Quantum nondemolition measurements via quantum counting

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For a harmonic-oscillator gravitational-wave detector, we show that a quantum nondemolition measurement of the square of the number operator may be made by coupling the detector to an oscillator readout via a quadratic interaction, as in optical four-wave mixing. Explicit evaluation of the effect of a meter readout on the detector demonstrates the possibility of arbitrarily accurate instantaneous measurements for sufficiently large coupling strength.

Recently a class of measurement schemes intended to determine the effect of a gravitational wave on a large mass have required such accuracy of resolution that the detector must be described entirely within a quantum-mechanical framework.[1–5]

In these schemes one wishes to make a sequence of measurements on a single system the results of which must be entirely predictable in the absence of the gravitational wave. However, it is possible that if we were working at a level where the quantum nature of the detector was manifest, such a sequence of measurements would not generally lead to a determinate sequence of results in the absence of the gravitational wave. However, a quantum-mechanical description of the measurement process does not preclude such a determinate sequence if the measurement scheme is chosen appropriately. Such measurement schemes have come to be known as "quantum nondemolition" (QND) measurements.

In a QND measurement scheme one must choose carefully the observable to be measured, the so-called QND variable of the detector. Furthermore, the coupling of the detector to any subsequent readout stage must be quite specific; it must be "back-action evading."

Caves[5] has given a precise prescription for determining the QND variables and a back-action-evading readout-meter coupling. Specifically, for an operator $\hat{A}(t)$ to correspond to a QND variable we require that

$$[\hat{A}(t),\hat{A}(t')]=0\ , \tag{1}$$

which means that the uncertainties in $\hat{A}$ are isolated from the uncertainties in variables it does not commute with.

For a measurement scheme to be back-action evading we require that the QND observable be the only detector observable to appear in the detector-meter interaction Hamiltonian. This ensures that the QND observable remains isolated from observables it does not commute with. The back-action-evading criteria may be written as

$$[\hat{A},H_I]=0\ , \tag{2}$$

where H_I is the detector-meter interaction Hamiltonian.

When conditions (1) and (2) are met we can make a sequence of measurements of $\hat{A}$, the results of which can be predicted with certainty given an initial sufficiently precise measurement. Such a sequence constitutes a QND measurement of $\hat{A}$.

An analysis of a QND measurement process may be divided into two stages. The first stage involves solving for the time-dependent unitary evolution of the coupled detector-meter system. During this stage correlations between the state of detector and meter build up. At some point the free evolution is suspended and a readout of the meter is made, whereupon the meter state is reduced. The second stage of the analysis then involves a determination of the nonunitary effect of meter-state reduction upon the detector.

Quantum counting formed the basis of one of the earliest QND measurement proposals.[2] Unruh[3] pointed out that this would require a quadratic coupling to the oscillator coordinate. Such a coupling would give rise to an interaction Hamiltonian that commutes with the detector number operator. Unruh[4] has suggested such a coupling scheme using an LC circuit model. We have previously considered linear-coupling schemes between harmonic oscillators[6] (see also Hillery and Scully[7]). In this paper we consider a quadratic coupling scheme based on a quantum-optical four-wave-mixing interaction. An analysis of the nonunitary effect of the readout system for this nonlinear coupling scheme is presented.

We consider two harmonic oscillators coupled via the Hamiltonian

$$H=\hbar\omega_a a^\dagger a+\hbar\omega_b b^\dagger b+\hbar\chi' a^\dagger a[b\epsilon(t)+b^\dagger\epsilon^*(t)]\ , \tag{3}$$

where a and b obey boson commutation relations and represent the detector and meter, respectively. Such a Hamiltonian could represent two electromagnetic field modes coupled by a third-order susceptibility as in a four-wave-mixing process[8] with one mode in a highly populated coherent excitation and treated classically. Alternatively it may be viewed as representing two mechanical oscillators with a time-dependent coupling. We shall assume that $\epsilon(t)$ has the form $\epsilon(t)=\epsilon e^{i\omega_b t}$, that is, the classical field is resonant with the ω_b mode.

