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Interpretation of quantum jump and diffusion processes illustrated on the Bloch sphere

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It is shown that the evolution of an open quantum system whose density operator obeys a Markovian master equation can in some cases be meaningfully described in terms of stochastic Schrödinger equations (SSE's) for its state vector. A necessary condition for this is that the information carried away from the system by the bath (source of the irreversibility) be recoverable. The primary field of application is quantum optics, where the bath consists of the continuum of electromagnetic modes. The information lost from the system can be recovered using a perfect photodetector. The state of the system conditioned on the photodetections undergoes stochastic quantum jumps. Alternative measurement schemes on the outgoing light (homodyne and heterodyne detection) are shown to give rise to SSE's with diffusive terms. These three detection schemes are illustrated on a simple quantum system, the two-level atom, giving new perspectives on the interpretation of measurement results. The reality of these and other stochastic processes for state vectors is discussed.

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I. INTRODUCTION

Conventionally, irreversible quantum processes (which typically arise through coupling to an unbounded environment) have been described in terms of a master equation (ME) for a density operator. The density operator represents a classical ensemble of quantum pure states, represented by rays in Hilbert space. Recently, there has been considerable interest in the "unraveling" of such master equations for density operators into stochastic trajectories for state vectors [1–14]. At any point in time, the ensemble of state vectors generated by a stochastic Schrödinger equation (SSE) reproduces the density operator generated by the original ME. Just as different ensembles of state vectors may be represented by one density operator, one ME may be decomposed in many different ways into SSE's. Some approaches, involving jump processes [1–7,12–14], were originally motivated by experimental observations of such processes [15], while others involving diffusion processes [8–10] have there origins in quantum measurement theory.

Most of the theoretical work in quantum jumps is based on a master equation of the form

$$\dot{\rho} = \mathcal{D}[c]\rho + \mathcal{L}_0\rho, \tag{1.1}$$

where $\mathcal{L}_0$ is a Liouville superoperator and $\mathcal{D}[c]$ is a superoperator defined by

$$\mathcal{D}[c]\rho = c\rho c^\dagger - \tfrac{1}{2}[c^\dagger c\rho + \rho c^\dagger c], \tag{1.2}$$

where c is some Schrödinger picture system operator. We now define a "jump" superoperator by

$$\mathcal{J}\rho = c\rho c^\dagger. \tag{1.3}$$

In the usual treatment, one then expresses the solution of (1.1) as a generalized Dyson expansion in $\mathcal{J}$ [3, 6, 11]. However, we can arrive at the SSE more directly by considering the following stochastic master equation for the density operator $\rho_c(t)$:

$$d\rho_c(t) = dN_c(t)\left(\frac{\mathcal{J}\rho_c(t)}{\langle n\rangle_c(t)} - \rho_c(t)\right) + dt\left\{\langle n\rangle_c(t)\rho_c(t) - \tfrac{1}{2}[n\rho_c(t) + \rho(t)_c n] + \mathcal{L}_0\rho_c(t)\right\}, \tag{1.4}$$

where $n = c^\dagger c$ and $dN_c(t)$ is a random real variable defined by

$$E[dN_c(t)] = \langle n\rangle_c(t)dt = \mathrm{Tr}[\mathcal{J}\rho_c(t)] \tag{1.5a}$$

$$dN(t)^2 = dN(t). \tag{1.5b}$$

Here, E denotes an ensemble expectation value, not to be confused with quantum averages which are denoted by angular brackets, so that $\langle n\rangle_c(t) = \mathrm{Tr}[n\rho_c(t)]$. Equation (1.5b) simply indicates that $dN(t)$ equals either zero or one. That is to say, in an increment of time, the system jumps via the superoperator $\mathcal{J}$ with probability given by (1.5a); otherwise it evolves smoothly. The subscript c on an object indicates that the object is conditioned on the occurrence of these jumps in the past. To average over such stochastic histories, one simply replaces $dN_c(t)$ in Eq. (1.4) by its expectation value. Then it is easily seen that the ensemble average density operator $\rho(t) = E[\rho_c(t)]$ obeys the original master equation (1.1).

Now if $\mathcal{L}_0\rho = -i[H,\rho]$, where H is the Hamiltonian operator ($\hbar = 1$), then it is evident that the stochastic master equation (1.4) is equivalent to the following stochastic Schrödinger equation:

$$d|\psi_c(t)\rangle = \left[dN_c(t) \left(\frac{c}{\sqrt{\langle n \rangle_c(t)}} - 1 \right) + dt \left(\frac{\langle n \rangle_c(t)}{2} - \frac{n}{2} - iH \right) \right] |\psi_c(t)\rangle, \quad (1.6)$$

where now the quantum averages are defined by $\langle n \rangle_c(t) = \langle \psi_c(t)|n|\psi_c(t)\rangle$. Thus the original master equation is equivalent to the ensemble of trajectories generated by the SSE (1.6), provided that the initial density operator can be written as $\rho(0) = |\psi(0)\rangle\langle\psi(0)|$.

Alternatively, it is possible to unravel the master equation (1.1) as a diffusion process for state vectors. This is the approach of Gisin and Percival [9], who consider a SSE of the form

$$d|\tilde{\psi}_c(t)\rangle = \left\{ dt \left[\langle c^\dagger \rangle_c(t)\, c - \frac{n}{2} - iH \right] + dW(t)\, c \right\} |\tilde{\psi}_c(t)\rangle, \quad (1.7)$$

where dW is a complex Weiner increment [16] satisfying

$$E[dW(t)] = 0, \quad (1.8a)$$

$$dW(t)^* dW(t) = dt, \quad (1.8b)$$

with all other second-order and higher moments vanishing. The tilde in (1.7) indicates that the conditioned state vector obeying this equation is not normalized. The only implication for this is that the quantum averages are defined by, for example, $\langle c^\dagger \rangle_c(t) = \langle \tilde{\psi}_c(t)|c^\dagger|\tilde{\psi}_c(t)\rangle / \langle \tilde{\psi}_c(t)|\tilde{\psi}_c(t)\rangle$. It is easy to verify from (1.7) that $\rho(t) = E[|\tilde{\psi}_c(t)\rangle\langle\tilde{\psi}_c(t)| / \langle\tilde{\psi}_c(t)|\tilde{\psi}_c(t)\rangle]$ satisfies the original master equation (1.1).

A third and quite different approach is that of Teich and Mahler [13]. One first solves the master equation [which is not limited to the form of (1.1)], and determines the (in general, time-dependent) basis states in which the density operator is diagonal. These change smoothly in time, and constitute the "coherent" part of the dynamics. The "incoherent" part is manifest by the system jumping between these orthogonal basis states according to readily derived rate equations. This scheme is superficially appealing because an eigenvalue of the density matrix can be regarded as the probability of the system actually being in the corresponding eigenstate. For stationary processes, the eigenstates are fixed in time and only jump processes occur.

A major advantage of using a SSE in place of its corresponding master equation is that less memory is required for computation of the density operator, as a state vector in N-dimensional Hilbert space is specified by $2N$ real numbers, compared to N^2 for the density operator. However, the decreased memory requirement is offset by the necessity to do many simulations in order to obtain a reliable ensemble average. What is of more interest is the question of the interpretation of the stochastic evolution of state vectors. This is especially so with the scheme of Teich and Mahler, because it does not offer the computational advantages of the other two approaches. The reality of such state vectors is thus a matter of contention.

In Sec. II of this paper, we show that, under certain circumstances, the state vectors generated by the jump SSE (1.6) and the diffusion SSE (1.7) do have a physical interpretation as the conditioned state of the system given the history of measurement results. The different SSE's result from different measurement schemes on outgoing electromagnetic radiation which carries information about the system's irreversible evolution. In the two cases mentioned, the relevant measurement schemes are direct and heterodyne photodetection, respectively. We also consider another optical measurement scheme, homodyne detection, which gives rise to a different diffusion equation from that of heterodyne detection.

In Sec. III we apply these stochastic equations to the simplest nontrivial quantum system: a classically driven, detuned, damped two-level atom. Here the homodyne and heterodyne measurement schemes give rise to diffusion on the Bloch sphere, while direct detection gives rise to quantum jumps. The evolution equation for the probability density function of state vectors is derived, and the stationary probability distribution is plotted numerically on the Bloch sphere for each case. In Sec. IV we discuss these results and present conclusions regarding the physical reality of the state generated by the various stochastic dynamical equations.

II. QUANTUM THEORY OF OPTICAL MEASUREMENTS

A. The quantum optical master equation

The irreversible dynamics of a quantum system is often treated in terms of a master equation for its density operator. A master equation can be derived when the system is small compared to a reservoir or bath to which it is coupled weakly over a large bandwidth. An obvious bath is the continuum of electromagnetic field modes. The requirements on the system can be satisfied by a typical localized (in space) coupling, such as at an atom, or at a good cavity mirror. The electromagnetic field in free space is equivalent to a continuum of harmonic oscillators with annihilation operators $b(\omega)$ satisfying the following commutation relation:

$$[b(\omega), b^\dagger(\omega')] = \delta(\omega - \omega'). \quad (2.1)$$

For simplicity, we consider a one-dimensional traveling wave only. This is appropriate for a cavity coupling. Assuming that the field carries significant energy only around a carrier frequency (which in practice will be some characteristic frequency of the system) with some bandwidth small in comparison with the carrier frequency, the electric field at some point in space-time is effectively [17]

$$\mathbf{E}(z,t) = \mathbf{E}_1 \left(b_{t-z/c} + b^\dagger_{t-z/c} \right), \quad (2.2)$$

where $\mathbf{E}_1$ is a constant vector representing in some sense the electric field per photon, c is the speed of light, and b_t is defined by

$$b_t = \int_0^\infty d\omega\, b(\omega) e^{-i\omega t}. \quad (2.3)$$

Under the above assumption, the lower limit of this integral can be extended to $-\infty$ with negligible error and the interaction picture operators b_t then satisfy the following commutation relation [18]:

$$[b(t), b^\dagger(t)] = \delta(t-t'). \tag{2.4}$$

Say the system is coupled to the bath at $z=0$. The electric field for $z<0$ then represents an incoming field and that for $z>0$ an outgoing one. If we now make the Markov approximation that the bandwidth of the coupling is large compared to dynamical frequencies (but small compared to the carrier frequency), then the Hamiltonian for the system and bath can be modeled as

$$H_t = H_{\rm sys} + i\sqrt{\gamma}\left[b_t^\dagger a - b_t a^\dagger\right], \tag{2.5}$$

where a is some system (usually lowering) operator and γ is the effective damping rate. Because of the singularity of the commutation relation (2.4), it is necessary to be careful in dealing with the above Hamiltonian. A convenient way to treat it is to interpret the operators b_t in the manner of Ito stochastic calculus [7]. The analog of the Weiner increment is

$$dB_t = b_t dt, \tag{2.6}$$

which satisfies

$$[dB_t, dB_t^\dagger] = dt. \tag{2.7}$$

It is thus necessary to expand infinitesimals to second order in dB_t. The infinitesimal unitary transformation arising from (2.5) is then

$$\begin{aligned} U(t+dt,t) = {} & 1 - iH_{\rm sys}dt + \sqrt{\gamma}[dB_t^\dagger a - dB_t a^\dagger] \\ & + \frac{\gamma}{2}[c^\dagger c dt + dB_t^\dagger dB_t(c^\dagger c + cc^\dagger) \\ & + dB_t^2 c^{\dagger 2} + dB_t^{2\dagger} c^2]. \end{aligned} \tag{2.8}$$

