Linear amplifiers with phase-sensitive noise

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We present a model for a linear amplifier which adds phase-dependent noise to the input signal. This is achieved by preparing the internal modes of the amplifier in a squeezed vacuum. Such a scheme could be used to amplify a squeezed-signal quadrature with reduced added noise compared with conventional schemes. The model discussed could be realized as nondegenerate parametric amplification.

Standard models[1–5] of linear amplifiers and attenuators indicate that such devices must add noise to the signal. The noise added arises from the coupling of the signal to the internal degrees of freedom of the amplifier. The nature of the added noise depends on the state of these internal modes. Friberg and Mandel[2] discuss the case of a linear amplifier formed by a group of partly excited two-level atoms. A similar model is presented in Ref. 4. In this case the noise added is essentially thermal in character. In Refs. 1 and 3 a number of results pertaining to amplifiers based on nonlinear optical processes are presented. In these cases the ultimate source of added noise arises from the vacuum fluctuations in the internal amplifier modes. Clearly the added noise in these cases is phase insensitive, that is an equal amount of noise is added to both quadrature phases of the signal. Recently, broadband squeezed states have been produced.[6–8] These states have phase-sensitive noise with the noise in one quadrature reduced below the vacuum level. It thus becomes of interest to ask how one may amplify a squeezed state without adding additional noise to the squeezed quadrature. The problem of the noise added by the internal modes may be approached in two different ways. One may employ a back-action evading scheme as discussed, for example, by Yurke.[9] Experiments using such schemes in optical fibers are presently being conducted by Levenson *et al.*[10] The scheme that we are proposing is to squeeze the internal modes of the amplifier so that less noise is fed into the amplified quadrature of the signal. We shall discuss a model for this amplification process below.

The most direct way to see how such an amplifier would work is to regard it as performing a linear transformation of the input and internal modes.[1,3] In the model considered we choose

$$\hat{X}_\theta^{\text{out}}=G^{1/2}\hat{X}_\theta^{\text{in}}+(G-1)^{1/2}\hat{Y}_\theta \ , \tag{1}$$

$$\hat{X}_{\theta+\pi/2}^{\text{out}}=G^{1/2}\hat{X}_{\theta+\pi/2}^{\text{in}}-(G-1)^{1/2}\hat{Y}_{\theta+\pi/2} \ , \tag{2}$$

where G is the gain and

$$\hat{X}_\theta=ae^{-i\theta}+a^\dagger e^{i\theta} \tag{3}$$

is a quadrature phase for the signal mode with creation and annihilation operators $a^\dagger,a$, while $\hat{Y}_\theta$ represents the corresponding quadrature of the multimode internal field of the amplifier. In a nondegenerate parametric amplification process $\hat{Y}_\theta$ would represent the idler field.[3] Using Eqs. (1) and (2) one may establish that

$$\text{Var}(\hat{X}_\theta^{\text{out}})=G\text{Var}(\hat{X}_\theta^{\text{in}})+(G-1)\text{Var}(\hat{Y}_\theta) \ , \tag{4}$$

$$\text{Var}(\hat{X}_{\theta+\pi/2}^{\text{out}})=G\text{Var}(\hat{X}_{\theta+\pi/2}^{\text{in}})+(G-1)\text{Var}(\hat{Y}_{\theta+\pi/2}) \ , \tag{5}$$

where "Var" indicates the variance. The maximum gain consistent with any squeezing at the output is

$$G_{\max}=\frac{1+\text{Var}(\hat{Y}_\theta)}{Var(\hat{X}_\theta^{\text{in}})+\text{Var}(\hat{Y}_\theta)} \ . \tag{6}$$

If the internal modes are in the vacuum state, $\text{Var}(\hat{Y}_\theta)=1$ and

$$G_{\max}=\frac{2}{1+\text{Var}(\hat{X}_\theta^{\text{in}})} \ , \tag{7}$$

giving a maximum gain of 2 for a highly squeezed input. This result is also found in (4). We suggest that the "idler" field ($\hat{Y}_\theta$) also be prepared in a squeezed state, $\text{Var}(\hat{Y}_\theta)<1$, which allows the amplifier gain to exceed the limit of 2.

A convenient measure of the noise added by the amplifier is given by the signal-to-noise ratio at the output,

$$\frac{\langle \hat{X}_\theta^{\text{out}}\rangle^2}{\text{Var}(\hat{X}_\theta^{\text{out}})}=\frac{\langle \hat{X}_\theta^{\text{in}}\rangle^2}{\text{Var}(\hat{X}_\theta^{\text{in}})+\left[1-\frac{1}{G}\right]\text{Var}(\hat{Y}_\theta)} \ , \tag{8}$$

where we have used $\langle \hat{Y}_\theta\rangle=0$. The second term in the denominator represents the noise added by the amplifier to the $\hat{X}_\theta$ quadrature of the signal. This added noise may be reduced by squeezing the idler field [i.e., reducing $\text{Var}(\hat{Y}_\theta)$ below the vacuum level of one].

