbf{F}(\mathbf{x},t) = \nabla_x\Phi(\mathbf{x},t) \tag{7.2}$$

with some potential $\Phi \in W_2^1(Q)$ *such that* $(\partial/\partial t)\nabla_x \Phi \in L^2(Q)$*. Then the following bound holds:*

$$\begin{aligned}
&\frac{1}{2}\operatorname*{ess\,sup}_{t\in[0,T]} \alpha_\tau \alpha_\lambda \Big\| \sqrt{\bar\rho}\frac{\partial \mathbf{w}}{\partial t}(t)\Big\|_{2,\Omega}^2 + \alpha_\nu \alpha_\lambda \Big\| \bar\chi \operatorname*{div}_x \frac{\partial \mathbf{w}}{\partial t}\Big\|_{2,Q}^2 \\
&+ \alpha_\mu \alpha_\lambda \Big\| \bar\chi \mathbf{D}\Big(x, \frac{\partial \mathbf{w}}{\partial t}\Big)\Big\|_{2,Q}^2 + \frac{1}{4}\frac{\alpha_\eta}{\alpha_\lambda}\operatorname*{ess\,sup}_{t\in[0,T]}\Big\|(1-\bar\chi)\alpha_\lambda \operatorname*{div}_x \mathbf{w}(t)\Big\|_{2,\Omega}^2 \\
&+ \frac{1}{4}\frac{\alpha_p}{\alpha_\lambda}\operatorname*{ess\,sup}_{t\in[0,T]}\Big\|\bar\chi \alpha_\lambda \operatorname*{div}_x \mathbf{w}(t)\Big\|_{2,\Omega}^2 + \frac{1}{4}\operatorname*{ess\,sup}_{t\in[0,T]}\|(1-\bar\chi)\alpha_\lambda \mathbf{D}(x,\mathbf{w}(t))\|_{2,\Omega}^2 \\
&\le C_{\mathrm{sol}}\Big(1 + \frac{\alpha_\lambda}{\alpha_\eta} + \frac{\alpha_\lambda}{\alpha_p}\Big)\Big(\|\Phi\|_{W_2^1(Q)}^2 + \Big\|\frac{\partial}{\partial t}\nabla_x \Phi\Big\|_{2,Q}^2 + \|\Psi\|_{2,Q}^2 + \Big\|\frac{\partial \Psi}{\partial t}\Big\|_{2,Q}^2\Big),
\end{aligned} \tag{7.3}$$

where $C_{\mathrm{sol}} = C_{\mathrm{sol}}(T, \Omega, \bar\chi, \alpha_\theta, \alpha_F, \bar\rho, \bar c_p)$.

Remark 7.4. We emphasize that the constant C_{sol} does not depend on α_p, α_η, and α_λ. Thus estimate (7.3) becomes uniform in $\alpha_\lambda \nearrow \infty$, whenever it is assumed that α_η and α_p also grow infinitely such that

$$\alpha_\eta = O(\alpha_\lambda), \quad \alpha_p = O(\alpha_\lambda) \quad \text{as} \quad \alpha_\lambda \nearrow \infty. \tag{7.4}$$

Remark 7.5. On the strength of (3.9)–(3.11) and energy estimate (4.9), we observe that

$$\begin{aligned}
\frac{\alpha_p}{\alpha_\lambda}\operatorname*{ess\,sup}_{t\in[0,T]}\Big\|\bar\chi \alpha_\lambda \operatorname*{div}_x \mathbf{w}(t)\Big\|_{2,\Omega}^2 &= \frac{\alpha_\lambda}{\alpha_p}\operatorname*{ess\,sup}_{t\in[0,T]}\|p(t)\|_{2,\Omega}^2, \\
\frac{\alpha_\eta}{\alpha_\lambda}\operatorname*{ess,\,sup}_{t\in[0,T]}\Big\|(1-\bar\chi)\alpha_\lambda \operatorname*{div}_x \mathbf{w}(t)\Big\|_{2,\Omega}^2 &= \frac{\alpha_\lambda}{\alpha_\eta}\operatorname*{ess\,sup}_{t\in[0,T]}\|\pi(t)\|_{2,\Omega}^2, \\
\|q\|_{2,Q} \le \|p\|_{2,Q} \le \|p\|_{2,Q} + \alpha_\nu \Big\|\bar\chi \operatorname*{div}_x \frac{\partial \mathbf{w}}{\partial t}\Big\|_{2,Q} &\le \|p\|_{2,Q} + \sqrt{\alpha_\nu C_{\mathrm{en}}(T)}.
\end{aligned}$$

These formulas and estimate (7.3) yield that the pressures p, q, and π stay bounded in $L^2(Q)$ as α_λ, α_η, and α_p grow infinitely such that limiting relations (7.4) hold.

Proof of Theorem 7.3. Substitute (3.7), (3.9)–(3.11), (6.1) into (4.3) to get

$$\begin{aligned}
&\int_0^\tau \int_\Omega \Big(\alpha_\tau \bar\rho \frac{\partial \mathbf{w}}{\partial t}\cdot\frac{\partial \varphi}{\partial t} - \bar\chi \alpha_p \operatorname*{div}_x \mathbf{w}\cdot \operatorname*{div}_x \varphi - \bar\chi \alpha_\nu \operatorname*{div}_x \frac{\partial \mathbf{w}}{\partial t}\cdot \operatorname*{div}_x \varphi \\
&- \bar\chi \alpha_\mu \mathbf{D}\Big(x, \frac{\partial \mathbf{w}}{\partial t}\Big) : \mathbf{D}(x,\varphi) - (1-\bar\chi)\alpha_\eta \operatorname*{div}_x \mathbf{w}\cdot \operatorname*{div}_x \varphi \\
&- (1-\bar\chi)\alpha_\lambda \mathbf{D}(x,\mathbf{w}) : \mathbf{D}(x,\varphi) + \bar\alpha_\theta \theta \operatorname*{div}_x \varphi - \alpha_F \bar\rho \mathbf{F}\cdot\varphi\Big) d\mathbf{x}dt \\
&= \int_\Omega \alpha_\tau \bar\rho \frac{\partial \mathbf{w}}{\partial t}(\mathbf{x},\tau)\cdot\varphi(\mathbf{x},\tau)d\mathbf{x}, \quad \forall\, \tau \in [0,T].
\end{aligned} \tag{7.5}$$

