# Squeezing via feedback

H.M. Wiseman and G.J. Milburn
*Department of Physics, University of Queensland, Queensland 4072, Australia*
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We present the quantum theory of optical cavity feedback mediated by homodyne detection, with an arbitrary time delay. We apply this theory to a system with nonclassical dynamics, a sub-Poissonian pumped laser. By using the feedback to phase lock the laser it is possible to produce output light which exhibits perfect quadrature squeezing on resonance, rather than just sub-Poissonian intensity statistics. However, we also show that feedback mediated by homodyne detection (or any other extracavity measurement) cannot produce nonclassical light unless the cavity dynamics can do so without feedback. Futhermore, in systems which already exhibit squeezing, such feedback can only degrade the squeezing in the output. With feedback mediated by an intracavity measurement, these theorems do not apply. We show that an (admittedly unrealistic) intracavity quantum nondemolition quadrature measurement allows arbitrary squeezing to be produced by controlling the amplitude of a coherent driving field.

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## I. INTRODUCTION

Experimentalists in the field of quantum optics have long used feedback to reduce noise in lasers and other optical cavity systems. If not ignored theoretically, such feedback has generally been treated classically. That is, the quantum nature of the system has not played a role. This is no great problem if the optical system can be described classically. However, the advent of practical sources of nonclassical (in particular, squeezed) light [1] has made the lack of a practical quantum theory of optical cavity feedback an obvious deficiency. "What are the quantum limits to optical noise reduction via feedback?" is a question which needs to be answered. In this paper we provide an answer applicable to systems in which noise is reduced in one quadrature at the expense of the other; that is to say, to squeezed systems.

In order to derive a quantum theory of feedback, it is first necessary to have a quantum theory for the measurement step in the feedback loop. The practical application of quantum measurement theory has come a long way since Dirac's projection postulate [2], especially in quantum optics. The fundamental measurement process in quantum optics is the collapse which occurs when a photon is detected [3–5]. The quantum nature of this event is evident in that the intensity of the light at the photodetector represents the probability that a photodetection will occur per unit time. In general, it is difficult to deal with this sort of measurement, in which the state of the system undergoes jumps and the measurement result (the photocurrent) consists of a series of $\delta$-function spikes, other than numerically. This is perhaps part of the explanation as to why an adequate quantum theory of optical feedback has been elusive.

For this reason, the feedback theory presented here does not deal with direct photodetection, but rather with homodyne detection. Homodyne detection is also the more appropriate measurement scheme for controlling squeezing. Of course, homodyne detection still relies on photodetection; the only alteration from direct detection is that a local oscillator amplitude is added to the output field of the cavity before it enters the photodetector. However, by letting the local oscillator amplitude go to infinity, it is possible to approximate the discrete photocount by a continuous (although not differentiable) photocurrent. Furthermore, it is possible to derive a stochastic differential equation giving the evolution of the quantum state of the system, conditioned on the measured homodyne photocurrent [3,6]. The great advantage of these stochastic equations over the jump process of direct detection is that the noise involved is Gaussian white noise, which is easy to treat theoretically. It is so easy to treat that, in the limit when the time delay in the feedback loop is negligible, it is possible to derive a simple, exact master equation describing homodyne-mediated feedback in which the nature of the feedback forces is left completely arbitrary. The case of the finite time delay is somewhat more difficult, and is solved only for a certain class of systems encompassing most sources of squeezed light.

One consequence of the theory can be stated simply: homodyne-mediated feedback cannot produce nonclassical light. That is to say, if the internal dynamics of the cavity (including the dynamics which are influenced by the feedback) generate a state which has a positive Glauber-Sudarshan $P$ function, then completing the feedback loop will not alter this fact. In fact, this theorem is true of any feedback which relies on measurements external to the cavity, contrary to previous claims based on a model of feedback which is obviously inappropriate in hindsight [7,8]. On the other hand, if the free dynamics of the cavity already produce a squeezed state (which is nonclassical), then adding feedback can make that state more squeezed. This is of little practical use, however, for we also show that diverting part of the cavity output into a feedback loop can only degrade the squeezing in the re-

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mainder of the output. These "no-go" theorems indicate that extracavity feedback cannot increase the nonclassicality of the output light (although feedback can turn a nonclassical but unsqueezed output into a squeezed output, as we explore in Sec. IV). None of these theorems apply to intracavity measurements, which of course must add additional terms to the master equation. In particular, we show that a feedback loop using a quantum nondemolition (QND) measurement of one quadrature can in principle produce arbitrary squeezing.

The organization of this paper is as follows. In Sec. II we derive a stochastic equation describing selective evolution under homodyne-mediated feedback with an arbitrary time delay. Assuming the time delay to be small allows the noise in the photocurrent to to be averaged over, giving a relatively simple master equation. In Sec. III we linearize this master equation to apply it to squeezed systems. The "no-go" theorems mentioned above are derived. Section IV presents an application of the feedback master equation to a system with nonclassical dynamics: the sub-Poissonian pumped laser. There we show that phase-locking feedback can produce squeezing in the sense that the output light without feedback is nonclassical, but not quadrature squeezed. Section V explains how the feedback works by looking in detail at the selective evolution of the system. This theory is used in Sec. VI where we return to the case of the finite time delay, and show that this increases the noise in the output, as expected. Section VII deals with QND-mediated feedback, and shows how this can produce nonclassical light. Finally, Sec. VIII is a discussion of the most important aspects of the theory, with future generalizations and applications.

## II. QUANTUM THEORY OF HOMODYNE FEEDBACK

In this section we derive a master equation which describes quantum-limited feedback of a photocurrent onto the source cavity. The photocurrent to be considered comes from an extracavity homodyne measurement of the $X$ quadrature of the cavity field. The manner by which this current is used to change the cavity dynamics is left open. The master equation which we derive is valid in the limit when the delay time in the feedback loop is negligible compared to the characteristic response time of the system. Typically, this characteristic time will be the inverse of the damping rate of the cavity. Such damping to the external continuum of field modes is necessary to do homodyne detection. For simplicity, we measure time in inverse units of the cavity linewidth. Then we can write the master equation for the nonselective (ignoring measurement) evolution of the state matrix as $\dot{\rho} = \mathcal{L}\rho$ where

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{D}[a], \tag{2.1}$$

where $\mathcal{L}_0$ generates the internal evolution and $\mathcal{D}[a]$ is the standard damping term

$$\mathcal{D}[a] = \mathcal{J}[a] - \mathcal{A}[a], \tag{2.2}$$

where

$$\mathcal{J}[a]\rho = a\rho a^\dagger, \tag{2.3}$$

$$\mathcal{A}[a]\rho = \tfrac{1}{2}(a^\dagger a\rho + \rho a^\dagger a). \tag{2.4}$$

These definitions (2.2)–(2.4) are to be taken to apply to any operator $a$. In this case, $a$ is the annihilation operator for the cavity mode. We are assuming that the bath of external electromagnetic modes is practically at zero temperature for reasons explained in a previous paper [4].

Now we need an equation which describes how the state of the cavity mode is conditioned by the measured homodyne photocurrent. The general solution to this problem has been published in a previous paper [6]. We consider ideal measurements in the sense that the local oscillator amplitude is assumed arbitrarily large, but we allow for finite efficiency detection. The efficiency $\eta$ is the fraction of the output light from the cavity which enters the detector, multiplied by the quantum efficiency of the detector. Let us use the subscript $c$ to denote conditioning, so $\rho_c(t)$ is the state of the system conditioned on the photocurrent history up to time $t$. The instantaneous homodyne photocurrent is (within a constant of proportionality)

$$I_c(t) = \eta\langle a + a^\dagger\rangle_c(t) + \sqrt{\eta}\xi(t), \tag{2.5}$$

where $\langle a + a^\dagger\rangle_c(t)$ is the conditioned quantum average $\mathrm{Tr}[(a+a^\dagger)\rho_c(t)]$, and $\xi(t)$ is real white noise [9] satisfying

$$E[\xi(t)\xi(t')] = \delta(t-t'), \tag{2.6}$$

where $E$ indicates an ensemble expectation value. This noise can be interpreted as local oscillator shot noise, and arises as the Gaussian limit of a Poisson process as the number of photocounts in a small interval of time goes to infinity.

This result comes from the consideration of quantum jumps [3,6,5] caused by photodetection of the output light after it has been combined with the local oscillator at the beam splitter. In the limit of infinite local oscillator amplitude, the rate of these jumps goes to infinity, but the effect on the system becomes infinitesimal (because almost all of the photodetections are due to the local oscillator in some sense.) This allows the point process photocount to be replaced by the continuous, noisy photocurrent of Eq. (2.5), and the jumps of the system to be replaced by diffusive evolution. Specifically, the effect of the measurement of the homodyne photocurrent on the conditioned system state is given by the following stochastic equation [6]:

$$\dot{\rho}_c = [\mathcal{L} + \sqrt{\eta}\xi(t)\mathcal{H}]\rho_c, \tag{2.7}$$

where $\mathcal{H}$ is a nonlinear superoperator defined by

$$\mathcal{H}\rho = a\rho + \rho a^\dagger - \mathrm{Tr}(a\rho + \rho a^\dagger)\rho. \tag{2.8}$$

In Eq. (2.7), the stochastic term is to be interpreted in the Ito sense [9]. That is to say, the noise term $\xi(t)$ is statistically independent of the state of the system $\rho_c(t)$

at that time. Thus, to ignore the result of the measurement, one simply averages over $\xi(t)$ in Eq. (2.7). This restores the original master equation evolution (2.1) for the ensemble average state matrix $\rho = E[\rho_c]$.

One important feature of the stochastic evolution under Eq. (2.7) is that the positivity of the Glauber-Sudarshan $P$ function [13] is preserved, providing that $\mathcal{L}$ itself has this property. That is to say, unless the nonselective master equation produces nonclassical unconditioned states, adding homodyne measurement will not produce nonclassical conditioned states. In particular, the effect of homodyne measurements on a coherent state vanishes. Another important special case of (2.7) is when the detector efficiency is one, and $\mathcal{L}_0$ is generated by a Hamiltonian $H$. In that case, the measurement results provide complete knowldege about the nonunitary evolution of the system. Then it is possible to use a state vector rather than a state matrix to describe the system, and replace the stochastic master equation (2.7) by the stochastic Schrödinger equation

$$|\dot{\psi}_c(t)\rangle = \left\{-iH/\hbar - \tfrac{1}{2}\left[a^\dagger a - 2\langle X\rangle_c(t)a + \langle X\rangle_c^2(t)\right] + \xi(t)\left[a - \langle X\rangle_c(t)\right]\right\}|\psi_c(t)\rangle, \tag{2.9}$$

where $\langle X\rangle_c(t) = \frac{1}{2}\langle\psi_c(t)|a + a^\dagger|\psi_c(t)\rangle$. Note that the use of a state vector rather than a density operator does not imply that the evolution is unitary. A discussion of the interpretation of this and other nonlinear stochastic equations for state vectors may be found in Ref. [4].

An obvious use for these equations is to consider feeding back the homodyne photocurrent to change the system dynamics. It is reasonable to assume that the strength of the feedback is linear in the photocurrent, since higher powers of (2.5) are ill defined because of the white-noise term. We thus propose adding a feedback term of the form

$$[\dot{\rho}_c(t)]_{\text{fb}} = [I_c(t-\tau)/\eta]\mathcal{K}\rho_c(t) \tag{2.10}$$

$$= [\langle a + a^\dagger\rangle_c(t-\tau) + \xi(t-\tau)/\sqrt{\eta}]\mathcal{K}\rho_c(t), \tag{2.11}$$

where $\tau$ is the time delay in the feedback loop and $\mathcal{K}$ is an arbitrary Liouville superoperator. It would seem that we also require that $-\mathcal{K}$ be a valid Liouville superoperator, because the homodyne photocurrent $I_c(t)$ may be negative. This restriction would imply that $\mathcal{K}$ be of the Hamiltonian form

$$\mathcal{K}\rho = -i[A, \rho], \tag{2.12}$$

where $A$ is an arbitrary Hermitian operator. Examples include $A \propto a^\dagger a$ (detuning) and $A \propto i(a - a^\dagger)$ (driving). In practice, this could be achieved by using the current to drive an electro-optic device with variable refractive index or transmittivity. However, non-Hamiltonian feedback (such as controlling the pump rate of a laser) is also possible providing the feedback control produces only small variations of a large positive coefficient multiplying the non-Hamiltonian Liouville superoperator $\mathcal{K}$.

