On the Motion of a Uniformly Heated Drop in a Viscous Liquid under Gravity

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Abstract—The motion of a uniformly heated spherical drop under gravity is theoretically studied within the Stokes approximation. The Stokes and Hadamard–Rybchinsky formulas are generalized so that the temperature dependence of the viscosity can be found in a wide temperature range. Also, the drag force and the velocity of gravity fall are calculated for an arbitrary temperature difference between the surface of the drop and distant points.

STATEMENT OF THE PROBLEM

We consider the motion of a uniformly heated hydrosol drop (particle) with a surface temperature T_s in a viscous incompressible liquid under gravity. The liquid occupies the entire space, does not mix with the drop, and is at rest at infinity. A particle is considered to be heated (cooled) if its surface temperature differs from the temperature far away from it. Uniform heating can be associated with heat liberation during chemical reactions on its surface, the radioactive decay of the material, external effects, etc. If, for example, the particle is subjected to a monochromatic radiation of wavelength λ_0 and intensity I_0, it absorbs energy $\pi R^2 I_0 K_n$ (where R is the radius of the drop and K_n is the absorption factor [1, 2]), which is uniformly distributed over its volume. This statement is valid if the thermal conductivity of the drop is much higher than that of the environment and $\lambda_0 \gg R$. All the processes in the drop-liquid medium system are quasi-stationary, because the thermal relaxation time of the system is small.

The heated surface of the drop influences the thermal physical characteristics of the surrounding liquid and, eventually, the velocity and pressure fields in its neighborhood.

Unlike [3–6], the author generalizes the Stokes and Hadamard–Rybchinsky formulas for the case of a uniformly heated spherical drop steadily moving in a viscous incompressible liquid. The temperature difference between the surface of the drop and distant sites, as well as the temperature dependence of the viscosity of the liquid, is assumed to be arbitrary.

Among all the parameters of liquid transport, the viscosity depends on temperature to the greatest extent, exponentially decreasing with growing temperature [7, 8]. The review of the available semi-empirical formulas and experimental data shows that the temperature dependence of the liquid viscosity μ in a wide temperature range and with any desired accuracy can be described by the formula

$$\mu_e = \mu_\infty\left[1 + \sum_{n=1}^{\infty} F_n\left(\frac{T_e}{T_\infty} - 1\right)^n\right]\exp\left\{-A\left(\frac{T_e}{T_\infty} - 1\right)\right\}, \quad (1)$$

where A and F_n are constants, $\mu_\infty = \mu_e(T_\infty)$, and T_∞ is the liquid temperature far away from the particle (at $F_n = 0$, this formula reduces to the well-known Reynolds expression [7]). Hereafter, the subscripts e and i refer to the viscous liquid and heated particle, respectively; the subscript ∞ designates the parameters of the undisturbed flow at infinity; and the subscript s refers to the parameters taken at the mean surface temperature T_s. For water, $A = 5.779$, $F_1 = 2.318$, and $F_2 = 9.118$ with an accuracy of 2% or higher at temperatures between 273 and 363 K ($T_\infty = 273$ K).

It is assumed that the densities, thermal conductivities, and specific heat capacities of the liquid and the drop are constant. The drop moves slowly (small Reynolds and Peclet numbers) and retains the spherical shape. The latter statement is valid if the surface tension forces at the drop–environment interface far exceed the drag forces, which tend to distort the sphere. Analytically, the shape conservation condition is written as the inequality [9] $\sigma/R \gg \mu_e|\mathbf{U}_e|/R$, where σ is the surface tension coefficient at the drop–environment interface and $\mathbf{U}_e$ is the velocity of the particle. This inequality holds true for most liquids.

It is appropriate to relate the frame of reference to the center of the moving particle (the problem is reduced to the analysis of an infinite parallel flow with a velocity $\mathbf{U}_\infty$ to be determined over the particle). The velocity and temperature distributions are symmetric about the 0z axis, which passes through the center of the particle and has the same direction as the incoming flow velocity. Therefore, we use the spherical coordinate system where the radius r is counted from the center of the drop and the angle Θ, from the incoming flow direction.

