Motion through a Viscous Liquid of a Heated Spheroidal Solid Particle under Conditions of Uniform Internal Heat Release

N. V. Malaĭ, E. R. Shchukin, and Yu. I. Yalamov

Belgorod State University, Belgorod, Russia

e-mail: malay@bsu.edu.ru

Abstract—An expression has been derived allowing the drag to be estimated on a spheroidal hydrosol particle moving in a liquid under conditions of an arbitrary temperature difference between the particle surface and a separate region, which takes into account the temperature dependence of the liquid viscosity presented in the form of an exponential power series.

FORMULATION OF THE PROBLEM

Consider a spheroidal particle of a solid containing uniformly distributed constant-power heat sources (sinks), which is immersed in an incompressible viscous liquid occupying the whole space. The motion of heated spheroidal particles through viscous liquid and gaseous media was studied in a number of papers [1–5]. A particle is considered heated if the average temperature of its surface is well above the ambient temperature. Heating of the particle surface may be due, for example, to a chemical reaction inside the particle, radioactive decay of the particle matter, and so on. The heated particle significantly influences the thermophysical characteristics of a medium and may appreciably affect the velocity field and the pressure distribution in its vicinity.

The motion of nonspherical particles through liquid and gaseous media under conditions of small relative differences of temperature was considered in papers [6–8].

In this paper, in the framework of the Stokes approximation, an analytical expression for the hydrodynamic force acting on a uniformly heated spheroidal particle has been obtained, which takes into account the temperature dependence of the liquid viscosity represented in the form of an exponential power series, for an arbitrary difference between the temperatures of the particle surface and a separate region.

In the frame of reference with respect to the center of mass of the particle, the problem is reduced to that of a heated immovable oblate (prolate) spheroid placed in a plane-parallel flow of liquid having a velocity $\mathbf{U}_\infty$ (parallel to the symmetry axis of the spheroid). It is assumed that the density, thermal conductivity, and thermal capacity of the liquid and the particle are constant, and the thermal conductivity of the particle is much greater than that of the ambient liquid.

Of the parameters governing the liquid flow, only the dynamic viscosity coefficient strongly depends on temperature [9]. This dependence will be taken into account using the expression

$$\mu_{\text{liq}} = \mu_\infty\left[1 + \sum_{n=1}^{\infty} F_n\left(\frac{T_{\text{liq}}}{T_\infty} - 1\right)^n\right] \times \exp\left\{-A\left(\frac{T_{\text{liq}}}{T_\infty} - 1\right)\right\} \tag{1}$$

(at $F_n = 0$ this formula is reduced to the well-known Reynolds relation [9]). Here $A = \text{const}$; $\mu_\infty = \mu_{\text{liq}}(T_\infty)$; and T_∞ is the liquid temperature away from the particle; the indices “liq” and “p” here and below refer to the parameters of the ambient liquid and the particle, respectively. It is known that the viscosity of a liquid drops exponentially with temperature [9]. An analysis of the known semiempirical formulas has shown that expression (1) best describes viscosity variations in a wide temperature range with any required accuracy.

The flow about the spheroid is presented in spheroidal coordinates (ε, η, φ) with the origin at the center of the hydrosol particle. The curvilinear coordinates ε, η, and φ are related to the Cartesian coordinates by the following expressions [10]

$$x = c\sinh\varepsilon\sin\eta\cos\varphi, \quad y = c\sinh\varepsilon\sin\eta\sin\varphi, \quad z = c\cosh\varepsilon\cos\eta \tag{2}$$

or

$$x = c\cosh\varepsilon\sin\eta\cos\varphi, \quad y = c\cosh\varepsilon\sin\eta\sin\varphi, \quad z = c\sinh\varepsilon\cos\eta, \tag{3}$$

where $c = \sqrt{b_0^2 - a_0^2}$ in the case of a prolate spheroid ($a_0 < b_0$, formulas (2)) and $c = \sqrt{a_0^2 - b_0^2}$ for an oblate spheroid ($a_0 > b_0$, formulas (3)), a_0 and b_0 are the semi-axes of the spheroid, and the z axis of the particle Cartesian coordinate system coincides with the spheroid's axis of symmetry.

