Hyperbolic phase and squeeze-parameter estimation

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We define a new representation, the hyperbolic phase representation, which enables optimal estimation of a squeeze parameter in the sense of quantum estimation theory. We compare the signal-to-noise ratio for such measurements, with conventional measurement based on photon counting and homodyne detection. The signal-to-noise ratio for hyperbolic phase measurements is shown to increase quadratically with the squeezing parameter for fixed input power.

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I. INTRODUCTION

How efficiently can one estimate a squeeze parameter? An initial state $|\psi_i\rangle$ enters a device that transforms it according to

$$|\psi_0\rangle = e^{-ir\hat{G}}|\psi_i\rangle, \tag{1}$$

where $\hat{G}$ is the generator of squeezing;

$$\hat{G} = \frac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q}) \tag{2}$$

and $\hat{q}, \hat{p}$ are canonical position and momentum variables that satisfy $[\hat{q}, \hat{p}] = i$. The parameter r is the squeeze parameter. This transformation can be realized in quantum optics by parametric amplification [1]. Our objective is then to find the measurement on $|\psi_0\rangle$ that permits the best estimate of the parameter r, as defined below. Formally we wish to find a representation $|r\rangle$ such that the conditional distribution

$$P(\tilde{r}|r) = |\langle\tilde{r}|e^{-ir\hat{G}}|\psi\rangle|^2 \tag{3}$$

when sampled provides a good estimate of the parameter.

The question posed above is suggestive of the much debated question of how best to estimate a phase-shift parameter in quantum mechanics [2]. Indeed the analogy is quite strong. To see this, note that the classical analog of the transformation in Eq. (2) translates phase-space points along hyperbolic curves defined by qp = const. In the case of phase transformations the generator is $(\hat{q}^2+\hat{p}^2)/2$, the classical analog of which translates phase-space points around the circle $q^2 + p^2$ = const. We are thus motivated to refer to the parameter r as the hyperbolic phase.

The general parameter-estimation problem has been discussed by Helstrom [3] and Holevo [4]. We sketch briefly its solution for the covariant measurement scenario described in Eq. (1), with maximum-likelihood parameter estimation. Assume the relation between the input and output states is given by

$$\hat{\rho}(\beta) = e^{i\hat{A}\beta}\hat{\rho}_0 e^{-i\hat{A}\beta}. \tag{4}$$

Our objective is to find a positive operator valued measure (POM) $d\hat{\Pi}(\tilde{\beta})$ such that

$$p(\tilde{\beta}|\beta)d\tilde{\beta} = \mathrm{tr}[\hat{\rho}(\beta)d\hat{\Pi}(\tilde{\beta})] \tag{5}$$

provides an optimal estimate of the parameter β. The optimal Bayes-cost theory of Helstrom assigns a *cost function* $C(\beta,\tilde{\beta})$ and the quantity

$$\bar{C}(\hat{\Pi}) = \int C(\beta,\tilde{\beta})p(\tilde{\beta}|\beta)p_0(\beta)d\beta d\tilde{\beta} \tag{6}$$

as an outcome-averaged measure of performance for ingoing states distributed according to the prior $p_0(\beta)$. Minimization of $\bar{C}(\hat{\Pi})$ according to the choice

$$C(\beta,\tilde{\beta}) = -\delta(\beta-\tilde{\beta}) \tag{7}$$

identifies the strategy of least probable error in a maximum-likelihood data analysis scheme (a robust and common choice). An intriguing method of solution, due to Holevo, allows us to verify, with confidence, whether a candidate POM is optimal, but does not in general tell us how to find it. In the covariant case an optimal POM can be constructed.