In the spherical coordinate system r, Θ, φ with regard for our assumptions, the equations and the boundary conditions for the velocities $\mathbf{U}$, pressures P, and temperatures T_e outside and inside the drop in the Stokes approximation are written in the form [5, 10]

$$\nabla P_e = \mu_e \nabla^2 \mathbf{U}_e + 2(\nabla \mu_e)(\nabla \mathbf{U}_e) + (\nabla \mu_e) \times (\nabla \times \mathbf{U}_e) + \mathbf{F}_g,$$

$$\operatorname{div} \mathbf{U}_e = 0, \tag{2}$$

$$\mu_i \Delta \mathbf{U}_i = \nabla P_i, \quad \operatorname{div} \mathbf{U}_i = 0, \tag{3}$$

$$\Delta T_e = 0, \tag{4}$$

$$r = R, \quad U_r^e = U_r^i = 0, \quad U_\Theta^e = U_\Theta^i, \quad T_e = T_s, \tag{5}$$

$$r \longrightarrow \infty, \quad \mathbf{U_e} \longrightarrow U_\infty \cos\Theta \mathbf{e}_r - U_\infty \sin\Theta \mathbf{e}_\Theta, \quad T_e \longrightarrow T_\infty, \quad P_e \longrightarrow P_\infty, \tag{6}$$

$$r \longrightarrow 0, \quad |\mathbf{U}_i| \neq \infty, \quad P_i \neq \infty. \tag{7}$$

Here, U_r and U_Θ are the radial and tangential components of the mass velocity $\mathbf{U}$ of the liquid in the spherical coordinate system; $\mathbf{F}_g$ is the vector of the gravitational forces; $U_\infty = |\mathbf{U}_\infty|$; U_∞ is the incoming flow velocity, which is to be determined from the condition of vanishing the total force acting on the particle (i.e., U_∞ and $|\mathbf{F}_g|$ should be so related that the total force acting on the particle vanishes); $\mathbf{e}_r$ and $\mathbf{e}_\Theta$ are the unit vectors in the spherical coordinate system; T_s is the mean surface temperature of the drop; ∇ is the del operator; Δ is Laplacian; and $(\nabla \mathbf{U}_e)$ is the scalar product.

Conditions (5) on the surface of the drop imply the impermeability and continuity conditions for the normal and tangential components of the mass velocity, respectively, as well as the constancy of the surface temperature of the particle. As the boundary conditions at infinity, i.e., far away from the particle, we take conditions (6), and the finiteness of the physical quantities characterizing the particle at $r \longrightarrow 0$ is included by (7).

To state the problem in closed form, the boundary conditions on the surface of a uniformly heated drop must be complemented by the continuity conditions for the stress tensor (normal and tangential) components [9, 10]

$$-P_e + 2\mu_e \frac{\partial U_r^e}{\partial r} = -P_i + 2\mu_i \frac{\partial U_r^i}{\partial r} + 2\frac{\sigma}{R}, \tag{8}$$

$$\mu_e \left(\frac{1}{r} \frac{\partial U_r^e}{\partial \Theta} + \frac{\partial U_\Theta^e}{\partial r} - \frac{U_\Theta^e}{r} \right) = \mu_i \left(\frac{1}{r} \frac{\partial U_r^i}{\partial \Theta} + \frac{\partial U_\Theta^i}{\partial r} - \frac{U_\Theta^i}{r} \right). \tag{9}$$

VELOCITY AND TEMPERATURE FIELDS. DRIFT VELOCITY OF THE DROP

To find the rate of fall of the uniformly heated drop, one should know the temperature, velocity, and pressure distributions in its vicinity. The general solution of heat conduction equation (4) that satisfies the appropriate boundary conditions has the form

$$t_e = 1 + \frac{\gamma}{y}, \quad \gamma = (T_s - T_\infty)/T_\infty, \tag{10}$$

where $y = r/R$ is the dimensionless radial coordinate, $t_e = T_e/T_\infty$, and γ is the dimensional parameter that characterizes the temperature difference between the surface of the particle and distant points.

Substituting (10) into (1) yields the expression for the dynamic viscosity

$$\mu_e = \mu_\infty \left[1 + \sum_{n=1}^{\infty} F_n \frac{\gamma^n}{y^n} \right] \exp\left\{ -A \frac{\gamma}{y} \right\}. \tag{11}$$

Formula (11) will further be used to find the velocity and pressure fields near the uniformly heated drop. Boundary conditions (5)–(9) admit the separation of variables upon solving the hydrodynamic equations. The components of the mass velocity and pressure were found in the form

$$U_r(r, \Theta) = U_\infty G(r) \cos\Theta,$$

$$U_\Theta(r, \Theta) = -U_\infty g(r) \sin\Theta, \quad P(r, \theta) = P_0 + h(r)\cos\Theta,$$

where $G(r)$, $g(r)$, and $h(r)$ are arbitrary functions depending on the radial coordinate r.