For small Reynolds numbers, the distributions of velocity $\mathbf{U}_{\text{liq}}$, pressure P_{liq}, and temperature T_{liq} are described by the following set of equations [11]:

$$\nabla P_{\text{liq}} = \mu_{\text{liq}}\Delta\mathbf{U}_{\text{liq}} + 2(\nabla\mu_{\text{liq}}\nabla)\mathbf{U}_{\text{liq}} + [\nabla\mu_{\text{liq}} \times \operatorname{rot}\mathbf{U}_{\text{liq}}], \quad \operatorname{div}\mathbf{U}_{\text{liq}} = 0, \tag{4}$$

$$\Delta T_{\text{liq}} = 0, \quad \Delta T_{\text{p}} = -q_p/\lambda_{\text{p}}. \tag{5}$$

Equations (4) and (5) are solved with the following boundary conditions:

$$\mathbf{U}_{\text{liq}} = 0, \quad T_{\text{liq}} = T_p, \quad \lambda_{\text{liq}}\frac{\partial T_{\text{liq}}}{\partial \varepsilon} = \lambda_p\frac{\partial T_p}{\partial \varepsilon} \quad \text{at} \quad \varepsilon = \varepsilon_0, \tag{6}$$

$$\mathbf{U}_{\text{liq}} \longrightarrow U_\infty\cos\eta\,\mathbf{e}_\varepsilon - U_\infty\sin\eta\,\mathbf{e}_\eta, \quad T_{\text{liq}} \longrightarrow T_\infty, \quad P_{\text{liq}} \longrightarrow P_\infty \quad \text{at} \quad \varepsilon \longrightarrow \infty, \tag{7}$$

$$T_{\text{p}} \neq \infty \quad \text{at} \quad \varepsilon \longrightarrow 0. \tag{8}$$

Here, $\mathbf{e}_\varepsilon$ and $\mathbf{e}_\eta$ are the unit vectors of the spheroidal coordinate system, λ is the thermal conductivity, $U_\infty = |\mathbf{U}_\infty|$, and q_p is the (constant) power of the heat sources (sinks) per particle unit volume. Boundary conditions (6) allow for a zero liquid velocity condition at the particle surface, equality of the particle surface and the liquid temperatures, and continuity of the heat fluxes through the particle surface. The particle surface is specified by the coordinate ε_0. At a large distance from the particle ($\varepsilon \longrightarrow \infty$), boundary conditions (7) are valid, and the finiteness of the physical parameters of the particle at $\varepsilon \longrightarrow 0$ is taken into account in (8).

The force exerted by the flow on the particle is defined by the formula

$$F_z = \int\limits_S\left(-P_{\text{liq}}\cos\eta + \sigma_{\varepsilon\varepsilon}\cos\eta - \frac{\sinh\varepsilon}{\cosh\varepsilon}\sigma_{\varepsilon\eta}\sin\eta\right)dS, \tag{9}$$

where $dS = c^2\cosh^2\varepsilon\sin\eta d\eta d\varphi$ is a differential surface element, and $\sigma_{\varepsilon\varepsilon}$ and $\sigma_{\varepsilon\eta}$ are the components of the stress tensor in a spheroidal coordinate system [11].

THE FIELD OF VELOCITIES AND THE TEMPERATURE DISTRIBUTION: DETERMINATION OF THE DRAG

To find the force exerted by a liquid on a heated spheroidal solid particle, it is necessary to know the temperature field around the particle. Integrating Eq. (5) under the corresponding boundary conditions gives

$$t_{\text{liq}} = 1 + \frac{\gamma}{c}a_0\operatorname{arccot}\lambda, \tag{10}$$

$$t_{\text{p}} = B + \frac{\lambda_{\text{liq}}\gamma a_0}{\lambda_{\text{p}}\, c}\operatorname{arccot}\lambda + \int\limits_{\lambda_0}^{\lambda}\frac{\operatorname{arccot}\lambda}{c}f d\lambda - \frac{\operatorname{arccot}\lambda}{c}\int\limits_{\lambda_0}^{\lambda} f d\lambda. \tag{11}$$

Here, $\lambda = \sinh\varepsilon$; $t = T/T_\infty$; $\gamma = t_{\text{s}} - 1$ is a dimensionless parameter characterizing the heating of the particle surface; $t_s = T_s/T_\infty$; and T_s is the average surface temperature of the heated spheroid defined by the formula

$$T_s = T_\infty + \frac{a_0 b_0}{3\lambda_{\text{liq}}}q_p, \tag{12}$$

$$B = 1 + \left(1 - \frac{\lambda_{\text{liq}}}{\lambda p}\right)\gamma\sqrt{1+\lambda^2}\operatorname{arccot}\lambda_0,$$

$$\lambda_0 = \sinh\varepsilon_0,$$

$$f = -\frac{c^2}{2\lambda_{\text{p}}T_\infty}\int\limits_{-1}^{+1} q_{\text{p}}(\lambda^2 + x^2)dx; \quad x = \cos\eta.$$