We will assume that the parameter β takes values on the real line and further that

$$d\hat{\Pi}(\tilde{\beta}) = \hat{\xi}(\beta)d\beta, \tag{8}$$

where

$$\hat{\xi}(\beta) = e^{i\hat{A}\beta}\xi_0 e^{-i\hat{A}\beta}. \tag{9}$$

The optimization problem then involves (i) finding the observable $\hat{B}$, with eigenvalues β, that commutes with this POM, and (ii) choosing ξ_0 correctly. With the choice Eq. (8) and Eq. (9), one easily shows that the conditional distribution in Eq. (5) is a function of $\tilde{\beta}-\beta$ alone and is thus shift invariant. We now define the "optimal" parameter determination as that measurement which results in a conditional distribution for which the maximum-likelihood estimator is optimal. We will assume a uniform prior distribution for β (maximum initial ignorance). Helstrom [3] has shown that in this case the

POM must satisfy

$$[\hat{\rho}(\beta) - \hat{\Upsilon}]d\hat{\Pi}(\tilde{\beta}) = 0, \quad (10)$$

$$\hat{\Upsilon} - \hat{\rho}(\beta) \geq 0, \quad (11)$$

where

$$\hat{\Upsilon} = \int_{-\infty}^{\infty} \hat{\rho}(\beta)d\hat{\Pi}(\tilde{\beta}). \quad (12)$$

Let $\hat{A}$ have the spectral resolution

$$\hat{A} = \int_{-\infty}^{\infty} \alpha|\alpha\rangle\langle\alpha|d\alpha. \quad (13)$$

For $d\hat{\Pi}(\tilde{\beta})$ to be a POM we require that

$$\int_{-\infty}^{\infty} d\hat{\Pi}(\tilde{\beta}) = \hat{1}, \quad (14)$$

which, from Eq. (8), requires that

$$\int_{-\infty}^{\infty} \hat{\xi}(\beta)d\beta = \hat{1}, \quad (15)$$

where $\hat{1}$ is the identity operator. Let us now assume that

$$\hat{\xi}_0 = |\phi\rangle\langle\phi|. \quad (16)$$

Then Eq. (15) requires that in the basis $|\alpha\rangle$

$$|\langle\alpha|\phi\rangle| = 1. \quad (17)$$

Furthermore we can show that

$$[\hat{\Upsilon}, \hat{A}] = 0. \quad (18)$$

In this case the constraint equations (10) and (11) become

$$\left(\hat{\rho}_0 - \hat{\Upsilon}\right)\hat{\xi}_0 = 0, \quad (19)$$

$$\left(\hat{\Upsilon} - \hat{\rho}_0\right) \geq 0. \quad (20)$$

It is easy to show that Eq. (19) is satisfied given the choice of $\hat{\xi}_0$ in Eq. (16). Equation (20) will be satisfied if we choose

$$\langle\alpha|\phi\rangle = e^{i\theta}, \quad (21)$$

where θ is the phase of $\langle\alpha|\phi\rangle$.

Using the results in Eqs. (8), (9), (16), and (21) in the definition Eq. (5) we find that

$$p(\tilde{\beta}|\beta) = \left|\int d\alpha e^{-i\alpha(\beta-\tilde{\beta})}|\langle\phi|\alpha\rangle|\right|^2. \quad (22)$$

The above results can be summarized in the following way. If we choose the initial states $|\psi\rangle$ such that in the eigenstates $|\alpha\rangle$ of the generator $\hat{A}$, we have that $\langle\alpha|\psi\rangle = |\langle\alpha|\psi\rangle|$. Then the optimal estimation of the parameter results for ideal measurements of the physical quantity $\hat{B}$ with eigenstates $|\beta\rangle$ given by

$$|\beta\rangle = \int d\alpha e^{-i\alpha\beta}|\alpha\rangle\langle\alpha|. \quad (23)$$

Then Eq. (22) becomes

$$p(\tilde{\beta}|\beta) = |\langle\psi|e^{-i\hat{A}\beta}|\tilde{\beta}\rangle\ |^2. \quad (24)$$

It is then easy to see that in the $|\beta\rangle$ representation $\hat{A}$ is a pure differential operator and thus $\hat{A}$ and $\hat{B}$ are canonically conjugate variables. For example, if $\hat{A} = a^\dagger a$ where a is the annihilation operator of a simple harmonic oscillator, then the above construction shows that we need to measure a quantity diagonal in the Susskind-Glogower phase states [5]. The resulting POM is not particularly well behaved (it projects onto states of infinite energy), but it can be considered as the limit of a sequence of well behaved physical POM's [6].