On the strength of Lemma 6.1, $\varphi = \alpha_\lambda \partial \mathbf{w}/\partial t$ is a valid test function for (7.5). Inserting it and (7.2) into (7.5), we arrive at the equality

$$\begin{aligned}
&\frac{1}{2}\alpha_\tau\alpha_\lambda\Big\|\sqrt{\bar\rho}\frac{\partial \mathbf{w}}{\partial t}(\tau)\Big\|^2_{2,\Omega} + \frac{1}{2}\alpha_\lambda\alpha_p\|\bar\chi \operatorname*{div}_x \mathbf{w}(\tau)\|^2_{2,\Omega} \\
&+ \alpha_\nu\alpha_\lambda\Big\|\bar\chi \operatorname*{div}_x \frac{\partial \mathbf{w}}{\partial t}\Big\|^2_{2,\Omega\times(0,\tau)} + \alpha_\mu\alpha_\lambda\Big\|\bar\chi\mathbf{D}\Big(x,\frac{\partial \mathbf{w}}{\partial t}\Big)\Big\|^2_{2,\Omega\times(0,\tau)} \\
&+ \frac{1}{2}\alpha_\eta\alpha_\lambda\|(1-\bar\chi)\operatorname*{div}_x \mathbf{w}(\tau)\|^2_{2,\Omega} + \frac{1}{2}\|(1-\bar\chi)\alpha_\lambda\mathbf{D}(x,\mathbf{w}(\tau))\|^2_{2,\Omega} \\
&= -\int_0^\tau\int_\Omega\Big(\alpha_F\alpha_\lambda\bar\rho\nabla_x\Phi\cdot\frac{\partial \mathbf{w}}{\partial t} - \alpha_\lambda\bar\alpha_\theta\theta\operatorname*{div}_x\frac{\partial \mathbf{w}}{\partial t}\Big)d\mathbf{x}dt, \quad \forall\,\tau\in[0,T].
\end{aligned}\tag{7.6}$$

Using integration by parts on the right-hand side of this equality, we have

$$\begin{aligned}
&\frac{1}{2}\alpha_\tau\alpha_\lambda\Big\|\sqrt{\bar\rho}\frac{\partial \mathbf{w}}{\partial t}(\tau)\Big\|^2_{2,\Omega} + \alpha_\nu\alpha_\lambda\Big\|\bar\chi \operatorname*{div}_x \frac{\partial \mathbf{w}}{\partial t}\Big\|^2_{2,\Omega\times(0,\tau)} \\
&+ \alpha_\mu\alpha_\lambda\Big\|\bar\chi\mathbf{D}\Big(x,\frac{\partial \mathbf{w}}{\partial t}\Big)\Big\|^2_{2,\Omega\times(0,\tau)} + \frac{1}{2}\alpha_\lambda\alpha_p\|\bar\chi \operatorname*{div}_x \mathbf{w}(\tau)\|^2_{2,\Omega} \\
&+ \frac{1}{2}\alpha_\lambda\alpha_\eta\|(1-\bar\chi)\operatorname*{div}_x \mathbf{w}(\tau)\|^2_{2,\Omega} + \frac{1}{2}\|(1-\bar\chi)\alpha_\lambda\mathbf{D}(x,\mathbf{w}(\tau))\|^2_{2,\Omega} \\
&= \int_\Omega \alpha_F\alpha_\lambda\rho_f\Phi(\mathbf{x},\tau)\operatorname*{div}_x\mathbf{w}(\mathbf{x},\tau)d\mathbf{x} - \int_0^\tau\int_\Omega\alpha_F\alpha_\lambda\rho_f\frac{\partial\Phi}{\partial t}\operatorname*{div}_x\mathbf{w}d\mathbf{x}dt \\
&\quad - \int_\Omega\alpha_F\alpha_\lambda(1-\bar\chi(\mathbf{x}))(\rho_s-\rho_f)\nabla_x\Phi(\mathbf{x},\tau)\cdot\mathbf{w}(\mathbf{x},\tau)d\mathbf{x} \\
&\quad + \int_0^\tau\int_\Omega\alpha_F\alpha_\lambda(1-\bar\chi)(\rho_s-\rho_f)\Big(\frac{\partial}{\partial t}\nabla_x\Phi\Big)\cdot\mathbf{w}d\mathbf{x}dt \\
&\quad + \int_\Omega\alpha_\lambda\bar\alpha_\theta(\mathbf{x})\theta(\mathbf{x},\tau)\operatorname*{div}_x\mathbf{w}(\mathbf{x},\tau)d\mathbf{x} \\
&\quad - \int_0^\tau\int_\Omega\alpha_\lambda\bar\alpha_\theta\frac{\partial\theta}{\partial t}\operatorname*{div}_x\mathbf{w}d\mathbf{x}dt, \quad \forall\,\tau\in[0,T].
\end{aligned}\tag{7.7}$$