The question now arises as to which stochastic calculus is appropriate for Eq. (2.10), Ito or Stratonovich [9]? Equation (2.7) was derived using the Ito stochastic calculus, assuming that the noise $\xi(t)$ is an anticipating function of time [so that it is independent of $\rho_c(t)$]. In contrast, Eq. (2.10) has been postulated rather than derived, and so its interpretation is open. There are a number of reasons to prefer the Stratonovich treatment [in which $\xi(t)$ is nonanticipating] of Eq. (2.10). A physical photocurrent is finite at all times and only approaches the white-noise expression (2.5) in an idealized limit. In considering the effect of such a noisy photocurrent on the cavity, it is sensible to assume that it is a sufficiently smooth function of time that the rules of regular calculus apply, and then to take the appropriate limit. This is a standard procedure [9] and amounts to treating Eq. (2.10) as a Stratonovich equation. Although the rules of Stratonovich calculus are the same as that of regular calculus, they are harder to apply than those of Ito calculus because one cannot assume that the noise is independent of the system state. For this reason, we wish to convert Eq. (2.10) from a Stratonovich to an Ito equation. The latter is found by

$$[d\rho_c(t)]_{\text{fb}} = [I_c(t-\tau)/\eta]dt\mathcal{K}\left[\rho_c(t) + \tfrac{1}{2}d\rho_c(t)\right] \tag{2.13}$$

$$= [\langle a + a^\dagger\rangle_c(t-\tau)dt + dW(t-\tau)/\sqrt{\eta}]\mathcal{K}\rho_c(t) + \frac{1}{2\eta}\mathcal{K}^2\rho_c(t)dt, \tag{2.14}$$

where $dW(t) = \xi(t)dt$ satisfies $dW^2(t) = dt$ under the rules of Ito stochastic calculus. Adding the remaining terms (2.7) gives the following Ito stochastic equation for the evolution of the conditioned system state under homodyne feedback:

$$\dot{\rho}_c(t) = \left\{\mathcal{L} + \sqrt{\eta}\xi(t)\mathcal{H} + [\langle a + a^\dagger\rangle_c(t-\tau) + \xi(t-\tau)/\sqrt{\eta}]\mathcal{K} + \frac{1}{2\eta}\mathcal{K}^2\right\}\rho_c(t). \tag{2.15}$$

As it stands, this equation is of limited use. It gives an algorithm for simulating an ensemble of quantum trajectories representing the different possible histories of the system under feedback. It would be preferable to have a single master equation which gives the evolution of the entire ensemble, as represented by the ensemble average state matrix $\rho = E[\rho_c(t)]$. In practice, an experimenter cannot keep track of the exact photocurrent, and so such an ensemble average does represent the state of system. The difficulty in trying to derive a master equation is that feedback is a non-Markovian process. In particular, in order to determine the future behavior of the system, it is necessary to know its past in complete detail. We aim to overcome these difficulties by assuming that the feedback time delay $\tau$ is sufficiently small for a Markovian approximation to be valid. Thus we forge ahead and take the ensemble average of the time derivative in the conditioned state at time $t$ [Eq. (2.15)], assuming that the state at time $t-\tau$, and all previous times, is known.

$$E[\dot{\rho}_c(t)] = \left\{ \mathcal{L} + \langle a + a^\dagger \rangle_c(t-\tau)\mathcal{K} + \frac{1}{2\eta}\mathcal{K}^2 \right\} \rho(t) + \frac{1}{\sqrt{\eta}} \mathcal{K} E[\xi(t-\tau)\rho_c(t)]. \qquad (2.16)$$

Taking the ensemble average has eliminated the conditioned subscript on $\rho_c(t)$ since $\rho(t) = E[\rho_c(t)]$. Of course $\rho(t)$ is still conditioned on the particular history up to the time $t-\tau$, but not conditioned on the stochastic evolution between $t-\tau$ and $t$. The ensemble average $E[\rho_c(t)\langle a + a^\dagger\rangle_c(t-\tau)]$ factorizes because $\rho_c(t-\tau)$ is assumed known. The ensemble average $E[\xi(t-\tau)\rho_c(t)]$ still remains to be evaluated. It is shown in Appendix A that if we assume that $\tau \ll 1$, then

$$E[\xi(t-\tau)\rho_c(t)] = \sqrt{\eta}\Big[1 + O(\tau)\Big]\Big[a\rho(t) + \rho(t)a^\dagger - \langle a + a^\dagger\rangle_c(t-\tau)\rho(t)\Big]. \qquad (2.17)$$

Substituting this into Eq. (2.16) gives

$$E[\dot{\rho}_c(t)] = \mathcal{L}\rho(t) + \mathcal{K}[a\rho(t) + \rho(t)a^\dagger] + \frac{1}{2\eta}\mathcal{K}^2\rho(t) + O(\tau). \qquad (2.18)$$

If the delay $\tau$ is sufficiently small then we can ignore the final term in this equation and thus arrive at the master equation for the unconditioned (nonselective) state of the system under feedback from homodyne detection

$$\dot{\rho} = \mathcal{L}\rho + \mathcal{K}(a\rho + \rho a^\dagger) + \frac{1}{2\eta}\mathcal{K}^2\rho. \qquad (2.19)$$

Equation (2.19) shares much in common with a feedback master equation derived from the idealized continuous position measurement model of Caves and Milburn [10]. It is important to note that all terms containing conditional averages such as $\langle a + a^\dagger\rangle_c(t)$ have been eliminated. This is necessary because the foundations of probability theory imply that the generator of motion of a complete statistical representation of a system [here the unconditioned density operator $\rho(t)$] must be linear [11]. This theorem does not apply to the conditioned density operator $\rho_c(t)$, because this is not a complete statistical representation of the system. It is incomplete because there is additional information needed to specify the evolution of the system, namely, the noise $\xi(t)$ in the photocurrent, which is independent of $\rho_c(t)$. Thus a stochastic master equation can be nonlinear, but not a deterministic one such as Eq. (2.19). In this equation, the term which was nonlinear in $\rho$ has been transformed into the first feedback term. This is the desired feedback effect, and is nonunitary and may be nonlinear in the usual sense of giving a nonlinear equation of motion for the field amplitude $a$. The second feedback term represents diffusion which can be attributed to the inevitable introduction of noise by the measurement step in a quantum-limited feedback loop. These features are perhaps more easily seen by specifying $\mathcal{K}\rho = -i[A, \rho]$ to give

$$\dot{\rho} = \mathcal{L}\rho - i[A, a\rho + \rho a^\dagger] - \frac{1}{2\eta}[A, [A, \rho]]. \qquad (2.20)$$

The final term causes the variable conjugate to $A$ to diffuse. The lower the efficiency of the detecting system, the larger this diffusion becomes. This is because the feedback has been defined [Eq. (2.10)] so that the signal in the feedback photocurrent remains constant, and so the noise increases, as the efficiency decreases. The no-feedback limit is $A \to 0$, rather than $\eta \to 0$, so the divergence in the diffusion in the latter case is not an unphysical result.

In a previous work [12], we derived Eq. (2.19) somewhat differently, by taking the feedback delay time to be zero from the start. One advantage of that method was that it produced a selective evolution equation describing instantaneous feedback. Equation (2.15) fails in this respect because if we put $\tau = 0$ we obtain

$$\dot{\rho}_c(t) = \Big\{ \mathcal{L} + \langle a + a^\dagger\rangle_c(t)\mathcal{K} + \frac{1}{2\eta}\mathcal{K}^2 + \xi(t)[\sqrt{\eta}\mathcal{H} + \mathcal{K}/\sqrt{\eta}] \Big\} \rho_c(t). \qquad (2.21)$$

Averaging over the noise would give a deterministic equation nonlinear in $\rho$, which is forbidden. This problem arises from taking the limit $\tau \to 0$ before taking into account that the feedback via $\mathcal{K}$ must act after the conditioning via $\mathcal{H}$. Taking the limit properly gives the conditioning equation

$$\dot{\rho}_c(t) = \mathcal{L}\rho_c(t) + \mathcal{K}[a\rho_c(t) + \rho_c(t)a^\dagger] + \frac{1}{2\eta}\mathcal{K}^2\rho_c(t) + \xi(t)[\sqrt{\eta}\mathcal{H} + \mathcal{K}/\sqrt{\eta}]\rho_c(t). \qquad (2.22)$$

In this case, averaging over the noise gives the correct master equation (2.19). With unit efficiency detection, we know that the stochastic master equation for the state matrix describing homodyne detection (2.7) is equivalent to a stochastic Schrödinger equation for the state vector (2.9), providing that the internal dynamics are unitary. If in addition the feedback is unitary as in Eq. (2.20), then the selective feedback master equation (2.22) can also be replaced by the stochastic Schrödinger equation

$$\begin{aligned}|\dot{\psi}_c(t)\rangle = \{ & -iH/\hbar + \xi(t)[a - \langle X\rangle_c(t) - iA] \\ & - \tfrac{1}{2}[a^\dagger a + \langle X\rangle_c(t)^2 + A^2] \\ & - [iAa - \langle X\rangle_c(t)a + iA\langle X\rangle_c(t)]\}|\psi_c(t)\rangle. \end{aligned} \qquad (2.23)$$

We include this equation for completeness only; it will not be used in this work. However, we will later use the more general Eq. (2.22) to give insight into how the feedback works.

An important special case of Eq. (2.19) is when the feedback superoperator $\mathcal{K}$ corresponds to a "classical" process. By this, we mean a process which gives an evolution equation for the Glauber-Sudarshan $P(\alpha, \alpha^*)$

function which has first order derivatives only, so that a coherent state is simply translated [13]. Examples are driving and detuning as mentioned above, and damping for which $\mathcal{K} \propto \mathcal{D}[a]$. In such cases, the first feedback term in Eq. (2.19) will contribute to the evolution equation for the $P$ function a term which also has only first order derivatives. This is because the normally ordered $a\rho + \rho a^\dagger$ simply becomes $(\alpha + \alpha^*)P(\alpha, \alpha^*)$. The second feedback term will give second order derivatives, but the diffusion matrix will be positive definite, because it comes explicitly from the outer-product square of the drift vector from $\mathcal{K}$. That is, classical feedback from homodyne detection gives a true Fokker-Planck equation for the $P$ function. Thus we have shown that, providing the superoperator $\mathcal{L}$ does not generate nonclassical field states, the addition of feedback as described will not enable one to produce such states. This is related to the fact that homodyne feedback does not produce nonclassical conditioned states, as we will explore in Sec. V. In the next section, we will investigate the case when $\mathcal{L}$ can produce nonclassical states, and find the effect of feedback on squeezing.

