Reduction in laser-intensity fluctuations by a feedback-controlled output mirror

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We present the theory of a laser in which the transmittivity of one output mirror is controlled by a current derived from a photodetector illuminated by the output light from that end of the cavity. That is, one output port of the laser is controlled by a feedback loop. We calculate the photon statistics inside the cavity. We also calculate the spectrum of intensity fluctuations for the light leaving the cavity through the output mirror not controlled by feedback. We show that intensity fluctuations inside the cavity may be reduced to 50% below the Poissonian limit while outside the cavity the reduction is at best 25% of the shot-noise limit. These optimum results, however, are not achieved under the same operating conditions.

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I. INTRODUCTION

The experimental demonstration by Machida and Yamamoto [1] of the feedback intensity fluctuations in a semiconductor laser has engendered a great deal of interest in related schemes to reduce the intensity fluctuations below the shot-noise limit [2–7]. Shapiro *et al.* [3] have shown that the intensity fluctuations on a traveling-wave field may be reduced below the shot-noise limit by passing the beam through a current-controlled beam splitter, the current being derived from a photodetector illuminated by the beam transmitted by the beam splitter. As they point out, the resulting stochastic process for the photon count realizes a self-exciting point process [8]. In this paper we consider a similar scheme in which the feedback-controlled beam splitter forms the output port from a laser cavity. In other words, we present a cavity feedback analog of the traveling-wave feedback model of Shapiro *et al.*

In Sec. II we present a quantum theory of a feedback-controlled cavity field. This is a quantum generalization of the theory of self-exciting point processes [8,9], combined with a general theory of quantum counting given by Srinvivas and Davies [10]. The essential result is a master equation for the intracavity field, which describes multiple photon absorption to all orders. Thus feedback is shown to be equivalent to intracavity nonlinear absorbers. That intracavity nonlinear absorption can reduce laser-intensity fluctuations has been demonstrated by Ritsch [6] and Walls, Collett, and Lane [7].

In Sec. III we consider the situation where the cavity contains a laser gain medium and a second (standard) output mirror. We use the Scully-Lamb laser master equation for the photon number. We show that in the region of saturated gain the photon number fluctuations inside the cavity may be reduced below the Poissonian level by $\frac{1}{2}$. Outside the cavity, intensity fluctuations may also be reduced below the shot-noise level. However, the parameter region for maximum reduction in fluctuations outside the cavity does not coincide with the conditions for maximum reduction of intensity fluctuations inside the cavity. In fact, we show that at the optimum operating point for reducing intensity fluctuations in the output field, reduction in the intracavity fluctuations is half of the maximum level that can be achieved in the model. This is in fact true of nonlinear absorption in general [11].

II. QUANTUM THEORY OF A FEEDBACK-CONTROLLED CAVITY

A single-mode cavity field is coupled to the many modes external to the cavity through the cavity mirrors. This external field may be monitored by a photoelectron counter. The input/output theory of Gardiner and Collett [12] shows that the resulting average count rate is directly proportional to the photon number in the cavity,

$$\left\langle \frac{dN(t)}{dt} \right\rangle = \gamma \langle a^\dagger a \rangle \ , \tag{2.1}$$

where a is the annihilation operator for the field inside the cavity and γ is the damping rate through the end of the cavity. We will assume all photons lost from the cavity are counted and that the only source of loss in the cavity is through the end mirrors. For present purposes we will further assume that the cavity has only one output mirror. Under these assumptions the state of the field inside the cavity evolves according to the master equation

$$\frac{d\rho}{dt} = \frac{\gamma}{2}(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) \ . \tag{2.2}$$

Srinivas and Davies [10] have shown that Eq. (2.2) is a particular example of a more general description of the photon counting process. We will summarize here as much of this theory as is needed for our purposes. The photon-counting process is completely characterized by a set of operations or superoperators $\mathcal{N}_m(\tau)$. This determines the probability of counting m photons in the time interval $(0,\tau)$ by

$$P_m(t) = \mathrm{tr}[\mathcal{N}_m(t)\rho(0)] \ , \tag{2.3}$$

where $\rho(0)$ is the initial state of the cavity field, and also determines the final state of the field conditioned on the tally m by

$$\rho(\tau)=\mathcal{N}_m(\tau)\rho(0)/P_m \ . \tag{2.4}$$

For canonical counting processes [10], this set of operations can be derived from a single-count superoperator Λ defined by

$$\lim_{t\to 0}\frac{1}{t}[\mathcal{N}_1(t)\rho]=\Lambda\rho \ . \tag{2.5}$$

This represents the act of counting one photon. The average count rate is then determined by

$$\left\langle \frac{dN(t)}{dt}\right\rangle = \mathrm{tr}(\Lambda\rho) \ . \tag{2.6}$$