The obvious choice for the detector QND variable is $a^\dagger a$, which in fact is a constant of the motion, and thus, clearly satisfies Eq. (1). Furthermore, we see that the interaction Hamiltonian is back-action evading.

Solving the Heisenberg equations of motion in the interaction picture we have

$$\hat{N}_b(t)=(\chi t)^2\hat{G}_a+i\chi t\hat{N}_a[b(0)-b^\dagger(0)] +b^\dagger(0)b(0)\ , \tag{4}$$

where

$$\hat{N}_b(t)\equiv b^\dagger(t)b(t)\ ,$$

$$\hat{N}_a\equiv a^\dagger a\ ,$$

$$\hat{G}_a\equiv(a^\dagger a)^2\ ,$$

$$\chi=\chi'\epsilon\ .$$

If we assume the meter is initially in a number state $|n_b\rangle$, perhaps resulting from a previous measurement of $\hat{N}_b(t)$, we have

$$\langle\hat{N}_b(t)\rangle=(\chi t)^2\langle\hat{G}_a\rangle+n_b(0)\ . \tag{5}$$

This equation indicates that the most natural detector observable to attempt to measure is $\hat{G}_a$ rather than $\hat{N}_a$. Clearly, $\hat{G}_a$ is also a QND variable. From a measurement of $\hat{N}_b(t)$ at time t with result $n_b(t)$ we may infer a value g_a for $\hat{G}_a$ given by

$$g_a=[n_b(t)-n_b(0)]/(\chi t)^2\ . \tag{6}$$

The possible error in this inferred value is given by Δg_a, where $\Delta g_a=\Delta n_b(t)/(\chi t)^2$ and $\Delta n_b(t)=\text{var}(\hat{N}_b(t))$, where $\text{var}(\hat{N}_b(t))$ is the variance in $\hat{N}_b(t)$.

For the assumed initial meter state $|n_b(0)\rangle$ we find

$$(\Delta g_a)^2=\text{var}(\hat{G}_a)+\frac{2\langle\hat{G}_a\rangle}{(\chi t)^2}(n_b+\tfrac{1}{2})\ . \tag{7}$$

Equation (7) indicates that we cannot infer with certainty, a value for $\hat{G}_a$ by measurement of $\hat{N}_b(t)$ even if the detector happens to be an eigenstate of $\hat{G}_a$ [i.e., $\text{var}(\hat{G}_a)=0$] prior to the measurement. However, by ensuring that χt is sufficiently large the inference may be made with arbitrary certainty. This is the usual limit for arbitrarily accurate instantaneous quantum measurements.[1]

So far we have considered only the reversible unitary evolution of the coupled detector-meter system. However, if we are to be able to consider a sequence of measurements we must now consider the nonunitary change of the detector upon readout of the meter variable. In particular, we wish to calculate the distribution of $\hat{G}_a$ after a measurement has taken place. Clearly this is identical to the number distribution of the detector after a readout of the meter.

If $\rho(t)$ is the density operator of the coupled detector-meter system, the density operator for the total system after readout $\rho(t)$ is given by[9]

$$\bar{\rho}(t)=\mathcal{N}\hat{P}_{\hat{N}_b}(n_b(t))\rho(t)\hat{P}_{\hat{N}_b}(n_b(t))\ , \tag{8}$$

where $\hat{P}_{\hat{N}_b}(n_b(t))$ is a projector onto the one-dimensional subspace spanned by $|n_b(t)\rangle$ and $\mathcal{N}^{-1}=\text{Tr}[\rho(t)\hat{P}_{\hat{N}_b}(n_b(t))]$.

The state of the detector after readout is then obtained by tracing out over meter variables

$$\bar{\rho}_D(t)=\text{Tr}_M[\bar{\rho}(t)]\ . \tag{9}$$

The unitary time-evolution operator for the coupled system in the interaction picture is

$$U(t,0)=\exp\{\hat{N}_a[\xi(t)b^\dagger-\xi^*(t)b]\}\ , \tag{10}$$

where $\xi(t)=-i\chi t$.

We assume the initial state of the system is given by $\rho(0)=|\psi\rangle|n_b\rangle\langle n_b|\langle\psi|$, where ψ refers to the state of the detector and n_b refers to the initial state of the meter.