The above model obeys a fundamental theorem for the noise added by a linear amplifier. If we define the total noise in the signal quadrature by

$$N=\mathrm{Var}(\hat{X}_\theta)+\mathrm{Var}(\hat{X}_{\theta+\pi/2}) , \tag{9}$$

then

$$N_{\mathrm{out}}=G(N_{\mathrm{in}}+A) , \tag{10}$$

where

$$A=\left[1-\frac{1}{G}\right][\mathrm{Var}(\hat{Y}_\theta)+\mathrm{Var}(\hat{Y}_{\theta+\pi/2})]\geq 2\left[1-\frac{1}{G}\right] . \tag{11}$$

This is equivalent to the fundamental theorem of Caves[1] for the noise added by a linear amplifier.

It is instructive to compare this amplifier model to another model which has received attention recently, the back-action evading amplifier.[9] The quadrature transformations defining this process are

$$\hat{X}_\theta^{\mathrm{out}}=\hat{X}_\theta^{\mathrm{in}} , \tag{12}$$

$$\hat{X}_{\theta+\pi/2}^{\mathrm{out}}=\hat{X}_{\theta+\pi/2}^{\mathrm{in}}+G\hat{Y}_\theta^{\mathrm{in}} , \tag{13}$$

$$\hat{Y}_{\theta+\pi/2}^{\mathrm{out}}=\hat{Y}_{\theta+\pi/2}^{\mathrm{in}}+G\hat{X}_\theta^{\mathrm{in}} . \tag{14}$$

The signal quadrature $\hat{X}_\theta$ is not amplified and thus receives no added noise. In this case one monitors $\hat{Y}_{\theta+\pi/2}^{\mathrm{out}}$ to obtain information about $\hat{X}_\theta^{\mathrm{in}}$. We find that

$$\frac{\langle \hat{Y}_{\theta+\pi/2}^{\mathrm{out}}\rangle^2}{V(\hat{Y}_{\theta+\pi/2}^{\mathrm{out}})}=\frac{\langle \hat{X}_\theta^{\mathrm{in}}\rangle^2}{V(\hat{X}_\theta^{\mathrm{in}})+\dfrac{1}{G^2}V(\hat{Y}_{\theta+\pi/2}^{\mathrm{in}})} . \tag{15}$$

Comparing this with Eq. (8) we see that in this case a squeezed internal quadrature $\hat{Y}^{\mathrm{in}}$ is not necessary. One need only make the gain large.

We now show how these results may be obtained from an explicit model of an amplifier. Consider the interaction Hamiltonian

$$H_1=ha^\dagger\Gamma^\dagger+ha\Gamma . \tag{16}$$

This represents an amplifier model either of the type considered by Glauber[11] where Γ is a bath of inverted harmonic oscillators, or a Raman amplifier considered by Walls.[12] Stenholm[5] has recently used this model of an amplifier to consider the limits to amplification of squeezed states. In terms of our previous treatment, Γ represents the internal boson modes of the amplifier. We allow for the bath to be a squeezed vacuum with the correlation functions[13]

$$\langle \Gamma^\dagger(t)\Gamma(t')\rangle=N\delta(t-t') , \tag{17}$$

$$\langle \Gamma(t)\Gamma(t')\rangle=M\delta(t-t') , \tag{18}$$

where $|M|^2=N(N+1)$ for a minimum uncertainty squeezed state. The master equation for the signal field "a" may then be derived as in Collett and Gardiner:[13]

$$\begin{aligned}\frac{\partial\rho}{\partial t}=&\frac{\gamma}{2}(2a^\dagger\rho a-aa^\dagger\rho-\rho aa^\dagger)\\&+N\gamma(a\rho a^\dagger+a^\dagger\rho a-aa^\dagger\rho-\rho a^\dagger a)\\&-\frac{\gamma}{2}M^*(2a\rho a-aa\rho-\rho aa)\\&-\frac{\gamma}{2}M(2a^\dagger\rho a^\dagger-a^\dagger a^\dagger\rho-\rho a^\dagger a^\dagger) ,\end{aligned} \tag{19}$$

where γ is an amplification constant. This may be written in terms of a Fokker-Planck equation for the Q function, $Q(\alpha)=\langle\alpha|\rho|\alpha\rangle$,[14]

