Effect of dissipation and measurement on a tunneling system

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We consider a parametrically driven Kerr medium in which the pumping may be sinusoidally varied. It has been previously found that this system exhibits coherent tunneling between two fixed points which can be either enhanced or suppressed by altering the driving frequency and strength. We numerically investigate the dynamics when damping is included. This is done both by solving a master equation and using the quantum-trajectory method. In the latter case it is also possible to model the result of a continuous heterodyne measurement of the cavity output. The dissipation destroys the coherences which give rise to the tunneling, causing the sinusoidal oscillation of the mean to give way to a stochastic jumping between the fixed points, manifested as a random telegraph signal. In the quantum-trajectory picture we show that the coherences responsible for tunneling are an exponentially decreasing function of the signal-to-noise ratio for heterodyne measurements. However, evidence of both the bare tunneling rate and the driving modified tunneling rate are still apparent in the random telegraph signal.

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I. INTRODUCTION

A. The model

The Cassinian oscillator consists of a cavity field coupled to a medium with an intensity dependent refractive index and subject to parametric pumping generated by a second order optical nonlinearity. In previous work [1,2] a system consisting of a Kerr medium pumped by a parametric oscillator which has its amplitude periodically perturbed was presented. The Hamiltonian for this system (in the interaction picture) is

$$H_0(a,a^\dagger,t)=\frac{\hbar\chi}{2}(a^\dagger a)^2-\frac{\hbar\kappa(1+\epsilon\cos\Omega t)}{2}(a^2+a^{\dagger 2}) \tag{1}$$

where a is the annihilation operator for the cavity field, χ is proportional to the third order nonlinear susceptibility, κ is proportional to the product of the amplitude of the parametric pump field and the second order susceptibility, ϵ is the strength of the perturbation, and Ω is the frequency of the driving.

The classical version of the system was shown to follow the equations of motion

$$\dot{x}=\chi(x^2+p^2)p+\kappa(1+\epsilon\cos\Omega t)p, \tag{2}$$

$$\dot{p}=-\chi(x^2+p^2)x+\kappa(1+\epsilon\cos\Omega t)x \tag{3}$$

with x and p commuting real numbers which replace the quantum operators defined by $x=\frac{1}{2}(a+a^\dagger)$ and $p=(-i/2)(a-a^\dagger)$. The overdot represents differentiation with respect to time.

The trajectories generated by these equations, when ϵ is set to zero, possess two stable fixed points at $x=\pm\sqrt{\kappa/\chi},\ p=0$ and an unstable fixed point at the origin. The separatrix is a lemniscate. The curves of motion with constant energy, in phase space, are ovals of Cassini and so the system is called the Cassinian oscillator.

The quantum mechanical system exhibits tunneling between localized states positioned at the classical fixed points, which is the principal reason for studying this system. This tunneling is well described in terms of the two lowest energy eigenstates [1].

B. Time dependent behavior

A number of authors have considered the effect of time dependent driving on quantum tunneling in bistable systems such as the Cassinian oscillator [3–8]. We consider a similar phenomenon in our model by sinusoidally varying the strength of the parametric pumping with time (ϵ is nonzero). With this driving included, the classical motion of the system becomes chaotic. In the quantum system, for small variations in the driving strength, the two-state approximation for the tunneling still holds for most driving frequencies but the frequency of the tunneling is either enhanced or suppressed, depending on the driving frequency. This behavior is due to the interaction of one of the two lowest states with a higher energy eigenstate [2]. A perturbative treatment of the system to calculate the Floquet states [2] reveals that the corrected frequencies of the quasienergy eigenstates are given by

$$\omega_m=\omega_m^{(0)}+\epsilon^2\omega_m^{(2)} \tag{4}$$

in which $\omega_m^{(n)}$ is the nth order correction to the mth eigenstate and

$$\omega_m^{(2)}=\frac{\kappa^2}{8}\sum_n\frac{|\langle\omega_m^{(0)}|(a^2+a^{\dagger 2})|\omega_n^{(0)}\rangle|^2(\omega_m^{(0)}-\omega_n^{(0)})}{(\omega_m^{(0)}-\omega_n^{(0)})^2-\Omega^2}, \tag{5}$$

and where $|\omega_m^{(0)}\rangle$ is the unperturbed form of the mth eigenstate. As can be seen from a study of the above equation, there exist resonances where the driving frequency equals the difference in frequency between one of the tunneling states and a higher energy state. At these resonances the correction to the tunneling frequency goes to infinity, and the two-state approximation breaks down. One of the tunneling states gains a large contribution from one of the higher energy levels and so the tunneling can no longer be characterized by a single frequency.

C. Summary

In this paper we investigate tunneling in the Cassinian oscillator including damping. The two main sources of damping on the system are simple linear damping, which arises from the escape of photons out of the cavity mode, and nonlinear (two-photon) damping, due to parametric up-conversion from the cavity field to the parametric pump. The modeling of the linear damping process by a master equation is standard. The result is the usual zero-temperature quantum-optical master equation. The two-photon damping terms are found by performing adiabatic elimination on the pump mode.

The effect of linear damping on the classical system is briefly discussed, followed by a discussion of the effect of the damping on the quantum system. An analytic calculation is done with a simplified form of the system consisting of just two states. It has been found that this is a good approximation of the present system. Within this approximation we find that the two-photon damping terms have no effect. This is because the terms depend on a^2 and so conserve parity producing no coupling between the two, opposite parity, states. The effect of the linear dissipation superoperator is the same as that of the superoperator describing a quantum non-demolition measurement of coarse-grained quadrature phase. To perform an exact analysis of the behavior of the system under damping numerical methods are used. There are two ways in which this is done: using the quantum-trajectory method [9] and using a direct solution of the master equation in the number state basis.

The results of solving the master equation numerically by Runge-Kutta methods are presented. Two-photon damping is found to have only a small effect for the values of damping constant used, as predicted by the two-level approximation. The analysis also shows that under linear damping the tunneling frequencies of the system are preserved while the mean position of the system ceases to oscillate between the fixed points.