## III. LINEARIZED RESULTS FOR SQUEEZING

In this section, we wish to consider the effect of homodyne-mediated feedback on systems which ordinarily produce squeezed light [1]. This means that inside the cavity, the quantum uncertainty in one field quadrature is less than that of a coherent state. Since a coherent state is a minimum-uncertainty state (in fact, the unique classical minimum-uncertainty state), then by Heisenberg's uncertainty principle, the uncertainty in the conjugate quadrature must be increased above that of a coherent state. If we denote our two quadratures $X = \frac{1}{2}(a + a^\dagger)$ and $Y = \frac{1}{2}(-ia + ia^\dagger)$, then Heisenberg's relation is

$$V(X)V(Y) \geq \tfrac{1}{16}, \tag{3.1}$$

where $V$ stands for the quantum variance. As the feedback theory in Sec. II was developed for homodyne measurements of the $X$ quadrature, it is natural to look at squeezing in this quadrature. To solve the master equation, we choose to use the Wigner function representation of the density operator [13]. This is defined by

$$W(x,y) = \frac{1}{4\pi^2}\int du \int dv \mathrm{Tr}\left\{\rho \exp[iu(x-X) + iv(y-Y)]\right\}. \tag{3.2}$$

The Wigner function has the property of a joint distribution in $X$ and $Y$ in that

$$W(x) = \int dy W(x,y) \tag{3.3}$$

is a true marginal probability distribution for $X$.

We will now assume that the no-feedback master equation can be written as a linearized Fokker-Planck equation for the Wigner function, and furthermore that this equation has a factorizable solution $W(x,y) = W(x)W(y)$. All of these assumptions are approximately valid for many systems which produce viable squeezing, such as the phase-locked regularly pumped laser considered in Sec. IV, and the parametric oscillator both above and below threshold [13]. Linearizing the Fokker-Planck equation for $W(x)$ around the semiclassical steady-state $x_0$ gives the general Ornstein-Uhlenbeck equation

$$\dot{W}(x) = \left[\partial_x k(x - x_0) + \frac{1}{2}\partial_x^2 \frac{1+l}{4}\right] W(x), \tag{3.4}$$

where $\partial_x = \frac{\partial}{\partial x}$. Here $k > 0$ and $l \geq 0$ are constants found from the Liouville superoperator $\mathcal{L}$. The 1/4 term in the diffusion constant comes from the damping to the external modes. Recall that we are measuring all rates in units of the cavity linewidth. In fact, for the ideal source of squeezed light, a classically pumped degenerate parametric oscillator below threshold, this equation is exact. If we denote the threshold parameter by $\kappa \in [0, 1)$, then we have $x_0 = 0$, $l = 0$, and $k = (1 \pm \kappa)/2$ for $X$ being the squeezed (stretched) quadrature. In future, we will always take $x_0 = 0$ without loss of generality.

Equation (3.4) has a Gaussian stationary solution with variance

$$V(X) = \frac{1+l}{8k}. \tag{3.5}$$

If this is less than 1/4, then the system exhibits squeezing. It is convenient to define a parameter which more clearly characterizes the presence or absence of squeezing. In analogy with Mandel's $Q$ parameter [14] which characterizes sub-Poissonian light, we define a squeezing $Q$ parameter

$$Q = 4V - 1. \tag{3.6}$$

This is negative for squeezed light and is bounded below by $-1$. In the absence of feedback we have

$$Q_0 = \frac{1+l}{2k} - 1. \tag{3.7}$$

For the ideal parametric oscillator described above, $Q_0 \to -\frac{1}{2}$ as $\kappa \to 1$. That is, the best intracavity squeezing is half of the theoretical minimum.

It is of more practical use to consider the noise properties of the output light of the cavity, represented by the operator $b(t)$. Since this operates on a continuum of modes, it is necessary to consider a noise spectrum. If we are interested in the squeezing in the $X$ quadrature, the appropriate spectrum is

$$S(\omega) = \int_{-\infty}^{\infty} dt' \langle [b(t+t') + b^\dagger(t+t')], [b(t) + b^\dagger(t)] \rangle e^{-i\omega t'}, \tag{3.8}$$

where $\langle A, B\rangle \equiv \langle AB\rangle - \langle A\rangle\langle B\rangle$. It can be shown [13] that this is equal to

$$S(\omega) = 1 + 2\int_0^\infty dt' \mathrm{Tr}[(a+a^\dagger)e^{\mathcal{L}t'}(a\rho + \rho a^\dagger)]\cos\omega t', \tag{3.9}$$

where $\rho$ is the steady-state density operator. For the linearized evolution defined as above (3.5), $S(\omega)$ has an extremum at $\omega = 0$ (on resonance). In that case, output squeezing is best characterized by the parameter

$$R = S(0) - S(\infty) = S(0) - 1. \tag{3.10}$$

Once again, this is greater than or equal to $-1$, and greater than or equal to zero classically. It is easy to show that $R$ is related to $Q$ by

$$R = 2Q/k. \tag{3.11}$$

This is easily seen to be bounded below by $-1$ provided that $l \geq 0$, as we assumed earlier. In the case of the just below threshold parametric oscillator, we find $R \to -1$ so that there is perfect squeezing on resonance.

Now we add feedback to try to reduce the fluctuations in $X$. Restricting ourselves to classical feedback we choose the driving superoperator

$$\mathcal{K}\rho = \frac{\lambda}{2}[a - a^\dagger, \rho]. \tag{3.12}$$

By itself, this superoperator translates a state in the negative $X$ direction for $\lambda$ positive. We thus have the ability to change the statistics for $X$ and perhaps achieve better squeezing. Substituting Eq. (3.12) into the general homodyne feedback master equation (2.19) gives

$$\dot\rho = \mathcal{L}\rho + \frac{\lambda}{2}[a - a^\dagger, a\rho + \rho a^\dagger] + \frac{\lambda^2}{8\eta}[a - a^\dagger, [a - a^\dagger, \rho]]. \tag{3.13}$$

The corresponding Fokker-Planck equation for the Wigner function is

$$\dot W(x) = \left[\partial_x(k+\lambda)x + \frac{1}{2}\partial_x^2\left(\frac{1+l}{4} + \frac{\lambda}{2} + \frac{\lambda^2}{4\eta}\right)\right]W(x). \tag{3.14}$$

This has the same form as Eq. (3.4). The diffusion constant will always be non-negative because $l \geq 0$.

There exists a stable Gaussian solution of Eq. (3.14) provided that $k + \lambda > 0$. The intracavity squeezing parameter is

$$Q_\lambda = (k+\lambda)^{-1}\left(kQ_0 + \frac{\lambda^2}{2\eta}\right). \tag{3.15}$$

An immediate consequence of Eq. (3.15) is that $Q_\lambda$ can only be negative if $Q_0$ is. This is a consequence of the theorem proved in Sec. II, that this sort of feedback cannot in itself produce nonclassical states. We minimize $Q_\lambda$ with respect to $\lambda$ to find

$$Q_{\min} = \eta^{-1}\left(-k + \sqrt{k^2 + 2\eta k Q_0}\right), \tag{3.16}$$

when

$$\lambda = -k + \sqrt{k^2 + 2\eta k Q_0}. \tag{3.17}$$

Note that this $\lambda$ has the same sign as $Q_0$. That is to say, if the system produces squeezed light, then the best way to enhance the squeezing is to add a force which displaces the state in the direction of the difference between the measured photocurrent and the desired mean photocurrent. This is the opposite of what would be expected classically, and is due to the effect of homodyne measurement on squeezed states, as will be explained in Sec. V. Obviously, the best intracavity squeezing will be when $\eta = 1$, which gives

$$Q_{\min} = k\left(-1 + \sqrt{1 + R_0}\right). \tag{3.18}$$

The fact that $R_0$ by definition (3.10) is bounded below by $-1$ ensures that one can always take the square root in the above expressions.

Next, we prove that $Q_{\min} \leq Q_0$, with equality only if $\eta = 0$ or $Q_0 = 0$. We use the result $\sqrt{1+R_0} \leq 1 + R_0/2$ since $R_0 \geq -1$. Recalling that $R_0 = 2Q_0/k$, and comparing to Eq. (3.18) gives

$$Q_{\min} \leq Q_0 \tag{3.19}$$

for $\eta = 1$. Using the mean value theorem, it is easy to show that this is true for all $\eta$. This result implies that the intracavity variance in $X$ can always be reduced by classical homodyne-mediated feedback, unless it is at the classical minimum. In particular, intracavity squeezing can always be enhanced. For the parametric oscillator defined above, we find $Q_{\min} = -\kappa/\eta$. For $\eta = 1$, we have an $X$ variance of $(1-\kappa)/4$. The $Y$ variance, which is unaffected by feedback, is seen from Eq. (3.4) to be $1/4(1-\kappa)$. Thus, with perfect detection, it is possible to produce a minimum-uncertainty squeezed state with arbitrarily high squeezing as $\kappa \to 1$. This is not unexpected as a parametric amplifier (in an undamped cavity) also produces minimum-uncertainty squeezed states. The feedback removes the noise which was added by the damping which is necessary to do the measurement used in the feedback.

Intracavity squeezing is of limited use experimentally; the output light from the system is what is usually of interest. Here it must be remembered that the feedback loop is part of the system. Assuming perfect efficiency detectors in the feedback loop, the fraction of light emitted by the cavity used in the loop is $\eta$. Thus the fraction $\theta$ of cavity output available as an output of the system is at best $1-\eta$. The amount of squeezing which is actually available is $\theta Q_\lambda$. We will show that

$$\theta Q_{\min} \geq Q_0 \text{ for } Q_0 < 0. \tag{3.20}$$

That is, dividing the cavity output and using some in a feedback loop produces worse squeezing in the remaining output than was present in the original, undivided output. Note, however, that if the cavity output is inherently divided (which is often the case, with two output mirrors), then using one output in the feedback loop will enhance squeezing in the other output. This is because the squeezing in the system output has changed from $\theta Q_0$ to $\theta Q_{\rm min}$.

The proof of Eq. (3.20) is as follows. Since $Q_0 = 0 \Longrightarrow Q_{\rm min} = 0$, we consider the case $Q_0 \neq 0$. If the detectors are not perfect, then the results can only be worse, so we take the best case $\theta = 1 - \eta$. The condition $(1-\eta)Q_{\rm min} = Q_0$ gives $\sqrt{1+\eta R_0} = 1 + \eta R_0/[2(1-\eta)]$. For $R_0 \geq -1$ this has only one solution, namely, $\eta = 0$. The condition $(1-\eta)Q_{\rm min} = 0$ implies $\eta = 1$. Thus, by the intermediate value theorem, $(1-\eta)Q_{\rm min}$ lies between 0 and $Q_0$. Equation (3.20) follows when $Q_0 < 0$. For the other case, when $Q_0 > 0$, the result implies that the feedback does reduce the noise in the output. Part of this reduction is simply due to reducing the fraction of light used as the system output, but part is due to the reduction of $Q_0$ to $Q_{\rm min}$.

The quantity $\theta Q_\lambda$ represents the noise spectrum (3.8), minus the shot noise, integrated over all frequencies. As explained above, experimentalists are often more interested in the minimum noise reduction, on resonance. With no feedback, this is given by $R_0$. With feedback, it is given by

$$R_\lambda = \theta 2Q_\lambda/(k+\lambda) = \frac{\theta}{(k+\lambda)^2}\left(2kQ_0 + \lambda^2/\eta\right). \quad (3.21)$$

In general, $R_\lambda$ is minimized for a different value of $\lambda$ from that which minimizes $Q_\lambda$. We find

$$R_{\rm min} = R_0\frac{1-\eta}{1+R_0\eta} \quad (3.22)$$

when

$$\lambda = 2\eta Q_0. \quad (3.23)$$

Again, $\lambda$ has the same sign as $Q_0$. It follows immediately from Eq. (3.22) that, since $R_0 \geq -1$,

$$R_{\rm min} \geq R_0 \ \text{ for } R_0 < 0. \quad (3.24)$$

That is to say, dividing the cavity output to add a homodyne-mediated classical feedback loop cannot produce better output squeezing at any frequency than would be available from an undivided output with no feedback. These "no-go" theorems are perhaps not surprising given the approximate traveling-wave results of Shapiro *et al.* [16], and the experimental results of Yamamoto, Imoto, and Machida [15].