The density operator for the system and bath (denoted by W) thus obeys the following evolution equation:

$$W(t+dt) = U(t+dt,t)W(t)U^\dagger(t+dt,t). \tag{2.9}$$

Now, if we wish to derive a master equation for the system density operator $\rho(t) = {\rm Tr}_B[W(t)]$, it is necessary to make the Born approximation, which is that the bath is not affected by the system. That is to say, if the initial combined density operator at time $t=0$ factorizes as $W(0) = \rho(0)\otimes\rho_B(0)$, then it will factorize for all times as $W(t) = \rho(t)\otimes\rho_B(0)$. This approximation can be justified by a system size expansion for certain initial bath states [19]. Essentially, the q numbers dB_t in (2.8) have to be replaced by c numbers. This is well understood for two classes of bath states: coherent and thermal [18]. For a coherent bath of bandwidth λ and central frequency ω, we have

$$\begin{aligned} \langle dB_t\rangle &= {\rm Tr}[dB_t\rho_B(0)] = \sqrt{\lambda}\,\beta_t dt \\ &= \sqrt{\lambda}\,\beta_0 e^{-(\lambda/2+i\omega)t}dt, \end{aligned} \tag{2.10}$$

and the second order averages in (2.8) vanish to first order in dt. In a thermal bath, the first order moments vanish while the nonzero second-order moment in (2.8) is given by

$$\langle dB_t^\dagger dB_t\rangle = N\,dt. \tag{2.11}$$

From Eq. (2.9), one then derives the following master equation for the system:

$$\begin{aligned} \dot\rho(t) = {} & -i[H_{\rm sys} + i\gamma(\epsilon_t^* a - \epsilon_t a^\dagger), \rho(t)] \\ & + (N+1)\gamma\mathcal{D}[a]\rho(t) + \gamma N\mathcal{D}[a^\dagger]\rho(t), \end{aligned} \tag{2.12}$$

where $\epsilon_t = \sqrt{\lambda/\gamma}\,\beta_t$, and $\mathcal{D}[c]$ is the superoperator defined in the Introduction (1.2). For reasons given in Appendix A, we will ignore the thermal terms by putting $N=0$, which gives

$$\begin{aligned} \dot\rho(t) &= -i[H_{\rm sys} + i\gamma(\epsilon_t^* a - \epsilon_t a^\dagger), \rho(t)] + \gamma\mathcal{D}[a]\rho(t) \\ &\equiv \mathcal{L}_t\rho(t). \end{aligned} \tag{2.13}$$

Here the bath coherently drives the system, and also damps it.

Having sketched the derivation of the optical master equation, we now justify its special significance. Briefly, the interaction of the optical bath with its physical surroundings can be controlled experimentally. This is in contrast to, for example, the particulate Brownian motion master equation [19] whereby the particles constituting the bath are presumably subject to the same sort of random fluctuations as the system. In that case, the information regarding the change in the system is irretrievably lost into the universe at large, whereas the outgoing light from the system obeying the optical master equation can be collected and the information regained. However, in making the Born approximation to derive the master equation, one has discarded this information by assuming that the system does not affect the bath. Thus to explicitly see the bath carrying away information about the system, it is necessary to return to Eq. (2.9) before the Born approximation has been made.

For example, we calculate the intensity of the outgoing light at time t at a distance z from the system. From (2.2), this is given by (in units of photon flux)

$$I(z,t) = {\rm Tr}\left[b_{t-z/c}W(t)b_{t-z/c}^\dagger\right]. \tag{2.14}$$

Naively applying the Born approximation at this stage yields a constant, equal to ${\rm Tr}\left[b_{t-z/c}\rho_B(0)b_{t-z/c}^\dagger\right]$. However, using the unitary evolution operator (2.8), it can be rewritten as

$${\rm Tr}\left[b_{t-z/c}U(t,t-z/c)W(t-z/c)U^\dagger(t,t-z/c)b_{t-z/c}^\dagger\right]. \tag{2.15}$$

Using the commutation relation (2.7), this is easily shown [7] to be equal to

$${\rm Tr}\left\{\left[b_{t-z/c} + \sqrt{\gamma}a\right]W(t-z/c)\left[b_{t-z/c} + \sqrt{\gamma}a\right]^\dagger\right\}. \tag{2.16}$$

Now restoring the Born approximation and assuming a coherent bath as above gives

$$I(z,t) = \gamma \mathrm{Tr}\left\{ \left[\epsilon_{t-z/c} + a\right] \rho(t-z/c) \left[\epsilon^*_{t-z/c} + a^\dagger\right] \right\}. \tag{2.17}$$

This can be simply interpreted in the case of a damped optical cavity, with a being the annihilation operator of the cavity mode. If $\epsilon_t = 0$, the intensity is simply the rate of loss of photons from the cavity at the time $t - z/c$, as expected. If $\epsilon_t \neq 0$, there is interference between the field transmitted through the output mirror and the incoming light reflected there. Thus, for the purposes of intensity measurements (in which vacuum fluctuations may be ignored), the annihilation operator for the output field relevant to the system state at time t is $\sqrt{\gamma}(\epsilon_t + a)$.

At first sight, the theory presented here would seem to be limited in applicability to systems in which the irreversibility arises from a direct coupling to the continuum of the electromagnetic field. In fact, this is not the case; there are a large number of quantum optical intracavity measurement schemes to which this theory would apply [20–22, 11]. Specifically, say the cavity field (represented by annihilation operator c) is damped to a bath in the vacuum state at rate κ. Let the cavity mode be coupled to another intracavity system (such as a second mode or an atom) by the interaction picture coupling

$$V = \frac{g}{2}(ac^\dagger + a^\dagger c). \tag{2.18}$$

In practice, this may require a nonlinear medium, and other driving fields in the cavity if a is not a lowering operator near cavity resonance. If $\kappa \gg g$, the first cavity mode can be adiabatically eliminated [11], giving the following master equation for the second system

$$\dot{\rho}(t) = \gamma \mathcal{D}[a]\rho(t), \tag{2.19}$$

where $\gamma = g^2/\kappa$. This equation is identical to that which would have arisen if this system were directly coupled to the optical bath. Furthermore, the output light is the same as in the direct coupling case, so the following theory applies equally to such indirect couplings.

B. Direct photodetection

We have shown that the electromagnetic bath carries away the information about the irreversible evolution of the system. To take advantage of this, we have to obtain this information by measuring this outgoing light. There are many schemes of measurement which are used experimentally, but most ultimately rely upon photodetection. We will not be using a specific physical model of a photodetector, although the formulas we use can be justified by such models [19]. This is because, however sophisticated the model may be, at some stage it is necessary to invoke the projection postulate (or some other postulate of quantum measurement theory) in order to talk about the result of the measurement. Unlike optical beams, material measurement devices are strongly coupled to their environment, so that irrevocable loss of information is unavoidable. Once this has taken place, the location of the division between the quantum system and the classical measuring apparatus (the so-called Heisenberg cut [23]) is essentially arbitrary. For our purposes, it is sufficiently accurate to place the cut between the outgoing light and the photodetector, and to define the latter by the following postulates.

Consider a photodetector of efficiency η at position z (as defined in the preceding subsection). In the infinitesimal interval of time $(t, t+dt)$, the detector will either detect one photon or no photons. The probability that it will detect one photon is given by

$$P(z,t)dt = \eta I(z,t)dt. \tag{2.20}$$

That is to say, the rate of photodetections is equal to the photon flux times the efficiency. With a coherent bath as before, this becomes

$$P(z,t)dt = \mathrm{Tr}\left[\mathcal{J}_{t-z/c}\rho(t-z/c)\right] dt, \tag{2.21}$$

where we have defined a jump superoperator

$$\mathcal{J}_t\rho = \eta\gamma\left[\epsilon_t + a\right]\rho\left[\epsilon^*_t + a^\dagger\right]. \tag{2.22}$$

Given that a photodetection does occur, the new system state conditioned on this is given by

$$\rho_c(t - z/c + dt) = \frac{\mathcal{J}_{t-z/c}\rho(t-z/c)}{P(z,t)}. \tag{2.23}$$

Note that a photodetection at time t causes a change in the system at time $t - z/c$. Despite appearances, this does not violate causality, for the same reason that the experiment of Einstein, Podolsky, and Rosen [24] does not. The change in the system (which we will call a "quantum jump") is more akin to a change in knowledge than it is to Bohr's original concept of a quantum jump between stationary atomic states [25]. This change in knowledge takes place locally, within the observer, and does not imply any physical mechanism causing a transition in the system.