From the continuity equation, a relation between the functions $G(r)$ and $g(r)$ were determined. Finally, all the parameters and relations found were substituted into the appropriate Stokes equations. Eventually, we obtained the fourth-order ordinary differential equation for the function $G(r)$ that is similar to that derived in [11]. Its solution was sought in the form of generalized power series. For the components of the mass velocity and pressure, we found

$$U_r^e(y, \Theta) = U_\infty \cos\Theta [1 + A_1 G_1(y) + A_2 G_2(y)], \tag{12}$$

$$U_\Theta^e(y, \Theta) = -U_\infty \sin\Theta [1 + A_1 G_3(y) + A_2 G_4(y)], \tag{13}$$

$$P_e(y, \Theta) = P_\infty + \frac{\mu_e U_\infty}{R} \cos\Theta [A_1 G_5 + A_2 G_6], \tag{14}$$

$$U_r^i(y, \Theta) = U_\infty \cos\Theta (A_3 + A_4 y^2), \tag{15}$$

$$U_\Theta^i(y, \Theta) = -U_\infty \sin\Theta (A_3 + 2A_4 y^2), \tag{16}$$

$$P_i(y, \Theta) = P_0 + 10 \frac{\mu_i U_\infty}{R} A_4 \cos\Theta. \tag{17}$$

Here,

$$G_1(y) = -\frac{1}{y^3}\sum_{n=0}^{\infty}\frac{\Delta_n^{(1)}}{(n+3)y^n}, \quad G_3(y) = G_1(y) + \frac{1}{2}yG_1^I,$$

$$G_2(y) = -\frac{1}{y}\sum_{n=0}^{\infty}\frac{\Delta_n^{(2)}}{(n+1)y^n}$$

$$-\frac{\alpha}{y^3}\sum_{n=0}^{\infty}\left[(n+3)\ln\frac{1}{y}-1\right]\frac{\Delta_n^{(1)}}{(n+3)^2y^n},$$

$$G_5(y) = \frac{1}{2}yG_1^{III} + y\left(3+\frac{1}{2}\sum_{n=0}^{\infty}s_n\frac{\gamma^n}{y^n}\right)G_1^{II}$$

$$+\left(2+\sum_{n=0}^{\infty}s_n\frac{\gamma^n}{y^n}\right)G_1^I,$$

$$G_6(y) = \frac{1}{2}yG_2^{III} + y\left(3+\frac{1}{2}\sum_{n=0}^{\infty}s_n\frac{\gamma^n}{y^n}\right)G_2^{II}$$

$$+\left(2+\sum_{n=0}^{\infty}s_n\frac{\gamma^n}{y^n}\right)G_2^I,$$

$$s_n = AF_{n-1} - nF_n - \sum_{k=1}^{n}s_{n-k}F_k, \quad F_0 = 1,$$

and $G_4(y) = G_2(y) + \frac{1}{2}yG_2^I$, G_1^I, G_1^{II}, G_1^{III}, G_2^I, G_2^{II}, and G_2^{III} are the respective first, second, and third derivatives with respect of y. Also,

$$\Delta_n^{(1)} = -\frac{1}{n(n+5)}\sum_{k=1}^{n}[(n+4-k)$$
$$\times(\alpha_k^{(1)}(n+5-k)-\alpha_k^{(2)})+\alpha_k^{(3)}]\gamma^k\Delta_{n-k}^{(1)} \ (n\geq 1), \tag{18}$$

$$\Delta_n^{(2)} = -\frac{1}{(n+3)(n-2)}\Bigg[-6\alpha_n^{(4)}\gamma^n + \sum_{k=1}^{n}[(n+2-k)$$
$$\times\{(n+3-k)\alpha_k^{(1)}-\alpha_k^{(2)}\}+\alpha_k^{(3)}]\gamma^k\Delta_{n-k}^{(2)} \tag{19}$$
$$+\alpha\sum_{k=0}^{n}[(2n+5-2k)\alpha_k^{(1)}-\alpha_k^{(2)}]\gamma^k\Delta_{n-k-2}^{(1)}\Bigg] \ (n\geq 3).$$