There is one case in which adding feedback does not degrade the output squeezing: if the no-feedback output noise is zero ($R_0 = -1$). In that case, the noise added by dividing the output can be exactly offset by the feedback, providing that the detection efficiency is unity. However, the bandwidth of the squeezing is reduce from $k$ to $k + 2\eta Q_0$. For example, in the parametric oscillator just below threshold, the bandwidth of squeezing in the system output would approach zero. In all other cases ($1 < R_0 < 0$), the noise added is greater and squeezing is degraded. Of course, if the original output had classical noise ($R_0 > 0$), then the feedback can reduce these fluctuations. It is a matter of minor interest that, regardless of the system state, the following relation holds:

$$R_\lambda = R_{\rm min} \Longrightarrow Q_\lambda = Q_0. \quad (3.25)$$

That is to say, when the output noise is minimized, there is no change in the intracavity noise from the no-feedback case.

## IV. APPLICATION TO A REGULARLY PUMPED LASER

In this section we present a useful application for homodyne-mediated feedback: phase locking a regularly pumped laser. The steady state of an ideal Poissonian pumped laser is a mixture of equal amplitude coherent states, and the photon statistics of its output are Poissonian. A more regularly pumped laser has a nonclassical steady state, and its output photon statistics are sub-Poissonian. The steady-state phase of the output is completely undefined, due to the finite linewidth which all lasers have. By using some of the cavity output light in a homodyne-mediated feedback loop, one can phase lock the laser and hence reduce the phase noise relative to a local oscillator. For a perfectly regularly pumped laser we show that it is possible to produce a near minimum-uncertainty squeezed state in the cavity, and perfect squeezing on resonance in the output. In this sense the feedback does produce squeezing. It must be emphasized, however, that the feedback is not responsible for the nonclassicality; that arises from the intracavity dynamics. What the phase locking does achieve is to define a preferred quadrature to be squeezed, and to reduce the noise in the unsqueezed quadrature to near the minimum required by the uncertainty relations.

We use the model for the regularly pumped laser derived by one of us previously [17]. In this model, only processes essential for ideal laser operation are included. In particular, there are no excess noise terms, so this model represents the true quantum noise limit to an incoherently pumped single mode laser. The statistics generated by our model agree with those of Louisell [18] in the same limits. The fundamental nature of the laser process can be seen from the simplicity of the master equation

$$\dot\rho = \mu\left\{\mathcal{E}[a^\dagger] + \frac{q}{2}\mathcal{E}[a^\dagger]^2\right\}\rho + \mathcal{D}[a]\rho. \quad (4.1)$$

Here the damping superoperator $\mathcal{D}[a]$ is as defined in Sec. II [Eq. (2.2)], while the excitation superoperator is

$$\mathcal{E}[a^\dagger] = \mathcal{J}[a^\dagger]\mathcal{A}[a^\dagger]^{-1} - 1, \quad (4.2)$$

where $\mathcal{J}$ and $\mathcal{A}$ are defined in Eqs. (2.3) and (2.4), respectively. The pumping rate (in units of the cavity linewidth as always) is $\mu$, and $q$ is the Mandel $Q$ parameter [14] for the pump noise [19] . A standard (Poisson) ideal laser has $q = 0$, and a perfectly regularly pumped laser has $q = -1$. The mean photon number in the cavity is $\mu$ (which must be assumed to be very large) irrespective of $q$. The intracavity photon number variance is $\mu(1 + q/2)$, which is nonclassical for $q < 0$.

As shown in Ref. [17], the above master equation is approximately equivalent to the following Fokker-Planck equation for the Wigner function $W(\alpha,\alpha^*)$ :

$$\dot{W} = \left\{\frac{\partial}{\partial\alpha}\left[\frac{-\mu}{2\alpha^*}+\frac{\alpha}{2}+\left(1+\frac{q}{2}\right)\frac{\mu}{4|\alpha|^2\alpha^*}\right]+\text{c.c.}\right\}W$$
$$+\frac{1}{2}\left\{\frac{\partial^2}{\partial\alpha^2}\frac{q\mu}{4\alpha^{*2}}+\frac{\partial^2}{\partial\alpha^{*2}}\frac{q\mu}{4\alpha^2}\right.$$
$$\left.+\frac{\partial^2}{\partial\alpha\partial\alpha^*}\left[1+\left(1+\frac{q}{2}\right)\frac{\mu}{|\alpha|^2}\right]\right\}W. \qquad (4.3)$$

Here $\alpha = x+iy$ where $x, y$ are as used in the preceding section. Now we anticipate the addition of phase-locking feedback by assuming that $W(\alpha,\alpha*)$ is localized around $\alpha = i\sqrt{\mu}$. That is, we intend to lock the phase of the laser to $\pi/2$ (relative to the local oscillator), while the photon number remains close to its mean $\mu$. Then we can assume that $x$ and $y-\sqrt{\mu}$ are of order unity. Linearizing Eq. (4.3) gives

$$\dot{W}(x,y) = \left[\partial_y(y-\sqrt{\mu})+\frac{1}{2}\partial_y^2\frac{2+q}{4}+\frac{1}{2}\partial_x^2\frac{1}{2}\right]W(x,y). \qquad (4.4)$$

As claimed in Sec. III, this equation factorizes into two equations, for $W(x)$ and $W(y)$. The mean of the $Y$ (amplitude) quadrature is obviously $\sqrt{\mu}$, and its variance is

$$V(Y) = \frac{2+q}{8}. \qquad (4.5)$$

For a Poisson laser, this is equal to the classical minimum, while for a perfectly regular laser the intracavity variance is squeezed to half of this value. With no feedback, the variance in the $X$ (phase) quadrature grows linearly in time so eventually the assumptions leading to Eq. (4.4) will fail. This is the manifestation of the linewidth of the laser as the rate of phase diffusion, $1/2\mu$. Note that the phase dynamics are independent of the pump regularity.

To counteract this phase diffusion, we add feedback as in Sec. II with the superoperator

$$\mathcal{K}\rho = i\frac{\lambda}{2\sqrt{\mu}}[a^\dagger a,\rho], \qquad (4.6)$$

where $\lambda$, the feedback strength, is of order $\mu^0$. This represents adding a detuning linear in the instantaneous $X$ homodyne photocurrent. To justify the assumption of instantaneous feedback, the time delay would have to be much less than the inverse of the cavity linewidth (that is, submicrosecond). In practice this could be achieved by changing the optical path length of the cavity, perhaps by an electro-optic modulator. For small changes in the path length, this is well modeled by Eq. (4.6). The effect of this is to add a restoring force which will tend to lock the phase at $\pi/2$, as well as noise. The full feedback master equation is

$$\dot{\rho} = \mu\left\{\mathcal{E}[a^\dagger]+\frac{q}{2}\mathcal{E}[a^\dagger]^2\right\}\rho+\mathcal{D}[a]\rho$$
$$+i\frac{\lambda}{2\sqrt{\mu}}[a^\dagger a, a\rho+\rho a^\dagger]+\frac{\lambda^2}{4\eta\mu}\mathcal{D}[a^\dagger a]\rho. \qquad (4.7)$$

Within the approximations used above, the addition of feedback will have no effect on the statistics of the $Y$ quadrature. The effect of the detuning on the $X$ quadrature will be identical to that obtained by driving in the preceding section, with $W(x)$ satisfying

$$\dot{W}(x) = \left(\partial_x\lambda x+\frac{1}{2}\partial_x^2\frac{1+\nu+\lambda^2/2\eta}{2}\right)W(x). \qquad (4.8)$$

Here we have added a new non-negative parameter $\nu$ to represent any excess phase noise above the quantum-limited laser limit.

Thus, under phase-locking feedback, the mean $X$ is zero and the variance is

$$V(X)_\lambda = \frac{1+\nu+\lambda^2/2\eta}{4\lambda}. \qquad (4.9)$$

Minimizing with respect to the feedback strength $\lambda$ gives

$$V(X)_{\min} = \frac{1}{4}\left(1+\sqrt{\frac{2(1+\nu)}{\eta}}\right) \qquad (4.10)$$

when $\lambda = \sqrt{2\eta(1+\nu)}$, which agrees with Eqs. (3.16) and (3.17). This variance is always significantly above the classical limit of 1/4. Under ideal conditions ($\eta = 1, \nu = 0$), we have $V(x)_{\min} = (1+\sqrt{2})/4$, while the $Y$ variance is unchanged from Eq. (4.5). For a perfectly regular laser pump, we get a Wigner phase space area of

$$\sqrt{V(X)_{\min}V(Y)} = \sqrt{\frac{1+\sqrt{2}}{32}} \simeq 0.2747. \qquad (4.11)$$

This is less than 10% greater than the minimum required by the Heisenberg relation (3.1). In other words, the intracavity state is an almost minimum-uncertainty squeezed state. In Fig. 1 we show the one-standard-deviation contour of the Wigner function for this state,

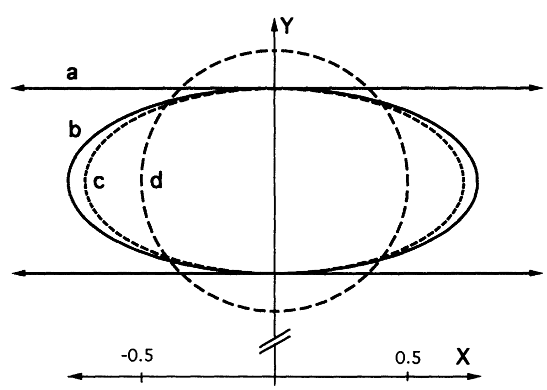


FIG. 1. Error ellipses (one standard deviation) for the Wigner function of various states pertaining to the sub-Poissonian pumped laser: (a) the steady state of the free running laser (which is actually an annulus with very large radius); (b) the steady state with optimum noise reduction via phase-locking feedback; (c) the minimum-uncertainty state with squeezing equal to that in (b) for comparison; and (d) a coherent state for comparison also.

the corresponding minimum-uncertainty squeezed state, the no-feedback state, and a coherent state.

We turn now to the squeezing in the output of the system. As noted in the preceding section, diverting some of the cavity emission into the feedback loop degrades squeezing by a multiplicative factor of $\theta$ which is less than or equal to $1-\eta$. From Eq. (4.4), the noise in the $Y$ quadrature is

$$S_Y(\omega) = 1 + \frac{\theta q}{1+\omega^2}, \tag{4.12}$$

which is below the shot-noise limit for $q < 0$. Equation (4.8) gives the $X$ quadrature noise spectrum

$$S_X(\omega) = 1 + \frac{\theta[\lambda^2/\eta + 2(1+\nu)]}{\lambda^2+\omega^2}. \tag{4.13}$$

This is always above the shot-noise limit. To minimize the low-frequency noise in $X$ we want $\lambda \gg 1$. Inside the cavity, this causes the $X$ variance to become very large $(\lambda/8\eta)$. The output noise spectrum $S_X$ remains finite, but becomes almost flat,

$$S_X(\omega) \simeq 1 + \theta/\eta. \tag{4.14}$$

This value is not affected by the excess phase diffusion $\nu$. The best results are obtained for $\theta = 1-\eta$ and $q=-1$:

$$S_Y(0) = \eta\ ,\quad S_X(0) = 1/\eta. \tag{4.15}$$

That is, the low-frequency noise properties of the output light show the two characteristics of perfect squeezing on resonance: (i) the noise spectra satisfy $S_X(0)S_Y(0)=1$, a minimum-uncertainty relation; (ii) zero noise is attainable in the $Y$ quadrature as $\eta \to 0$. Note that this limit ($\eta \to 0$) does not imply no feedback, as we have already assumed that $\lambda \gg 1$. With no phase locking, the spectrum of fluctuations in $X$ would be practically unbounded. The feedback is effective in turning intensity noise reduction into perfect quadrature squeezing by reducing noise which is above the classical limit.