A rate operator (not a superoperator) R can then be uniquely defined [10] by

$$\mathrm{tr}(\rho R)=\mathrm{tr}(\Lambda\rho) \ . \tag{2.7}$$

The action of this operator represents the change in the field when no photons are counted. It can then be shown that the state of the cavity field evolves according to

$$\frac{d\rho(t)}{dt}=\Lambda\rho(t)-\tfrac{1}{2}[R\rho(t)+\rho(t)R] \ . \tag{2.8}$$

The standard photon counting theory then results with the definitions

$$\Lambda=\gamma\mathcal{J} \ , \tag{2.9}$$

$$\mathcal{J}\rho=a\rho a^{\dagger} \ , \tag{2.10}$$

$$R=\gamma a^{\dagger}a \ . \tag{2.11}$$

We are now in a position to present a theory of photodetection in which the count rate depends on the counting history. The count rate is given by Eq. (2.6), but now

$$\Lambda(t)=\Lambda_0\left[1+A\int_{-\infty}^{t} e^{-\beta(t-u)}\Lambda(u)du\right] , \tag{2.12}$$

where A is a dimensionless parameter, $1/\beta$ is a measure of how far back in time the weighted average of the previous counts is taken, and $\Lambda_0=\gamma\mathcal{J}$, as in (2.9). Here we consider only the case $A>0$, corresponding to positive feedback for the cavity loss rate. We have chosen an exponentially decaying memory function for two reasons: (i) it leads to $\Lambda(t)$ being independent of time; and (ii) the corresponding classical model is a Markov self-exciting point process [9]. To see the first point, we simply iterate Eq. (2.12) to get

$$\Lambda=\Lambda_0+\frac{A}{\beta}\Lambda_0^2+\left[\frac{A}{\beta}\right]^2\Lambda_0^3+\cdots \ . \tag{2.13}$$

If $\Lambda(t)=\Lambda$ is independent of time, then we are implicitly assuming that the process is stationary. This assumption is obviously only valid if the cavity field is at steady state. Such a stable steady state is shown to exist in Sec. III, where a laser gain medium is added to the cavity. This does not prove that the above assumption is valid, and the formalism is not sophisticated enough for the stability to be investigated, nor for the effects of a time delay in feedback to be taken into account. Thus, the analysis which follows must be considered as part of a best case scenario, in which the feedback system is stable. In that case, we have

$$\Lambda=\gamma\mathcal{J}\left[1+\frac{A}{\beta}\Lambda\right] \tag{2.14}$$

or

$$\Lambda=\gamma\mathcal{J}\left[1-\mathcal{J}\chi\right]^{-1} , \tag{2.15}$$

where $\chi=\gamma A/\beta$. This last expression could also have been obtained from the above expansion (2.13). For this expansion to be meaningful we need the norm of $\mathcal{J}\chi$ to be less than unity in some sense. Using $\mathrm{tr}(|\chi|\mathcal{J}\rho)$ as the norm, we thus need

$$|\chi|\bar{n}<1 \ . \tag{2.16}$$

Then we may write Λ as

$$\Lambda=\gamma\mathcal{J}\sum_{n=0}^{\infty}\chi^n\mathcal{J}^n \ . \tag{2.17}$$

The system then obeys a master equation which follows directly from the Srinivas-Davies method outlined above, with the count operation being given by (2.16) and the rate operator given by

$$R=\gamma\sum_{n=0}^{\infty}\chi^n(a^{\dagger})^n a^{\dagger}a(a)^n \ . \tag{2.18}$$

The state of the cavity mode in the presence of the feedback is then determined by

$$\frac{d\rho}{dt}=\frac{\gamma}{2}\sum_{n=0}^{\infty}\chi^n(2a^{n+1}\rho a^{\dagger n+1}-a^{\dagger n+1}a^{n+1}\rho -\rho a^{\dagger n+1}a^{n+1}) \ . \tag{2.19}$$