The quantum-trajectory method replaces the master equation by a stochastic equation for the system state. Only linear damping is investigated by this method. The effect of measurement on the system is of primary interest in the quantum-trajectory calculations and the two-photon output cannot be used to measure the state of the system as it does not cause dissipation in the system. A solution to the stochastic Schrödinger equation gives a conditional pure state, which depends on the particular noise process used. The solution to the original master equation is then obtained by ensemble averaging over many conditional state trajectories. Particular classes of trajectories may be taken to represent the conditional evolution of the system state as the results of measurements, made on the output light, become known. The method is described in more detail below.

The specific quantum-trajectory equations which model heterodyne measurements on the system are derived in [9]. We solve these equations numerically using a semi-implicit method. The results of this solution are presented. They show that as the damping constant is increased more light escapes from the cavity and so the signal-to-noise ratio of the heterodyne measurement increases. However, as the damping increases, the coherences which lead to the tunneling are destroyed and so the system's motion ceases to be sinusoidal tunneling and is replaced by random telegraph switching from one classically stable well to the other. If the damping is small, so that the tunneling persists, the signal-to-noise ratio of the measurement is found to be too low to permit observation of the tunneling. On the other hand, if the signal-to-noise ratio is large enough to permit accurate observation of the internal behavior of the system, the tunneling is replaced by incoherent jumping between the fixed points. So observation of the system tunneling and hence a measurement of the tunneling rate appear difficult.

In the random telegraph regime the rate of switching of the system between the two states is found to be influenced by the original undriven tunneling rate. When this system is subjected to driving it can be modeled as a two-state system with different energy eigenvalues. The behavior of the switching rate might be expected to follow the tunneling rate, increasing where it is enhanced and decreasing where it is suppressed. The switching rate does nothing of the sort. Although its magnitude roughly follows that of the tunneling frequency it exhibits none of the resonances or the switching between enhancement and suppression which the tunneling frequency does. The enhancement of tunneling in the presence of driving occurs when a state of high energy contributes coherently to the dynamics along with the nearly degenerate two lowest states. However, damping destroys coherence between the low energy tunneling pair and the higher states. So a suppression of the observed effects of resonant enhancement of tunneling might be expected.

II. INITIAL HAMILTONIAN

In this section the complete master equation, including the pump mode, will be given. The pump mode will then be adiabatically eliminated, resulting in a master equation including both linear damping terms and two-photon damping terms.

The system described in the Introduction is derived from a Hamiltonian of the form

$$H=\frac{\hbar\chi}{2}(a^{\dagger}a)^2-\frac{\hbar g}{2}(a^2b^{\dagger}+ba^{\dagger 2})-\frac{i\hbar\varepsilon(1+\epsilon\cos\Omega t)}{2}\times(b-b^{\dagger}). \tag{6}$$

The a operator describes the signal mode which is of interest in this paper; the b operator describes the pump mode which will later be adiabatically eliminated. The first term in the Hamiltonian describes the Kerr nonlinearity in the signal mode which has strength χ. The second term describes the parametric interaction between the pump and signal modes, of strength g. The last term describes the classical coherent driving of the pump mode which has a base amplitude of ε

and a time dependent variation of strength ϵ with a frequency Ω. Both modes of the system described by this Hamiltonian are linearly damped. The effect of this is described by representing the system as a density operator w following a master equation [14],

$$\dot{w}=-\frac{i}{\hbar}[H,w]+\gamma\mathscr{D}[a]w+\gamma_p\mathscr{D}[b]w. \tag{7}$$

Here γ and γ_p are the coefficients of damping for the a and b modes, respectively, and $\mathscr{D}[a]$ is a superoperator which is given by

$$\mathscr{D}[a]\rho=a\rho a^\dagger-\tfrac{1}{2}a^\dagger a\rho-\tfrac{1}{2}\rho a^\dagger a, \tag{8}$$

which describes the effect of linear damping on the mode [14].

Adiabatic elimination of the pump mode

In this section the quantum mechanical mode operator b is replaced with a classical amplitude and the master equation for the reduced density operator ρ, which only depends on the a mode, is calculated. The following argument follows closely that in [9]. To remove the b mode it is necessary to derive the master equation for the density operator ρ, where ρ only includes the a mode. This is done first by displacing the density operator w and the master equation describing its evolution to zero in the b mode. The new density operator can be expanded as a power series in b mode projectors, the coefficients being a mode density operators. This is then substituted back into the original master equation and after some further approximations the new master equation is found.

It is assumed that the b mode, being both driven and damped, comes to equilibrium on a time scale of $1/\gamma_p$ which for γ_p sufficiently large is short compared to both the dynamics of the a mode and the time scale $2\pi/\Omega$. The equilibrium value is

$$\beta=\frac{\varepsilon(1+\epsilon\cos\Omega t)}{\gamma_p}. \tag{9}$$

The system is then transformed to a new set of co-ordinates in which the coherent amplitude of the b mode is displaced to zero. The displacement operator

$$D_b(\beta)=e^{(\beta b^\dagger-\beta^* b)} \tag{10}$$

is used to effect this transformation. So the new density operator is $v=D_b(-\beta)wD_b(\beta)$. This procedure gives as the new master equation for v

$$\begin{aligned}\dot{v}=&-i\left[\frac{\hbar\chi}{2}(a^\dagger a)^2,v\right]-i\left[\frac{g}{2}(a^{\dagger 2}\beta+a^2\beta^*),v\right]\\&-i\left[\frac{g}{2}(a^{\dagger 2}b+a^2b^\dagger),v\right]-\frac{\varepsilon(1+\epsilon\cos\Omega t)}{2}[(b-b^\dagger),v]\\&+\gamma\mathscr{D}[a]v+\gamma_p[\mathscr{D}[b]v+\tfrac{1}{2}[\beta^*b-\beta b^\dagger,v]].\end{aligned} \tag{11}$$