Taking (10) into account, expression (1) takes the form

$$\mu_{\text{liq}} = \mu_\infty\left[1 + \sum_{n=1}^{\infty} F_n\gamma_0^n(\operatorname{arccot}\lambda)^n\right]\exp\{-\gamma_0\operatorname{arccot}\lambda\}, \tag{13}$$

where

$$\gamma_0 = \frac{A\gamma}{c}a_0.$$

Making use of the fact that the viscosity depends only on the radial coordinate λ, we solve Eq. (4) by the method of the separation of variables, resolving the velocity and pressure fields into Legendre and Gegenbauer polynomials [10]. In particular, for the components of the mass velocity $\mathbf{U}$ the following expressions satisfying boundary conditions (7) are obtained:

$$U_\varepsilon(\varepsilon, \eta) = \frac{U_\infty}{cH_\varepsilon}\cos\eta[c^2 + A_1G_1 + A_2G_2], \tag{14}$$

$$U_\eta(\varepsilon, \eta) = -\frac{U_\infty}{cH_\varepsilon}\sin\eta[c^2 + A_1G_3 + A_2G_4], \tag{15}$$

where

$$G_1 = -\frac{1}{\lambda^3}\sum_{n=0}^{\infty}\frac{\Theta_n^{(1)}}{(n+3)\lambda^n},$$

$$G_2 = -\frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{\Theta_n^{(2)}}{(n+1)\lambda^n}$$

$$-\frac{\beta}{\lambda^3}\sum_{n=0}^{\infty}\frac{\Theta_n^{(1)}}{(n+3)\lambda^n}\left[(n+3)\ln\frac{\lambda_0}{\lambda}-1\right],$$

$$G_3 = G_1 + \frac{1+\lambda^2}{2\lambda}G_1^{\mathrm{I}}, \quad G_4 = G_2 + \frac{1+\lambda^2}{2\lambda}G_2^{\mathrm{I}},$$

$$\Theta_n^{(1)} = -\frac{1}{n(n+5)}\sum_{k=1}^{n}[(n+4-k)$$

$$\times\{(n+1-k)\alpha_l^{(1)} + \alpha_k^{(2)}\} + \alpha_k^{(3)}]\Theta_{n-k}^{(1)} \quad (n \ge 1),$$

$$\Theta_n^{(2)} = -\frac{1}{(n-2)(n+3)}\left[\sum_{k=1}^{n}\{(n+2-k)\right.$$

$$\times[(n+1-k)\alpha_k^{(1)} + \alpha_k^{(2)}] + \alpha_k^{(3)}\}\Theta_{n-k}^{(2)}$$

$$\left.+\beta\sum_{k=0}^{n}[(2n-2k-3)\alpha_k^{(1)} + \alpha_k^{(1)}]\Theta_{n-k-2}^{(1)} - 6\alpha_n^{(4)}\right]$$

$$(n \ge 3), \quad H_\varepsilon = c\sqrt{\cosh^2\varepsilon - \sin^2\eta},$$

$$\Theta_1^{(2)} = -\frac{1}{4}[2(\alpha_0^{(1)} + \alpha_2^{(2)}) + \alpha_1^{(3)} + 6\alpha_1^{(4)}],$$

$$\Theta_2^{(2)} = 1, \quad \Theta_0^{(1)} = -1, \quad \Theta_0^{(2)} = -1,$$

$$\beta = -\frac{1}{5}[\{3(2\alpha_1^{(1)} + \alpha_1^{(2)}) + \alpha_1^{(3)}\}\Theta_1^2$$

$$-2(\alpha_2^{(1)} + \alpha_2^{(2)}) - \alpha_2^{(3)} - 6\alpha_2^{(4)}],$$

$$\alpha_n^{(1)} = C_n + 12\sum_{k=0}^{\left[\frac{n-2}{2}\right]}(-1)^k\frac{C_{n-2k-2}}{(2k+1)(2k+3)(2k+5)}, \tag{16}$$

$$\Delta_0 = 1,$$

$$\alpha_n^{(2)} = (n-2)C_n - \gamma_0 C_{n-1}$$

$$+12\sum_{k=0}^{\left[\frac{n-2}{2}\right]}(-1)^k\frac{(4k+5)C_{n-2k-2}}{(2k+1)(2k+3)(2k+5)}$$

$$-3\sum_{k=0}^{\left[\frac{n-3}{2}\right]}(-1)^k\frac{1}{(2k+3)(2k+5)}[(n-2k-2)C_{n-2k-2}$$