III. LASER WITH FEEDBACK

Consider the laser system depicted in Fig. 1. A single-mode field is contained in an optical cavity and interacts with an atomic gain medium. The field is damped through loss at each end of the cavity. However, the loss rate γ at one output port of the cavity is controlled by a photodetection feedback loop of the kind discussed in the

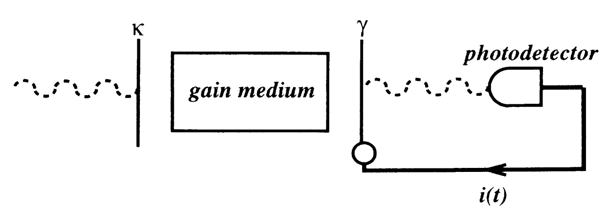

FIG. 1. Schematic representation of a laser with one output port controlled by a feedback loop. The mirror labeled γ is a current-controlled beam splitter. The current is derived from a photodetector illuminated by the light leaving that end of the cavity.

preceding section, with χ positive to reduce intensity fluctuations. The loss rate at the other output port is κ. We wish to calculate the statistics both inside the cavity and at the output port not controlled by the feedback loop. To describe the effect of the atomic gain medium we will use the Scully-Lamb model [13]. This model assumes that the decay rates of the atomic medium are much greater than the field decay rates and thus the atomic dynamics may be adiabatically eliminated to give an equation for the field state alone. It is parametrized by a gain rate G, and a saturation photon number n_s.

The photon-number distribution in the cavity $P_n = \langle n|\rho|n\rangle$ obeys the master equation

$$\frac{dP_n}{dt} = -Gn_s\left[\frac{n+1}{n_s+n+1}P_n - \frac{n}{n_s+n}P_{n-1}\right] + \kappa[(n+1)P_{n+1} - nP_n] + \sum_{k=0}^{\infty}\gamma\chi^k\frac{(n+k+1)!}{n!}P_{k+n+1} - \sum_{k=0}^{n-1}\gamma\chi^k\frac{n!}{(n-k-1)!}P_n \ . \tag{3.1}$$

We adopt the procedure used in Ref. [4] to convert this equation to a Fokker-Planck equation. The short-time solution with the initial condition $P_n(0)=\delta_{nm}$ is

$$P_n(t) = \delta_{nm}\left[1 - t\left[\kappa m + \frac{\gamma}{\chi}\sum_{k=1}^{m}\chi^k\frac{m!}{(m-k)!} - Gn_s\frac{m+1}{m+1+n_s}\right]\right] + \delta_{n,m-1}\kappa t m + \delta_{n,m+1}Gn_s t\frac{m+1}{m+1+n_s} + \frac{\gamma t}{\chi}\sum_{k=1}^{m}\delta_{n,m-k}\frac{m!}{(m-k)!}\chi^k \ . \tag{3.2}$$

Now for m large and $m\chi$ finitely less than one, we can replace the sums in this equation by $\sum_{k=1}^{\infty}(\chi m)^k\delta_{n,m-k}$. We can also ignore 1 compared to m. Thus the drift and diffusion coefficients can be written

$$d(m) = \frac{1}{t}\langle n-m\rangle = -\kappa m - \frac{\gamma m}{(1-m\chi)^2} + Gn_s\frac{m}{m+n_s} \ , \tag{3.3}$$

$$D(m) = \frac{1}{t}\langle (n-m)^2\rangle = \kappa m + \gamma m\frac{1+m\chi}{(1-m\chi)^3} + Gn_s\frac{m}{m+n_s} \ . \tag{3.4}$$

We now define the following scaled parameters

$$x = m\chi < 1 \ , \tag{3.5}$$

$$\nu = \frac{G}{\kappa}n_s \ , \tag{3.6}$$

$$y = \nu\chi \ , \tag{3.7}$$

$$s = \frac{n_s}{\nu} = \frac{\kappa}{G} < 1 \ , \tag{3.8}$$

$$r = \frac{\gamma}{\kappa} \ . \tag{3.9}$$

Note that ν is simply the mean photon number for a laser well above threshold in the absence of the feedback. The parameters y and r define the feedback, which must have χ positive. The parameter s is a measure of how far above threshold the laser would operate in the absence of feedback; well above threshold, s approaches zero.