Since the driving $\varepsilon(1+\epsilon\cos\Omega t)=\gamma_p\beta$ the last term will cancel the term of the form $\varepsilon(1+\epsilon\cos\Omega t)/2[b-b^\dagger,v]$. This leaves a master equation in which the b mode is of zero amplitude. This is

$$\begin{aligned}\dot{v}=&-i\left[\frac{\hbar\chi}{2}(a^\dagger a)^2,v\right]-i\left[\frac{g\varepsilon(1+\epsilon\cos\Omega t)}{2\gamma_p}(a^{\dagger 2}+a^2),v\right]\\&-i\left[\frac{g}{2}(a^{\dagger 2}b+a^2b^\dagger),v\right]+\gamma\mathscr{D}[a]v+\gamma_p\mathscr{D}[b]v.\end{aligned} \tag{12}$$

The constant $g\varepsilon/\gamma_p$ is called κ both in previous sections and in what follows. This term then subsumes the role of both ϵ and g in determining the strength of the parametric pumping of the b mode; however, the strength of the two-photon damping, which depends on up-conversion from the a mode followed by damping of the b mode, will depend on both g and γ_p. Until this is calculated g will still occur in these equations. The master equation is written as

$$\dot{v}=\mathscr{L}_0[a]v-i\left[\frac{g}{2}(a^{\dagger 2}b+a^2b^\dagger),v\right]+\gamma_p\mathscr{D}[b]v \tag{13}$$

in which all a only dependent terms have been included in the superoperator $\mathscr{L}_0[a]$.

Since the amplitude of mode b is small, a partial expansion of the density matrix v in terms of the b mode number states need only be carried out to small photon numbers. So

$$\begin{aligned}v=&\rho_0|0\rangle\langle 0|+(\rho_1|1\rangle\langle 0|+\text{ H.c.})+\rho_2|1\rangle\langle 1|\\&+(\rho_2'|2\rangle\langle 0|+\text{ H.c.})+O(\lambda^3).\end{aligned} \tag{14}$$

This is substituted into the master equation Eq. (13) which is expanded and terms multiplying equal sets of b mode number projectors are gathered together to get a set of four equations. Terms of greater than second order are neglected. These equations are

$$\dot{\rho}_0=\mathscr{L}_0[a]\rho_0+\frac{-ig}{2}[a^{\dagger 2}\rho_1-\rho_1^\dagger a^2]+\gamma_p\rho_2, \tag{15}$$

$$\dot{\rho}_1=\mathscr{L}_0[a]\rho_1+\frac{-ig}{2}[a^2\rho_0+\sqrt{2}a^{\dagger 2}\rho_2'-a^2\rho_2]-\frac{\gamma_p}{2}\rho_1, \tag{16}$$

$$\dot{\rho}_2=\mathscr{L}_0[a]\rho_2+\frac{-ig}{2}[a^2\rho_1^\dagger-\rho_1a^{\dagger 2}]-\gamma_p\rho_2, \tag{17}$$

$$\dot{\rho}_2'=\mathscr{L}_0[a]\rho_2'+\frac{-ig}{2}[\sqrt{2}a^2\rho_1]-\gamma_p\rho_2'. \tag{18}$$

The approximation that the time scale imposed by γ_p is greater than the time scale of evolution of the a mode results in the relations $\gamma_p \gg \langle \mathscr{L}_0[a] \rangle$ and $\gamma_p \gg g\langle a^2\rangle$. Then we make the assumption that both $\dot{\rho}_1=0$ and $\dot{\rho}'_2=0$ so that by using the second and fourth of Eq. (18) the values of ρ_1 and ρ'_2 are found to be

$$\rho_1=\frac{-ig}{\gamma_p}[a^2\rho_0-a^2\rho_2],$$

$$\rho'_2=\frac{ig}{\sqrt{2}\gamma_p}a^2\rho_1. \tag{19}$$

These are then substituted into the first and third equations of set (18) which become

$$\dot{\rho}_0=\mathscr{L}_0[a]\rho_0-\frac{g^2}{2\gamma_p}[a^{\dagger 2}a^2\rho_0+\rho_0a^{\dagger 2}a^2-a^{\dagger 2}a^2\rho_2 -\rho_2a^{\dagger 2}a^2]+\gamma_p\rho_2,$$

$$\dot{\rho}_2=\mathscr{L}_0[a]\rho_2+\frac{g^2}{\gamma_p}[a^2\rho_0a^{\dagger 2}-a^2\rho_2a^{\dagger 2}]-\gamma_p\rho_2. \tag{20}$$

Adding these two equations together and neglecting the ρ_2 terms give for the final master equation of the system

$$\dot{\rho}=\mathscr{L}_0[a]\rho+\frac{g^2}{\gamma_p}\mathscr{D}[a^2]\rho. \tag{21}$$

The coefficient g^2/γ_p describes the strength of the two-photon damping by which the system loses energy by up-conversion of photons from the a mode into the b mode, from which they are linearly damped. The dependence on a^2 is because the photons are lost in pairs. The dependence on g is because as the coupling increases between the modes more photons will be lost this way. Having γ_p as the denominator may seem unusual. Increasing this might be expected to increase the damping out of the system mode while decreasing it would lead to the photons remaining in the pump mode and hence being more likely to be coherently returned to the system mode. This argument is flawed, however: the state of the pump mode is not calculated and so a photon is lost as soon as it enters the pump mode, not when it leaves it. This constant will be called γ_2.

The term $\mathscr{L}_0[a]\rho$ is expanded to give the master equation as

$$\dot{\rho}=-i\left[\frac{\hbar\chi}{2}(a^\dagger a)^2,\rho\right]-i\left[\frac{\kappa(1+\epsilon\cos\Omega t)}{2}(a^{\dagger 2}+a^2),\rho\right] +\gamma\mathscr{D}[a]\rho+\gamma_2\mathscr{D}[a^2]\rho. \tag{22}$$

It is this master equation and the Hamiltonian

$$\hat{H}=\frac{\hbar\chi}{2}(a^\dagger a)^2-\frac{\hbar\kappa(1+\epsilon\cos\Omega t)}{2}(a^2+a^{\dagger 2}) \tag{23}$$

which describes its undamped evolution which is the model principally discussed in this paper.