$$-\gamma_0 C_{n-2k-3} + (n+2k-4)C_{n-2k-4}] \quad (n \ge 1),$$

$$\alpha_n^{(3)} = -2(n+2)C_n + 2\gamma_0 C_{n-1} - 2(n-2)C_{n-2}$$

$$+12\sum_{k=0}^{\left[\frac{n-2}{n}\right]}(-1)^k\frac{C_{n-2k-2}}{(2k+5)} + 6\sum_{k=0}^{\left[\frac{n-3}{2}\right]}(-1)^k$$

$$\times\frac{(k+2)(4k+5)}{(2k+3)(2k+5)}[(n-2k-2)C_{n-2k-2}$$

$$-\gamma_0 C_{n-2k-3} + (n-2k-4)C_{n-2k-4}] \quad (n \ge 1),$$

$$\alpha_n^{(4)} = \frac{1}{n!}[\gamma_0\Delta_{n-1} - (n-1)(n-2)\Delta_{n-2}] \quad (n \ge 1),$$

$$C_k = \sum_{l_1+3l_3+5l_5+\ldots+sl_s=k}\frac{l!}{l_1!l_3!l_5!\ldots l_s!}$$

$$\times F_l f_1^{l_1} f_3^{l_3} f_5^{l_5}\ldots f_s^{l_s},$$

$$s = k - \frac{1+(-1)^k}{2},$$

$$l = l_1 + l_3 + l_5 + \ldots + l_s,$$

$$f_{2k-1} = (-1)^{k-1}\frac{\gamma a_0}{c(2k-1)} \quad (k \ge 1),$$

$[k/2]$ denotes the integer part of $k/2$.

The force acting on the spheroid because of viscous stresses is determined by integrating expression (9) over the spheroid surface and, taking into account (14) and (15), is equal to

$$\mathbf{F}_z = -4\pi\frac{\mu_\infty U_\infty}{c}A_2\exp\left\{-\frac{A\gamma}{c}a_0\operatorname{arccot}\lambda_0\right\}\mathbf{n}_z, \tag{17}$$

where $\mathbf{n}_z$ is the unit vector along the z axis.

Note, that expression (17) for the force was obtained under the assumption of uniform particle motion, which is only possible when the total force acting on a particle is zero. Since force (17) is proportional to the velocity and becomes zero only together with it,

Variation of the coefficient K with the average surface temperature of the spheroid and the ratio of its semiaxes

	T_s, K				a_0/b_0
	293	313	331	353	
K	2.066	2.023	1.818	1.704	1.2
K	0.978	0.649	0.367	0.173	1.4

another force has to be present to balance (17) for uniform motion to be possible.

The integration constants A_1 and A_2 appearing in the expressions for the components of mass velocity are determined from the boundary conditions at the spheroid surface. Then, expression (17) can be represented in the form

$$\mathbf{F}_z = 6\pi a_0 \mu_\infty K U_\infty \mathbf{n}_z, \tag{18}$$

where

$$K = \frac{2}{3}\frac{G_1^{\mathrm{I}}}{\sqrt{1+\lambda_0^2}[G_2G_1^{\mathrm{I}} - G_1G_2^{\mathrm{I}}]}$$
$$\times \exp\left\{-\frac{A\gamma}{c}a_0 \operatorname{arccot}\lambda_0\right\},$$

G_1^{I} and G_2^{I} are the first derivatives of the corresponding functions with respect to λ; and $\boldsymbol{n}_z$ is the unit vector along the z axis.

To obtain an expression for the hydrodynamic drag on an aprolate spheroid it is necessary to substitute $i\lambda$ for λ and $-ic$ for c in formula (18) (i is an imaginary unit).

Thus, formula (18) allows one to estimate the hydrodynamic force acting on a spheroidal particle containing uniformly distributed constant-power heat sources (sinks). This estimation takes into account the dependence of viscosity on temperature expressed in the form of an exponential power series for an arbitrary difference between the temperatures at the particle surface and away from it.

As an example, the table gives the calculation results of the dependence of K on the average temperature of the spheroid surface and the ratio of spheroid semiaxes for granite particles of a radius $R = 2 \times 10^{-5}$ m suspended in water (T_∞ = 293 K, A = 6.095, F_n = 0, $n \geq 1$).

In the limit $\gamma \longrightarrow 0$ (small temperature gradients in the spheroid's vicinity), $a_0 = R$, $K = 1$, and formula (18) transforms into the Stokes formula [10].

Let us consider the motion of a spheroidal particle in a gravitational field. A particle sinking under the action of gravitational force in a viscous liquid ultimately acquires a constant velocity, such that gravity is balanced by hydrodynamic forces.