To find the mean photon number (scaled as x) we put $d(m)=0$, which gives

$$1 + \frac{r}{(1-x)^2} = \frac{y}{sy+x} \ . \tag{3.10}$$

For any value of y, this equation has a solution x satisfying $0<x<1$ and $x<y(1-s)$, providing that $s<(1+r)^{-1}$. This last inequality translates as $G>\gamma+\kappa$, which is simply the threshold condition for the laser. In Figs. 2(a) and 2(b) we plot x vs y for the solution less than unity. We see that as y increases x approaches unity. If the laser is well above threshold and the relative decay rate r is small, this limit is approached more quickly.

To calculate the photon-number statistics inside and outside the cavity we linearize the nonlinear Fokker-Planck process (3.1) about the steady state given by (3.10) to obtain an Ornstein-Uhlenbeck process [4]. The linearized drift and diffusion constants are

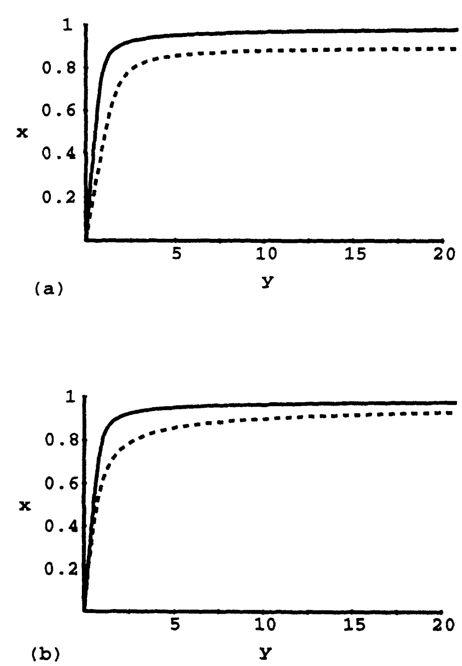

FIG. 2. Scaled mean photon number inside the cavity vs the feedback strength parameter. (a) The plot has $r=0.001$ with $s=0$ (solid) and $s=0.5$ (dashed). (b) The plot has $s=0$ with $r=0.01$ (solid) and $r=0.1$ (dashed).

$$k=-d'(\bar{n})=\kappa\left[1+\frac{r(1+x)}{(1-x)^3}-s\left[1+\frac{r}{(1-x)^2}\right]^2\right], \quad (3.11)$$

$$D=D(\bar{n})=2\kappa\bar{n}\left[1+\frac{r}{(1-x)^3}\right]. \quad (3.12)$$

It is simple to prove that

$$k>\kappa\frac{2xr}{(1-x)^3}, \quad (3.13)$$

so that the solution of (3.10) is stable provided that $x<1$. Thus we can immediately calculate the stationary photon-number variance inside the cavity,

$$\sigma^2=\frac{D}{2k}=\bar{n}\frac{1+r/(1-x)^3}{1+r[(1+x)/(1-x)^3]-s[1+r/(1-x)^2]^2}. \quad (3.14)$$

This is more conveniently represented by the Mandel Q parameter which measures the deviation of the intracavity photon statistics from a Poisson distribution,

$$Q=\frac{\sigma^2-\bar{n}}{\bar{n}}=\frac{-xr/(1-x)^3+s[1+r/(1-x)^2]^2}{1+r[(1+x)/(1-x)^3]-s[1+r/(1-x)^2]^2}. \quad (3.15)$$

Outside the cavity the intensity fluctuations are measured by the Fourier transform of the normalized photocurrent two-time correlation function [4,11],

$$\langle\delta i^2(\omega)\rangle=\kappa\bar{n}\left[1+R\frac{k^2}{k^2+\omega^2}\right], \quad (3.16)$$

where R is a measure of the deviation of the photocurrent fluctuations from the shot-noise limit and is given by

$$R=2Q\frac{\kappa}{k}=2\frac{-xr/(1-x)^3+s[1+r/(1-x)^2]^2}{\{1+r[(1+x)/(1-x)^3]-s[1+r/(1-x)^2]^2\}^2}. \quad (3.17)$$