III. CLASSICAL BEHAVIOR

When subjected to linear damping, the classical equations of motion, Eq. (3), acquire extra terms corresponding to the linear damping terms. These can be calculated from Eq. (22) as follows. The equations of motion for the mean position and momentum are calculated using Eq. (22). We then factorize all higher order moments, and replace averages by a corresponding position and momentum variable. The resulting classical equations of motion are

$$\dot{x}=\chi(x^2+p^2)p+\kappa p-\frac{\gamma}{2}x,$$

$$\dot{p}=-\chi(x^2+p^2)x+\kappa x-\frac{\gamma}{2}p. \tag{24}$$

For small damping and small parametric gain (κ), the fixed points become attractors and shift position slightly. Their position is computed analytically by setting $\dot{x}$ and $\dot{p}$ equal to zero in Eqs. (24). Then the distance of the fixed points from the origin, r, is found to be

$$r^2=\frac{\pm 2\sqrt{(\kappa^2-\gamma^2)}}{\chi} \tag{25}$$

and the angle made by the fixed points with the x axis, θ, is

$$\tan\theta=\frac{\gamma}{(\chi r^2+\kappa)}. \tag{26}$$

When the classical system is driven with sufficient strength the attracting fixed points lose stability and a strange attractor forms.

Two-state approximation

We now consider a two-state approximation in which it is assumed that only elements of the subspace consisting of the two lowest level energy eigenstates have nonzero amplitude. All the other states are then ignored in calculating the evolution of the system. Furthermore, numerical studies indicate that the two lowest states are approximated as the sum of the two coherent states centered at the classical fixed points which are a distance $\alpha=\sqrt{\kappa/\chi}$ from the origin (α is real). These coherent states will be written as $|\pm\rangle$. Thus,

$$|\pm\rangle=\frac{1}{\sqrt{2}}(|\omega_0\rangle\pm|\omega_1\rangle). \tag{27}$$

The effect of the annihilation operator in this subspace is particularly simple. Inverting Eq. (27) we see that

$$a|\omega_n\rangle=\alpha|\omega_{\bar{n}}\rangle, \tag{28}$$

where $n\in\{0,1\}$ and $\bar{n}$ is the complement of n. The effect of the annihilation operator is more like that of a spin operator, "flipping" the two-states, but unlike the spin operator a^2 is α^2, not zero.

The zero-temperature master equation Eq. (22) can be written in this two-state basis as

$$\dot{\rho}_{nm}=-i(\omega_n\rho_{nm}-\rho_{nm}\omega_m)+\gamma\sum_{rs}\langle\omega_n|a|\omega_r\rangle\rho_{rs}\langle\omega_s|a^\dagger|\omega_m\rangle-\frac{\gamma}{2}\left(\sum_r \alpha\langle\omega_{\bar{n}}|a|\omega_r\rangle\rho_{rm}+\sum_r \alpha\rho_{nr}\langle\omega_r|a^\dagger|\omega_{\bar{m}}\rangle\right)$$
$$+\gamma_2\sum_{rs}\langle\omega_n|a^2|\omega_r\rangle\rho_{rs}\langle\omega_s|a^{\dagger 2}|\omega_m\rangle-\frac{\gamma_2}{2}\left(\sum_r \alpha^2\langle\omega_n|a^2|\omega_r\rangle\rho_{rm}+\sum_r \alpha^2\rho_{nr}\langle\omega_r|a^{\dagger 2}|\omega_m\rangle\right), \tag{29}$$

where $\rho_{nm}=\langle\omega_n|\rho|\omega_m\rangle$. When one or both of the states in ρ_{nm} comes from outside the two-state subspace $\rho_{nm}=0$ in line with our assumptions. Using this and simplifying gives

$$\dot{\rho}_{nm}=-i(\omega_n-\omega_m)\rho_{nm}+\gamma\alpha^2\rho_{\bar{n}\bar{m}}-\gamma\alpha^2\rho_{nm}\,. \tag{30}$$

The terms coming from the two-photon damping terms cancel completely as they depend on a^2 and since they conserve parity will produce no coupling between the two-states in our approximation. Because of this two-photon damping will be ignored for the rest of this section. We take the initial state to be a coherent state centered on the positive fixed point $|\psi(0)\rangle=|+\rangle$.

The solutions for that case are

$$\rho_{00}(t)=\rho_{11}(t)=\tfrac{1}{2}, \tag{31}$$

$$\rho_{10}(t)=\frac{1}{2(\lambda_- - \lambda_+)}[(\lambda_- + i\Delta)e^{\lambda_+ t}-(\lambda_- + i\Delta)e^{\lambda_- t}], \tag{32}$$

where

$$\lambda_\pm=-\gamma\alpha^2\pm\sqrt{\gamma^2\alpha^4-\Delta^2} \tag{33}$$

and $\Delta=\omega_1-\omega_0$ is the tunnel frequency. The steady state solution is

$$\rho=\tfrac{1}{2}(|+\rangle\langle+|+|-\rangle\langle-|). \tag{34}$$

So the long term effect of the dissipation operator is to cause the density matrix to become a classical mixture of the localized states.

The tunnel splitting frequency is a rapidly decaying function of the fixed point separation which, for α sufficiently large, can be approximated by [1]

$$\Delta=\chi e^{-2\alpha^2}. \tag{35}$$

Thus the system of equations Eqs. (29) is very much overdamped when the separation of the fixed points is large, i.e., $\Delta\ll\gamma\alpha^2$. Within this approximation, the solution for an initial state $|\psi(0)\rangle=|+\rangle\langle+|$ is

$$\rho_{00}=\rho_{11}=\tfrac{1}{2}, \tag{36}$$

$$\rho_{01}=\rho_{10}^{*}=\tfrac{1}{2}\exp\left(-\frac{\Delta^2 t}{2\gamma\alpha^2}\right). \tag{37}$$

Note that the decay of coherence between the two lowest energy states depends on the square of the energy separation and is thus quite slow.