$$\frac{\partial Q}{\partial t}=\left[-\frac{\partial}{\partial x_i}A_i+\frac{1}{2}\frac{\partial^2}{\partial x_i\partial x_j}D_{ij}\right]Q , \tag{20}$$

where $\mathbf{x}=(x_1,x_2)=(\alpha,\alpha^*)$ and

$$A=\begin{bmatrix}-\frac{\gamma}{2} & 0\\ 0 & -\frac{\gamma}{2}\end{bmatrix} , \tag{21}$$

$$D=\gamma\begin{bmatrix}M^* & N\\ N & M\end{bmatrix} . \tag{22}$$

For an initial squeezed signal with complex amplitude α_0, the Q function at time t has the form

$$\begin{aligned}Q(t)=&\{4\pi^2\det\underset{\sim}{\sigma}(t)]\}^{-1/2}\\&\times\exp\left[-\tfrac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\underset{\sim}{\sigma}(t)^{-1}(\mathbf{x}-\boldsymbol{\mu})\right] ,\end{aligned} \tag{23}$$

where

$$\boldsymbol{\mu}=(\alpha_0e^{\gamma t/2},\alpha_0^*e^{\gamma t/2}) , \tag{24}$$

$$\underset{\sim}{\sigma}(t)=e^{-At}\underset{\sim}{\sigma}(0)e^{-At}+\underset{\sim}{\sigma}_1 , \tag{25}$$

$$\underset{\sim}{\sigma}_1=\begin{bmatrix}M^* & N\\ N & M\end{bmatrix}(e^{\gamma t}-1) , \tag{26}$$

and the initial covariance matrix is

$$\underset{\sim}{\sigma}(0)=\frac{1}{2}\begin{bmatrix}-\sinh(2r) & \cosh(2r+1)\\ \cosh(2r+1) & -\sinh(2r)\end{bmatrix} , \tag{27}$$

where r is the squeeze parameter.

The variance in the initially squeezed quadrature $\hat{X}_\theta$ is

$$\begin{aligned}\mathrm{Var}(\hat{X}_\theta)&=Ge^{-2r}+(G-1)(2N+2\,\mathrm{Re}M^*+1)\\&=Ge^{-2r}+(G-1)\mathrm{Var}(\hat{Y}_\theta) ,\end{aligned} \tag{28}$$

where $\hat{Y}_\theta$ is quadrature of the bath given by

$$\hat{Y}_\theta=\Gamma e^{-i\theta}+\Gamma^\dagger e^{+i\theta} \text{ and } G=e^{\gamma t} .$$

This reproduces the result of Eq. (4).

A linear attenuator may be described in a similar manner. In this case the master equation takes the form

$$\frac{\partial\rho}{dt}=\frac{\gamma}{2}(2a\rho a^\dagger-a^\dagger a\rho-\rho a^\dagger a)$$
$$+N\gamma(a^\dagger\rho a+a\rho a^\dagger-a^\dagger a\rho-\rho a a^\dagger)$$
$$-\frac{\gamma M}{2}(2a^\dagger\rho a^\dagger-a^\dagger a^\dagger\rho-\rho a^\dagger a^\dagger)$$
$$-\frac{\gamma M^*}{2}(2a\rho a-aa\rho-\rho aa)\ . \qquad (29)$$

Proceeding as before we find the variance of an initially squeezed quadrature to be given by

$$\mathrm{Var}(\hat{X}_\theta)=e^{-2r}e^{-\gamma t}+(1-e^{-\gamma t})\mathrm{Var}(\hat{Y}_\theta)\ . \qquad (30)$$

For long times $\mathrm{Var}(\hat{X}_\theta)\rightarrow\mathrm{Var}(\hat{Y}_\theta)$ which may be less than that of the vacuum if the bath modes are squeezed. This generalizes previous results for a squeezed state interacting with a thermal state.[11,15]

In this paper we have discussed a model in which one quadrature phase of a signal may be amplified with a reduction in the added noise. The scheme requires that the noise added by the amplifier be phase sensitive. This may be achieved by squeezing the internal modes of the amplifier. See *Note added in proof.* One particular realization of this scheme could be achieved in a parametric amplifier with a squeezed signal input and idler in a squeezed vacuum. Now that squeezed light with a 50% reduction in fluctuations has been generated in a parametric oscillator[8] this experiment becomes feasible.

Note added in proof. The advantage of squeezing the idler modes in a parametric amplifier had previously been noted by B. Yurke and J. S. Denker, Phys. Rev. A **29**, 1419 (1984).

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