The gravitational force on the particle, with allowance made for its buoyancy, is

$$F_{\mathrm{g}} = (\rho_p - \rho_{\mathrm{liq}})g\frac{4}{3}\pi a_0^2 b_0. \tag{19}$$

Equating (18) to (19), we obtain the velocity of the steady fall of a nonuniformly heated spheroidal particle

$$U_\infty = (\rho_{\mathrm{p}} - \rho_{\mathrm{liq}})g\sqrt{1+\lambda_0^2}\frac{a_0b_0}{3\mu_\infty}\frac{G_1G_2^{\mathrm{I}} - G_2G_1^{\mathrm{I}}}{G_1^{\mathrm{I}}}$$
$$\times \exp\left\{\frac{A\gamma}{c}a_0 \operatorname{arccot}\lambda_0\right\}.$$

Let us highlight some problems which can be solved using the results obtained. Consider the motion of a particle containing nonuniformly distributed heat sources (sinks) of density q_p. In this case, the average temperature of the spheroid surface is defined by the following relation:

$$\frac{T_s}{T_\infty} = 1 + \frac{1}{4\pi a_0 \lambda_{\mathrm{liq}} T_\infty}\int_V q_p dV,$$

where the integral is taken over the whole volume of the spheroidal particle.

Another case is that of heat sources (sinks) of constant intensity I_0 distributed not in the volume but over the particle surface. It is easy to show that the solution can be obtained if in the above relationships for the case of uniform internal heat release the following substitution is made

$$q_p = \frac{3}{4b_0}I_0\left[2 + \frac{b_0^2}{\varepsilon a_0^2}\ln\frac{1+\varepsilon}{1-\varepsilon}\right].$$

Here, ε is the spheroid's eccentricity. It is also possible to consider the motion of a uniformly heated particle with an average surface temperature T_s. In particular, if an electromagnetic radiation (having the wavelength λ_0) of intensity I_0 is incident on the spheroid, the absorbed energy is $\pi R^2 I_0 K_n$, where R is the semimajor axis of the spheroid and K_n is the absorption factor [12]. If $\lambda_0 \gg R$, the absorbed energy is uniformly distributed over the particle surface; that is, the particle can be considered uniformly heated. In this case, it is necessary to take that $q_p = 0$ and $T = T_s$ in the boundary conditions (6). The parameter γ takes the form

$$\gamma = \frac{1}{\sqrt{1+\lambda_0^2}}\frac{t_s - 1}{\operatorname{arccot}\lambda_0} \quad (t_s = T_s/T_\infty).$$

REFERENCES

1. D. R. Kassoy, T. C. Adamson, and A. F. Messiter, Phys. Fluids **9**, 671 (1966).
2. V. I. Naĭdenov, Prikl. Mat. Mekh. **38**, 162 (1974).
3. E. R. Shchukin and R. V. Malaĭ, Inzh.-Fiz. Zh. **54**, 630 (1988).

4. E. R. Shchukin, N. V. Malaĭ, and Yu. I. Yalamov, Teplofiz. Vys. Temp. **25**, 1020 (1988).

5. N. V. Malaĭ, Inzh.-Fiz. Zh. **73**, 728 (2000).

6. Yu. I. Yalamov and A. M. Afanas'ev, Zh. Tekh. Fiz. **47**, 1998 (1977) [Sov. Phys. Tech. Phys. **22**, 1163 (1977)].

7. Yu. I. Yalamov and A. M. Afanas'ev, Zh. Tekh. Fiz. **47**, 2001 (1977) [Sov. Phys. Tech. Phys. **22**, 1165 (1977)].

8. Yu. I. Yalamov, V M. Redchits, and M. N. Gaĭdukov, Zh. Tekh. Fiz. **49**, 1534 (1979) [Sov. Phys. Tech. Phys. **24**, 852 (1979)].

9. St. Bretsznajder, *Properties of Gases and Liquids. Engineering Methods of Calculation* (Wydawnictwo Naukowo-Techniczne, Warsaw, 1962; Moscow, 1966), translated from Polish.

10. J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics* (Prentice-Hall, Englewood Cliffs, 1965; Mir, Moscow, 1976).

11. L. D. Landau and E. M. Lifshitz, *Mechanics of Continuous Media* (Moscow, 1958).

12. C. F. Bohren and D. R. Huffman, *Absorption and Scattering of Light by Small Particles* (Wiley, New York, 1983; Mir, Moscow, 1986).

Translated by N. Mende