It is reasonable to ask, what is the advantage of this method of phase locking a laser over the much simpler method of injection phase locking? The latter consists of injecting the coherent field to which the laser phase is to be locked into the laser cavity, rather than using it as a local oscillator in the feedback detection step, as here. Assuming that the injection is made through a mirror with a loss rate much less than that of the output mirror, the effect of the injection can be modeled by adding the following term to the laser master equation (4.1):

$$\dot{\rho} = -i\epsilon[a + a^\dagger, \rho]. \tag{4.16}$$

Now if the laser is super-Poissonian ($q>0$), then obviously it would be preferable to simply use the coherent beam as the output, rather than the laser under consideration. Thus, to compare the two methods of phase locking (feedback and injection), it is necessary to consider a sub-Poissonian pumped laser. The clearest differences will occur for $q=-1$, so we consider only this case.

An arbitrarily small injected signal will produce phase locking, but the noise in the phase quadrature will be quite high. As the injected signal is increased, this phase noise decreases towards the shot-noise limit, but that of the amplitude quadrature increases towards that limit also, losing its nonclassical nature. This is the same behavior as in the feedback phase locking under ideal conditions, as the proportion of light $\eta$ used in the feedback loop is increased from zero to one [see Eq. (4.15)]. In fact, it is possible to define a parameter analogous to $\eta$ in the injection case

$$\beta = \frac{\epsilon^2}{\epsilon^2+\mu}. \tag{4.17}$$

It is shown in Appendix B that the on-resonance amplitude quadrature noise spectrum becomes

$$S_Y(0) = \beta, \tag{4.18}$$

while the bandwidth of squeezing is reduced from 1 to $(1+\sqrt{\beta})^{-1}$. The on-resonance noise in the phase quadrature is

$$S_X(0) = \frac{1+(1+2\nu)(1-\beta)}{\beta}. \tag{4.19}$$

From Eqs. (4.18) and (4.19) we see that the low-frequency noise spectra for the injection phase-locked regularly pumped laser do not satisfy a minimum-uncertainty relation, unlike those from the feedback phase-locked laser. Furthermore, the phase quadrature noise increases linearly with the excess phase noise in the laser (represented by $\nu$), whereas this was completely suppressed by the feedback phase locking. In practice, this is a very important issue because most lasers have many sources of phase noise above the minimum implied by the laser action. These results, combined with the fact that injection phase locking reduces the bandwidth of squeezing, justify the conclusion that the feedback scheme considered in this section is a superior method of phase locking for encoding low-frequency information in both quadratures of a sub-Poissonian laser.

## V. UNDERSTANDING FEEDBACK IN TERMS OF CONDITIONING

Section III established various results regarding the ability of homodyne-mediated classical feedback to produce squeezed states. In this section, we will give an explanation for some of these results, in terms of the conditioning of the system state by the feedback measurement. To do this, we must return to the selective stochastic master equation (2.22) for the conditioned state matrix $\rho_c(t)$:

$$\begin{aligned}\dot{\rho}_c(t) &= \mathcal{L}\rho_c(t) + \mathcal{K}[a\rho_c(t) + \rho_c(t)a^\dagger] + \frac{1}{2\eta}\mathcal{K}^2\rho_c(t)\\ &\quad + \xi(t)[\sqrt{\eta}\mathcal{H} + \mathcal{K}/\sqrt{\eta}]\rho_c(t).\end{aligned} \tag{5.1}$$

Changing this to a stochastic Liouville equation for the conditioned Wigner function, and using the expressions for $\mathcal{K}$ and $\mathcal{L}$ from Sec. III gives

$$\dot{W}_c(x) = \left[\partial_x(k+\lambda)x + \frac{1}{2}\partial_x^2\left(\frac{1+l}{4} + \frac{\lambda}{2} + \frac{\lambda^2}{4\eta}\right) + \xi(t)\left(\sqrt{\eta}\left\{2[x - \bar{x}_c(t)] + \frac{1}{2}\partial_x\right\} + \frac{\lambda}{2\sqrt{\eta}}\partial_x\right)\right] \times W_c(x), \quad (5.2)$$

where $\bar{x}_c(t)$ is the mean of the distribution $W_c(x)$.

This equation is obviously no longer a simple Ornstein-Uhlenbeck equation. Nevertheless, it still has a Gaussian as an exact solution. This is shown in a previous work by one of us [17]. With $\bar{x}_c$ the mean and $V_c$ the variance of the conditioned distribution, we find

$$\dot{\bar{x}}_c = -(k+\lambda)\bar{x}_c + \xi(t)\left[\sqrt{\eta}\left(2V_c - \frac{1}{2}\right) - \frac{\lambda}{2\sqrt{\eta}}\right], \quad (5.3)$$

$$\dot{V}_c = -2kV_c + \frac{1+l}{4} - \eta\left(2V_c - \frac{1}{2}\right)^2. \quad (5.4)$$

Two points about the evolution equation for $V_c$ are worth noting. It is completely deterministic (no noise terms), and it is uninfluenced by the presence of feedback. Furthermore, for this linear system, it is independent of $\bar{x}_c$. Thus the stochasticity and feedback terms in the equation for the mean do not even enter that for the variance indirectly.

The equation for the conditioned variance is more simply written in terms of the conditioned parameter $Q_c = 4V_c - 1$:

$$\dot{Q}_c = -2kQ_c - 2k + 1 + l - \eta Q_c^2. \quad (5.5)$$

On a time scale as short as a cavity linewidth, $Q_c$ will approach its stable steady-state value of

$$Q_c^{\mathrm{SS}} = \eta^{-1}\left[-k + \sqrt{k^2 + \eta(-2k+1+l)}\right]. \quad (5.6)$$

Note that this is equal to the minimum unconditioned $Q_{\min}$ (3.16) with feedback. The explanation for this will become evident shortly. We substitute the steady-state conditioned variance into Eq. (5.3) to get

$$\dot{\bar{x}}_c = -(k+\lambda)\bar{x}_c + \xi(t)\frac{1}{2\sqrt{\eta}}\left[-k + \sqrt{k^2 + \eta(-2k+1+l)} - \lambda\right]. \quad (5.7)$$

If we chose $\lambda = -k + \sqrt{k^2 + \eta(-2k+1+l)}$ then there is no noise at all in the conditioned mean and so we can set $\bar{x}_c = 0$. This value of $\lambda$ is precisely that value derived in Sec. III to minimize the unconditioned variance under feedback. Here we see that this unconditioned variance is equal to the conditioned variance. The feedback works simply by suppressing the fluctuations in the conditioned mean.

In general, the unconditioned variance will consist of two terms, the conditioned quantum variance in $X$ plus the classical (ensemble) average variance in the conditioned mean of $X$:

$$Q_\lambda = Q_c^{\mathrm{SS}} + E[\bar{x}_c^2]. \quad (5.8)$$

The latter term is found from Eq. (5.7) to be

$$E[\bar{x}_c^2] = \eta^{-1}\frac{1}{2(k+\lambda)} \times \left[-(k+\lambda) + \sqrt{k^2 + \eta(-2k+1+l)}\right]^2. \quad (5.9)$$

Adding Eq. (5.6) gives

$$Q_\lambda = \eta^{-1}\frac{1}{2(k+\lambda)}[\lambda^2 + \eta(-2k+1+l)]. \quad (5.10)$$

It can easily be verified that this is identical to the expression (3.15) derived in Sec. III using the unconditioned master equation. In this context, the explanation for the feedback is obvious. The homodyne measurement reduces the conditioned variance (except when it is equal to the classical minimum of 1/4). The more efficient the measurement, the greater the reduction. Ordinarily, this reduced variance is not evident because the measurement gives a random shift to the conditional mean of $X$, with the randomness arising from the shot noise of the photocurrent. By appropriately feeding back this photocurrent, it is possible to precisely counteract this shift.

The sign of the feedback parameter $\lambda$ is determined by the sign of the shift which the measurement gives. For classical statistics ($Q \geq 0$) , a higher than average photocurrent reading [$\xi(t) > 0$] leads to the conditioned mean $X$ increasing (except if $Q = 0$ in which case the measurement has no effect). This is what would be expected from a classical theory in which $Q$ represents the noise in $X$. However, for nonclassical states with $Q < 0$, the classical intuition fails as a positive photocurrent fluctuation causes $\bar{x}_c$ to decrease. This explains the counterintuitive negative value of $\lambda$ required in squeezed systems, which naively would be thought to destabilize the system and increase fluctuations. In fact this effect of homodyne measurements on squeezed states explains the very characteristic which makes their nonclassicality experimentally observable: a two-time correlation function for the homodyne photocurrent which drops below zero. If a high photocurrent is measured at one instant, the mean $X$ of the state is shifted lower, and so it is more likely to measure a lower photocurrent at the next instant. In this sense, the phenomenon of a homodyne measurement shifting the state of the system in the "wrong" direction is observed routinely.

Succinctly, we can state that feedback is conditioning made practical. The noise reduction produced by classical feedback can be precisely as good as that produced by conditioning. This gives a simple explanation as to why our homodyne-mediated classical feedback model cannot produce nonclassical states: because homodyne detection does not [6]. This can be understood by considering our original feedback term (2.10). Irrespective of whence the photocurrent came, it is merely a $c$ number in this equation, although it is necessary to be careful in treating it because it is so noisy. Thus we should not expect it to be able to reduce the variance in the conditioned state if it is coupled to a classical superoperator. Nonclassical feedback (such as using the photocurrent to influence nonlinear intracavity elements) may produce nonclassical states, but such elements can produce nonclassical states without feedback, so this is hardly surprising.

Although we have thus far only analyzed homodyne measurement, the intuition we have gained about the mechanism of feedback applies to other forms of measurement. In a previous paper [4], we considered two other extracavity detection schemes: direct photodetection and heterodyne detection. Both of these measurement schemes were shown not to produce nonclassical conditioned states. Hence, by the arguments given in the preceding paragraph, it is obvious that feedback based on these schemes will also fail to produce nonclassical states. In fact, feedback schemes based on *any* sort of extracavity detection will fail in this way. This can be seen from considering the simple classical cavity dynamics of driving, detuning, and damping. In that case, the stationary intracavity state is a coherent state, which is of course pure. That means that any measurement which does not change the master equation (which is what we mean by an extracavity measurement) cannot affect the state at all. If it did, then the ensemble average state would be a mixture, which it is not. Any other classical intracavity process will simply add noise to some quantities, and not help to reduce noise.

Thus we can conclude that feedback based on external detection cannot produce nonclassical light. This result is in contradiction to the predictions of an earlier, crude model of quantum-limited feedback used by Wiseman and Milburn [7] and Liebman and Milburn [8], motivated by an analogy with nonlinear absorption. It is apparent that such models do not represent external photodetection-mediated feedback. It is conceivable that they do model some intracavity measurement process, for we can now see that it is necessary to perform intracavity measurements in order to achieve squeezing via feedback. Such measurements (in particular, quantum nondemolition measurements) are not limited by the random process of damping to the external continuum. The extra term which the measurement introduces into the nonselective master equation will not produce nonclassical states, but may allow the measurement to produce nonclassical *conditioned* states. We thus expect that intracavity QND measurements will enable feedback which overcomes the classical limit. In Sec. VII, we show this explicitly.