Now, on the right-hand side apply the Cauchy-Schwartz and Korn's inequality (4.13) (with Ω_s substituted for Ω) along with a simple estimate

$$\|\phi\|^2_{2,\Omega\times(0,\tau)} \le \tau\operatorname*{ess\,sup}_{t\in[0,\tau]}\|\phi(t)\|^2_{2,\Omega}, \quad \forall\,\phi\in L^\infty(0,T;L^2(\Omega)) \tag{7.8}$$

to obtain

$$\begin{aligned}
&\frac{1}{2}\operatorname*{ess\,sup}_{t\in[0,\tau]}\alpha_\tau\alpha_\lambda\Big\|\sqrt{\bar\rho}\frac{\partial \mathbf{w}}{\partial t}(t)\Big\|^2_{2,\Omega} + \alpha_\nu\alpha_\lambda\Big\|\bar\chi \operatorname*{div}_x \frac{\partial \mathbf{w}}{\partial t}\Big\|^2_{2,\Omega\times(0,\tau)} \\
&+ \alpha_\mu\alpha_\lambda\Big\|\bar\chi\mathbf{D}\Big(x,\frac{\partial \mathbf{w}}{\partial t}\Big)\Big\|^2_{2,\Omega\times(0,\tau)} + \frac{1}{2}\operatorname*{ess\,sup}_{t\in[0,\tau]}\alpha_\lambda\alpha_\eta\|(1-\bar\chi)\operatorname*{div}_x \mathbf{w}(t)\|^2_{2,\Omega} \\
&+ \frac{1}{2}\operatorname*{ess\,sup}_{t\in[0,\tau]}\alpha_\lambda\alpha_p\|\bar\chi \operatorname*{div}_x \mathbf{w}(t)\|^2_{2,\Omega} + \frac{1}{2}\operatorname*{ess\,sup}_{t\in[0,\tau]}\|(1-\bar\chi)\alpha_\lambda\mathbf{D}(x,\mathbf{w}(t))\|^2_{2,\Omega} \\
&\le \frac{\alpha_F^2\rho_f^2}{2\varepsilon_0}\operatorname*{ess\,sup}_{t\in[0,\tau]}\|(1-\bar\chi)\Phi(t)\|^2_{2,\Omega} + \frac{\alpha_F^2\rho_f^2}{2\varepsilon_1}\operatorname*{ess\,sup}_{t\in[0,\tau]}\|\bar\chi\Phi(t)\|^2_{2,\Omega} \\
&\quad + \frac{\alpha_F^2\rho_f^2}{2\varepsilon_2}\Big\|(1-\bar\chi)\frac{\partial\Phi}{\partial t}\Big\|^2_{2,\Omega\times(0,\tau)} + \frac{\alpha_F^2\rho_f^2}{2\varepsilon_3}\Big\|\bar\chi\frac{\partial\Phi}{\partial t}\Big\|^2_{2,\Omega\times(0,\tau)}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha_F^2(\rho_s-\rho_f)^2}{2\varepsilon_4}\operatorname*{ess\,sup}_{t\in[0,\tau]}\|(1-\bar\chi)\nabla_x\Phi(t)\|_{2,\Omega}^2\\
&+\frac{\alpha_F^2(\rho_s-\rho_f)^2}{2\varepsilon_5}\Big\|(1-\bar\chi)\frac{\partial}{\partial t}\nabla_x\Phi\Big\|_{2,\Omega\times(0,\tau)}^2\\
&+\frac{\alpha_{\theta s}^2}{2\varepsilon_6}\operatorname*{ess\,sup}_{t\in[0,\tau]}\|(1-\bar\chi)\theta(t)\|_{2,\Omega}^2+\frac{\alpha_{\theta f}^2}{2\varepsilon_7}\operatorname*{ess\,sup}_{t\in[0,\tau]}\|\bar\chi\theta(t)\|_{2,\Omega}^2\\
&+\frac{\alpha_{\theta s}^2}{2\varepsilon_8}\Big\|(1-\bar\chi)\frac{\partial\theta}{\partial t}\Big\|_{2,\Omega\times(0,\tau)}^2+\frac{\alpha_{\theta f}^2}{2\varepsilon_9}\Big\|\bar\chi\frac{\partial\theta}{\partial t}\Big\|_{2,\Omega\times(0,\tau)}^2\\
&+\frac{\alpha_\lambda^2}{2}(\varepsilon_0+\tau\varepsilon_2+\varepsilon_6+\tau\varepsilon_8)\operatorname*{ess\,sup}_{t\in[0,\tau]}\|(1-\bar\chi)\operatorname*{div}_x\mathbf{w}(t)\|_{2,\Omega}^2\\
&+\frac{\alpha_\lambda^2}{2}(\varepsilon_1+\tau\varepsilon_3+\varepsilon_7+\tau\varepsilon_9)\operatorname*{ess\,sup}_{t\in[0,\tau]}\|\bar\chi\operatorname*{div}_x\mathbf{w}(t)\|_{2,\Omega}^2\\
&+\frac{(C_k(\Omega_s))^2}{2}(\varepsilon_4+\tau\varepsilon_5)\operatorname*{ess\,sup}_{t\in[0,\tau]}\|(1-\bar\chi)\alpha_\lambda^2\mathbf{D}(x,\mathbf{w}(t))\|_{2,\Omega}^2,\quad\forall\,\tau\in[0,T].
\end{aligned}
$$

Choose

$$
\varepsilon_0=\varepsilon_6=\frac{\alpha_\eta}{8\alpha_\lambda},\quad \varepsilon_1=\varepsilon_7=\frac{\alpha_p}{8\alpha_\lambda},\quad \varepsilon_2=\varepsilon_8=\frac{\alpha_\eta}{8\tau\alpha_\lambda},
$$
$$
\varepsilon_3=\varepsilon_9=\frac{\alpha_p}{8\tau\alpha_\lambda},\quad \varepsilon_4=\frac{1}{4(C_k(\Omega_s))^2},\quad \varepsilon_5=\frac{1}{4\tau(C_k(\Omega_s))^2},
$$

and set $\tau=T$. Applying energy inequality (4.9) and Lemma 6.1 in order to estimate the norms of θ and $\partial\theta/\partial t$ in the above estimate,we arrive finally at inequality (7.3). $\square$

8. Incompressibility Limits

In this section we prove the theorem, which explains the limiting behavior of solutions of *Model A*, provided with homogeneous initial data, as the coefficients α_p and α_η grow infinitely:

Theorem 8.1. *Let* $\mathbf{F}$, $\partial\mathbf{F}/\partial t$, Ψ, $\partial\Psi/\partial t\in L^2(Q)$, *and initial data be homogeneous, i.e., satisfy* (6.1). *Suppose that in* Model A *the positive constants* ρ_f, ρ_s, α_τ, α_ν, α_μ, α_λ, $\alpha_{\theta f}$, $\alpha_{\theta s}$, α_F, c_{pf}, c_{ps}, $\varkappa_f$, *and* $\varkappa_s$ *are fixed and the coefficients* α_p *and* α_η *depend on a small parameter* $\varepsilon>0$: $\alpha_p=\alpha_p^\varepsilon$, $\alpha_\eta=\alpha_\eta^\varepsilon$. *Let*

$$
\lim_{\varepsilon\searrow0}\alpha_p^\varepsilon=\alpha_p^0,\quad \lim_{\varepsilon\searrow0}\alpha_\eta^\varepsilon=\alpha_\eta^0. \tag{8.1}
$$

For a fixed $\varepsilon>0$ *by* $(\mathbf{w}^\varepsilon,\theta^\varepsilon,p^\varepsilon,q^\varepsilon,\pi^\varepsilon)$ *denote the generalized solution of* Model A *such that the pressures* p^ε, q^ε, *and* π^ε *are normalized, i.e., satisfy equations* (6.4)–(6.6) *instead of* (3.9)–(3.11).

Then the following assertions hold:

1) If $\alpha_p^0\in(0,\infty)$ *and* $\alpha_\eta^0=\infty$, *then the sequence* $(\mathbf{w}^\varepsilon,\theta^\varepsilon,p^\varepsilon,q^\varepsilon,\pi^\varepsilon)$ *is convergent such that*

$$
\mathbf{w}^\varepsilon\underset{\varepsilon\searrow0}{\longrightarrow}\mathbf{w},\ \frac{\partial\mathbf{w}^\varepsilon}{\partial t}\underset{\varepsilon\searrow0}{\longrightarrow}\frac{\partial\mathbf{w}}{\partial t}\quad \text{weakly* in } L^\infty(0,T;\overset{\circ}{W}{}_2^1(\Omega)), \tag{8.2}
$$

$$
\theta^\varepsilon\underset{\varepsilon\searrow0}{\longrightarrow}\theta\quad \text{weakly in } L^2(0,T;\overset{\circ}{W}{}_2^1(\Omega)),\quad \text{weakly* in } L^\infty(0,T;L^2(\Omega)), \tag{8.3}
$$

$$p^\varepsilon \underset{\varepsilon\searrow 0}{\longrightarrow} p,\ q^\varepsilon \underset{\varepsilon\searrow 0}{\longrightarrow} q,\ \pi^\varepsilon \underset{\varepsilon\searrow 0}{\longrightarrow} \pi \quad \textit{weakly in } L^2(Q), \tag{8.4}$$

and the five limiting functions $(\mathbf{w}, \theta, p, q, \pi)$ *are the generalized solution of* Model B1 *with* $\alpha_p = \alpha_p^0$. *(Statement of* Model B1 *and the notion of its generalized solution are given immediately after formulation of the theorem.)*

2) If $\alpha_p^0 = \infty$ *and* $\alpha_\eta^0 \in (0, \infty)$, *then the sequence* $(\mathbf{w}^\varepsilon, \theta^\varepsilon, p^\varepsilon, q^\varepsilon, \pi^\varepsilon)$ *is convergent such that the limiting relations* (8.2)–(8.4) *hold true and, moreover,* $p = q$ *and the four limiting functions* $(\mathbf{w}, \theta, q, \pi)$ *are the generalized solution of* Model B2 *with* $\alpha_\eta = \alpha_\eta^0$. *(Statement of* Model B2 *and the notion of its generalized solution are given after statement of the definition of generalized solution of* Model B1.*)*

3) If $\alpha_p^0 = \alpha_\eta^0 = \infty$, *then the sequence* $(\mathbf{w}^\varepsilon, \theta^\varepsilon, p^\varepsilon, q^\varepsilon, \pi^\varepsilon)$ *is convergent such that the limiting relations* (8.2)–(8.4) *hold true and, moreover,* $p = q$ *and the triple of limiting functions* $(\mathbf{w}, \theta, \mathfrak{H} := q + \pi)$ *is the generalized solution of* Model B3. *(Statement of* Model B3 *and the notion of its generalized solution are given after statement of the definition of generalized solution of* Model B2.*)*

Statement of Model B1. In the space-time cylinder Q it is necessary to find a displacement vector $\mathbf{w}$, a temperature distribution θ, and distributions of pressures p, q, and π, which satisfy equations (3.6)–(3.8), (6.4), and (6.6), incompressibility condition in the solid phase

$$(1 - \bar{\chi}) \operatorname*{div}_x \mathbf{w} = 0, \quad (\mathbf{x}, t) \in Q, \tag{8.5}$$

homogeneous initial data (6.1), and homogeneous boundary conditions (3.13).

Definition 8.2. The set of functions $(\mathbf{w}, \theta, p, q, \pi)$ is called a generalized solution of *Model B1* if they satisfy regularity conditions (3.14), condition $\pi \in L^2(Q)$, initial data (6.1), integral equalities (3.15) and (3.16), and equations (3.7), (6.4), (6.6), and (8.5) a.e. in Q.

Statement of Model B2. In the space-time cylinder Q it is necessary to find a displacement vector $\mathbf{w}$, a temperature distribution θ, and distributions of pressures q, and π, which satisfy equations (3.6)–(3.8) and (6.5), incompressibility condition in the liquid phase

$$\bar{\chi} \operatorname*{div}_x \mathbf{w} = 0, \quad (\mathbf{x}, t) \in Q, \tag{8.6}$$

homogeneous initial data (6.1), and homogeneous boundary conditions (3.13).

Definition 8.3. The set of functions $(\mathbf{w}, \theta, q, \pi)$ is called a generalized solution of *Model B2* if they satisfy regularity conditions (3.14), condition $q \in L^2(Q)$, initial data (6.1), integral equalities (3.15) and (3.16), and equations (3.7), (6.5), and (8.6) a.e. in Q.