The initial number distribution of the detector is $P(n_a)=|\langle n_a|\psi\rangle|^2$ while after readout it is given by $\bar{P}(n_a)=\langle n_a|\bar{\rho}_a(t)|n_a\rangle$. Using Eq. (10) we find

$$\bar{P}(n_a)=\mathcal{N}|\langle n_b(t)|\exp\{n_a[\xi(t)b^\dagger-\xi^*(t)b]\}|n_b\rangle|^2 \times P(n_a)\ . \tag{11}$$

Using a result [Eq. (2.26)] of Ref. 1, Eq. (11) becomes

$$\bar{P}(n_a)=\mathcal{N}\frac{N!}{M!}[L_N^K(x)]^2x^ke^{-x}P(n_a)\ , \tag{12}$$

where

$$M=n_b(t),\ \ N=n_b(0),$$

$$K=M-N=g_a(\chi t)^2,\ \ x=(n_a\chi t)^2\ ,$$

and $L_N^K(x)$ is the generalized Laguerre polynomial.

The function prefacing $P(n_a)$ in Eq. (12) in general has multiple roots, indicating that $\bar{P}(n_a)$ is multipeaked. However, as $(\chi t)^2$ becomes large these roots converge to one root at $x=K$, that is, $n_a=\sqrt{g_a}$. Then $\bar{P}(n_a)$ is peaked around $n_a=\sqrt{g_a}$ and becomes sharper as χt is increased.

Providing we choose χt sufficiently large the detector after readout is in the eigenstate $|\sqrt{g_a}\rangle$ of $\hat{G}_a$, with eigenvalue equal to the measured result. We showed previously [Eq. (7)] that an accurate determination of $\hat{G}_a$ may be made in the limit $\chi t\to\infty$. We now see that such an arbitrarily accurate measurement of $\hat{G}_a$ leaves the detector in an eigenstate of $\hat{G}_a$.

In the simplest case we may prepare the state of the meter so that $n_b(0)=0$. $\bar{P}(n_a)$ then has only one peak for all

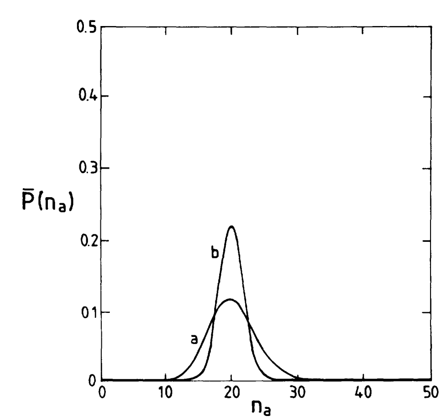

FIG. 1. Detector number distribution $\bar{P}(n_a)$ after readout. Parameters (a) $n_b(t)=4$, $n_b(0)=0$, $\chi t=0.1$, and $\bar{n}_a=20$. (b) $n_b(t)=25$, $n_b(0)=0$, $\chi t=0.25$, and $\bar{n}_a=20$.

values of χt and becomes more sharply peaked as χt increases.

This behavior is evident in Fig. 1 where we have plotted $\bar{P}(n_a)$ vs n_a for two different values of χt. For the values of the parameters chosen $\sqrt{g_a}=20$. We see that as χt increases the postreadout distribution becomes more narrowly concentrated on $n_a=\sqrt{g_a}$.

Since $a^\dagger a$ is a constant of motion the detector once placed in a near eigenstate of $\hat{G}_a$ will remain there. Thus a subsequent measurement of $\hat{G}_a$ will obtain a result g_a, equal to the previous measurement, if χt is large enough, providing no external force has acted. Any departure from this result may be taken as evidence of the presence of an external force.

We have shown that a sequence of determinate results for the measurement of the number operator for a harmonic oscillator may be made by coupling the oscillator to a second readout oscillator, via a four-wave-mixing interaction. Such an interaction is quadratic in the detector variables. Any departure from a determinate sequence of results may be attributed to the presence of an external classical driving force, for example, a gravitational wave acting on the detector.[1]

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