So far we have not specified how the system changes when there is no photodetection in the interval $(t, t + dt)$. This change is determined by the master equation (2.13) which the system obeys and the jump superoperator $\mathcal{J}$. (Contrast this approach with that of Srinivas and Davies [26], whereby the jump superoperator determines the master equation.) Denote the increment in the photocount of the detector in the time interval $(t + z/c, t + z/c + dt)$ by $dN_c(t)$. As in the Introduction, this is a random variable which satisfies

$$E[dN_c(t)] = \mathrm{Tr}\left[\mathcal{J}_t\rho_c(t)\right] dt = P_c(t)dt, \tag{2.24a}$$

$$dN_c(t)^2 = dN_c(t). \tag{2.24b}$$

Here, the c subscript refers to the state of the system conditioned on the previous history of photodetections. When a photodetection occurs $[dN_c(t - z/c) = 1]$, the system state changes discontinuously via (2.23). Requiring that $E[\rho_c(t)]$ (the ensemble average of the conditioned density operator) obey the master equation $\dot{\rho}(t) = \mathcal{L}_t\rho(t)$ (2.13) leads immediately to the stochastic quantum jump equation

$$\rho_c(t+dt) = dN_c(t)\left(\frac{\mathcal{J}_t\rho_c(t)}{P_c(t)} - \rho_c(t)\right) + \mathcal{S}_t(dt)\rho_c(t), \tag{2.25}$$

where the smooth evolution superoperator is given by

$$\mathcal{S}_t(dt) = 1 + [\mathcal{L}_t - \mathcal{J}_t + P_c(t)]\,dt. \tag{2.26}$$

With $\mathcal{L}_t$ as defined in (2.13), the nonjump evolution becomes

$$\mathcal{S}_t(dt)\rho(t) = \rho(t) - i\left[H_{\rm sys} - i\eta\gamma\left(\tfrac{1}{2}a^\dagger a + \epsilon_t a^\dagger\right), \rho(t)\right]_* + (1-\eta)\gamma\left\{[\epsilon_t^* a - \epsilon_t a^\dagger, \rho_c(t)] + \mathcal{D}[a]\rho_c(t)\right\}, \tag{2.27}$$

where we have introduced a variation on the commutator brackets via the notation

$$[A,B]_* = (AB - B^\dagger A^\dagger) - \mathrm{Tr}(AB - B^\dagger A^\dagger), \tag{2.28}$$

where A and B are arbitrary operators. This star commutator is necessarily anti-Hermitian and of zero trace. It reduces to the normal commutator when A and B are Hermitian, provided that $\mathrm{Tr}(AB)$ exists.

If and only if $\eta = 1$, the stochastic master equation for the conditioned density operator (2.27) is equivalent to the following stochastic Schrödinger equation (SSE) for the conditioned ket

$$d|\psi_c(t)\rangle = \left[dN_c(t)\left(\frac{\sqrt{\gamma}(a+\epsilon_t)}{\sqrt{P_c(t)}} - 1\right) - dt\left(\gamma\left\{\tfrac{1}{2}[a^\dagger a - \langle a^\dagger a\rangle_c(t)] + \epsilon_t a^\dagger - \tfrac{1}{2}\langle \epsilon_t a^\dagger + \epsilon_t^* a\rangle_c(t)\right\} + iH_{\rm sys}\right)\right]|\psi_c(t)\rangle, \tag{2.29}$$

where the quantum averages are defined by $\langle a^\dagger a\rangle_c(t) = \langle\psi_c(t)|a^\dagger a|\psi_c(t)\rangle$. The interpretation of the $\eta = 1$ condition is obvious; in order to continuously describe a system by a ket (rather than a density operator), it is necessary (and sufficient) to have maximal knowledge of its change of state. This requires unit efficiency in the detection of the outgoing light which contains the information lost by the system when it evolves nonunitarily. With zero efficiency detection, the stochastic master equation (2.27) reduces to the standard master equation (2.13). This interpretation highlights the fact that a density operator description of a quantum state is only necessary when information is lost irretrievably.

When the coherent driving field (ϵ_t) is zero, the master equation becomes the vacuum optical master equation (VOME):

$$\dot\rho(t) = -i[H_{\rm sys}, \rho(t)] + \gamma\mathcal{D}[a]\rho(t) \equiv \mathcal{L}_0\rho(t). \tag{2.30}$$

In this case, the direct photodetection SSE (2.29) is simply

$$d|\psi_c(t)\rangle = \left[dN_c(t)\left(\frac{a}{\sqrt{\langle a^\dagger a\rangle_c(t)}} - 1\right) - dt\left(\frac{\gamma}{2}[a^\dagger a - \langle a^\dagger a\rangle_c(t)] + iH_{\rm sys}\right)\right]|\psi_c(t)\rangle, \tag{2.31}$$

which is identical in form to the quantum jump SSE (1.6) given in the Introduction. However, the latter does not have a physical interpretation as the evolution equation for a ket conditioned on the results of photodetections; it is merely an algorithm for computing density operator evolution. Adding coherent driving would simply change the Hamiltonian H in Eq. (1.6). For a correct interpretation in terms of photodetections, changing just the Hamiltonian in Eq. (2.31) would be valid only if the coherent driving modes were (i) to cause negligible damping of the system, and (ii) not monitored by the photodetector. This is the situation considered in Sec. III. In the remainder of this paper we will always take $\epsilon_t = 0$ and so consider only the VOME.

In this case, it is easy to show (see Appendix B) that the ensemble expectation value of the two-time correlation function for the normalized photocurrent is

$$E[I_c(t+\tau)I(t)] = \gamma^2\eta^2\mathrm{Tr}\{a^\dagger a e^{\mathcal{L}_0\tau}[a\rho(t)a^\dagger]\} + \gamma\eta\mathrm{Tr}[a^\dagger a\rho(t)]\delta(\tau), \tag{2.32}$$

where $\mathcal{L}_0$ is defined in (2.30), the photocurrent is defined by $I_c(t) = dN_c(t)/dt$, and we have restored η the quantum efficiency of the detector. There is no conditioned subscript on $I(t)$ in Eq. (2.32) because it is determined only by $\rho(t)$, which is assumed given. This initial system state may be conditioned on some previous measurement result, but that is irrelevant. The photocurrent at the later time $I_c(t+\tau)$ is conditioned on the stochastic history of the photocurrent over time interval $[t, t+\tau)$. Changing to the Heisenberg picture for the operators gives

$$E[I_c(t+\tau)I(t)] = \gamma^2\eta^2\langle : n(t+\tau)n(t) :\rangle + \gamma\eta\langle n(t)\rangle\delta(\tau), \tag{2.33}$$

where $n = a^\dagger a$ and $::$ denotes normal ordering of the enclosed operators (i.e., $a^\dagger$'s before a's). This is the standard expression for the second-order coherence function $G^{(2)}(t, t+\tau)$ plus shot noise. Here we see that it can be derived explicitly as an ensemble average of products of classical photocurrents, which is how it is determined experimentally. This confirms that the above theory correctly models optical measurements and provides a perhaps more intuitive understanding of the relation between theory and experiment than that provided by standard derivations of autocorrelation functions [27] and the like.

C. Homodyne photodetection

The unraveling of the VOME (2.30) as a quantum jump SSE in terms of photodetections of the output light seems so natural that it is tempting to do away with the detector and to conclude that the quantum jumps of the source are objective events, resulting in the emission of a photon into the bath. To do this, however, is to place the Heisenberg cut too soon. The optical bath is still a quantum system in that it may exhibit interference effects (for example, Young's experiment). A system may only be considered as a classical object only when it strongly couples to its environment so as to destroy quantum coherences [28]. The outgoing light from the system may be channeled into a more elaborate measurement apparatus than that of a simple photodetector. Different measurement schemes result in different sorts of quantum trajectories for the system state. The simplest alternative to direct photodetection is homodyne photodetection. This has been considered in detail in a previous paper [11], and so the theory we will present here is a summary of a slightly more general case.

Consider the apparatus for balanced homodyne (or heterodyne) detection shown in Fig. 1. The beam splitter located a distance z_0 from the source has transmittivity $\frac{1}{2}$. We treat the local oscillator as a classical coherent field with amplitude $i\sqrt{\gamma}\beta_t$ at distance z_0 before the input to the beam splitter. Say $\beta_t = \exp[-i(\omega+\Omega)t]$, where ω is the frequency at which a rotates under the free Hamiltonian of the system. Moving to the interaction picture eliminates this free Hamiltonian, allowing us to replace $H_{\rm sys}$ by interaction picture V_t (where the time dependence is usually trivial), and to write $\beta_t = \beta\exp[-i\Omega t]$. The effective annihilation operators for the field at the detectors D_1 (at z_1) and D_2 (at z_2) at times $t+z_1/c$ and $t+z_2/c$, respectively, are

$$C_t^1 = \sqrt{\gamma/2}\,(a-\beta_t)e^{-i\omega t} \tag{2.34a}$$

$$C_t^2 = \sqrt{\gamma/2}\,i(a+\beta_t)e^{-i\omega t}. \tag{2.34b}$$

The appropriate jump superoperators for the (assumed unit efficiency) photodetectors are then defined by $\mathcal{J}_t^k\rho = C_t^k\rho C_t^{k\dagger}$ $(k=1,2)$. Now we have two photocount increments $dN_c^k(t)$ with means $P_c^k(t)dt = \langle C_t^{k\dagger}C_t^k\rangle_c(t)dt$, and using the method of the preceding subsection the stochastic master equation is found to be

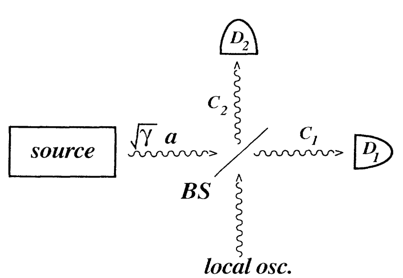

FIG. 1. Schematic diagram of balanced homodyne detection. The beam splitter (BS) has transmittivity 1/2. The photodetectors are denoted D_1 and D_2.

$$\begin{aligned}\dot\rho_c(t) = {}& dN_c^1(t)\left(\frac{\mathcal{J}_t^1\rho_c(t)}{P_c^1(t)}-\rho_c(t)\right)\\ &+dN_c^2(t)\left(\frac{\mathcal{J}_t^2\rho_c(t)}{P_c^2(t)}-\rho_c(t)\right)\\ &-i\left[V_t - i\frac{\gamma}{2}a^\dagger a,\rho_c(t)\right]_*\,,\end{aligned} \tag{2.35}$$

where the star-commutator is as defined previously in Eq. (2.28).