Statement of Model B3. In the space-time cylinder Q it is necessary to find a displacement vector $\mathbf{w}$, a temperature distribution θ, and a distribution of pressure $\mathfrak{H}$, which satisfy equations (3.6), (3.8), and

$$\mathbf{P} = -(\mathfrak{H} + \bar{\alpha}_\theta \theta)\mathbf{I} + \bar{\chi}\alpha_\mu \mathbf{D}\Big(x, \frac{\partial \mathbf{w}}{\partial t}\Big) + (1 - \bar{\chi})\alpha_\lambda \mathbf{D}(x, \mathbf{w}), \quad (\mathbf{x}, t) \in Q, \tag{8.7}$$

incompressibility condition in the both phases

$$\operatorname*{div}_x \mathbf{w} = 0, \quad (\mathbf{x}, t) \in Q, \tag{8.8}$$

homogeneous initial data (6.1), and homogeneous boundary conditions (3.13).

Definition 8.4. The triple $(\mathbf{w}, \theta, \mathfrak{H})$ is called a generalized solution of *Model B3* if it satisfies regularity conditions (3.14), condition $\mathfrak{H} \in L^2(Q)$, initial data (6.1), integral equalities (3.15) and (3.16), and equations (8.7) and (8.8) a.e. in Q.

Remark 8.5. Clearly Theorem 8.1 provides the existence results for generalized solutions to *Models B1–B3*, as a byproduct. At the same time, existence and uniqueness of generalized solutions of *Models B1–B3* may be justified without considering *Model A* firstly, but starting from Definitions 8.2–8.4, introducing the proper Galerkin's approximations, and keeping track of the proof of Theorem 3.4. Thus, *Models B1–B3* are well-posed.

Remark 8.6. Equations (8.5), (8.6), and (8.8) are indeed the conditions of incompressibility, since, on the strength of continuity equations (2.29) and (2.33), they imply that $\rho \equiv \rho_s$ in $\Omega_s \times (0,T)$, $\rho \equiv \rho_f$ in $\Omega_f \times (0,T)$, and $\rho \equiv \bar{\rho}$ in Q, respectively. Consequently, *Models B1, B2*, and *B3* are respective solid, liquid, and total incompressibility limits of *Model A*.

Proof of Theorem 8.1. We verify assertion 3 of the theorem only. Justification of assertions 1 and 2 is quite similar.

If $\alpha_p^\varepsilon, \alpha_\eta^\varepsilon \underset{\varepsilon\searrow 0}{\longrightarrow} \infty$ then there exist a subsequence $(\mathbf{w}^\varepsilon, \theta^\varepsilon, p^\varepsilon, q^\varepsilon, \pi^\varepsilon)$ and a limiting set of functions $(\mathbf{w}, \theta, p, q, \pi)$ such that limiting relations (8.2)–(8.4) hold true, due to energy estimate (4.9), Lemma 6.1, Corollary 4.4, Theorem 6.3, and homogeneous initial conditions (6.1). Energy estimate (4.9) yields that

$$\operatorname{ess\,sup} t \in [0,T] \| \bar{\chi} \operatorname*{div}_x \mathbf{w}^\varepsilon \|_{2,\Omega}^2 \leq 2(C_{\mathrm{en}}(T)/\alpha_p^\varepsilon) \underset{\varepsilon\searrow 0}{\longrightarrow} 0,$$
$$\operatorname{ess\,sup} t \in [0,T] \| (1-\bar{\chi}) \operatorname*{div}_x \mathbf{w}^\varepsilon \|_{2,\Omega}^2 \leq 2(C_{\mathrm{en}}(T)/\alpha_\eta^\varepsilon) \underset{\varepsilon\searrow 0}{\longrightarrow} 0,$$

since initial data are homogeneous and therefore $C_{\mathrm{en}}(T)$ does not depend on ε. Hence (8.8) holds true for the limiting function $\mathbf{w}$.

Next, substitute (6.4) into (6.6) and pass to the limit as $\varepsilon \searrow 0$, using limiting relations (8.2) and (8.4), to get

$$q = p - \bar{\chi}\alpha_\nu (\partial/\partial t) \operatorname*{div}_x \mathbf{w}.$$

By (8.8) this equality yields $q = p$ on Q.

Now, from the limiting relations (8.2)–(8.4), passing to the limit in (3.15) and (3.16) as $\varepsilon \searrow 0$, we conclude that the triple $(\mathbf{w}, \theta, q + \pi)$ is a generalized solution of *Model B3*.

To complete the proof of assertion 3, it remains to notice that due to Remark 8.5 the solution of *Model B3* is unique. Hence the sequence $(\mathbf{w}^\varepsilon, \theta^\varepsilon, p^\varepsilon, q^\varepsilon, \pi^\varepsilon)$ has exactly one (weak) limiting point and therefore converges entirely, and there is no need to shift to a subsequence. □

9. Solidification Limits

In this section we observe limiting behavior of solutions of *Model A*, as the coefficient α_λ grows infinitely. First, we prove the following result.

Theorem 9.1. *Let* $\mathbf{F}$, $\partial\mathbf{F}/\partial t$, Ψ, $\partial\Psi/\partial t \in L^2(Q)$, *and initial data be homogeneous, i.e., satisfy* (6.1)*. Let Assumption 7.2 hold.*

Suppose that in Model A *the positive constants* ρ_f, ρ_s, α_τ, α_p, α_η, α_ν, α_μ, $\alpha_{\theta f}$, $\alpha_{\theta s}$, α_F, c_{pf}, c_{ps}, $\varkappa_f$, *and* $\varkappa_s$ *are fixed and the coefficient* α_λ *depends on a small parameter* $\varepsilon > 0$, $\alpha_\lambda = \alpha_\lambda^\varepsilon$. *Let*

$$\lim_{\varepsilon\searrow 0} \alpha_\lambda^\varepsilon = \infty. \tag{9.1}$$

By $(\mathbf{w}^\varepsilon, \theta^\varepsilon, p^\varepsilon, q^\varepsilon, \pi^\varepsilon)$ *denote the generalized solution of* Model A *corresponding to a fixed* $\varepsilon > 0$.