Now, if x is close to 1, the series in (2.18) will converge slowly. This means that the terms representing nth-order photon absorption will be significant for n large. Such highly nonlinear processes have been shown to produce the best noise reduction [6,11]. Thus, in seeking solutions of (3.10) which give Q and hence R negative, it is convenient to change variables to $\epsilon=1-x$. We anticipate that later ϵ will be assumed small. Rewriting (3.10), we have

$$r\epsilon^{-2}=\frac{y}{sy+1-\epsilon}-1. \quad (3.18)$$

ϵ can certainly be made small by having $r\ll 1$, and letting $y\to\infty$, so that

$$r\epsilon^{-2}=\frac{1-s}{s}=O(1). \quad (3.19)$$

Substituting this into (3.15) gives

$$Q=\frac{-1+s+\epsilon(2-s)}{2(1-s)(1-\epsilon)}, \quad (3.20)$$

and assuming that ϵ is small we get

$$Q_{\text{in}}=-\frac{1}{2}\left[1-\frac{\epsilon}{1-s}\right]+O(\epsilon^2). \quad (3.21)$$

The "in" subscript here indicates that the laser regime being considered is that which minimizes the intensity fluctuations "in" the cavity, rather than in the output. As $\epsilon\to 0$, $Q_{\text{in}}\to-\frac{1}{2}$, which is the expected best intracavity noise reduction for multiple photon absorption [6,11]. Note that the $1-s$ in the denominator shows that this limit is more easily achieved well above threshold, also as expected. As is true generally for nonlinear damping processes [11], this regime is not the optimum one for noise reduction in the laser output. Under the conditions (3.19) with ϵ small, the output noise reduction parameter R is given by

$$R_{\text{in}}=-\frac{s}{2(1-s)}\epsilon+O(\epsilon^2), \quad (3.22)$$

and as $\epsilon\to 0$, $R_{\text{in}}\to 0$. That is, there is no significant output noise reduction below the shot-noise limit.

However, there are alternate solutions to (3.18) in which the right-hand side equals $O(\epsilon)$, rather than $O(1)$ as in (3.19). It is here that we find the optimum regime for output noise reduction. We still assume that r is small, but not that y is very large. If the left side of (3.18) is of order ϵ, we can define a new parameter of order unity, $\xi=\xi(r,y)$, by

$$\xi r=\epsilon^3. \quad (3.23)$$

In terms of ξ we find

$$Q=\frac{-1+s\xi+\epsilon(1+2s)}{2+(1-s)\xi-\epsilon(1+2s)}+O(\epsilon^2) \quad (3.24)$$

and

$$R=\frac{-2\xi[1-s\xi-\epsilon(1+2s)]}{[2+(1-s)\xi-\epsilon(1+2s)]^2}+O(\epsilon^2). \quad (3.25)$$

To find the minimum R with r fixed, we must solve $\partial R/\partial\xi=0$. This is messy in general, and the limiting result ($\epsilon\to 0$) may be obtained by ignoring terms of $O(\epsilon)$ in (3.25) from the start. That is, we use

$$R=\frac{-2\xi(1-s\xi)}{[2+(1-s)\xi]^2}+O(\epsilon). \quad (3.26)$$

This has a minimum of

$$R=R_{\text{out}}=-\frac{1}{4(1+s)}+O(\epsilon) \quad (3.27)$$

at

$$\xi_{\mathrm{out}}=\frac{2}{1+3s}+O(\epsilon)\ . \tag{3.28}$$

The "out" subscript indicates the value of the parameter when the laser is operating under optimal conditions for reducing output noise (minimizing R). Note that the best output noise reduction is limited by how far above threshold the laser operates. Only for s negligible does R approach $-\frac{1}{4}$, its theoretical minimum for multiple photon absorption [11]. Nevertheless, for any s, sub-shot-noise output is achievable. In this regime, the intracavity Mandel Q parameter is given by

$$Q=Q_{\mathrm{out}}=-\tfrac{1}{4}+O(\epsilon)\ . \tag{3.29}$$

As expected we have $Q_{\mathrm{out}}=\frac{1}{2}Q_{\mathrm{in}}$, as is true for general nonlinear damping processes [11]. For r fixed and small, the optimum value of y for output noise reduction is found by substituting (3.28) into (3.18), giving

$$y_{\mathrm{out}}=\frac{1}{1-s}\left[1-\frac{\epsilon(1-2s)}{2(1-s)}\right]+O(\epsilon^2)\ . \tag{3.30}$$

Thus, the actual mean photon number in this regime is given by

$$\bar{n}_{\mathrm{out}}=\nu\frac{x}{y}=n_s\left[\frac{G}{\kappa}-1-\frac{\epsilon}{2}\right], \tag{3.31}$$

which is only marginally below the standard (no feedback) result. Again, this is as expected [11].