Because of the perfect detection, this stochastic master equation is equivalent to the following SSE:

$$\begin{aligned}d|\psi_c(t)\rangle = \Bigg[& dN_c^1(t)\left(\frac{C_t^1}{\sqrt{P_c^1(t)}}-1\right)\\ &+dN_c^2(t)\left(\frac{C_t^2}{\sqrt{P_c^2(t)}}-1\right)\\ &-dt\left\{iV_t+\frac{\gamma}{2}[a^\dagger a-\langle a^\dagger a\rangle_c(t)]\right\}\Bigg]|\psi_c(t)\rangle\end{aligned} \tag{2.36}$$

This explicitly shows that the direct photodetection SSE (2.31) is not the only way to unravel the VOME with a simple interpretation. It is still, however, in the form of a quantum jump SSE. We can change this into a diffusion process by considering a continuum limit in which $|\beta|\to\infty$, so that the rate of jumps goes to infinity as the size of the jumps goes to zero. Individual photodetections are replaced by a classical photocurrent. In balanced detection as we are considering here, the difference between the photocurrents of the two detectors is the signal current. In this subsection we want this to be a homodyne signal (proportional to one of the quadratures of the system), so we put the detuning Ω of the local oscillator equal to zero. This allows the subscripts t to be dropped from β_t.

Consider $|\beta|^2 \gg \langle a^\dagger a\rangle$ and a time Δt such that the number of photodetections $\sim \gamma|\beta|^2\Delta t$ is very large, but that the change in the system is very small. Then it can be shown [11] that the number of photodetections at detector D_k can be approximated by a Gaussian random variable:

$$\begin{aligned}m_k = {}& \frac{\gamma}{2}|\beta^2|\left[1+(-1)^k\langle a/\beta + a^\dagger/\beta^*\rangle_c(t)\right]\Delta t\\ &+\sqrt{\frac{\gamma}{2}}|\beta|\Delta W_k,\end{aligned} \tag{2.37}$$

where ΔW_k are independent real Weiner increments [16]. The infinitesimally evolved system state given the photocounts m_k is approximately

$$\begin{aligned}|\tilde\psi_c(t+\Delta t)\rangle = {}& \exp\left[-\left(\frac{\gamma}{2}a^\dagger a + iV_t\right)\Delta t\right]\\ &\times\left(1+\frac{a}{\beta}\right)^{m_2}\left(1-\frac{a}{\beta}\right)^{m_1}|\psi_c(t)\rangle,\end{aligned} \tag{2.38}$$

where the tilde denotes that we are ignoring normalization. Substituting in the stochastic expression for the photocounts (2.37) and taking the continuum limit

$|\beta| \to \infty$, $[\arg(\beta) = \varphi]$, $\Delta t \to dt$, and $\Delta W_k \to dW_k$ gives the following SSE for the unnormalized conditioned ket:

$$|\tilde{\psi}_c(t+dt)\rangle = \left\{ 1 - \left(\frac{\gamma}{2}a^\dagger a + iV_t\right) dt + \left[2\gamma dt \langle X_\varphi\rangle_c(t) + \sqrt{\gamma} dW(t)\right] e^{-i\varphi} a \right\} |\psi_c(t)\rangle. \quad (2.39)$$

Here, $X_\varphi = \frac{1}{2}(e^{-i\varphi}a + e^{i\varphi}a^\dagger)$ is the quadrature of the system aligned parallel to the local oscillator, and $dW = (dW_2 - dW_1)/\sqrt{2}$ is a real infinitesimal Weiner increment.

In the continuum limit, the photocounts m_k are replaced by the photocurrents given by Eq. (2.37). The signal photocurrent is the difference $I_c^2(t) - I_c^1(t)$

$$I_c(t) = 2\gamma|\beta|\langle X_\varphi\rangle_c(t) + \sqrt{\gamma}|\beta|\xi(t), \quad (2.40)$$

where $\xi(t) = dW(t)/dt$ represents Gaussian white noise and satisfies

$$E[\xi(t)\xi(t')] = \delta(t-t'). \quad (2.41)$$

The deterministic part of this photocurrent is proportional to the X_φ quadrature of the system, as desired from homodyne detection. It is shown in Ref. [11] that this expression and the homodyne SSE (2.39) yield the standard expression for the X_φ quadrature squeezing autocorrelation function:

$$E[I_c(t+\tau)I(t)] = 4\gamma^2\langle : X_\varphi(t+\tau)X_\varphi(t) : \rangle + \gamma\delta(\tau), \quad (2.42)$$

where $X_\varphi(t)$ is a standard Heisenberg picture operator. (The generalization for nonunit efficiency detectors is also derived in Ref. [11].) Furthermore, the above expression for the photocurrent allows the homodyne SSE to be written in the following concise form:

$$\frac{d}{dt}|\tilde{\psi}_c(t)\rangle = \left[-\frac{\gamma}{2}a^\dagger a - iV_t + \frac{I_c(t)}{\beta}a\right]|\tilde{\psi}_c(t)\rangle. \quad (2.43)$$

This clearly shows how the change in the system is conditioned on the stochastic result of the measurement.

D. Heterodyne detection

For heterodyne detection, the detection scheme is the same as that for homodyne detection, but the local oscillator is detuned from the system (i.e., Ω is finite). Then the SSE for the conditioned state vector becomes

$$|\tilde{\psi}_c(t+dt)\rangle = \left\{ 1 - \left(\frac{\gamma}{2}a^\dagger a + iV_t\right) dt + \left[\gamma dt\langle a^\dagger + a e^{-2i(\varphi-\Omega t)}\rangle_c(t) + \sqrt{\gamma}dW(t)e^{-i(\varphi-\Omega t)}\right] a \right\} |\psi_c(t)\rangle, \quad (2.44)$$

while the signal photocurrent is given by

$$\begin{aligned} I_c(t) &= \gamma|\beta|\langle a e^{-i(\varphi-\Omega t)} + a^\dagger e^{i(\varphi-\Omega t)}\rangle_c(t) + \sqrt{\gamma}|\beta| dW(t) \\ &= 2\gamma|\beta|\langle X_\varphi\cos(\Omega t) - X_{\varphi+\pi/2}\sin(\Omega t)\rangle_c(t) \\ &\quad + \sqrt{\gamma}|\beta| dW(t). \end{aligned} \quad (2.45)$$

From this expression it is evident that by Fourier analyzing the photocurrent, a simultaneous readout of the approximate values of the two quadratures can be made. However, this will only work if the detuning frequency is much higher than the characteristic system frequencies (other than the free evolution frequency which has been eliminated). That is, we require $\Omega \gg \gamma \sim V_t$. Then we can consider a time scale Δt such that $\Omega\Delta t \gg 1$ but $\gamma\Delta t \ll 1$. Over this time, we can integrate Eq. (2.44) and ignore terms of second order in $\gamma\Delta t$ to obtain

$$|\tilde{\psi}_c(t+\Delta t)\rangle = \left\{ 1 - \left(\frac{\gamma}{2}a^\dagger a + iV_t\right)\Delta t + \left[\gamma\Delta t\langle a^\dagger\rangle_c(t) + \gamma\langle a\rangle_c(t)\int_t^{t+\Delta t} e^{-2i(\varphi-\Omega s)}ds + \sqrt{\gamma}\int_t^{t+\Delta t} e^{-i(\varphi-\Omega s)}dW(s)\right] a \right\} |\psi_c(t)\rangle. \quad (2.46)$$

Now the first integral in (2.46) is of order γ/Ω. The second integral will evidently be a new complex random variable of zero mean which we denote by

$$\Delta W_\Omega(t) = \int_t^{t+\Delta t} e^{-i(\varphi-\Omega s)}dW(s). \quad (2.47)$$

It is easy to show that

$$\mathrm{E}[\Delta W_\Omega^*(t)\Delta W_\Omega(t')] = (\Delta t - |t-t'|)H(\Delta t - |t-t'|), \quad (2.48)$$

where H is the Heaviside function which is zero when its argument is negative and one when its argument is positive. Other second-order moments are of order γ/Ω. In taking the continuum limit of $\gamma/\Omega \to 0$, $\Delta t \to dt$, we can take $\Delta W_\Omega(t) \to dW_\Omega(t) = \xi_\Omega(t)dt$, where $\xi_\Omega(t)$ represents Gaussian complex white noise satisfying

$$\mathrm{E}[\xi_\Omega^*(t)\xi_\Omega(t')] = \delta(t-t'), \quad (2.49)$$

with other second-order moments vanishing. The conditioned ket then obeys the following SSE:

$$\frac{d}{dt}|\tilde{\psi}_c(t)\rangle = \left\{ -\frac{\gamma}{2}a^\dagger a - iV_t + \left[\gamma\langle a^\dagger\rangle_c(t) + \sqrt{\gamma}\xi_\Omega(t)\right] a \right\}|\tilde{\psi}_c(t)\rangle. \quad (2.50)$$

This SSE is precisely the same as that proposed by Gisin and Percival [9], presented as Eq. (1.7) in the Introduction. Here we see its interpretation in terms of heterodyne measurements on the output light.

We turn now to the Fourier analysis of the heterodyne photocurrent (2.45). Using an obvious notation we find as expected

$$I_c^{\cos}(t) = (\Delta t)^{-1}\int_t^{t+\Delta t} I_c(s)\cos(\Omega s)ds \simeq \gamma|\beta|\langle X_\varphi\rangle_c(t) + \sqrt{\gamma}\,\mathrm{Re}[\beta\xi_\Omega(t)], \quad (2.51a)$$

$$I_c^{\sin}(t) = (\Delta t)^{-1}\int_t^{t+\Delta t} I_c(s)\sin(\Omega s)ds \simeq -\gamma|\beta|\langle X_{\varphi+\pi/2}\rangle_c(t) + \sqrt{\gamma}\,\mathrm{Im}[\beta\xi_\Omega(t)], \quad (2.51b)$$

where Re and Im denote real and imaginary parts, respectively. Defining a "complex current"

$$I_c^\Omega(t) = I_c^{\cos}(t) + iI_c^{\sin}(t) = \gamma\beta\langle a^\dagger\rangle_c(t) + \sqrt{\gamma}\beta\xi_\Omega(t), \quad (2.52)$$

we find (using the method of Ref. [11]) that the two-time correlation function

$$E[I_c^\Omega(t+\tau)I^\Omega(t)^*] = \gamma^2\langle a^\dagger(t+\tau)a(t)\rangle + \gamma\delta(\tau) \quad (2.53)$$

gives the first-order coherence function $G^{(1)}(t, t+\tau)$ plus the shot-noise term. The Fourier transform of this quantity may be used to determine the spectrum of the source. Also, the complex photocurrent allows the heterodyne SSE (2.50) to be written in the same form as the homodyne SSE (2.43):

$$\frac{d}{dt}|\tilde{\psi}_c(t)\rangle = \left[-\frac{\gamma}{2}a^\dagger a - iV_t + \frac{I_c^\Omega(t)}{\beta}a\right]|\tilde{\psi}_c(t)\rangle. \quad (2.54)$$

Finally, it is worth noting that the heterodyne scheme outlined here is completely equivalent to the so-called eight-port homodyne detection [29]. In this scheme, half of the output light is channeled into a balanced homodyne measurement of X_φ and half into a measurement of $X_{\varphi+\pi/2}$. The former gives rise to the photocurrent $I_c^{\cos}(t)$, while the latter gives $I_c^{\sin}(t)$, and the SSE obeyed by the system is again given by (2.54).