## VI. SQUEEZING SPECTRA WITH FINITE TIME DELAY

The results obtained so far are valid only if the feedback delay time is negligible. The original motivation for this was to enable a general nonselective master equation to be derived describing any feedback based on external homodyne detection. However, as the preceding section showed, it is quite simple to understand the mechanism of the feedback selectively (i.e., keeping track of the feedback photocurrent) for the particular case of linearized squeezing. This suggests that we may be able to treat this simple case with a finite time delay in the feedback loop included. Although we cannot describe the nonselective evolution of the state of the cavity mode, we can calculate the squeezing spectrum for the system output, which is what is easily accessible experimentally. It is shown that, as expected, a finite delay increases the overall output noise.

The squeezing in the system output is measured operationally by a homodyne measurement. An ideal homodyne measurement enables the squeezing spectrum (3.8) to be calculated exactly, as shown previously [6]. Since the finite time delay makes it necessary to use conditioned states, the easiest way to proceed is to make the system state conditioned on the detection of its output, as well as on the detection in the feedback loop. That is, we consider a two-sided cavity, with a homodyne measurement of the $X$ quadrature at both ends, as shown in Fig. 2. The two photocurrents are

$$I_c(t) = \eta[\langle a + a^\dagger\rangle_c(t) + \xi(t)/\sqrt{\eta}], \tag{6.1}$$

$$J_c(t) = \theta[\langle a + a^\dagger\rangle_c(t) + \zeta(t)/\sqrt{\theta}], \tag{6.2}$$

where $\xi(t)$ and $\zeta(t)$ are two independent Gaussian white-noise terms, and the efficiencies $\eta$ and $\theta$ must sum to unity or less. Using the photocurrent $I_c(t)$ in a feedback loop with time delay $\tau$ gives the general conditioning master equation for the cavity mode,

$$\begin{aligned}\dot{\rho}_c(t) = \Big\{&\mathcal{L} + [\sqrt{\eta}\xi(t) + \sqrt{\theta}\zeta(t)]\mathcal{H} \\ &+[\langle a + a^\dagger\rangle_c(t-\tau) + \xi(t-\tau)/\sqrt{\eta}]\mathcal{K} \\ &+\frac{1}{2\eta}\mathcal{K}^2\Big\}\rho_c(t). \end{aligned}\tag{6.3}$$

The intent is to use this equation to calculate the spectrum of fluctuations in the output photocurrent $J_c(t)$.

Such a calculation would be very difficult in general. However, for a system which produces squeezing as considered in Sec. III, it turns out to be quite tractable. In this case, the stochastic master equation (6.3) can be replaced by the stochastic Wigner function equation

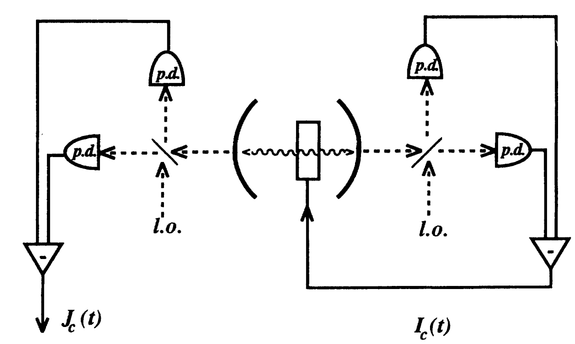


FIG. 2. Schematic diagram of a general feedback scheme with a second output for measuring the squeezing spectrum, as discussed in the text. The initials l.o. denote a local oscillator and p.d. a photodetector. Light beams are indicated by dashed lines and electronics by solid lines.

$$\dot{W}_c(x,t) = \left\{ \partial_x kx + \frac{1}{2}\partial_x^2 \frac{1+l}{4} + \left[\sqrt{\eta}\xi(t) + \sqrt{\theta}\zeta(t)\right]\left[2x - 2\bar{x}_c(t) + \frac{1}{2}\partial_x\right] + \left[2\bar{x}_c(t-\tau) + \xi(t-\tau)/\sqrt{\eta}\right]\frac{\lambda}{2}\partial_x + \frac{1}{2}\frac{\lambda^2}{4\eta}\partial_x^2 \right\} W_c(x,t), \tag{6.4}$$

where $\bar{x}_c(t)$ is the mean of $x$ computed from the conditional probability distribution $W_c(x,t)$. Using the usual Gaussian ansatz, we find the following equations for the conditional mean and variance in $X$:

$$\dot{\bar{x}}_c(t) = -k\bar{x}_c(t) + \left[\sqrt{\eta}\xi(t) + \sqrt{\theta}\zeta(t)\right]\left[2V_c(t) - \frac{1}{2}\right] -\lambda\left[\bar{x}_c(t-\tau) + \xi(t-\tau)/(2\sqrt{\eta})\right], \tag{6.5}$$

$$\dot{V}_c(t) = -2kV_c(t) + \frac{1+l}{4} - (\eta+\theta)\left[2V_c(t) - \frac{1}{2}\right]^2. \tag{6.6}$$

Once again, the differential equation for the conditioned variance (6.6) is closed and deterministic, and in addition it is Markovian. In fact, it is identical to Eq. (5.4) with the replacement of $\eta$ by $\eta+\theta$. The steady-state solution is defined as follows, using $Q_c = 4V_c - 1$ :

$$(\eta+\theta)Q_c^2 + 2kQ_c = 2kQ_0, \tag{6.7}$$

where $Q_0$ represents the no-feedback noise in $X$ (3.7). Substituting the steady-state solution into Eq. (6.5) gives the closed, stochastic, non-Markovian differential equation for the conditioned mean,

$$\dot{\bar{x}}_c(t) = -k\bar{x}_c(t) + \left[\sqrt{\eta}\xi(t) + \sqrt{\theta}\zeta(t)\right]\frac{Q_c}{2} - \lambda\left[\bar{x}_c(t-\tau) + \xi(t-\tau)/(2\sqrt{\eta})\right]. \tag{6.8}$$

To solve this equation we convert to the frequency domain. Denoting Fourier transforms by a tilde, so

$$\tilde{x}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t}\bar{x}_c(t)dt, \tag{6.9}$$

Eq. (6.8) gives

$$\tilde{x}(\omega) = \frac{(\sqrt{\eta}Q_c - \lambda e^{i\omega\tau}/\sqrt{\eta})\tilde{\xi}(\omega) + \sqrt{\theta}Q_c\tilde{\zeta}(\omega)}{2(-i\omega + k + \lambda e^{i\omega\tau})}. \tag{6.10}$$

Here $\tilde{\xi}(\omega)$ is a complex white-noise term satisfying

$$\tilde{\xi}(-\omega) = \tilde{\xi}(\omega)^*, \tag{6.11}$$

$$E[\tilde{\xi}(\omega)\tilde{\xi}(\omega')] = 2\pi\delta(\omega+\omega'), \tag{6.12}$$

and $\tilde{\zeta}(\omega)$ is an independent white-noise term satisfying identical relations.

Now, the output squeezing spectrum for $X$ which we wish to determine is

$$S(\omega) = 2\int_0^{\infty} dt' \cos\omega t' E[J_c(t+t')J_c(t)]. \tag{6.13}$$

By the Fourier transform theorem, this can be written as

$$S(\omega) = e^{-i\omega t}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' e^{i\omega' t} E[\tilde{J}(\omega)\tilde{J}(\omega')^*], \tag{6.14}$$

where, from Eq. (6.2),

$$\tilde{J}(\omega) = 2\theta\frac{(\sqrt{\eta}Q_c - \lambda e^{i\omega\tau}/\sqrt{\eta})\tilde{\xi}(\omega) + \sqrt{\theta}Q_c\tilde{\zeta}(\omega)}{2(-i\omega + k + \lambda e^{i\omega\tau})} + \sqrt{\theta}\tilde{\zeta}(\omega). \tag{6.15}$$

Using the relations (6.11) and (6.12) gives

$$S(\omega) = \theta + \theta^2\frac{(\eta+\theta)Q_c^2 + \lambda^2/\eta + 2kQ_c}{(k+\lambda\cos\omega\tau)^2 + (\omega - \lambda\sin\omega\tau)^2}. \tag{6.16}$$

Substituting the expression for $Q_c$ (6.7) gives finally

$$\frac{S(\omega)}{\theta} = 1 + \theta\frac{2kQ_0 + \lambda^2/\eta}{(k+\lambda\cos\omega\tau)^2 + (\omega - \lambda\sin\omega\tau)^2}. \tag{6.17}$$

Putting $\tau = 0$ into this formula gives the correct Lorentzian noise spectrum which would be generated by the Ornstein-Uhlenbeck process (3.14). Indeed, the zero-frequency value (3.21) is unaffected by the time delay, since it represents the long time fluctuations. The effect of the time delay is to introduce more noise at higher frequencies. This can be seen by considering the small delay limit $\tau \ll 1$. To first order in $\tau$ we have

$$\frac{S(\omega)}{\theta} = 1 + \theta\frac{2kQ_0 + \lambda^2/\eta}{(k+\lambda)^2 + [\omega(1-\lambda\tau)]^2} \tag{6.18}$$

for $\omega\tau \ll 1$, which will be satisfied by all values of $\omega$ for which the fraction is significant. In this limit, the noise spectrum is still approximately Lorentzian, but with the bandwidth changed to

$$(k+\lambda)(1+\lambda\tau). \tag{6.19}$$

This shows that the condition for the zero time delay feedback theory of Sec. II to be valid is $|\lambda\tau| \ll 1$. For squeezed systems, in which $\lambda$ is negative for best noise reduction, the bandwidth of squeezing is reduced, so that the total noise reduction is less than in the $\tau = 0$ case. For unsqueezed systems we have $\lambda > 0$ and the bandwidth of excess noise (above the shot-noise limit) is increasesd. In either case, the time delay introduces more noise as expected. In Fig. 3 we show the output noise spectrum for the squeezed quadrature of a classically pumped below threshold parametric oscillator with threshold parameter $\kappa = 2/3$ under various conditions: with an undivided output, with an output divided in half but the feedback loop not completed, with the feedback loop completed but no time delay, and with the feedback loop completed and a time delay of $\tau = 2.5$.

In this section, we have considered feeding back a time-delayed, but otherwise unchanged photocurrent. In fact, analogous results hold for the feedback of any signal linear in the photocurrent. If we assume a feedback term of the form

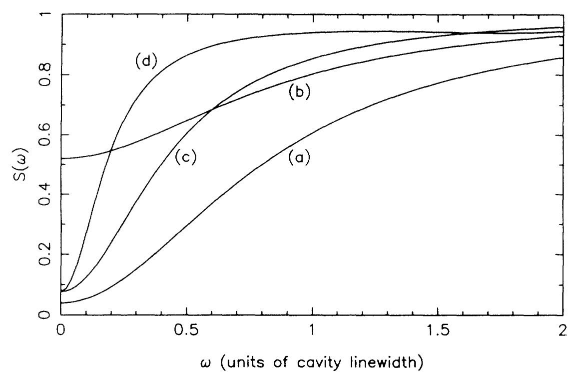


FIG. 3. Squeezing spectrum for a parametric oscillator with a classical pump amplitude of two-thirds the threshold amplitude, under various conditions: (a) free running in a single sided cavity; (b) free running in cavity with two mirrors of equal transmittivities (so that the squeezing is degraded by a factor of one-half); (c) in such a two-sided cavity with the instantaneous homodyne photocurrent from one end used optimally to control the amplitude of a coherent driving field (showing that much of the squeezing is restored, especially near resonance); and (d) as in (c) but with a finite time delay in the feedback loop of 2.5 times the cavity lifetime (showing that the bandwidth of the squeezing is reduced and the spectrum is non-Lorentzian).

$$[\dot{\rho}_c(t)]_{\rm fb} = \mathcal{K}\rho_c(t)\eta^{-1}\int_{-\infty}^{t} g(t-t')I_c(t')dt', \qquad (6.20)$$

where $\int_0^\infty g(t)dt = 1$, then the equation for the stationary conditioned variance (6.7) is unchanged, while the conditioned mean obeys

$$\begin{aligned}\dot{\bar{x}}_c(t) = & -k\bar{x}_c(t) + \left[\sqrt{\eta}\xi(t) + \sqrt{\theta}\zeta(t)\right]\frac{Q_c}{2} \\ & -\lambda\int_{-\infty}^{t} g(t-t')\left[\bar{x}_c(t') + \xi(t')/(2\sqrt{\eta})\right]dt'.\end{aligned} \qquad (6.21)$$

Proceeding as above yields the free spectrum

$$\frac{S(\omega)}{\theta} = 1 + \theta\frac{2kQ_0 + \lambda^2|\tilde{g}(\omega)|^2/\eta}{|-i\omega + k + \lambda\tilde{g}(\omega)|^2}, \qquad (6.22)$$

as expected.