Then the sequence $(\mathbf{w}^\varepsilon, \theta^\varepsilon, p^\varepsilon, q^\varepsilon, \pi^\varepsilon)$ *is convergent such that*

$$\mathbf{w}^\varepsilon \underset{\varepsilon\searrow 0}{\longrightarrow} \mathbf{w},\ \frac{\partial \mathbf{w}^\varepsilon}{\partial t} \underset{\varepsilon\searrow 0}{\longrightarrow} \frac{\partial \mathbf{w}}{\partial t} \quad \textit{weakly* in } L^\infty(0,T;\overset{\circ}{W}{}_2^1(\Omega)), \tag{9.2}$$

$$(1-\bar{\chi})\nabla_x \mathbf{w}^\varepsilon \underset{\varepsilon\searrow 0}{\longrightarrow} 0,\ \pi^\varepsilon \underset{\varepsilon\searrow 0}{\longrightarrow} 0 \quad \textit{strongly in } L^\infty(0,T;L^2(\Omega)), \tag{9.3}$$

$$\theta^\varepsilon \underset{\varepsilon\searrow 0}{\longrightarrow} \theta \quad \textit{weakly in } L^2(0,T;\overset{\circ}{W}{}_2^1(\Omega)), \quad \textit{weakly* in } L^\infty(0,T;L^2(\Omega)), \tag{9.4}$$

$$p^\varepsilon \underset{\varepsilon\searrow 0}{\longrightarrow} p,\ q^\varepsilon \underset{\varepsilon\searrow 0}{\longrightarrow} q \quad \textit{weakly in } L^2(Q), \tag{9.5}$$

and the four limiting functions $(\mathbf{w}, \theta, p, q)$ *are the generalized solution of* Model C1.

Statement of Model C1. In the space-time cylinder Q it is necessary to find a displacement vector $\mathbf{w}$, a temperature distribution θ, and distributions of pressures p and q, which satisfy the equations

$$\alpha_\tau \rho_f \frac{\partial^2 \mathbf{w}}{\partial t^2} = \operatorname*{div}_x \mathbf{P}_f + \alpha_F \rho_f \mathbf{F}, \quad \text{in } \Omega_f \times (0,T), \tag{9.6}$$

$$\mathbf{P}_f = -(q + \alpha_{\theta f}\theta)\mathbf{I} + \alpha_\mu \mathbf{D}\Big(x, \frac{\partial \mathbf{w}}{\partial t}\Big)\Big), \quad \text{in } \Omega_f \times (0,T), \tag{9.7}$$

$$(1-\bar{\chi})\mathbf{w} = 0, \quad \text{in } Q, \tag{9.8}$$

$$\bar{c}_p \frac{\partial \theta}{\partial t} = \operatorname*{div}_x(\bar{\varkappa}\nabla_x \theta) - \bar{\chi}\alpha_{\theta f}\frac{\partial}{\partial t}\operatorname*{div}_x \mathbf{w} + \Psi, \quad \text{in } Q, \tag{9.9}$$

$$p + \bar{\chi}\alpha_p \operatorname*{div}_x \mathbf{w} = 0, \quad \text{in } Q, \tag{9.10}$$

$$q = p + \frac{\alpha_\nu}{\alpha_p}\frac{\partial p}{\partial t}, \quad \text{in } Q, \tag{9.11}$$

homogeneous initial data (6.1), and homogeneous boundary conditions

$$\mathbf{w} = 0, \quad \text{for } \mathbf{x} \in \partial\Omega_f,\ t \geq 0; \quad \theta = 0, \quad \text{for } \mathbf{x} \in \partial\Omega,\ t \geq 0. \tag{9.12}$$

Definition 9.2. A set of functions $(\mathbf{w}, \theta, p, q)$ is called a generalized solution of *Model C1* if they satisfy the regularity conditions

$$\mathbf{w},\ \frac{\partial \mathbf{w}}{\partial t},\ \nabla_x \mathbf{w},\ \theta,\ \nabla_x \theta \in L^2(Q), \tag{9.13}$$

equations (9.7), (9.8), (9.10), and (9.11) a.e. in Q, and the integral equalities

$$\int_0^T \int_{\Omega_f} \Big(\alpha_\tau \rho_f \frac{\partial \mathbf{w}}{\partial t}\cdot\frac{\partial \varphi}{\partial t} - \mathbf{P}_f : \nabla_x \varphi + \alpha_F \rho_f \mathbf{F}\cdot\varphi\Big) d\mathbf{x}dt = 0 \tag{9.14}$$

for all smooth $\varphi = \varphi(\mathbf{x},t)$ such that $\varphi|_{\partial\Omega_f} = \varphi|_{t=T} = 0$ and

$$\int_Q \Big(\bar{c}_p \theta \frac{\partial \psi}{\partial t} - \bar{\varkappa}\nabla_x \theta\cdot\nabla_x \psi + \bar{\chi}\alpha_{\theta f}(\operatorname*{div}_x \mathbf{w})\frac{\partial \psi}{\partial t} + \Psi\psi\Big) d\mathbf{x}dt = 0 \tag{9.15}$$

for all smooth $\psi = \psi(\mathbf{x},t)$ such that $\psi|_{\partial\Omega} = \psi|_{t=T} = 0$.

Remark 9.3. As in Remark 8.5, we notice that Theorem 9.1 provides the existence result for *Model C1* as a byproduct, and that existence and uniqueness of generalized solutions of *Model C1* may be justified independently of Theorem 9.1 by considerations similar to the proof of Theorem 3.4.

Proof of Theorem 9.1. Firstly, we have that limiting relations (9.2), (9.4), and (9.5) hold true for some subsequence $(\mathbf{w}^\varepsilon, \theta^\varepsilon, p^\varepsilon, q^\varepsilon)$ due to energy estimate (4.9) and Lemma 6.1.