On the other hand, suppose the system starts in one of the energy eigenstates, say $|\omega_0\rangle$. This is in fact a superposition of the two localized states at the fixed points, which for α large can be a macroscopic superposition. If we now consider the rate of coherence decay between the localized states, that is, the matrix element $\rho_{+-}=\langle+|\rho|-\rangle$, we find

$$\rho_{+-}\approx-\tfrac{1}{2}e^{-2\gamma\alpha^2 t}. \tag{38}$$

For macroscopically distinct states this is very fast and indicates that an initially nonlocalized state such as $|\omega_0\rangle$ will rapidly become a mixture of states localized on the two fixed points. Note that the rate of decay of coherence between the localized states depends on the square of the separation of these states. This is the expected rate for coherence decay between macroscopically distinct states for Markov open systems [12,13].

This last point suggests that the master equation in the two-state approximation, Eq. (29), may be conveniently written in terms of the (very) coarse-grained position operator

$$q=|+\rangle\langle+|-|-\rangle\langle-|. \tag{39}$$

The eigenvalues of this operator are ± 1, with the plus (negative) sign indicating the system has been found at the positive (negative) fixed point. It is easy to verify that Eq. (29) is equivalent to

$$\frac{d\rho}{dt}=-i[H_0,\rho]-\Gamma[q,[q,\rho]] \tag{40}$$

where $H_0=\omega_0|\omega_0\rangle\langle\omega_0|+\omega_1|\omega_1\rangle\langle\omega_1|$ and

$$\Gamma=\frac{\gamma\alpha^2}{2}\,. \tag{41}$$

This equation is in the form which would result for a QND measurement of the approximate position operator q [12].

The master equation can also be solved numerically and thus the validity of the two-state approximation can be examined. To solve Eq. (22) numerically we use the number basis to write the master equation as an ordinary differential equation for the matrix element $\langle n|\rho|m\rangle$. The basis was truncated at $n=40$ and the equation was solved numerically using a Runge-Kutta method. We set the initial state of the system to be a coherent state centered at the positive classical fixed point. To compare the approximate and the numerical results, we compute the probability $Q(t)$ to find the system at the initial coherent state $|+\rangle$. For two-photon damping the two-state approximation predicts that this value should remain unchanged when the damping is changed. That this is approximately true is illustrated in Fig. 1. For trajectory (d) the two-photon damping parameter γ_2 has a value large enough for the two-state approximation to break down. In this case not only does the system spread out into the two

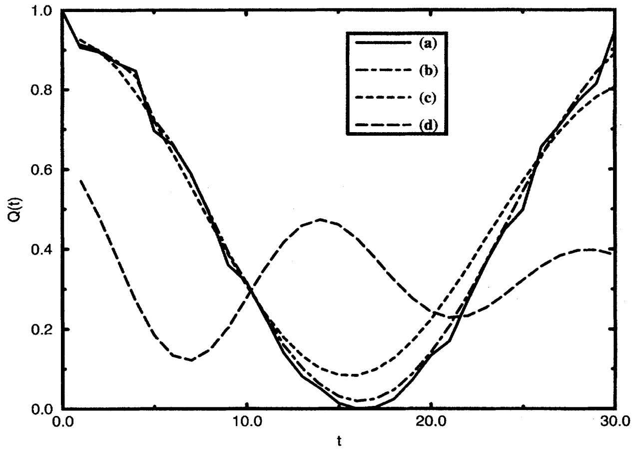

FIG. 1. Comparison of the Q function $[\langle + | \rho(t) | + \rangle]$ versus time for the system with no linear damping and two-photon damping of (a) $\gamma_2 = 0.0$, (b) $\gamma_2 = 0.04$, (c) $\gamma_2 = 0.4$, and (d) $\gamma_2 = 4.0$. In all cases $\epsilon = 0.0$, $\kappa = 12$, and $\chi = 4$.

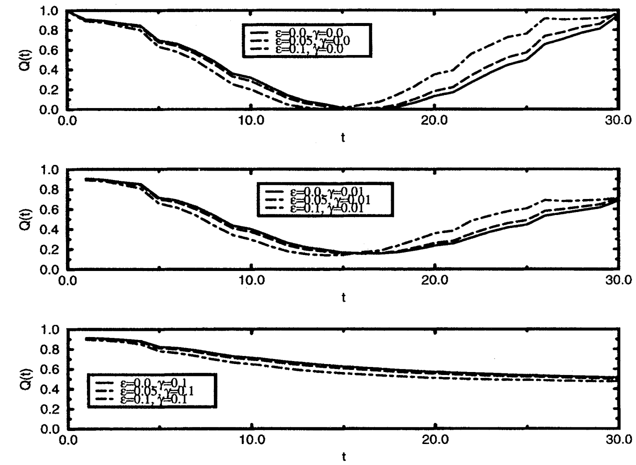

FIG. 3. The Q function $[\langle + | \rho(t) | + \rangle]$ as a function of time, with a driving frequency of $\Omega = 70$, and values of $\gamma = 0.0, 0.05, 0.1$ and $\epsilon = 0.0, 0.05, 0.1$.

wells but the frequency of the tunneling decreases. In the case of linear damping, where the two-state approximation does predict an effect, the Q function is given by $Q(t) = \rho_{++}$ (which is just the Q function evaluated at the initial coherent state). The two-state approximation, for an initial coherent state at the positive fixed point, gives

$$Q(t) = 1 + \tfrac{1}{2}[\rho_{01}(t) + \rho_{10}(t)], \tag{42}$$

where ρ_{01} is given by Eq. (37). A comparison of the approximate result, Eq. (42), and the numerical result is presented in Fig. 2, for the case $\kappa = 12$, $\chi = 4$. In this case $\alpha^2 = 3$, which is not sufficiently large to use the approximate tunnel frequency, so in Fig. 2 we use the numerically determined energy separation between the two tunneling states to obtain Δ. The agreement is quite good.