III. APPLICATION TO A TWO-LEVEL ATOM

A. The optical Bloch equations

We are now in a position to apply the theory of the Sec. II to a simple quantum system: a classically driven, damped, detuned two-level atom. Denoting the upper and lower states $|2\rangle$ and $|1\rangle$, respectively, we use the following operators:

$$\hat{x} = |2\rangle\langle 1| + |1\rangle\langle 2|, \quad (3.1a)$$

$$\hat{y} = -i|2\rangle\langle 1| + i|1\rangle\langle 2|, \quad (3.1b)$$

$$\hat{z} = |2\rangle\langle 2| - |1\rangle\langle 1|, \quad (3.1c)$$

$$\sigma = |1\rangle\langle 2| = \tfrac{1}{2}(\hat{x} - i\hat{y}). \quad (3.1d)$$

The master equation for the atom is

$$\dot{\rho} = -i[H,\rho] + \gamma\mathcal{D}[\sigma]\rho, \quad (3.2)$$

where γ is the spontaneous emission rate and

$$H = \frac{\alpha}{2}\hat{x} + \frac{\Delta}{2}\hat{z}, \quad (3.3)$$

where α is the Rabi frequency (proportional to the classical field amplitude by the dipole coupling constant) and Δ is the atomic frequency minus the classical field frequency. If we denote the averages of the operators $\hat{x}, \hat{y}, \hat{z}$ by x, y, z, respectively, the density operator for the atom can be simply expressed in terms of the Bloch vector (x, y, z) as

$$\rho(t) = \tfrac{1}{2}[1 + x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}]. \quad (3.4)$$

The master equation (3.2) can then be written in the following succinct form:

$$\dot{x} = -\Delta y - \frac{\gamma}{2}x, \quad (3.5a)$$

$$\dot{y} = -\alpha z + \Delta x - \frac{\gamma}{2}y, \quad (3.5b)$$

$$\dot{z} = +\alpha y - \gamma(z+1). \quad (3.5c)$$

The stationary solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{ss}} = \begin{pmatrix} -4\Delta\alpha \\ 2\alpha\gamma \\ -\gamma^2 - 4\Delta^2 \end{pmatrix}\left(\gamma^2 + 2\alpha^2 + 4\Delta^2\right)^{-1}. \quad (3.6)$$

Anticipating the following subsections in which we unravel the master equation as stochastic trajectories for a state vector, we note that when ρ is pure, the Bloch vector is confined to the unit sphere $x^2 + y^2 + z^2 = 1$. In this case, it is possible to parametrize the state of the system by two Euler angles on the unit sphere, (θ, ϕ). These angles will obey coupled stochastic differential equations. Nevertheless, the probability distribution of states on the sphere surface $p(\phi, \theta, t)$ will obey a deterministic evolution equation derived from these stochastic equations. Such an equation is equivalent to the master equation in that

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \int_0^\pi d\theta \int_0^{2\pi} d\phi \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} p(\phi,\theta,t). \quad (3.7)$$

However, different unravelings of the master equation will give rise to different evolution equations for $p(\phi, \theta, t)$. For the three measurement schemes considered here, these evolution equations are members of the class of two-dimensional differential Chapman-Kolmogorov equations [16].

B. Direct photodetection

The state of a classically driven two-level atom conditioned on direct photodetection of its resonance fluorescence has been considered in detail before [1, 4, 5]. The treatment here is thus restricted to formulating the stochastic evolution in the manner presented above, and giving a closed-form expression for the stationary probability distribution of states on the Bloch sphere, which has not been done before.

Consider a two-level atom situated in an experimental apparatus such that the light it emits is all collected and enters a detector (in principle this could be achieved by placing the atom at the focus of a large parabolic mirror). The annihilation operator for the field at the photodetector is effectively $\sqrt{\gamma}\sigma$. From the theory in Secs. II B and III A, the state vector of the atom conditioned on the photodetector count obeys the following SSE:

$$d|\psi_c(t)\rangle = \left[dN_c(t)\left(\frac{\sigma}{\sqrt{\langle\sigma^\dagger\sigma\rangle_c(t)}} - 1\right) - dt\left(\frac{\gamma}{2}[\sigma^\dagger\sigma - \langle\sigma^\dagger\sigma\rangle_c(t)] - iH\right)\right]|\psi_c(t)\rangle, \tag{3.8}$$

where H is as defined in Eq. (3.3) and the photocount increment $dN_c(t)$ satisfies $E[dN_c(t)] = \gamma\langle\sigma^\dagger\sigma\rangle_c(t)dt$. With the conditioned subscript understood, we can write the conditioned state in terms of the Euler angles (ϕ,θ) as defined in the preceding subsection. These parameters then obey the following coupled nonlinear stochastic differential equations (SDE's):

$$d\phi(t) = A_\phi[\phi(t),\theta(t)]dt, \tag{3.9a}$$

$$d\theta(t) = A_\theta[\phi(t),\theta(t)]dt + \frac{\gamma}{2}\sin\theta(t)dt + [\pi - \theta(t)]dN(t), \tag{3.9b}$$

where the Hamiltonian drift terms are defined by

$$A_\phi(\phi,\theta) = -\alpha\cot\theta\cos\phi + \Delta, \tag{3.10a}$$

$$A_\theta(\phi,\theta) = -\alpha\sin\phi, \tag{3.10b}$$

and

$$E[dN(t)] = \gamma\cos^2[\theta(t)/2]dt. \tag{3.11}$$

Now to write these SDE's as a differential Chapman-Kolmogorov equation for the probability distribution $p(\phi,\theta,t)$ we need the following non-negative function to exist:

$$W(\phi',\theta'|\phi,\theta) = \lim_{\Delta t\to 0} p(\phi',\theta',t+\Delta t|\phi,\theta,t)/\Delta t, \tag{3.12}$$

where $p(\phi',\theta',t+\Delta t|\phi,\theta,t)$ has its obvious meaning [16], and (ϕ',θ') is finitely separated from (ϕ,θ). In the above equations (3.9a) and (3.9b), the jump process is particularly simple: it always takes the state to the south pole (level $|1\rangle$ of the atom), with probability $E[dN(t)]$. Thus we have

$$W(\phi',\theta'|\phi,\theta) = \frac{\delta(\pi-\theta')}{2\pi}\gamma\cos^2[\theta/2], \tag{3.13}$$

where the δ function is defined so as to give unity when integrated on a finite interval closed below at zero. From this, the equation for $p(\phi,\theta,t)$ under direct photodetection is seen to be

$$\dot{p}(\phi,\theta,t) = \left\{-\frac{\partial}{\partial\phi}A_\phi(\phi,\theta) - \frac{\partial}{\partial\theta}\left[A_\theta(\phi,\theta) + \frac{\gamma}{2}\sin\theta\right] - \gamma\cos^2(\theta/2)\right\}p(\phi,\theta,t) + \gamma\frac{\delta(\pi-\theta)}{2\pi}\int_0^{2\pi}d\phi'\int_0^{\pi}d\theta'\cos^2(\theta'/2)p(\phi',\theta',t). \tag{3.14}$$

In practice, it is easier to find the steady-state solution of this equation by returning to the SSE (3.8), and ignoring normalization terms. We consider the evolution of the system following a photodetection at time $t=0$ so that $|\psi(0)\rangle = |1\rangle$. Assuming that no further photodetections take place, and omitting the normalization terms in Eq. (3.8), the state evolves via

$$\frac{d}{dt}|\tilde{\psi}_c(t)\rangle = -\left(\frac{\gamma}{2}\sigma^\dagger\sigma + iH\right)|\tilde{\psi}_c(t)\rangle. \tag{3.15}$$

Writing the unnormalized conditioned state vector as

$$|\tilde{\psi}_c(t)\rangle = \tilde{c}_1(t)|1\rangle + \tilde{c}_2(t)|2\rangle, \tag{3.16}$$

we easily find the solution satisfying $\tilde{c}_j(0) = \delta_{j,1}$ to be

$$\tilde{c}_1(t) = \cos(\zeta t) + \frac{\gamma/4 + i\Delta/2}{\zeta}\sin\zeta t, \tag{3.17a}$$

$$\tilde{c}_2(t) = -i\frac{\alpha/2}{\zeta}\sin(\zeta), \tag{3.17b}$$

where

$$\zeta = \left[\left(\frac{\Delta}{2} - i\frac{\gamma}{4}\right)^2 + \left(\frac{\alpha}{2}\right)^2\right]^{1/2} \tag{3.18}$$

is a complex number which reduces to half the detuned Rabi frequency as $\gamma\to 0$. From these we can define the time-dependent angle variables

$$\phi(t) = \arg[\tilde{c}_1(t)\tilde{c}_2^*(t)], \tag{3.19a}$$

$$\theta(t) = 2\arctan[|\tilde{c}_1(t)/\tilde{c}_2(t)|]. \tag{3.19b}$$

Note that it has not been necessary to introduce normalization. Now we denote the probability that there have been no photodetections in the interval $(0,t)$ by $S(t)$. The decrease in this survival probability from t to $t+dt$

is equal to the probability for a photodetection to occur (given that none have occurred so far) times the probability that none have occurred so far:

$$dS(t) = -E[dN(t)]S(t). \tag{3.20}$$

With the initial condition $S(0) = 1$, the solution is thus

$$S(t) = \exp\left\{-\gamma \int_0^t ds \cos^2[\theta(s)/2]\right\}. \tag{3.21}$$

It can be verified that $S(t)$ is also given by the modulus squared of the unnormalized state:

$$S(t) = \langle\tilde{\psi}_c(t)|\tilde{\psi}_c(t)\rangle = |\tilde{c}_1(t)|^2 + |\tilde{c}_2(t)|^2, \tag{3.22}$$

as expected since the decay in the norm of the conditioned state is due to the discarding of that component which arises from a photodetection having taken place.