Secondly, bound (7.1) immediately implies that the limiting displacement vector $\mathbf{w}$ satisfies the equality $(1-\bar{\chi})\mathbf{D}(x,\mathbf{w}) = 0$ in Q. On the strength of Remark 7.1 and Assumption 7.2, this equality yields that (9.8) holds. From bound (7.1), Assumption 7.2, homogeneity of initial data (see (6.1)), and Korn's inequality (4.13) (with Ω_s substituted for Ω) it follows that

$$\begin{aligned} &\operatorname*{ess\,sup}_{t\in[0,\tau]}\big(\|(1-\bar{\chi})\mathbf{w}^\varepsilon(t)\|^2_{2,\Omega} + \|(1-\bar{\chi})\nabla_x\mathbf{w}^\varepsilon(t)\|^2_{2,\Omega}\big) \\ &\leq (C_k(\Omega_s))^2 \operatorname*{ess\,sup}_{t\in[0,\tau]} \|(1-\bar{\chi})\mathbf{D}(x,\mathbf{w}^\varepsilon(t))\|^2_{2,\Omega} \underset{\varepsilon\searrow 0}{\longrightarrow} 0, \end{aligned}$$

which proves limiting relations (9.3). (By $C_k(\Omega_s)$ the constant in Korn's inequality is denoted.)

Next, from the limiting relations (9.2)–(9.5), passing to the limit in (3.15), (3.16), (3.9), and (3.10) as $\varepsilon \searrow 0$, we conclude that the four functions $(\mathbf{w},\theta,p,q)$ are a generalized solution of *Model C1*.

To complete the proof, it remains to notice that due to Remark 9.3 the solution of *Model C1* is unique. Hence the sequence $(\mathbf{w}^\varepsilon, \theta^\varepsilon, p^\varepsilon, q^\varepsilon)$ has exactly one (weak) limiting point and therefore converges entirely and there is no need to shift to a subsequence. $\square$

To complete this article, we derive a rather peculiar asymptotic behavior, which is observed thanks to additional bound (7.3). Namely, we prove the following result.

Theorem 9.4. *Let Assumption 7.2 hold, initial data be homogeneous, i.e., satisfy* (6.1)*, the given density of distributed mass forces have the form*

$$\mathbf{F}(\mathbf{x},t) = \nabla_x\Phi(\mathbf{x},t) \tag{9.16}$$

with some potential $\Phi \in W^1_2(Q)$ such that $(\partial/\partial t)\nabla_x\Phi \in L^2(Q)$, and the given volumetric density of exterior heat application be such that $\Psi, \partial\Psi/\partial t \in L^2(Q)$.

Suppose that in Model A *the positive constants ρ_f, ρ_s, α_τ, α_ν, α_μ, $\alpha_{\theta f}$, $\alpha_{\theta s}$, α_F, c_{pf}, c_{ps}, $\varkappa_f$, and $\varkappa_s$ are fixed and the coefficients α_λ, α_p, and α_η depend on a small parameter $\varepsilon > 0$: $\alpha_\lambda = \alpha^\varepsilon_\lambda$, $\alpha_p = \alpha^\varepsilon_p$, and $\alpha_\eta = \alpha^\varepsilon_\eta$. Let limiting relations* (7.4) *hold, or, equivalently,*

$$\lim_{\varepsilon\searrow 0}\alpha^\varepsilon_\lambda = \infty, \quad \lim_{\varepsilon\searrow 0}\frac{\alpha^\varepsilon_p}{\alpha^\varepsilon_\lambda} = \alpha^0_p \in (0,\infty), \quad \lim_{\varepsilon\searrow 0}\frac{\alpha^\varepsilon_\eta}{\alpha^\varepsilon_\lambda} = \alpha^0_\eta \in (0,\infty). \tag{9.17}$$

By $(\mathbf{w}^\varepsilon, \theta^\varepsilon, p^\varepsilon, q^\varepsilon, \pi^\varepsilon)$ denote the generalized solution of Model A *corresponding to a fixed $\varepsilon > 0$.*

Then the sequence $(\mathbf{w}^\varepsilon, \theta^\varepsilon, p^\varepsilon, q^\varepsilon, \pi^\varepsilon)$ is convergent and there exist functions $(\mathbf{u},\theta,p)$ such that

$$\mathbf{w}^\varepsilon \underset{\varepsilon\searrow 0}{\longrightarrow} 0, \ \frac{\partial\mathbf{w}^\varepsilon}{\partial t} \underset{\varepsilon\searrow 0}{\longrightarrow} 0 \quad \textit{strongly in } L^\infty(0,T;L^2(\Omega)), \tag{9.18}$$

$$\theta^\varepsilon \underset{\varepsilon\searrow 0}{\longrightarrow} \theta \quad \textit{weakly in } L^2(0,T;\overset{\circ}{W}{}^1_2(\Omega)), \quad \textit{weakly* in } L^\infty(0,T;L^2(\Omega)), \tag{9.19}$$

$$p^\varepsilon \underset{\varepsilon\searrow 0}{\longrightarrow} p, \; q^\varepsilon \underset{\varepsilon\searrow 0}{\longrightarrow} p, \tag{9.20}$$

$$\pi^\varepsilon \underset{\varepsilon\searrow 0}{\longrightarrow} -(1-\bar{\chi})\alpha_\eta^0 \operatorname*{div}_x \mathbf{u}, \tag{9.21}$$

$$(1-\bar{\chi})\alpha_\lambda^\varepsilon \mathbf{w}^\varepsilon \underset{\varepsilon\searrow 0}{\longrightarrow} \mathbf{u} \quad \textit{weakly* in } L^\infty(0,T;L^2(\Omega)), \tag{9.22}$$

$$(1-\bar{\chi})\mathbf{D}(x,\alpha_\lambda^\varepsilon \mathbf{w}^\varepsilon) \underset{\varepsilon\searrow 0}{\longrightarrow} (1-\bar{\chi})\mathbf{D}(x,\mathbf{u}) \quad \textit{weakly* in } L^\infty(0,T;L^2(\Omega)), \tag{9.23}$$

and the triple $(\mathbf{u},\theta,p)$ *is the generalized solution of* Model C2*, formulated below.*