II. OPTIMAL ESTIMATION OF A SQUEEZE PARAMETER

The general generator of squeezed states is given by [7]

$$\hat{G}(\theta) = \frac{1}{2}[a^2e^{2i\theta} + (a^\dagger)^2e^{-2i\theta}]. \quad (25)$$

If we define $a = (\hat{q} + i\hat{p})/\sqrt{2}$ and choose $\sin 2\theta = -1$ the generator takes the form given in Eq. (2). The corresponding unitary transformation then generates a squeezed vacuum state from the field ground state. We are thus asking for the optimal measurements to determine the degree of squeezing.

Using the results in the Introduction we see that we need to measure a quantity $\hat{R}$ in the eigenstates of which $\hat{G}$ becomes a pure differential operator. The first step in constructing $\hat{R}$ and the resulting conditional distribution $p(\tilde{r}|r)$ is to find the eigenstates of $\hat{G}$. These have been given by Bollini and Oxman [8]. They fall into two classes denoted by the labels + or −. Thus

$$\hat{G}|\mu\rangle_\pm = \mu|\mu\rangle_\pm, \quad (26)$$

where μ is real. The position representation of these states is

$$\langle x|\mu\rangle_\pm = \frac{1}{\sqrt{2\pi}}x_\pm^{i\mu-\frac{1}{2}}, \quad (27)$$

where the generalized functions $x_\pm^\lambda$ are defined by [9]

$$x_+^\lambda = \begin{cases} x^\lambda, & \text{if } x > 0 \\ 0, & \text{otherwise,} \end{cases} \quad (28)$$

$$x_-^\lambda = \begin{cases} 0, & \text{if } x > 0 \\ |x|^\lambda, & \text{otherwise.} \end{cases} \quad (29)$$

States from different classes are orthogonal; $_+\langle\mu|\mu'\rangle_- = 0$, while states within a class are orthonormal with δ function normalization. However, because $\langle x|\mu\rangle_+$ ($\langle x|\mu\rangle_-$) is nonzero only on the positive (negative) real line, these states do not form a complete basis for the entire Hilbert space of square integrable functions. However, each state is complete within a class. We can then expand an arbitary state $|\psi\rangle$ as

$$|\psi\rangle = \int_{-\infty}^{\infty} d\mu \, ({}_+\langle\mu|\psi\rangle|\mu\rangle_+ + {}_-\langle\mu|\psi\rangle|\mu\rangle_-) , \tag{30}$$

with the normalization condition

$$\int_{-\infty}^{\infty} d\mu(|{}_+\langle\mu|\psi\rangle|^2 + |{}_-\langle\mu|\psi\rangle|^2) = 1. \tag{31}$$

The conjugate representation to $|\mu\rangle_\pm$ is then defined as

$$|r\rangle_\pm = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\mu e^{-i\mu r} |\mu\rangle_\pm . \tag{32}$$

These states are also complete only on the positive and negative real line. We are thus led to define the hyperbolic phase representation for a state $|\psi\rangle$ as

$$P_\psi(r) = |{}_+\langle r|\psi\rangle|^2 + |{}_-\langle r|\psi\rangle|^2, \tag{33}$$

with the normalization condition

$$\int_{-\infty}^{\infty} dr P_\psi(r) = 1. \tag{34}$$

The distribution defined in Eq. (33) is the optimal distribution for squeeze-parameter estimation. That is to say, for suitable input states $|\psi_i\rangle$ and output states

$$|\psi(r)\rangle = e^{-ir\hat{G}}|\psi_i\rangle \tag{35}$$

the conditional distribution

$$p(\tilde{r}|r) = |{}_+\langle\tilde{r}|\psi(r)\rangle|^2 + |{}_-\langle\tilde{r}|\psi(r)\rangle|^2 \tag{36}$$

enables an optimal estimation of the squeeze paremeter r. For an arbitrary input state the hyperbolic phase distribution is expected to give the best estimation of a squeeze parameter, although if the initial state is poorly chosen even the hyperbolic phase distribution may be useless.