Since whenever a photodetection does occur, the system returns to its state at $t = 0$, the stationary probability distribution on the Bloch sphere is confined to the curve parametrized by $[\phi(t), \theta(t)]$, weighted by the survival probability $S(t)$. Explicitly,

$$p_{\rm SS}(\phi,\theta) = \left\{\int_0^\infty dt\, S(t)\delta(\theta-\theta(t))\delta(\phi-\phi(t))\right\} \times \left\{\int_0^\infty dt S(t)\right\}^{-1}. \tag{3.23}$$

In Fig. 2 we plot this probability distribution on the Bloch sphere for $\alpha = 3\gamma$, $\Delta = \gamma/2$. Here, as in subsequent subsections, we use an equal-area projection of the sphere onto the $\cos\theta, \phi$ plane. The solid curve (call it Γ) is a truncated representation of the one-dimensional submanifold to which $p_{\rm SS}(\phi,\theta)$ is confined. The probability density itself is approximated by a discrete distribution: the weight assigned to each small section of Γ is represented by the height of the line segment drawn orthogonal to Γ from the middle of that section. In fact, these line segments are drawn at regular intervals in time [the argument of the integral in (3.23)], so that their heights are simply given by $S(t)$. This figure thus also contains all of the information about the evolution of the atom following a photodetection and the waiting time distribution between photodetections. Using this distribution (with time increment $dt = 10^{-2}\gamma^{-1}$) to calculate the average of the steady-state Bloch vector confirms the analytic result (3.6) to four decimal places; in this case $(x, y, z)_{\rm SS} = (-0.3, 0.3, 0.1)$.

C. Homodyne detection

In this subsection we consider homodyne detection of the light emitted from the atom. Say the local oscillator has phase φ relative to the driving field (which is in phase with $\hat{x}$). Then, from Eq. (2.39), the system obeys the following SSE:

$$d|\tilde{\psi}_c(t)\rangle = \left\{ -\left(\frac{\gamma}{2}\sigma^\dagger\sigma + iH\right)dt + \left[\gamma dt\langle e^{-i\varphi}\sigma + e^{i\varphi}\sigma^\dagger\rangle_c(t) + \sqrt{\gamma}dW(t)\right]e^{-i\varphi}\sigma\right\}|\psi_c(t)\rangle, \tag{3.24}$$

where $dW(t)$ is a real infinitesimal Weiner increment. Transforming to the Euler angles gives the following set of SDE's:

$$d\phi(t) = A_\phi[\phi(t),\theta(t)]dt + \cos\theta(t)\frac{1+\cos\theta(t)}{1-\cos\theta(t)}\sin\tilde{\phi}(t)\cos\tilde{\phi}(t)\gamma\, dt - \frac{1+\cos\theta(t)}{\sin\theta(t)}\sin\tilde{\phi}(t)\sqrt{\gamma}dW \tag{3.25a}$$

$$d\theta(t) = A_\theta[\phi(t),\theta(t)]dt + \frac{1+\cos\theta(t)}{\sin\theta(t)}\left\{1 - \tfrac{1}{2}[1+\cos\theta(t)]\cos\theta(t)\cos^2\tilde{\phi}(t)\right\}\gamma\, dt + \cos\tilde{\phi}(t)[1+\cos\theta(t)]\sqrt{\gamma}dW, \tag{3.25b}$$

where $\tilde{\phi}(t) = \phi(t) + \varphi$. In this case the noise terms are diffusive rather than jump processes, so that the probability distribution obeys a Fokker-Planck equation

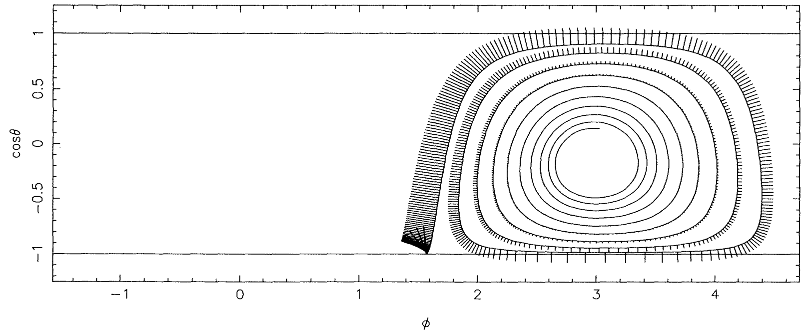

FIG. 2. Equal-area projection of the Bloch sphere showing the stationary probability distribution of the state vector for a driven, damped, detuned atom whose fluorescence is subject to direct photodetection. For detailed explanation, see text. The parameters here are (in units of the spontaneous emission rate γ) driving $\alpha = 3$ and detuning $\Delta = 0.5$.

$$\dot{p}(\phi,\theta,t) = \Bigg\{ -\frac{\partial}{\partial\phi}\left[A_\phi(\phi,\theta) + \gamma\cos\theta\frac{1+\cos\theta}{1-\cos\theta}\sin\tilde{\phi}\cos\tilde{\phi}\right] - \frac{\partial}{\partial\theta}\left[A_\theta(\phi,\theta) + \gamma\frac{1+\cos\theta}{\sin\theta}\left(1 - \frac{1+\cos\theta}{2}\cos\theta\cos^2\tilde{\phi}\right)\right] - \frac{1}{2}\frac{\partial^2}{\partial\phi^2}\gamma\left[\frac{1+\cos\theta}{1-\cos\theta}\sin^2\tilde{\phi}\right] - \frac{1}{2}\frac{\partial^2}{\partial\theta^2}\gamma\left[(1+\cos\theta)^2\cos^2\tilde{\phi}\right] - \frac{\partial^2}{\partial\phi\partial\theta}\gamma\left[\frac{(1+\cos\theta)^2}{\sin\theta}\sin\tilde{\phi}\cos\tilde{\phi}\right]\Bigg\} p(\phi,\theta,t), \tag{3.26}$$

where $\tilde{\phi} = \phi + \varphi$. The steady-state solution to this equation can be approximated numerically by a long-time average of trajectories on the Bloch sphere generated by the SDE's (3.25a) and (3.25b).

In Fig. 3 we plot this approximation to the stationary probability distribution for two values of $\varphi = 0$. The average is taken over a time of $5 \times 10^5\gamma^{-1}$, and a total of 20 000 points are plotted (one every $0.025\gamma^{-1}$). The two values of φ are 0 and $\pi/2$, corresponding to measuring the quadrature of the spontaneously emitted light in phase and in quadrature with the driving field, respectively. In both plots we take $\sqrt{2}\,\alpha = 7\gamma$ and $\Delta = 0$, so that the true distributions are symmetrical about the line $\phi = \pm\pi/2$. The effect of the measurement is dramatic and readily understandable. In terms of the Euler angles, the homodyne photocurrent from Eq. (2.40) is

$$I_c(t) = |\beta|[\gamma\sin\theta(t)\cos\tilde{\phi}(t) + \sqrt{\gamma}\xi(t)]. \tag{3.27}$$

When the local oscillator is in phase ($\varphi = 0$), the deterministic part of the photocurrent is proportional to $x(t)$. Under this measurement, the atom tends towards states with well-defined $\hat{x}$. The eigenstates of $\hat{x}$ are stationary states of the driving Hamiltonian and so this leads to the probability distribution in Fig. 3(a), which has two circumequatorial peaks near $\phi = 0$ and π. The steady-state Bloch vector (3.6) points slightly southwards and in the $\phi = \pi/2$ direction, so the two peaks are actually shifted somewhat in that direction also. In contrast, measuring the $\varphi = \pi/2$ quadrature tries to force the system into an eigenstate of $\hat{y}$. However, such an eigenstate will be rapidly spun around the sphere by the driving Hamiltonian. This effect is clearly seen on the steady-state distribution in Fig. 3(b), which is spread around the $\phi = \pm\pi/2$ great circle. As before, the $\phi = +\pi/2$ side is weighted somewhat more heavily. The analytic result (3.6) for the steady-state Bloch vector is $(x,y,z)_{\rm SS} = (0, 7\sqrt{2}, -2)/100$.

The above explanation for the stationary probability distributions are also useful for understanding the noise spectra of the quadrature photocurrents in Eq. (3.27). The spectrum of resonance fluorescence of a two-level atom has three peaks, the central one at the atomic frequency, and the two sidebands (of half the area) displaced by the Rabi frequency [31]. It is well known [30] that the spectrum of the in-phase homodyne photocurrent gives the central peak, while the quadrature photocurrent gives the two sidebands. This is readily explained qualitatively from the evolution of the atomic state under

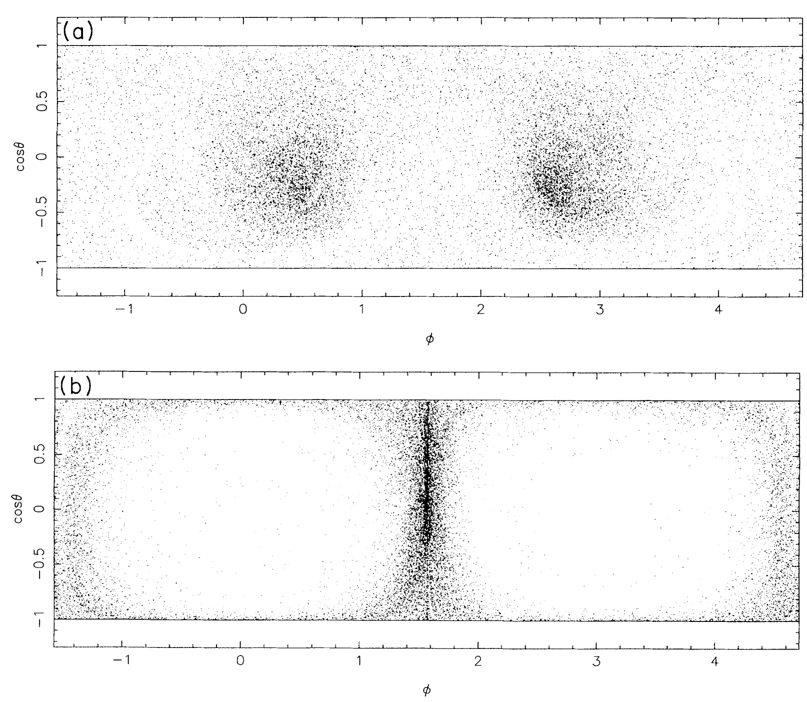

FIG. 3. Stationary probability distribution of the state vector of an atom whose fluorescence is subject to homodyne detection. The distribution is approximated by an ensemble of 20 000 points on the Bloch sphere. The phase of the local oscillator relative to the phase of the driving field (ϕ) is 0 in (a) and $\pi/2$ in (b). Here, $\alpha = \frac{7}{\sqrt{2}}\gamma$ and $\Delta = 0$. For further details, see text.

homodyne measurements. When $\hat{x}$ is being measured, it varies slowly, remaining near one eigenvalue on a time scale like γ^{-1}. This gives rise to a simply decaying autocorrelation function for the photocurrent (3.27), or a Lorentzian with width scaling as γ in the frequency domain. When $\hat{y}$ is measured, it undergoes rapid sinusoidal variation at frequency α, with noise added at a rate γ. This explains the side peaks.