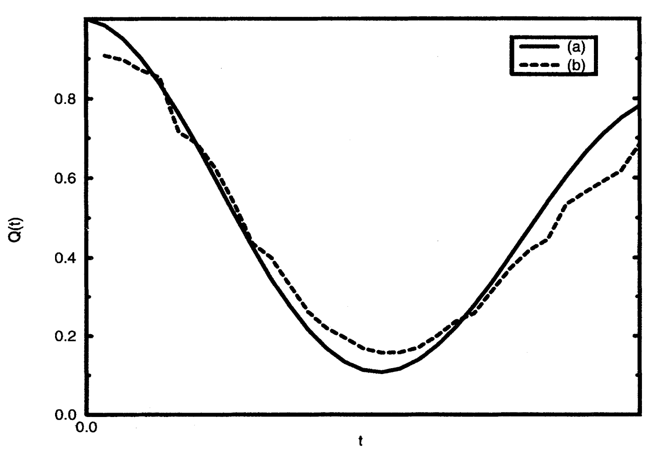

FIG. 2. Comparison of the Q functions $[\langle + | \rho(t) | + \rangle]$ versus time of the system (a) from the exact two-state solution and (b) from the numerical simulation. In both cases $\gamma = 0.01$, $\epsilon = 0.0$, $\kappa = 12$, and $\chi = 4$.

So, to summarize, by approximating the complete system as consisting of just two-states, which are approximated as even and odd superpositions of coherent states centered on the classical fixed points, the behavior of the system under damping can be modeled to a fair degree of accuracy. In particular, the effect of the damping operator on the system is identical to that of the operator describing a QND measurement.

The numerical solution can also be used to determine the effect of perturbation on the tunneling as it is reflected in the dynamics of the Q function defined above. In Fig. 3 we plot the Q function of the system as a function of time, with various choices for the damping and perturbation. The parameters used are $\kappa = 12, \chi = 4$ and the driving frequency is 70. For no damping, increasing the size of the perturbation is seen to increase the tunneling frequency, i.e., the tunneling is enhanced for this choice of parameters. As the damping is increased, the tunneling is destroyed and the system relaxes to an incoherent mixture at the fixed points. Increasing the perturbation then has very little effect.

IV. QUANTUM TRAJECTORIES

In this section we give a brief description of the method of quantum trajectories and point out the connection to measurement. A more rigourous introduction will be found in Ref. [15]. The method of quantum trajectories is at least a method to solve master equations by stochastic simulation, but it is more than that. A master equation describes an open quantum system, that is, a system which irreversibly loses information to its environment. This information can often be extracted by suitable measurements made directly on the environment. In general, the measurement produces a sequence of results with both a systematic and a stochastic component. The stochastic component can be viewed as noise added by the measurement apparatus; however, in a quantum world, it cannot be eliminated. As information about the state of the system is extracted from the environment, we must update our description of the system state. A quantum trajectory represents the evolution of the conditional state of the system, conditioned on the entire stochastic record of a particular sequence of measurements. Each realization of a sequence of measurement results will give a different conditional state. In special cases it may be possible to write down a stochastic Schrödinger equation describing the quantum trajectory [15].

If an ensemble is now formed from all possible conditional states corresponding to one particular kind of measurement, the state of that ensemble at the end of the measurement is simply the solution to the master equation for the source up to the time of the last readout. There may be many types of measurements that can be made on the environment, each with a corresponding class of quantum trajectories, but all the ensembles that these trajectories define are described by the same state; viz., the solution to the source master equation up to the time of the last measurement result.

In this paper the open system is the cavity field. It can be open in two ways: though the end mirrors (linear damping) and through parametric up-conversion to the parametric pump field (two-photon damping). We consider only the first of these cases. Light leaving the cavity through an end mirror may then be subject to direct photodetection or to homodyne-heterodyne detection. In this paper we will only consider the latter. In this case the measurement record is the photocurrent; the stochastic component of this signal is well approximated by classical white noise. This record is all that an experimentalist has access to. The state of the field in the cavity can only be deduced by suitable processing of this classical stochastic process. Our objective is to find evidence, in this photocurrent, of coherent tunneling between the fixed points of the Cassinian oscillator. In addition the quantum-trajectory method can be used to construct arbitrary source operator expectation values, by ensemble averaging the conditional expectation values of each quantum trajectory.

In this approach the system, if it starts off in a pure state, remains in a pure state, which then evolves according to a stochastic Schrödinger equation. In the case of simple photodetection the system either evolves, in any time interval, by making a jump corresponding to the detection of a photon (in which case the state vector is modified by multiplying it by the annihilation operator) or freely evolves in a nonunitary manner. In the case considered here the system is subject to a heterodyne measurement scheme, which leads to the state vector of the system following a continuous stochastic differential equation.

The measurement scheme applied to the system is a heterodyne measurement of the light field which escapes through a mirror with damping constant γ. The stochastic Schrödinger equation appropriate for this type of measurement is derived in [9]. It is

$$\frac{d|\tilde{\Psi}_c(t)\rangle}{dt}=\left[-\frac{\gamma}{2}a^\dagger a-\frac{i}{\hbar}H(t)+I_c(t)a\right]|\tilde{\Psi}_c(t)\rangle. \quad (43)$$

The tilde over the wave function indicates that it is not normalized while the subscript c indicates that it is conditioned on the previous stochastic evolution of the system. The stochasticity enters the system by way of the complex

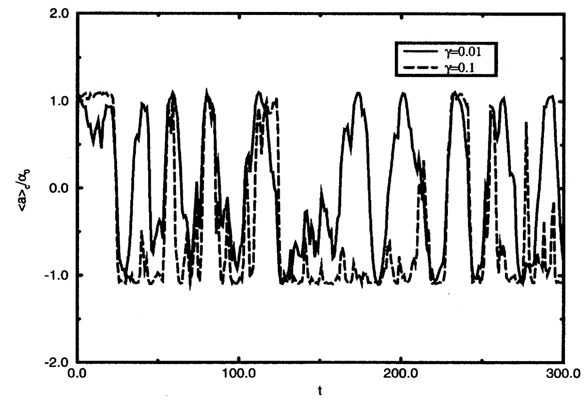

FIG. 4. Values of the conditional mean (normalized to unity) $\langle a\rangle/\alpha_0$ with γ of (a) 0.01 and (b) 0.1.

photocurrent $I_c(t)$ which is related to the mean value of the creation operator in the cavity at any time by

$$I_c(t)=\gamma\langle a^\dagger\rangle_c(t)+\sqrt{\gamma}\xi(t). \quad (44)$$

In this case $\xi_c(t)=dW(t)/dt$ represents Gaussian white noise and $W(t)$ is a Weiner function. The value of $I_c(t)$ simulates the complex heterodyne photocurrent, the real part of which corresponds to the in-phase quadrature, and the imaginary part corresponds to the out-of-phase quadrature.