To compute the hyperbolic phase distribution we need only give the wave function for the state of interest, as we now show. We first note that

$$\langle x|r\rangle_\pm = x_\pm^{-\frac{1}{2}} \delta(r - \ln x_\pm). \tag{37}$$

Then for an arbitary state $|\psi\rangle$ we find

$${}_\pm\langle r|\psi\rangle = \int_{-\infty}^{\infty} dx \psi(x) x_\pm^{-1/2} \delta(r - \ln x_\pm) \tag{38}$$

$$= e^{r/2}\psi(\pm e^r), \tag{39}$$

where $\psi(x) = \langle x|\psi\rangle$. For example, if we take the coherent state $|\alpha\rangle$ we find

$${}_\pm\langle r|\alpha\rangle = \pi^{-1/4} \exp\left(\frac{r}{2} - \frac{e^{2r}}{2} - \frac{1}{2}(|\alpha|^2 + \alpha^2) \pm \sqrt{2}\alpha e^r\right). \tag{40}$$

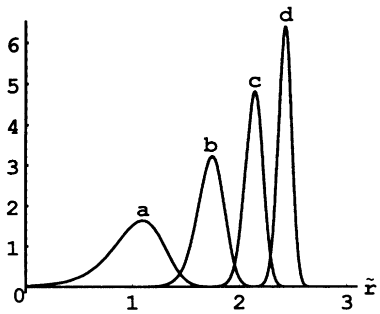

FIG. 1. A plot of the hyperbolic phase representation for a coherent state, for different values of α; (a) α=2, (b) α=4, (c) α=6, (d) α=8.

Thus the hyperbolic phase distribution is

$$P_\alpha(r) = 2\pi^{-1/2} \exp[-2\mathrm{Re}(\alpha)^2 + r - e^{2r}] \times \cosh[2\sqrt{2}\mathrm{Re}(\alpha)e^r]. \tag{41}$$

In Fig. 1 we plot $P_\alpha(r)$ for various values of α. For $\mathrm{Re}(\alpha) \gg 1$ the distribution is sharply peaked at $r = \ln\sqrt{2}\alpha$.

If the initial state is a vacuum state, the output state is a squeezed vacuum state $|0, r\rangle$. The resulting conditional hyperbolic phase distribution is

$$p(\tilde{r}|r) = 2\left(\frac{e^{-2r}}{\pi}\right)^{\frac{1}{2}} \exp[\tilde{r} - e^{2(\tilde{r}-r)}]. \tag{42}$$

The mean and variance of this distribution are given by

$$E(\tilde{r}) = r - (C + 2\ln 2)/2 \tag{43}$$

$$\approx r - 0.9817, \tag{44}$$

$$E(\Delta\tilde{r}^2) = \frac{1}{4}\zeta\left(2, \frac{1}{2}\right) \tag{45}$$

$$\approx 1.2337, \tag{46}$$

where C is the Euler gamma constant ($C = 0.577\,215\ldots$); $\zeta(x, y)$ denotes the Riemann zeta function, and $\Delta\tilde{r} = \tilde{r} - E(\tilde{r})$. Note that the noise is independent of the degree of squeezing. Thus the signal-to-noise ratio (S) defined by $S = \frac{E(\tilde{r})^2}{E(\Delta\tilde{r}^2)}$ increases quadratically with the squeeze parameter, away from zero at $r = 0.9817$. As a comparison we consider how the squeeze parameter would be estimated in this case using available measurement methods. As there is no coherent phase information in this state, the best we can do is to measure the photon number. The mean and variance of the photon number distribution for a squeezed vacuum state are $E(n) = \sinh^2 r$ and $E(\Delta n^2) = \cosh^2 r \sinh^2 r$. Thus the signal-to-noise ratio is $\tanh^2 r$, which is always less than 1 and approaches 1 only as $r \to \infty$.