As stated above, expressions for R and Q which are correct to $O(\epsilon)$ are messy in general. However, they are relatively simple in the case when s can be neglected ($G\gg\kappa$). Setting $\partial R/\partial\xi=0$, while remembering that $\partial\epsilon/\partial\xi=\epsilon/3\xi$, gives the minimum

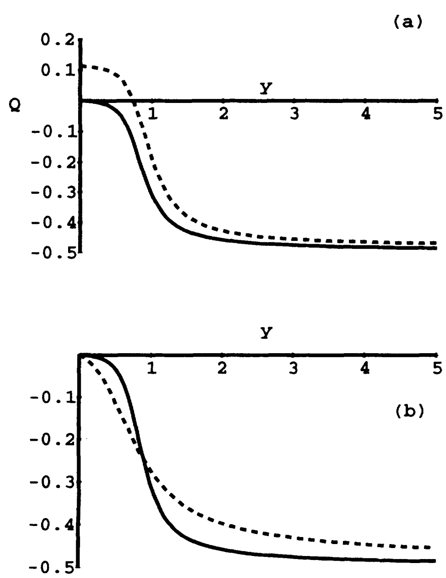

FIG. 3. The Mandel Q parameter vs the feedback strength parameter. (a) The plot has $r=0.01$ with $s=0$ (solid) and $s=0.1$ (dashed). (b) The plot has $s=0$ with $r=0.01$ (solid) and $r=0.1$ (dashed).

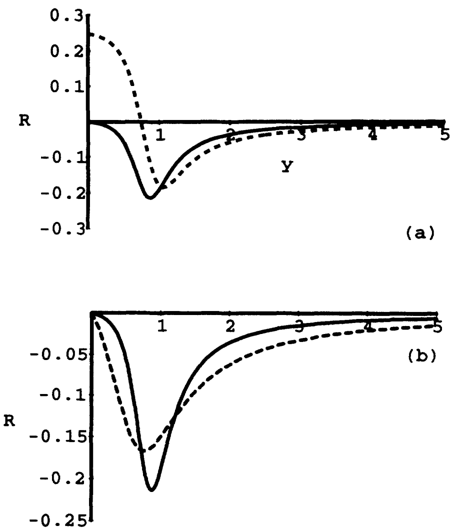

FIG. 4. The R parameter (which determines the low-frequency reduction in the intensity noise spectrum outside the cavity) vs the feedback strength parameter. (a) The plot has $r=0.01$ with $s=0$ (solid) and $s=0.1$ (dashed). (b) The plot has $s=0$ with $r=0.01$ (solid) and $r=0.1$ (dashed).

$$R_{\mathrm{out}}=-\frac{1}{4}\left[1-\frac{\epsilon}{2}\right]+O(\epsilon^2) \tag{3.32}$$

at

$$\xi_{\mathrm{out}}=2\left[1-\frac{5}{6}\epsilon\right]+O(\epsilon^2)\ , \tag{3.33}$$

giving

$$Q_{\mathrm{out}}=-\frac{1}{4}\left[1-\frac{\epsilon}{3}\right]+O(\epsilon^2)\ . \tag{3.34}$$

Thus, to approach the theoretical limits, it is indeed necessary that ϵ be small.

In Figs. 3(a)–4(b) we plot the Q parameter and the R parameter versus the feedback strength y. Clearly evident in Figs. 3(a) and 3(b) is the approach of the Q parameter to -0.5. If the laser operates far above threshold $s=0$ and the relative decay rate r is small this limiting value is approached more rapidly as a function of y. Close to threshold the feedback must overcome the large intensity fluctuations present in the ordinary laser. In Figs. 4(a) and 4(b) we see that the optimum value of R is achieved for a value of y less than infinity. However, as for the Q parameter it is advantageous to operate well above threshold and with as small a relative decay rate as possible, in which case R can be made close to -0.25.

IV. CONCLUSIONS

The feedback model presented in Sec. II predicts that sub-Poissonian and sub-shot-noise photon statistics can be achieved in the cavity and in its output, respectively.