In calculating the coefficients $\Delta_n^{(1)}$ and $\Delta_n^{(2)}$, it is necessary to take into account that

$$\Delta_0^{(1)} = -3, \quad \Delta_0^{(2)} = -1, \quad \Delta_2^{(2)} = 1,$$

$$\alpha_0^{(1)} = \alpha_0^{(4)} = 1, \quad \alpha_0^{(2)} = 4, \quad \alpha_0^{(3)} = -4,$$

$$\alpha_n^{(1)} = F_n, \quad \alpha_n^{(2)} = (4-n)F_n + AF_{n-1},$$

$$\alpha_n^{(3)} = 2AF_{n-1} - 2(2+n)AF_n,$$

$$\alpha_n^{(4)} = A^n/n!,$$

$$\alpha = -\frac{\gamma}{15}\{6\gamma^2\alpha_2^{(4)} - [3(4\alpha_1^{(1)}-\alpha_1^{(2)})+\alpha_1^{(3)}]\Delta_1^{(2)}$$
$$+[2(3\alpha_2^{(1)}-\alpha_2^{(2)})+\alpha_2^{(3)}]\gamma\},$$

$$\Delta_1^{(2)} = -\frac{\gamma}{4}[6\alpha_1^{(4)} + 2(3\alpha_1 - \alpha_1^{(2)}) + \alpha_1^{(3)}].$$

The constants of integration A_1, A_2, A_3, and A_4, which enter expressions (13)–(17), are determined by substituting them into the corresponding boundary conditions on the surface of the drop. Once they have been found, the force acting on the particle is found by integrating the stress tensor over the surface of the particle [12]:

$$\mathbf{F} = \int_{(S)}(-P_e\cos\Theta + P_{\mathrm{rr}}\cos\Theta - P_{0\mathrm{r}}\sin\Theta)$$
$$\times r^2\sin\Theta d\Theta d\varphi\mathbf{n}_z, \tag{20}$$

where

$$P_{\mathrm{rr}} = 2\mu_e\frac{\partial U_{\mathrm{r}}^e}{\partial r}, \quad P_{\mathrm{r}\Theta} = \mu_e\left(\frac{\partial U_\Theta^e}{\partial r} + \frac{1}{r}\frac{\partial U_{\mathrm{r}}^e}{\partial\Theta} - \frac{U_\Theta^e}{r}\right)$$

are the stress tensor components in the spherical coordinates and $\mathbf{n}_z$ is the unit vector directed along the z axis of the Cartesian coordinates.

Substituting (13) and (14) into (20) and integrating yields

$$\mathbf{F} = 4\pi R\mu_\infty U_\infty A_2\exp\{-A\gamma\}\mathbf{n}_z, \tag{21}$$

where

$$A_2 = -\left(N_3 + N_4\frac{\mu_e^{\mathrm{s}}}{3\mu_i^{\mathrm{s}}}\right)\bigg/\left(N_1 + N_2\frac{\mu_e^{\mathrm{s}}}{3\mu_i^{\mathrm{s}}}\right),$$

$$N_1|_{y=1} = [G_1G_2^I - G_2G_1^I],$$

$$N_2|_{y=1} = [G_2(2G_1^I + G_1^{II}) - G_1(2G_2^I + G_2^{II})],$$

$$N_3|_{y=1} = -G_1^I,$$

$$N_4|_{y=1} = [2G_1^I + G_1^{II}].$$

Substituting the coefficient A_2 into (21), we find the expression for the drag (viscous) force acting on a uniformly heated drop moving under gravity:

$$\mathbf{F} = -6\pi R\mu_\infty U_\infty f_\mu\mathbf{n}_z, \tag{22}$$

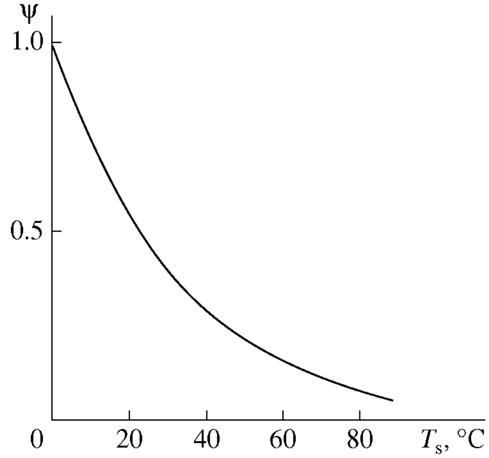

Fig. 1. ψ vs. mean surface temperature T_s.