## VII. SQUEEZING VIA QND FEEDBACK

As explained at the end of Sec. V, we expect that the "no-go" theorems for the production of squeezing via feedback could be overcome by using intracavity QND measurements. In this section, we verify that this is the case, and find the limits to squeezing via QND measurements. The natural choice of quantum nondemolition variable is the quadrature to be squeezed, say $X$ as before. A simple model for a QND measurement of $X$ has been given by us in a previous paper [6]. We assume that the cavity supports a second mode which is coupled to the system mode via the interaction

$$H_{\rm int} = \hbar\chi X\tfrac{1}{2}(-ib + ib^\dagger), \qquad (7.1)$$

where $b$ is the annihilation operator for the second mode and $\chi$ is real. The $\frac{1}{2}(b+b^\dagger)$ quadrature of the $b$ mode is driven by the $X$ quadrature of the $a$ mode. If we denote the combined density operator for the two modes by $D$, then this will obey the master equation

$$\dot{D} = \mathcal{L}D - \frac{\chi}{2}[X(b-b^\dagger), D] + \gamma\mathcal{D}[b]D, \qquad (7.2)$$

where $\gamma$ is the damping rate for the $b$ mode.

It is shown in Ref. [6] that, if $\gamma$ is sufficiently large, mode $b$ will almost always be in the vacuum state and its dynamics can be adiabatically eliminated. The reduced density operator $\rho = {\rm Tr}_b D$ for mode $a$ then obeys the master equation

$$\dot{\rho} = \mathcal{L}\rho + \Gamma\mathcal{D}[X]\rho, \qquad (7.3)$$

where the measurement strength parameter is

$$\Gamma = \chi^2/\gamma. \qquad (7.4)$$

The superoperator $\mathcal{D}[X]$ has the form of a double commutator which routinely arises in QND measurements. Furthermore, the full density operator was shown to be (to second order in $\chi/\gamma$)

$$\begin{aligned}D = & \rho\otimes|0\rangle_b\langle 0| + \frac{\chi}{\gamma}\left(X\rho\otimes|1\rangle_b\langle 0| + \rho X\otimes|0\rangle_b\langle 1|\right) \\ & +\frac{\chi^2}{\gamma^2}X\rho X\otimes|1\rangle_b\langle 1| \\ & +\frac{\chi^2}{\sqrt{2}\gamma^2}\left(X^2\rho\otimes|2\rangle_b\langle 0| + \rho X^2\otimes|0\rangle_b\langle 2|\right).\end{aligned} \qquad (7.5)$$

If we now add homodyne measurement of the $b$ mode with efficiency $\eta$, the conditioned density operator $D_c$ obeys

$$\begin{aligned}\dot{D}_c = & \mathcal{L}D_c - \frac{\chi}{2}[X(b-b^\dagger), D_c] + \gamma\mathcal{D}[b]D_c \\ & +\sqrt{\gamma\eta}\,\xi(t)(bD_c + D_cb^\dagger - \langle b+b^\dagger\rangle_c D_c).\end{aligned} \qquad (7.6)$$

Substituting in the solution (7.5) gives the conditioning master equation for $\rho_c$,

$$\dot{\rho}_c = \mathcal{L}\rho_c + \Gamma\mathcal{D}[X]\rho_c + \sqrt{h}\xi(t)\left(X\rho_c + \rho_c X - 2\langle X\rangle_c\rho_c\right), \qquad (7.7)$$

and the homodyne photocurrent

$$\begin{aligned}I_c(t) &= \eta\gamma\langle b+b^\dagger\rangle_c(t) + \sqrt{\eta\gamma}\,\xi(t) \qquad (7.8)\\ &= \eta\chi\left[2\langle X\rangle_c(t) + \xi(t)/\sqrt{h}\right], \qquad (7.9)\end{aligned}$$

where we have defined a new parameter

$$h = \eta\Gamma. \qquad (7.10)$$

We can use this photocurrent to feedback on the $a$ mode just as in Sec. II. A feedback term of the form

$$[\dot{\rho}_c]_{\rm fb} = \frac{I_c(t-\tau)}{\eta\chi}\mathcal{K}\rho_c \tag{7.11}$$

gives, in the limit $\tau \to 0$, the nonselective master equation

$$\dot{\rho} = \mathcal{L}\rho + \Gamma\mathcal{D}[X]\rho + \mathcal{K}[X\rho + \rho X] + \frac{1}{2h}\mathcal{K}^2\rho. \tag{7.12}$$

This differs from Eq. (2.19) in that it has a QND measurement term, and a different feedback drift term. The feedback diffusion term retains its previous form, but the efficiency of the feedback loop $\eta$ is replaced by $h$ (which is also dimensionless if measured in units of the cavity linewidth as is our convention). However, unlike $\eta$, $h$ is not confined to the interval $[0,1]$. In particular, $h$ can be arbitrarily large, which makes the noise associated with the feedback arbitrarily small. As we shall see soon, this allows the production of arbitrarily squeezed states. Of course, the quantum noise has not been eliminated but rather redistributed. For $h$ to be large requires $\Gamma$ to be large also [Eq. (7.10)], so that the variance in the unsqueezed quadrature is greatly increased by the measurement term in Eq. (7.12). This ensures that Heisenberg's uncertainty principle is not violated. In what follows, we are concerned only with the statistics of the squeezed quadrature $X$.

With superoperators $\mathcal{L}$ and $\mathcal{K}$ defined as in Sec. III, Eq. (7.12) gives the following Ornstein-Uhlenbeck equation for the probability distribution for $X$:

$$\dot{W}(x) = \left[\partial_x(k+\lambda)x + \frac{1}{2}\partial_x^2\left(\frac{1+l}{4} + \frac{\lambda^2}{4h}\right)\right]W(x). \tag{7.13}$$

Now we find the intracavity squeezing parameter

$$Q_\lambda = -1 + \frac{1+l+\lambda^2/h}{2(k+\lambda)}. \tag{7.14}$$

Minimizing with respect to $\lambda$ yields

$$Q_{\min} = -1 + h^{-1}\left[-k + \sqrt{k^2 + h(1+l)}\right] \tag{7.15}$$

when

$$\lambda = -k + \sqrt{k^2 + h(1+l)}. \tag{7.16}$$

In the limit $h \to 0$, Eq. (7.15) reduces to the no-feedback expression (3.7). In the other limit, $h \to \infty$, it is easy to see that $Q_{\min}$ approaches the theoretical minimum value of $-1$. That is, perfect squeezing can be produced inside the cavity by QND-mediated feedback. In this limit, one requires the feedback to be very strong, with $\lambda \simeq \sqrt{h(1+l)}$. Unlike the homodyne-mediated feedback case, $\lambda$ should always be positive, as in accord with classical intuition. Indeed, all of the features of QND-mediated feedback conform to a classical theory of feedback with measurements of finite accuracy (related to $h$). The quantum nature of the feedback is manifest only in the increased fluctuations in $Y$ due to the measurement backaction not present classically.

Outside the cavity, fluctuations in the $X$ quadrature are not necessarily suppressed to the same extent as they are inside; the degree of extracavity noise reduction depends on the intracavity dynamics. Once again, we use the parameter $R$ [Eq. (3.10)] to quantify output noise minus shot-noise. From Eq. (7.13), it is easy to find

$$R = \frac{-2(k+\lambda) + 1 + l + \lambda^2/h}{(k+\lambda)^2}. \tag{7.17}$$

Here we are assuming that all light lost from the cavity goes into its output, as the QND feedback loop does not consume emitted light. This expression (7.17) is minimized when

$$\lambda = \left(1 + \frac{k}{h}\right)^{-1}(-k+1+l). \tag{7.18}$$

For $h$ small, this gives $\lambda$ small and hence negligible noise reduction. For $h$ very large, we find the best possible on-resonance output noise reduction is

$$R_{\min} = -(1+l)^{-1}, \tag{7.19}$$

with $\lambda + k = 1 + l$. It is thus always possible to produce a sub-shot-noise $X$ homodyne photocurrent, but the degree of nonclassicality is determined by $l$, which measures the amount of diffusion in the $X$ quadrature in excess of that produced by the damping to the external continuum of modes. For example, simply driving the cavity does not introduce any excess noise, so $l = 0$ and it is possible to achieve perfect noise reduction on resonance with $R \to -1$. For a laser (now assumed to be phase locked so that $X$ represents its amplitude), we have $1 + l = 2 + q$ [see Eq. (4.4)]. Thus for an ideal standard laser (Poissonian pumped with $q = 0$), the output can be squeezed only to $R = -\frac{1}{2}$.

The above analysis could have been carried out using the selective evolution of the system, just as in Sec. V. The stochastic master equation for the conditioned density operator is

$$\begin{aligned}\dot{\rho}_c = {} & \mathcal{L}\rho_c + \Gamma\mathcal{D}[X]\rho_c + \mathcal{K}[X\rho_c + \rho_c X] + \frac{1}{2h}\mathcal{K}^2\rho_c \\ & + \xi(t)\left[\sqrt{h}(X\rho_c + \rho_c X - 2\langle X\rangle_c\rho_c) + \frac{\mathcal{K}}{\sqrt{h}}\rho_c\right].\end{aligned} \tag{7.20}$$

The probability distribution for the $X$ quadrature obeys

$$\begin{aligned}\dot{W}_c(x) = {} & \left[\partial_x(k+\lambda)x + \frac{1}{2}\partial_x^2\left(\frac{1+l}{4} + \frac{\lambda^2}{4\eta}\right)\right. \\ & \left. +\xi(t)\left(\sqrt{h}2[x - \bar{x}_c(t)] + \frac{\lambda}{2\sqrt{h}}\partial_x\right)\right]W_c(x),\end{aligned} \tag{7.21}$$

where we are assuming the usual expression for $\mathcal{L}$ and $\mathcal{K}$. The mean and variance of this conditioned distribution obey

$$\dot{\bar{x}}_c = -(k+\lambda)\bar{x}_c + \xi(t)\left(\sqrt{h}2V_c - \frac{\lambda}{2\sqrt{h}}\right), \qquad (7.22)$$

$$\dot{V}_c = -2kV_c + \frac{1+l}{4} - h4V_c^2. \qquad (7.23)$$

These equations are identical to the corresponding equations for homodyne-mediated feedback, (5.3) and (5.4), apart from the replacement of $V_c - \frac{1}{4}$ by $V_c$ and $\eta$ by $h$. In the limit $h \to \infty$, Eq. (7.23) predicts an arbitrarily small steady-state conditioned variance. This is characteristic of a good QND measurement. Choosing the value of $\lambda$ in Eq. (7.16) eliminates the stochastic element in Eq. (7.22). In this case, the nonconditioned variance [as given by Eq. (7.15)] agrees with the conditioned variance from Eq. (7.23). The output spectrum could also have been calculated from the selective evolution equation (7.21), and a finite delay (or more general frequency response) incorporated precisely as in Sec. VI. The result is

$$S(\omega) = 1 + \frac{-2\{k + \lambda \mathrm{Re}[\tilde{g}(\omega)]\} + 1 + l + \lambda^2|\tilde{g}(\omega)|^2/h}{|-i\omega + k + \lambda\tilde{g}(\omega)|^2}. \qquad (7.24)$$

The ability to produce the arbitrary squeezing described above depends on having $h$ large. From Eqs. (7.4) and (7.10), it can be seen that this amounts to having $\chi$ much greater than the cavity linewidth. In Ref. [6], we have shown that the Hamiltonian (7.1) could be achieved by using two pump modes and two $\chi^{(2)}$ nonlinearities. In this case, $\chi$ is proportional to the dimensionless intracavity pump amplitude multiplied by the magnitude of $\chi^{(2)}$. The large $\chi$ condition obviously could not be achieved in a normal laboratory experiment. Typical values of nonlinearity $\chi^{(2)} \sim 10^{-10}$ s$^{-1}$ and cavity linewidth $\gamma \sim 10^8$ s$^{-1}$ would require the number of pump photons in the cavity to be much greater than $10^{36}$. This compares to typical photon numbers of $10^{10}$. The reason we chose to analyze this model rather than other, more realizable QND schemes, is that it is simple and shows that the effectiveness of a QND scheme depends on one number only, $h$.