D. Heterodyne detection

If the atomic fluorescence enters a perfect heterodyne detection device, then from Eq. (2.50), the system evolves via the SSE

$$|\tilde{\psi}_c(t+dt)\rangle = \left\{ 1 - \left(\frac{\gamma}{2}\sigma^\dagger\sigma + iH\right) dt + \left[\gamma dt \langle\sigma^\dagger\rangle_c(t) + \sqrt{\gamma} dW_\Omega(t)\right]\sigma \right\} |\psi_c(t)\rangle, \tag{3.28}$$

where $dW_\Omega(t)$ is a complex infinitesimal Weiner increment with independent real and imaginary parts. This SSE is equivalent to the following coupled SDE's for the angles on the Bloch sphere:

$$d\phi(t) = A_\phi[\phi(t),\theta(t)]dt - \frac{1+\cos\theta(t)}{\sin\theta(t)}\sqrt{\frac{\gamma}{2}}[dW_1\sin\phi(t) - dW_2\cos\phi(t)], \tag{3.29a}$$

$$\begin{aligned} d\theta(t) = {} & A_\theta[\phi(t),\theta(t)]dt + \frac{1+\cos\theta(t)}{\sin\theta(t)}\left\{1 - \tfrac{1}{4}[1+\cos\theta(t)]\cos\theta(t)\right\}\gamma\, dt \\ & + [1+\cos\theta(t)]\sqrt{\frac{\gamma}{2}}[\cos\phi(t)dW_1 + \sin\phi(t)dW_2], \end{aligned} \tag{3.29b}$$

where $dW_1 = \sqrt{2}\mathrm{Re}\,(dW)$ and $dW_2 = \sqrt{2}\mathrm{Im}\,(dW)$. The Fokker-Planck equation for $p(\phi,\theta,t)$ is then

$$\begin{aligned} \dot{p}(\phi,\theta,t) = \Bigg\{ & -\frac{\partial}{\partial\phi}[A_\phi(\phi,\theta)] - \frac{\partial}{\partial\theta}\left[A_\theta(\phi,\theta) + \gamma\frac{1+\cos\theta}{\sin\theta}\left(1 - \frac{1+\cos\theta}{4}\cos\theta\right)\right] \\ & - \frac{1}{4}\frac{\partial^2}{\partial\phi^2}\gamma\left[\frac{1+\cos\theta}{1-\cos\theta}\right] - \frac{1}{4}\frac{\partial^2}{\partial\theta^2}\gamma\left[(1+\cos\theta)^2\right]\Bigg\} p(\phi,\theta,t). \end{aligned} \tag{3.30}$$

As for the homodyne detection case, we approximate the steady-state probability distribution by a time ensemble of points on the Bloch sphere found from the above SDE's. The result (using the same parameters as in the preceding subsection) is shown in Fig. 4. In this case, the stationary probability distribution is spread fairly well over the entire Bloch sphere. This can be understood as the result of the two competing measurements ($\hat{x}$ and $\hat{y}$) combined with the driving Hamiltonian causing rotation around the x axis. The complex photocurrent as defined in Eq. (2.52) is given simply in the Euler angles by

$$I_c^\Omega(t) = \beta\left\{\tfrac{1}{2}\gamma\sin\theta(t)\exp[i\phi(t)] + \sqrt{\gamma}\xi^\Omega(t)\right\}. \tag{3.31}$$

The spectrum of this photocurrent gives the complete Mollow triplet, as y is rotated at frequency α while x simply diffuses.

IV. DISCUSSION

In this paper we have examined three ways by which the quantum optical master equation (2.13) can be "unraveled" into stochastic quantum trajectories for state vectors. The first (which we have associated with direct photodetection) involves discontinuous quantum jumps in the system state, while the other two (homodyne and heterodyne detection) lead to continuous quantum tra-

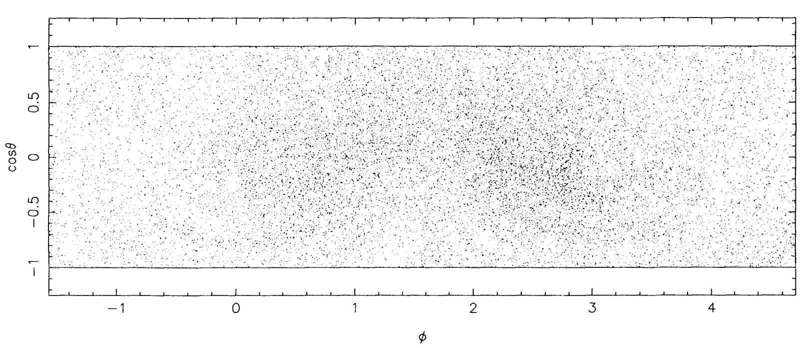

FIG. 4. Stationary probability distribution of the state vector of an atom whose fluorescence is subject to heterodyne detection. Other details are as in Fig. 3.

jectories with diffusive noise. In the sense that the density operator obtained from the ensemble average of the state vectors produced by any of the three stochastic Schrödinger equations obeys the original master equation, they are equivalent. However, the individual quantum trajectories have a quite different nature in each case. The "direct" SSE (2.31) has already been put to extensive use [2–7,12,14,32], usually with the attitude that it is merely an algorithm for simulating irreversible evolution which avoids the use of density operators. The "homodyne" SSE (2.39), discovered by Carmichael [3], has been discussed extensively previously [11], along with its interpretation. What we call the "heterodyne" SSE (2.50) was first put forward by Gisin and Percival [9], as a "... model for the motion of an [individual] quantum system in interaction with its environment." Referring to the "direct" SSE, Gisin and Percival go on to say that it "... provides a *different* insight [into the behavior of individual systems], and it remains to be seen which, if any, is preferable."

The main thrust of this paper is that it does not remain to be seen which model is preferable. We have shown that, under the conditions outlined in Sec. II A, all three models have an equally valid interpretation in terms of representing the evolution of an individual quantum system. The relevant model for a given experimental situation depends on the method by which information is to be extracted from the light leaving the system. That is to say, the state of a quantum system is always conditioned on (and in fact can be identified with) our knowledge of the system obtained from a measuring apparatus which effectively behaves classically. This lesson is almost as old as quantum mechanics. In the words of Bohr [33], "... these conditions [which define the possible types of predictions regarding the future behavior of the system] constitute an inherent element of the description of any phenomenon to which the term "physical reality" can be properly attached." The state vectors produced by the SSE's considered here are as real as anything in the quantum world.

In spite of the above remarks about the equality of all measurement schemes, it must be admitted that the direct and heterodyne SSE's are in a sense more natural ways to unravel the master equation than the homodyne SSE. The homodyne SSE requires the specification of the phase of the local oscillator and so is not unique. In the case of the driven two level atom, there were two natural choices (in phase and in quadrature with the driving field), but in other systems this may not be the case. The direct SSE results from measuring the intensity of the outgoing light, while the heterodyne SSE results from measuring its electric-field amplitude and phase. The former emphasizes the quantum nature of the dissipation (jumps due to individual photodetections), while the latter presents a more classical (diffusive) behavior. It might be expected that the heterodyne SSE would be a more general model, perhaps applicable to field measurements where photon detection is impractical (such as with microwaves).

It should be emphasized that the above SSE's are only derivable for perfect photodetectors, for which our knowledge of the system is maximal. For less than perfect detectors (which is the case in practice), quantum trajectories of the conditioned system state still exist, but the evolution equations will contain terms which do not preserve the purity of states [see, for example, Eq. (2.27)]. Thus it would be necessary to use a density matrix (indicating less than maximal knowledge) rather a state vector to represent the system. This makes the computation of a simulated measurement more difficult, and so it is for convenience only that we use perfect detectors. However, for some measurement schemes not considered here (e.g., spectral detection [34]), the density operator must be used even with perfect photodetectors. There is also a question as to how accurately one must know the output of the photodetector in order to reproduce well the conditioned system state. For direct photodetection the only issue is when the photodetections occur; for homodyne and heterodyne detection, the question is to how many significant figures the current is known. This issue remains to be investigated.

So far, we have discussed the first two master equation unravelings discussed in the Introduction and concluded that under some conditions they have a meaningful interpretation. The third unraveling, due to Teich and Mahler [13], has not been mentioned because it does not appear to correspond to any measurement scheme. Indeed, Teich and Mahler do not consider any measurement scheme, but treat their quantum jumps as objective processes "... connected with the spontaneous emission of a photon." It could be argued that we have not looked at enough measurement schemes. For instance, Teich and Mahler refer to the different frequencies of their scattered photons, so perhaps their scheme corresponds to spectral detection. However, this is not the case. We have examined spectral detection of the resonance fluorescence of a two-level atom in the high driving limit and found that the conditioned system state undergoes jumps between the atomic dressed states [34], as predicted by the dressed-atom model [35]. Teich and Mahler's model in the same limit predicts jumps between the diagonal states, which are completely different (in fact they are orthogonal to the dressed states in the Bloch sphere sense).