Statement of Model C2. Find successively the following:
(C2a) a temperature distribution θ, solving the initial-boundary value problem for the heat equation

$$\bar{c}_p \frac{\partial \theta}{\partial t} = \operatorname*{div}_x(\bar{\varkappa}\nabla_x \theta) + \Psi, \text{ in } Q, \tag{9.24}$$

$$\theta|_{t=0} = 0, \text{ on } \Omega, \quad \theta|_{\partial\Omega} = 0, \text{ for } t > 0; \tag{9.25}$$

(C2b) a hydrostatic pressure p in the liquid phase from the equilibrium equation

$$p = -\alpha_{\theta f}\theta + \alpha_F \rho_f \Phi, \text{ in } \Omega_f \times (0,T); \tag{9.26}$$

(C2c) an up-scaled displacement vector field $\mathbf{u}$, solving the mixed problem for the 3D system of stationary wave equations

$$\operatorname*{div}_x \mathbf{D}(x,\mathbf{u}) + \alpha_\eta^0 \nabla_x \operatorname*{div}_x \mathbf{u} = \alpha_{\theta s}\nabla_x\theta - \alpha_F \rho_s \nabla_x \Phi, \text{ in } \Omega_s, \tag{9.27}$$

$$\mathbf{D}(x,\mathbf{u})\cdot\mathbf{n} + \alpha_\eta^0(\operatorname*{div}_x \mathbf{u})\mathbf{n} = (\alpha_{\theta s}\theta - \alpha_F\rho_f\Phi)\mathbf{n}, \text{ on } \Gamma, \tag{9.28}$$

$$\mathbf{u} = 0, \text{ on } (\partial\Omega_s \setminus \Gamma). \tag{9.29}$$

(Here $\mathbf{n}$ is the outward unit normal to $\partial\Omega_s$. Variable t appears in the system as a parameter. Also, recall that $\Gamma := \partial\Omega_s \cap \partial\Omega_f$ and that the Lebesgue measure of $\partial\Omega_s \setminus \Gamma$ is positive due to Assumption 7.2.)

Definition 9.5. The triple $(\mathbf{u},\theta,p)$ is called a generalized solution of *Model C2* if it satisfies the regularity conditions

$$\theta, \, \nabla_x\theta, \, p, \, (1-\bar{\chi})\nabla_x \mathbf{u} \in L^\infty(0,T;L^2(\Omega)), \tag{9.30}$$

and the integral equalities

$$\begin{aligned}\int_Q \Big(\big(\bar{\chi}p - (1-\bar{\chi})\alpha_\eta^0 \operatorname*{div}_x \mathbf{u} + \bar{\alpha}_\theta \theta\big) \operatorname*{div}_x \varphi \\ - (1-\bar{\chi})\mathbf{D}(x,\mathbf{u}) : \mathbf{D}(x,\varphi) + \alpha_F \bar{\rho}\nabla_x\Phi\cdot\varphi\Big) d\mathbf{x}dt = 0\end{aligned} \tag{9.31}$$

for all smooth $\varphi = \varphi(\mathbf{x},t)$ such that $\varphi|_{\partial\Omega} = \varphi|_{t=T} = 0$ and

$$\int_Q \Big(\bar{c}_p \theta \frac{\partial \psi}{\partial t} - \bar{\varkappa}\nabla_x\theta\cdot\nabla_x\psi + \Psi\psi\Big) d\mathbf{x}dt = 0 \tag{9.32}$$

for all smooth $\psi = \psi(\mathbf{x},t)$ such that $\psi|_{\partial\Omega} = \psi|_{t=T} = 0$.

Remark 9.6. By the standard arguments it is easy to verify that integral equalities (9.31) and (9.32) are equivalent to equations (9.24)–(9.29) in the distributions sense. Due to the well-known theory of second-order elliptic and parabolic equations [6, 7] there exists exactly one generalized solution of *Model C2*, whenever Φ and Ψ satisfy conditions of Theorem 9.4 and whenever we prescribe some certain values to $(1-\bar{\chi})p$ and $\bar{\chi}\mathbf{u}$. (For example, we may simply set $(1-\bar{\chi})p = \bar{\chi}\mathbf{u} = 0$.)

Proof of Theorem 9.4. From energy estimate (4.9), bound (7.3), limiting relations (9.17), and Remark 7.5, we conclude that limiting relations (9.19)–(9.23) hold true, at least for some subsequence, and that limiting relation (9.18) and relations

$$\bar{\chi}\alpha_\nu \operatorname*{div}_x \frac{\partial \mathbf{w}^\varepsilon}{\partial t} \underset{\varepsilon\searrow 0}{\longrightarrow} 0, \quad \bar{\chi}\alpha_\mu \mathbf{D}\Big(x, \frac{\partial \mathbf{w}^\varepsilon}{\partial t}\Big) \underset{\varepsilon\searrow 0}{\longrightarrow} 0 \quad \text{strongly in } L^2(Q) \tag{9.33}$$

are fulfilled for the entire sequence $\varepsilon \searrow 0$.

The limiting functions $\mathbf{u}$, p, and θ satisfy regularity conditions (9.30) due to relations (9.19)–(9.23) and Korn's inequality

$$\|\mathbf{u}(t)\|_{W_2^1(\Omega_s)} \leq C_k(\Omega_s)\|\mathbf{D}(x,\mathbf{u}(t))\|_{2,\Omega_s}, \quad \forall\, t \in [0,T].$$

Substituting (3.9)–(3.11), (6.1), and (9.16) into integral equality (3.15), and passing to the limit as $\varepsilon \searrow 0$, we arrive at integral equality (9.31) due to relations (9.18)–(9.23) and (9.33). Analogously we justify that θ satisfies integral equality (9.32). Finally, it remains to notice that, since solution of *Model C2* is unique, the sequence $((1-\bar{\chi})\alpha_\chi^\varepsilon \mathbf{w}^\varepsilon,\ \theta^\varepsilon,\ p^\varepsilon,\ q^\varepsilon,\ \pi^\varepsilon)$ has exactly one (weak) limiting point and therefore converges entirely. $\square$

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