To numerically solve the resultant differential equation the normal methods, Runge-Kutta for example, are unsatisfactory. A discussion of the various methods of solution of stochastic differential equations can be found in [10]. The stochastic nature of the solution also precludes the use of adaptive algorithms. The method used here is a modified midpoint method. For each step the procedure is as follows.

First a value for $\xi(t)$ over the small time interval dt is calculated by generating a value for $w(t)$ with the appropriate distribution and multiplying by the dt for the interval. Then an estimate of the midpoint of of the step is found by a Euler step. This estimate is then improved by taking another Euler step, using the derivative at the estimated midpoint in the calculation. This procedure is then repeated a number of times, in the present case four, and finally the result at the end of the step is calculated, using the value of the derivative based on the value of the midpoint found. It is at this point that the state vector is normalized.

This method was used to find both trajectories for $\langle a\rangle$ and the in-phase component of $I_c(t)$, which are plotted in Fig. 4 for $\langle a\rangle$ and Fig. 5 for $I_c(t)$. As can be seen from Fig. 3 for a value of γ of 0.01 the tunneling is still present and the only effect of the dissipation is to add a small amount of noise to the system. From Fig. 5 it can be seen that for this value of γ the behavior of the in-phase component of the photocurrent is lost in the noise. As γ is increased to 0.1 the tunneling is suppressed and replaced by random jumping from one of the classical fixed points to the other. This random telegraph effect can also be glimpsed in Fig. 4, however, the tunneling cannot be seen at any stage.

So the conclusion drawn from modeling of the system by quantum trajectories is that, while the tunneling in the sys-

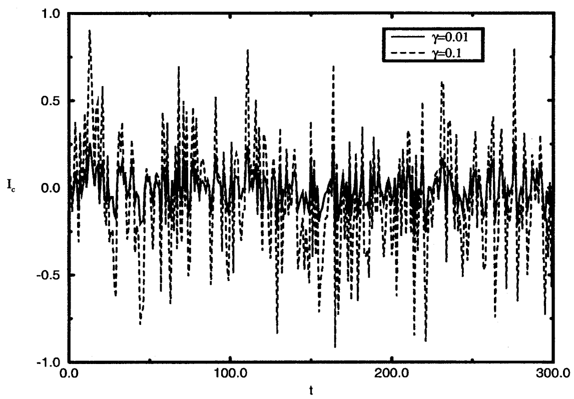

FIG. 5. Values of real part of the conditional heterodyne photocurrent $I_c(t)$ with γ of (a) 0.01 and (b) 0.1.

tem is gradually destroyed for values of the damping from $\gamma=0.01$ to $\gamma=0.1$, the amount of information that can be extracted for small γ is too low for tunneling to be visible. To elucidate this phenomenon we return to the two-state approximation.

If the system starts in a pure state, and we assume unit efficiency for photodetection, the conditional state is always a pure state. Within the two-state approximation this conditional state may be written as a linear combination of the two lowest energy eigenstates. Then Eq. (44) may be written in terms of the approximate position operator q within the two-state approximation as

$$I_c(t)=\alpha\gamma\langle q\rangle_c(t)+\sqrt{\gamma}\xi(t). \tag{45}$$

Rather than work with the stochastic current, we define a new stochastic variable given by the integral of the current over some small time interval ΔT,

$$\Delta N_c(t)=I_c(t)\Delta t=\gamma\alpha\langle q\rangle_c(t)\Delta T+\sqrt{\gamma}\Delta W(t), \tag{46}$$

where $\Delta W(t)$ is the classical Wiener process, with average $E(\Delta W(t))=0$ and $E([\Delta W(t)]^2)=\Delta T$, where we have used E to denote an average over a classical stochastic variable to distinguish it from a quantum average. Thus the mean and variance of the signal are

$$E(\Delta N_c(t))=\alpha\gamma\Delta T\langle q\rangle(t), \tag{47}$$

$$E([\Delta N_c(t)-E(\Delta N_c(t))]^2)=\gamma\Delta T \tag{48}$$

(note that the quantum average appearing here is an ensemble average, not a conditional average). The signal-to-noise ratio, defined as the ratio of the mean signal to the square root of the variance, is

$$S=G\langle q\rangle(t), \tag{49}$$

where the gain is defined by $G=\alpha\sqrt{\gamma\Delta T}$. It might seem a simple matter to improve the gain simply by increasing α; however, from Eq. (38), we see that in the time interval ΔT coherence between the tunneling states is decreased by a factor e^{-G^2}; thus attempting to extract sufficient signal from the noise to see tunneling must necessarily destroy the very phenomenon we seek.

We can see this more clearly as follows. To see tunneling in the first place we require that the integration time T be of the order $T\approx 1/\Delta$. We also need to avoid the overdamped region so that $\gamma\alpha^2/\Delta \ll 1$. Then the coherence decay factor is close to unity but unfortunately the gain $G=\alpha\sqrt{\gamma/\Delta} < 1$. So the signal is lost in the noise.

Observation of tunneling frequency in random telegraph regime

From the above considerations it appears that the coherent tunneling of the system is not directly observable. In the regime where the strength of the measurement is small enough not to destroy the coherent tunneling the signal-to-noise ratio is too small for any useful observation of the tunneling. In particular, the frequency of the tunneling, which had been previously predicted to be changed by the variation of the strength of the parametric driving, cannot be measured. When the measurement strength is increased the signal-to-noise ratio increases and the behavior of the system becomes apparent, but by this stage the coherences which result in the tunneling have been destroyed and the system behaves like a random telegraph process. Because of this it seems that the predicted relationship between the variation of the driving and the tunneling frequency cannot be tested.