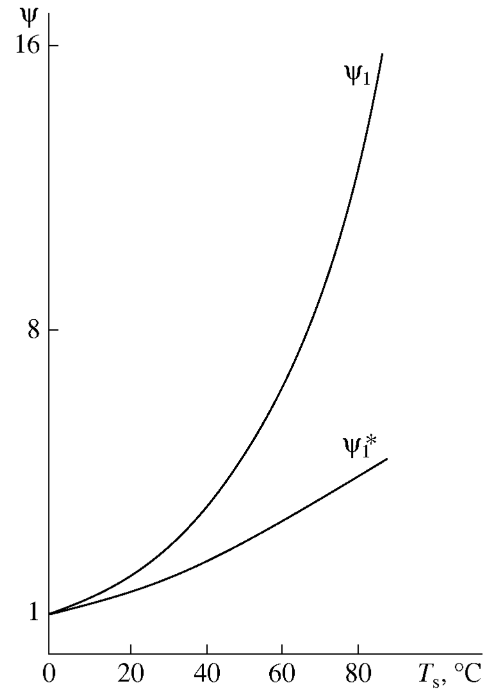

Fig. 2. ψ_1 and ψ_1^* vs. mean surface temperature T_s.

where

$$f_\mu = \frac{2}{3}\exp\{-A\gamma\}\left(N_3 + N_4\frac{\mu_e^s}{3\mu_i^s}\right)\Bigg/\left(N_1 + N_2\frac{\mu_e^s}{3\mu_i^s}\right).$$

Ultimately, a spherical drop moving under gravity in a liquid with viscosity acquires a constant velocity; that is, the gravity force is balanced out by the hydrodynamic forces. With regard for the buoyancy force, the gravity force acting on the particle is given by

$$\mathbf{F} = (\rho_i - \rho_e)g\frac{4}{3}\pi R^3\mathbf{n}_z, \qquad (23)$$

where g is the free-fall acceleration.

Equating (22) to (23), we obtain the rate of fall of a uniformly heated spherical drop (an analog of the Hadamard–Rybchinsky formula):

$$\mathbf{U}_e = h_\mu \mathbf{n}_z \quad \left(h_\mu = \frac{2}{9}R^2\frac{\rho_i - \rho_e}{\mu_\infty f_\mu}g\right). \qquad (24)$$

If $\mu_e^s/\mu_i^s \longrightarrow 0$ in (22), we come to the formula for the drag force acting on a uniformly heated solid particle (an analog of the Stokes formula).

If the surface of the drop is heated insignificantly, i.e., if the mean surface temperature of the drop differs from the environmental temperature at infinity only slightly [$\gamma = (T_s - T_\infty)/T_\infty \longrightarrow 0$], the temperature dependence of the dynamic coefficient viscosity can be neglected. Then, $G_1 = -1/3$, $G_1^I = 1$, $G_1^{II} = -4$, $G_1^{III} = 20$, $G_2 = -1$, $G_2^I = 1$, $G_2^{II} = -2$, $G_2^{III} = 6$, $N_1 = 2/3$, $N_2 = 2$, $N_3 = -1$, and $N_4 = -2$. In this case, formula (22) turns to the well-known expressions for a sphere that were obtained by Hadamard and also by Rybchinsky and Stokes [10].

The effect of heating the drop on the drag force and the rate of fall of the drop [i.e., the effect of the temperature dependence of the viscosity, formula (1)] is illustrated in Figs. 1 and 2. They plot, respectively, $\psi = f_\mu/f_\mu|_{T_s = 273\text{ K}}$ and $\psi_1 = h_\mu/h_\mu|_{T_s = 273\text{ K}}$ against T_s for large mercury drops of radius $R = 2 \times 10^{-5}$ m moving in water at $T_\infty = 273$ K. The curve ψ_1^* was constructed for small temperature differences ($\gamma \longrightarrow 0$) [10], but the molecular transport coefficients were taken for $T_e = T_s$. As follows from the curves, heating considerably affects both the drag force and the rate of gravity fall.

Thus, we generalized the Stokes and Hadamard–Rybchinsky expressions for the case of the steady-state motion of a uniformly heated solid spherical particle (drop) in an incompressible liquid under gravity at arbitrary temperature differences between the surface of the particle and distant points. In the analysis, the temperature dependence of the viscosity is represented as an exponential-power series.

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