If the squeezing were produced by parametric amplification of a coherent state, one would use homodyne detection [10] to determine the amplitude gain and thus the squeezing parameter. In this case a squeeze parameter is estimated by measuring a quadrature phase amplitude

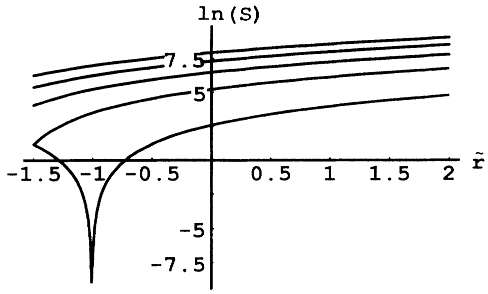

FIG. 2. A plot of the signal-to-noise ratio S defined as the ratio of the mean squared signal to the signal variance, for an initial coherent state with varying real amplitudes, plotted as a function of the imposed squeezing. The amplitudes increase from bottom to top as α=2,4,6,8,10.

variable, $x_\theta = ae^{i\theta} + a^\dagger e^{-i\theta}$. In order to compare this kind of measurement to the measurement of hyperbolic phase we consider the input state to the squeezing device to be a coherent state. The quadrature phase signal-to-noise ratio in this simple single mode treatment is then found to be unchanged from input to output as both the mean amplitude squared and the noise increase in the same way with the squeezing parameter.

The hyperbolic phase distribution for the squeezed coherent state (that is, a two photon coherent state [11]) with real amplitude, transformed according to Eq. (1), is

$$P(\tilde{r}|r) = 2\left(\frac{e^{-2r}}{\pi}\right)^{1/2} \exp[-2\alpha^2 + \tilde{r} - e^{2(\tilde{r}-r)}] \times \cosh(2\sqrt{2}\alpha e^{\tilde{r}-r}). \tag{47}$$

For large α we can approximate the cosh by an exponential and then it is easy to see that this distribution is peaked at $\tilde{r} \approx r + \ln(\sqrt{2}\alpha)$. This gives the approximate value of $\tilde{r}$ at which the mean goes to zero. The width of the distribution, like that for a squeezed vacuum state, is almost independent of the squeezing imposed on the state by the interaction. Thus the signal-to-noise ratio must increase with increasing squeezing, around an offset determined by $-\alpha$. In Fig. 2 we plot the log of the signal-to-noise ratio versus r for various values of α. It is clear that for fixed r the quality of the measurement may be made arbitrarily good by increasing the input power, that is, increasing $|\alpha|^2$. Clearly this is better than measurement of quadrature phase amplitude.

III. DISCUSSION AND CONCLUSION

In this paper we have defined a representation for the single-mode field called the hyperbolic phase representation. The resulting probability distributions obtain physical significance through the result that hyperbolic phase distributions enable optimal estimation of a squeeze parameter. This is the analog of the well-known result that the Susskind-Glogower phase distributions realize optimal estimation of a phase shift [2]. Unfortunately hyperbolic phase measurements must suffer from the same problems as Susskind-Glogower phase measurement. If one were to measure hyperbolic phase arbitrarily accurately the system would be left in a hyperbolic phase eigenstate, which is easily seen to be a state of infinite energy. One expects, however, that there are physical measurements that approximate arbitrarily closely the hyperbolic phase distribution. To determine what such measurements might be it is necessary to find the operator that is diagonal in the hyperbolic phase representation. This operator is $\hat{R} = \ln|\hat{x}|$ as can be seen as follows. Using Eq. (37) we find that, in the position representation,

$$\langle x|\hat{R}|r\rangle_\pm = r\langle x|r\rangle_\pm. \tag{48}$$

Thus $\hat{R}|r\rangle_\pm = r|r\rangle_\pm$.

It is not clear how to measure this operator directly in quantum optics. However, homodyne detection can, in principle, give the complete probability distribution for x, from which any moment of $\hat{R}$ could be constructed. This does not leave the system in an infinite energy eigenstate as homodyne measurements are made on a cavity field state as it damps through the end mirrors. The resulting distribution refers to the intial state inside the cavity. The final state in the cavity is the vacuum state. A related scheme is optical homodyne tomography [12], which also enables arbitrary moments of $\hat{R}$ to be constructed.

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