In Ref. [11] a system consisting of a particle confined in a double well potential subjected to continuous QND measurement of position was studied. It was found, as above, that for certain values of the parameters the behavior of the particle is that of a random telegraph. The rate of switching in a simplified two-level system is calculated to be, in their symbols,

$$R=\frac{\Delta^2}{32D\Gamma}, \tag{50}$$

where Δ is the tunneling frequency, Γ is the measurement parameter, and D is a parameter describing the position separation of the two wells. In our model these are given by $\Gamma=\gamma\alpha^2/2$ and $D=\alpha^2$; thus we expect

$$R\propto\frac{\Delta^2}{\gamma\alpha^4}. \tag{51}$$

While the exact relationship given above cannot be expected to persist in the more complicated situation of this paper, nevertheless, since many of the important features of the system in [11] exist, such as the effective QND nature of the measurements, one expects that the variation of switching frequency R with Ω at fixed ϵ might be related to the variation of tunneling frequency with Ω. This is what is found.

The quantity chosen to measure the rate of switching of the random telegraph, called R in the following, is simply the number of crossings of the zero axis by the photocurrent, divided by the time. In the case of a noiseless photocurrent this would give an estimate of the switching rate limited only by the sample time; however, the addition of noise increases

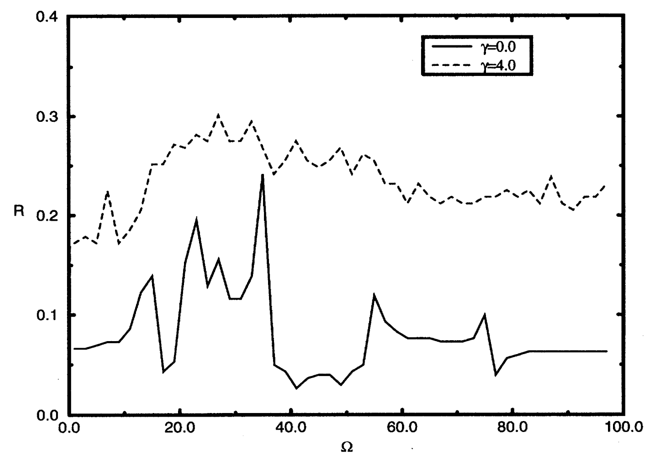

FIG. 6. The variation of the switching rate R with $\epsilon=0.1$ as a function of Ω. (a) $\gamma=0.0$ and (b) $\gamma=4.0$.

the rate as measured by this method by a fixed amount. Since this increase is fixed, and it is the change in switching rate which signifies the change in tunneling rate, the constant increase in measured rate due to noise can be easily subtracted. Both this additional noise and the switching rate are stochastic with a Poisson distribution. The lengths of the trajectories were chosen such that the fluctuations in R due to the stochastic nature of the trajectories would not be significant.

In the case where $\gamma=0.0$ the photocurrent is uniformly zero. To get a measure of the internal behavior of the cavity the value calculated for R when using $\langle a^\dagger \rangle$ in place of the photocurrent is used. As can be seen from Eq. (44), $\langle a^\dagger \rangle$ is similar to a noiseless photocurrent, except it is not scaled by γ. The values of R plotted for $\gamma=4.0$ also use $\langle a^\dagger \rangle$ so that a true comparison of the behavior of R can be done.

In Fig. 6 R is plotted out with $\epsilon=0.1$ and Ω varied from 0 to 100 for two curves with $\gamma=0.0$ and $\gamma=4.0$. The value of 4.0 is chosen because at that point γ has increased sufficiently for the telegraph switching to be seen in the photocurrent. In the $\gamma=0.0$ curve the enhancement and suppression of tunneling can readily be seen along with the resonances. In the $\gamma=4.0$ curve, however, only the magnitude of the frequency roughly follows the same pattern. Both the resonances and enhancement and suppression are gone.

This behavior appears intuitively obvious. The resonances and the switching between the enhancement and suppressing of tunneling are due to the system's coherences with higher energy eigenstates. One of the principle effects of dissipation will be to destroy these coherences. However, putting this argument into a mathematical format has proved more difficult and requires further research.

V. CONCLUSION

It is well known that dissipation tends to destroy coherent quantum tunneling. In this paper we have discussed the effect of dissipation on tunneling between two elliptic fixed points in a perturbed Cassinian oscillator. This may be realized as a nonlinear optical system with an intensity dependent refractive index and subject to parametric amplification. Tunneling then occurs between amplitudes π out of phase. The perturbation corresponds to periodic modulation of the parametric pump amplitude. The effect of damping is then modeled using a zero-temperature master equation to describe the system with additional terms which describe the two-photon damping effects. Solving this master equation we find the usual result; linear damping destroys coherent tunneling between the fixed points, and it is replaced by random telegraph switching. The two-photon damping, although it eventually also destroys the coherent oscillations, only does so at very high strengths compared to the linear effects.

However, the coupling to the external world responsible for damping is also the means by which the system is observed. Using the method of quantum trajectories we describe heterodyne detection of the output field from the cavity. We have shown that no evidence of coherent tunneling can be seen in the heterodyne current, when the signal-to-noise ratio is significantly greater than 1. This is a general feature of continuous measurement; the coherences responsible for tunneling are an exponentially decreasing function of the signal-to-noise ratio of the measurement that would enable tunneling to be seen. This result complements the standard result of coherence destruction in the master equation perspective. By numerical calculation of the density operator the changeover between the low damping and high damping behavior is found to occur for values of γ between 0.01 and 0.1. However, values of γ in excess of 1 are required to obtain a high enough signal-to-noise ratio to observe the random telegraphing by heterodyne measurement. The effect of the perturbation can still be seen in the switching rate of the random telegraph which replaces coherent tunneling when damping is included.

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