Quantitatively, we find that the best intracavity noise reduction is

$$Q = Q_{\text{in}} = -\tfrac{1}{2} + O(\sqrt{r}) , \tag{4.1}$$

and the best output noise reduction (under different operating conditions) is

$$R = R_{\text{out}} = -\frac{1}{4(1+s)} + O(\sqrt[3]{r}) , \tag{4.2}$$

where s is an inverse measure of gain saturation. The parameter r is the ratio of the loss from the feedback-controlled output mirror when the feedback is inoperative (γ) to that from the standard mirror (κ). Approaching the theoretical limits given by (4.1) and (4.2) depends on the smallness of r. Physically, r is limited by how small the basal loss rate from the feedback-controlled mirror can be made. Assuming that the reflectivity at the standard output mirror is 0.92 (low end of the range for which the linear loss model applies) and the reflectivity of the feedback mirror is 0.995 (near the present technical upper limit) gives $r = \frac{1}{16}$.

In this case, much of the analysis given in Sec. III is questionable. For example, the result (4.2) assumes that $\epsilon \simeq \sqrt[3]{2r} \ll 1$, yet $r = \frac{1}{16}$ gives $\epsilon \simeq \frac{1}{2}$. Instead, numerical techniques may be used to find the best noise reduction from Eqs. (3.15) and (3.17), with r and s fixed. Taking $r = \frac{1}{16}$ and $s = \frac{1}{8}$ (well above threshold) gives the following results:

$$Q_{\text{in}} = -0.4432 , \tag{4.3}$$

$$R_{\text{out}} = -0.1365 . \tag{4.4}$$

These results are below the Poissonian and shot-noise limit, respectively, but only modestly so for the output noise. However, it should be remembered that (3.4) and (3.5) should be compared with those values of a free running (no feedback) laser under the same saturation conditions, rather than with the idealized case where the gain is completely saturated and $s \to 0$. With $s = \frac{1}{8}$ as before, the standard laser has

$$Q_{\text{std}} = \frac{s}{1-s} = +0.1429 , \tag{4.5}$$

$$R_{\text{std}} = \frac{2s}{(1-s)^2} = +0.3265 . \tag{4.6}$$

In this context, it is seen that the noise reduction achieved by feedback is quite considerable.

Finally, we address the paradox that intensity fluctuations are reduced most when $r \to 0$, yet putting $r = 0$ means that there is no feedback and hence no noise reduction. The explanation is that in deriving the feedback master equation (2.17), it is necessary to assume that the scaled mean x is finitely less than one. That is, we must have ϵ finite, where $\epsilon = O(\sqrt{r})$ to minimize Q, and $\epsilon = O(\sqrt[3]{r})$ to minimize R. In fact, we must have $\epsilon \gg 1/\bar{n}$. As we have seen, this limitation is not important in practice, as $\bar{n}$ is typically 10^9. If r is strictly zero, there is no relation between r and ϵ. Indeed, x may take any value, being given simply by $x = y(1-s)$, from (3.10) where the feedback level y is arbitrary. The correct no feedback limit is not $r \to 0$, but $y \to 0$, although the parametrization (3.5–3.9) does not allow $y = 0$, as this would make x undefined

Nevertheless, it is counterintuitive that the best noise reduction is when $r \ll 1$, so that the feedback is apparently small. In fact, the ratio of loss from the feedback mirror to that from the standard mirror is not r, but $r/(1-x)^2 \simeq \frac{1}{2}\epsilon$ when the output noise parameter R is minimized. This is still small, but much larger than r. Furthermore, if we return to the variables at the start of the derivation of the feedback master equation in Sec. II, at the regime for optimum output where $y \simeq 1$ and $\bar{n} \simeq \nu$, we have

$$y = \chi \nu \simeq \gamma A \frac{1}{\beta} \bar{n} \simeq 1 , \tag{4.7}$$

where $1/\beta$ measures the time over which the average photon count is taken for the purposes of feedback, and A is the original dimensionless feedback parameter. Now the formalism places no limitations on β. However, it would seem physically reasonable to take β to be roughly the rate of photon counts at steady state. Thus, putting $\beta \simeq \gamma \bar{n} \epsilon^{-2}$, we get

$$A \simeq \epsilon^{-2} , \tag{4.8}$$

so that the feedback amplification is actually large for ϵ small.

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