## VIII. DISCUSSION

The most important point of this paper is that noise reduction via feedback is conditioning made practical. Intuitively, feedback can reduce noise in a particular quantity (call it $X$) by adding a restoring force which tends to prevent $X$ from wandering away from some predefined value $X_0$. To add such a force, which is usually linear in the separation $X - X_0$, it is of course necessary to know the value of $X$. Classically, the measurement process which provides this knowledge can be ignored, as it can be arbitrarily accurate and its effect on the system can be arbitrarily small. Thus classical feedback theory consists of little more than techniques of solving nonlinear differential (or difference) equations. Quantum mechanically, the measurement process with its inevitable stochasticity cannot be ignored. To derive a quantum theory of feedback, it is necessary to have a measurement theory which predicts precisely how the state of the system changes when it is conditioned on the result of the measurement.

In quantum optics, such measurement theories are often quite subtle mathematically, compared to the original theory based on the projection postulate [2]. In this paper for example, we have considered homodyne measurement in which the conditioned density operator obeys a nonlinear, stochastic (Ito) master equation. Nevertheless, all measurements of $X$ have one property in common with projective measurements in order to justify their being called measurements. This is that the variance of $X$ in any conditioned system is less than (or at worst, equal to) the unconditioned variance. The unconditioned (ensemble average) variance is greater than the conditioned variance because the members of the ensemble (the conditioned states) have different mean values of $X$. The smaller conditioned variance is not usually observed because it requires precise knowledge of the measured photocurrent. It is the quantum noise in this photocurrent which gives a random shift to the mean value of $X$, causing the distribution of means in the ensemble.

The way to make the smaller variance observable in practice is to feed back the photocurrent. By choosing the strength of the feedback carefully, it is possible to give a kick to the conditioned mean $X$ which precisely counteracts the earlier shift caused by the measurement of that photocurrent. If the time delay between the measurement and the feedback is sufficiently small, then the feedback can elimate the noise in $X$ caused by the measurement. Then the mean $X$ in all elements of the ensemble can be forced to the desired value $X_0$, and the only noise in $X$ is the conditioned variance. This is the quantum limit of feedback: the smallest variance in $X$ which can be produced is the variance which results from conditioning the system on the results of the measurement which is used to do the feedback. Feedback is conditioning made practical.

Obviously, the degree of noise reduction via feedback will depend heavily on the method by which information about $X$ is gathered. Specializing now to cavity quantum optics, it can be shown that any measurement of the light which has been lost from the cavity will not produce a nonclassical conditioned state inside the cavity, unless the internal dynamics of the cavity can already produce nonclassical states. It thus follows that feedback of information gained outside the cavity cannot produce a nonclassical cavity state. This means that noise reduction in this sort of feedback is limited to creating states which can be described as a classical mixture of coherent states. In particular, feedback mediated by a homodyne measurement of the $X$ quadrature of a field cannot produce squeezing (variance of $X$ less than that of a coherent state). If the intracavity dynamics already produce squeezing, then such feedback can enhance the squeezing inside the cavity. However, because extracavity feedback requires the use of a fraction of the light lost from the cavity, the squeezing in the remaining fraction of emitted light is degraded such that there is no advantage in adding feedback to a squeezed system if one is only interested in the output light. To achieve optimum noise reduction in squeezed systems, it is necessary for the sign of the feedback to be opposite that expected intuitively. This is directly traceable to the effect of homodyne detection on squeezed states, which also explains their sub-shot-noise photocurrent spectrum.

None of the above limitations apply to intracavity measurements. For example, we have shown that a quantum nondemolition measurement of the quadrature $X$ allows arbitrarily squeezed states to be produced via feedback. The degree of squeezing in the output depends on the intracavity dynamics, but the simplest dynamics give arbitrary noise reduction in the output. In order to achieve good noise reduction, we have shown that it is necessary for the measurement rate to be much greater than the cavity linewidth, as expected. For the model we have presented here, this would be extremely difficult to realize. We plan to present more realistic QND feedback schemes in future work. The practical application of QND measurements to produce squeezing should give extra impetus to the experimental and theoretical search to find measurement schemes of sufficient strength.

Although it is now known that extracavity measurements cannot produce squeezing via feedback, it is of interest to have a way of describing such schemes because they are already in use by experimentalists to achieve classical noise reduction. In this paper we have solved this problem (in the negligible-time-delay limit) for homodyne measurements. The result was a relatively simple master equation (2.19) in which the action of the feedback is arbitrary . It was possible to derive this master equation because of the simple form (Gaussian white noise) of the stochasticity in the photocurrent and in the conditioning master equation describing the effect of the homodyne measurement on the system. This was the principle reason for considering homodyne detection, although it is also the most appropriate measurement for controlling quadrature squeezing.

Heterodyne measurements have the same noise properties as homodyne measurements [4]. Indeed, a heterodyne measurement can be considered to consist of two homodyne measurements, one on each quadrature, each with half of the total efficiency. The quantum theory of feedback of a heterodyne photocurrent would thus be completely analagous to that presented here. Direct photodetection is a different matter. The theoretical record of measurement from direct detection is a series of photodetection spikes rather than a continuous photocurrent which arises from an infinitely large local oscillator. Similarly, the system changes by jumps, not smoothly. The properties of this stochastic process are not easy to treat in a realistic manner (when a photocurrent, rather than a series of individual photodetections, is the record). We will address this issue in future work.

## APPENDIX A: DERIVATION OF EQ. (2.17)

We wish to evaluate the ensemble average

$$E[\xi(t-\tau)\rho_c(t)], \tag{A1}$$

where $\rho_c(t)$ is the state of the system with homodyne-mediated feedback, obeying

$$\begin{aligned}\dot{\rho}_c(t) = \Big\{&\mathcal{L} + \sqrt{\eta}\xi(t)\mathcal{H}\\ &+[\langle a+a^\dagger\rangle_c(t-\tau) + \xi(t-\tau)/\sqrt{\eta}]\mathcal{K}\\ &+\frac{1}{2\eta}\mathcal{K}^2\Big\}\rho_c(t).\end{aligned} \tag{A2}$$

If we assume that $\tau$ is small, then the solution of (A2) can be written

$$\begin{aligned}\rho_c(t) &= [1+O(\tau)]\rho_c(t-\tau+dt)\\ &= [1+O(\tau)][1+\sqrt{\eta}dW(t-\tau)\mathcal{H}]\rho_c(t-\tau).\end{aligned} \tag{A3}$$

Strictly, the ignored evolution between $t-\tau$ and $t$ contains terms of order $\sqrt{\tau}$, but these are stochastic and independent of $\xi(t-\tau)$ and so will disappear when the ensemble average (A1) is taken. The only term which will not disappear is that due to the effect of the measurement at time $t-\tau$, acting via the superoperator $\mathcal{H}$. This gives

$$\begin{aligned}E[\xi(t-\tau)\rho_c(t)] &= [1+O(\tau)]\sqrt{\eta}\mathcal{H}\rho_c(t-\tau)\\ &= [1+O(\tau)]\sqrt{\eta}\\ &\quad\times[a\rho_c(t-\tau)+\rho_c(t-\tau)a^\dagger\\ &\quad-\langle a+a^\dagger\rangle_c(t-\tau)\rho_c(t-\tau)].\end{aligned} \tag{A4}$$

However, the ensemble average $\rho(t)$ differs from $\rho_c(t-\tau)$ only by terms of order $\tau$ because the ensemble average will again remove stochastic terms. Thus we can write

$$\begin{aligned}E[\xi(t-\tau)\rho_c(t)] = [1+O(\tau)]\sqrt{\eta}[&a\rho(t)+\rho(t)a^\dagger\\ &-\langle a+a^\dagger\rangle_c(t-\tau)\rho(t)],\end{aligned} \tag{A5}$$

which is the desired expression.

## APPENDIX B: SPECTRA FOR INJECTION PHASE-LOCKED LASER

The master equation for a laser with injected signal is

$$\dot{\rho} = \mu\left\{\mathcal{E}[a^\dagger] + \frac{q}{2}\mathcal{E}[a^\dagger]^2\right\}\rho + \mathcal{D}[a]\rho - i\epsilon[a+a^\dagger,\rho]. \tag{B1}$$

Using expression (4.3) for the Wigner function evolution equation for a laser without an injected signal and linearizing about the deterministic steady state gives the following Ornstein-Uhlenbeck equation for $W(x,y)$:

$$\dot{W}(x,y) = \left\{\frac{1}{2}\left(1+\frac{\mu}{y_0^2}\right)\partial_y(y-y_0) + \frac{1}{2}\left(1-\frac{\mu}{y_0^2}\right)\partial_x x\right. \tag{B2}$$

$$\left.+\frac{1}{8}\left[1+(1+q)\frac{\mu}{y_0^2}\right]\partial_y^2 + \frac{1}{8}\left[1+(1+2\nu)\frac{\mu}{y_0^2}\right]\partial_x^2\right\}W(x,y), \tag{B3}$$

where the deterministic field amplitude is

$$iy_0 = i\left(\epsilon + \sqrt{\epsilon^2 + \mu}\right), \tag{B4}$$

and we have included an excess phase noise parameter $\nu \geq 0$. The intracavity variances in $x$ and $y$ are given by $V = D/2k$, where $D$ and $k$ are the diffusion and drift constants, respectively. In terms of the $Q$ parameter ($Q = 4V - 1$),

$$Q_Y = \frac{q\mu/y_0^2}{1 + \mu/y_0^2}, \tag{B5}$$

$$Q_X = \frac{2(1+\nu)\mu/y_0^2}{1 - \mu/y_0^2}. \tag{B6}$$

The output noise is measured by $R = 2Q/k$,

$$R_Y = \frac{4q\mu/y_0^2}{(1 + \mu/y_0^2)^2}, \tag{B7}$$

$$R_X = \frac{8(1+\nu)\mu/y_0^2}{(1 - \mu/y_0^2)^2}. \tag{B8}$$

The on-resonance noise spectrum is given by $S(0) = 1 + R$. If $q = -1$ (a perfectly regularly pumped laser), then

$$S_Y(0) = \beta, \tag{B9}$$

$$S_X(0) = \frac{1 + (1+2\nu)(1-\beta)}{\beta}, \tag{B10}$$

where

$$\beta = \frac{\epsilon^2}{\mu + \epsilon^2}, \tag{B11}$$

and the linewidths of the output spectra are

$$k_Y(0) = \left(1 + \sqrt{\beta}\right)^{-1}, \tag{B12}$$

$$k_X(0) = 1 - \left(1 + \sqrt{\beta}\right)^{-1}. \tag{B13}$$


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