As well as having no apparent relation to experiment, the scheme of Teich and Mahler seems to be internally inconsistent. In their model, a system at steady state will always be in one of the diagonal states of the steady-state density operator and will jump between these states. Presumably an observer should be able to know which state the system is in, otherwise the meaning of the preceding claim is unclear. This state (call it $|\mu\rangle$) is thus objective knowledge. Another observer who does not know that the system has reached steady state will then treat $|\mu\rangle$ as the initial state of the system which (according to this second observer) will then relax to steady state by smooth evolution of the diagonal states accompanied by jumps between these changing states. It is easy to verify for a simple system (such as the two-level atom) that the diagonal states *do* change during this relaxation process, so that one observer's diagonal states are different from another's. This appears to contradict Teich and Mahler's claim that their model represents the stochastic dynamics of individual quantum systems.

In summary, we have presented three SSE's which enable the master equation dynamics to be solved using state vectors and which in addition can be interpreted as giving the evolution of the state of the system conditioned on the results of measurements. The SSE's give an insight into the behavior of an individual quantum system under different measurement schemes. They also allow the results of experiments to be simulated directly, and suggest new ways of understanding these results. These features were illustrated with a canonical quantum optical problem, the two-level atom. Applications for these SSE's include proposed experiments to test quantum measurement theory (for example [36]). In any such proposal, it is necessary to specify the measurement scheme, and use the appropriate dynamical model of state reduction. The SSE's presented here may also find use in models of feedback on quantum and semiclassical systems. Finally, they are valuable because they reemphasize one of the old lessons of quantum mechanics: that the state of open quantum systems can only be defined *relative* to the information we have about them.

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APPENDIX A: WHY THE THERMAL TERMS IN THE MASTER EQUATION WERE IGNORED

From the expression for $I(z,t)$ (2.16), it is evident that the case of the thermal bath could not be treated in the theory of optical measurements presented in this paper. This is because N in Eq. (2.11) represents the average number of photons per bath mode. Since we are dealing with a continuum of modes, the total photon flux is infinite. Obviously this is an idealization, and in any case the bandwidth of a real photodetector will not be infinite. Nevertheless, if the master equation appropriate to the thermal bath (2.12) were to apply, then the photon flux due to the thermal bath would be far greater than that due to the system for N finitely large. In that case, the occurrence of a photodetection effectively conveys no information about the system evolution. Photodetections will occur very rapidly, and will have Poissonian statistics on the time scale of the system. To lowest order, the system will evolve according to the master equation (2.12) at all times, and the concept of a quantum trajectory of the conditioned state is useless. Of course, it is possible to consider the case where, for instance, one end of a cavity is illuminated by broadband thermal light, giving rise to the usual master equation, but that the other end is illuminated by a vacuum and is subject to photodetection. The presence of this second mirror changes the total master equation for the system to one appropriate to a lower temperature. Finally, it is important to note that the above comments regarding the difficulties of treating photodetection in a thermal bath apply equally to a broadband squeezed vacuum [18].

APPENDIX B: DERIVATION OF EQ. (2.32)

Consider the following two-time correlation function for the photocount increment:

$$E[dN_c(t+\tau)dN(t)]. \tag{B1}$$

Here $dN_c(t)$ satisfies

$$E[dN_c(t)] = \mathrm{Tr}\,[\mathcal{J}\rho_c(t)]\,dt = P_c(t)dt, \tag{B2}$$

$$dN_c(t)^2 = dN_c(t), \tag{B3}$$

where $\mathcal{J}$ is the superoperator which gives the change in the system when $dN(t)=1$ via $\rho_c(t+dt) = \mathcal{J}\rho_c(t)/P_c(t)$. In the case of the VOME (2.30), the action of the jump superoperator is defined by $\mathcal{J}\rho = \gamma\eta a\rho a^\dagger$. First we consider the case when $\tau > 0$ (that is, $\tau \gg dt$, where dt is the minimum time step considered). Now $dN(t)$ is either zero or one. Thus, since if $dN(t)=0$ the function (B1) is automatically zero, we have

$$E[dN_c(t+\tau)dN(t)] = \mathrm{Prob}[dN(t)=1]E\left[dN_c(t+\tau)|_{dN(t)=1}\right], \tag{B4}$$

where the subscript to the vertical line is the condition for which the subscript c on $dN_c(t+\tau)$ exists. This can be rewritten as

$$[\gamma\eta\langle a^\dagger a\rangle(t)dt]\mathrm{Tr}\left\{\gamma\eta a^\dagger a dt E\left[\rho_c(t+\tau)|_{dN(t)=1}\right]\right\}, \tag{B5}$$

where $\langle a^\dagger a\rangle(t) = \mathrm{Tr}[a\rho(t)a^\dagger]$. Now from the action of $\mathcal{J}$ and the fact that $\rho(t) = E[\rho_c(t)]$ obeys the master equation $\dot\rho(t) = \mathcal{L}_0\rho(t)$, we have

$$E\left[\rho_c(t+\tau)|_{dN(t)=1}\right] = \exp[\mathcal{L}_0(\tau-dt)]a\rho(t)a^\dagger/\langle a^\dagger a\rangle(t). \tag{B6}$$

Thus to leading order in dt,

$$E[dN_c(t+\tau)dN(t)] = \gamma^2\eta^2 dt^2\mathrm{Tr}\{a^\dagger a e^{\mathcal{L}_0\tau}[a\rho(t)a^\dagger]\} \quad \text{for } \tau > 0. \tag{B7}$$

Now for $\tau = 0$ we have

$$E[dN(t)dN(t)] = E[dN(t)] = \gamma\eta dt\mathrm{Tr}[a^\dagger a\rho(t)]. \tag{B8}$$

For short times this term will be dominant, and $dN(t)/dt$ can be treated as δ-correlated noise for a suitably defined δ function. Thus we can write

$$E\left[\frac{dN_c}{dt}(t+\tau)\frac{dN}{dt}(t)\right] = \gamma^2\eta^2\mathrm{Tr}\{a^\dagger a e^{\mathcal{L}_0\tau}[a\rho(t)a^\dagger]\} + \gamma\eta\mathrm{Tr}[a^\dagger a\rho(t)]\delta(\tau), \tag{B9}$$

which is the desired result.

[1] H.J. Carmichael, S. Singh, R. Vyas, and P.R. Rice, Phys. Rev. A **39**, 1200 (1989).

[2] H.J. Carmichael and L. Tian, in *OSA Annual Meeting Technical Digest 1990*, Vol. 15 of *OSA Technical Digest Series* (Optical Society of America, Washington, DC, 1990).

[3] H.J. Carmichael, *An Open Systems Approach to Quantum Optics*, Lecture Notes in Physics (Springer-Verlag, Berlin, in press).

[4] J. Dalibard, Y. Castin, and K. Mølmer, Phys. Rev. Lett. **68**, 580 (1992).

[5] R. Dum, P. Zoller, and H. Ritsch, Phys. Rev. A **45**, 4879 (1992).

[6] R. Dum, A.S. Parkins, P. Zoller, and C.W. Gardiner, Phys. Rev. A **46**, 4382 (1992).

[7] C.W. Gardiner, A.S. Parkins, and P. Zoller, Phys. Rev. A **46**, 4363 (1992).

[8] L. Diosi, J. Phys. A **21**, 2885 (1988).

[9] N. Gisin and I. Percival, Phys. Lett. A **167**, 315 (1992); and J. Phys. A **25**, 5677 (1992).

[10] V.P. Belavkin and P. Staszewski, Phys. Rev. A **45**, 1347 (1992).

[11] H.M. Wiseman and G.J. Milburn, Phys. Rev. A **47**, 642 (1993).

[12] P. Alsing and H. J. Carmichael, Quantum Opt. **3**, 13 (1991).

[13] W.G. Teich and G. Mahler, Phys. Rev. A **45**, 3300 (1992).

[14] L. Tian and H.J. Carmichael, Phys. Rev. A **46**, R6801 (1992).

[15] R.J. Cook, Phys. Scr. **21**, 49 (1988).

[16] C.W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, Berlin, 1985).

[17] H.J. Carmichael, J. Opt. Soc. Am. B **4**, 1588 (1987).

[18] C.W. Gardiner and M.J. Collett, Phys. Rev. A **31**, 3761 (1985).

[19] C.W. Gardiner, *Quantum Noise* (Springer-Verlag, Berlin, 1991).

[20] D.F. Walls, M.J. Collett, and G.J. Milburn, Phys. Rev. D **32**, 3208 (1985).

[21] P. Alsing, G.J. Milburn, and D.F. Walls, Phys. Rev. A **37**, 2970 (1988).

[22] M.J. Gagen and G.J. Milburn, Phys. Rev. A **45**, 5228 (1992).

[23] W. Heisenberg, *The Physical Principles of Quantum Mechanics* (The University of Chicago Press, Chicago, 1930).

[24] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).

[25] N. Bohr, Philos. Mag. **26**, 1 (1913).

[26] M.D. Srinivas and E.B. Davies, Opt. Acta **28**, 981 (1981).

[27] R. Loudon, *The Quantum Theory of Light* (Oxford University Press, Oxford, 1983).

[28] H.D. Zeh, Found. Phys. **1**, 69 (1970).

[29] R. Loudon, in *Quantum Optics IV*, Proceedings of the International Symposium, Hamilton, New Zealand, edited by J.D. Harvey and D.F. Walls (Springer-Verlag, Berlin, 1986).

[30] M.J. Collett, D.F. Walls, and P. Zoller Opt. Commun. **52**, 145 (1984).

[31] B.R. Mollow, Phys. Rev. **188**, 1969 (1969).

[32] P. Marte, R. Dum, R. Taïeb and P. Zoller (unpublished).

[33] N. Bohr, Phys. Rev. **48**, 696 (1935).

[34] H.M. Wiseman and G.J. Milburn (unpublished).

[35] C. Cohen-Tannoudji and S. Reynaud, Philos. Trans. R. Soc. London Ser. A **293**, 223 (1979).

[36] P. Storey, M.J. Collett, and D.F. Walls, Phys. Rev. Lett **68**, 472 (1992).