Optimal Quantum Measurements for Phase Estimation

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Quantum information theory is applied to practical interferometer-based phase measurements to deduce the optimal phase measurement scheme with two optical modes. Optimal phase measurements, given ideal input states, reveal an asymptotic $1/n$ decrease in phase uncertainty $\Delta\theta$ for n the mean photon number of the input state. In contradistinction to previous schemes for realizing the number-phase uncertainty limit, the $1/n$ limit achieved here is *independent* of the interferometer phase shift; prior information about the expected phase shift is not necessary to attain this limit. These results apply more generally to su(2) and so(3) phase parameter estimation.

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The precise determination of phase shifts is an important issue for quantum measurement theory, both for applications such as gravitational wave detection [1] and because phase measurement sensitivity is fundamentally limited by the complementarity between photon number and phase [2]. In practice, phase difference measurements are performed using interferometers and optimizing the phase estimation has focused on adopting the ideal input states [3,4]; a photon number difference measurement of the two output fields from the interferometer yields an asymptotic $1/n$ decrease in phase uncertainty $\Delta\theta$ for *particular* values of the phase shift. Here we determine the optimal phase measurement scheme for two optical modes [4,5] in arbitrary input states, using quantum parameter estimation theory [6–8]. We determine the phase uncertainty $\Delta\theta$ for ideal input states (mean photon number n) which reveals $1/n$ decrease for large n; this uncertainty is *independent* of the choice of phase shift and is therefore the ideal measurement which corresponds to interferometric phase shift measurements at the Heisenberg uncertainty limit.

We introduce a scheme which might be realized using a Mach-Zehnder interferometer and a suitable two-mode input state. The two input modes are mixed at a beam splitter and propagate along (possibly distinct) optical paths to a final mixing beam splitter which produces the two output modes. All measurements are performed on the output modes, and we identify the particular joint observable on the output modes which yields an optimal determination of differential phase shifts between the two paths of the interferometer.

The two annihilation operators for the two input modes into an interferometer are designated $\hat{a}$ and $\hat{b}$, respectively. Using the Schwinger representation [3,9]

$$\hat{J}_x = (\hat{a}^\dagger\hat{b} + \hat{a}\hat{b}^\dagger)/2, \quad \hat{J}_y = (\hat{a}^\dagger\hat{b} - \hat{a}\hat{b}^\dagger)/2i,$$
$$\hat{J}_z = (\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b})/2, \quad \hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2, \qquad (1)$$

it follows that the common eigenstate of $\hat{J}_z$ and $\hat{J}^2$ is the two-mode Fock state

$$|j\mu\rangle_z = |j+\mu\rangle_a |j-\mu\rangle_b \qquad (2)$$

with eigenvalues μ and $j(j+1)$, respectively, where $|j+\mu\rangle_a$ is the Fock number state with $j+\mu$ photons entering port a, and $|j-\mu\rangle_b$ is the Fock state entering port b. Although $j \pm \mu$ must be integers, j and μ can both be integers or both be half-odd integers. An arbitrary pure input state can be expressed as

$$|\Psi\rangle = \sum_{m,n=0}^{\infty} \Psi_{mn}|m\rangle_a|n\rangle_b = \sum_{2j=0}^{\infty}\sum_{\mu=-j}^{j} \tilde{\Psi}_{j\mu}|j\mu\rangle_z, \qquad (3)$$

where the sum over $2j$ indicates that the sum includes j both integer and half-odd integer.

The 50/50 beam splitter and phase shift operators are given by [3,10]

$$\hat{\mathcal{B}}_\pm = \exp(\pm i\pi\hat{J}_x/2), \quad \hat{\mathcal{P}}(\phi) = \exp(i\phi\hat{J}_z), \qquad (4)$$

respectively, where the $\pm$ choice is an adjustable phase shift, and we assume that equal and opposite phase shifts exist in each arm of the interferometer. The interferometer transformation is thus given by

$$\hat{I}(\phi) = \hat{\mathcal{B}}_-\hat{\mathcal{P}}(\phi)\hat{\mathcal{B}}_+ = e^{-i\phi\hat{J}_y} \qquad (5)$$

and can be regarded as a device which induces a linear rotation of the input state by an angle ϕ about the $\hat{J}_y$ axis. In this paper we take the perspective of quantum parameter estimation [11]. Our objective is to determine the optimal measurement scheme, specified as a positive operator-valued measure, to estimate the phase shift parameter ϕ.

The interferometer matrix elements are given by

$$I^j_{\mu\nu}(\phi) = {}_z\langle j\mu|\hat{I}(\phi)|j\nu\rangle_z = 2^{-\mu}\left[\frac{(j-\mu)!}{(j-\nu)!}\frac{(j+\mu)}{(j+\nu)}\right]^{1/2}(1-\cos\phi)^{(\mu-\nu)/2}$$
$$\times (1+\cos\phi)^{(\mu+\nu)/2}P_{j-\mu}^{(\mu-\nu,\mu+\nu)}(\cos\phi), \quad \mu-\nu > -1, \quad \mu+\nu > -1, \qquad (6)$$

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where $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials [12], and symmetry rules produce the relations [13]

$$I^j_{\mu\nu}(\phi) = (-1)^{\mu-\nu} I^j_{\nu\mu}(\phi) = I^j_{-\nu,-\mu}(\phi). \quad (7)$$

Asymptotic limits for the rotation operator matrix elements have been studied [13–15], and the area of overlap technique produces the useful expression [15]

$$I^j_{\mu 0}(\pi/2) \approx (-1)^{j-\mu}\sqrt{\frac{2}{\pi}}\frac{\cos[(j-\mu)\pi/2]}{[j(j+1)-\mu^2]^{1/4}}, \quad (8)$$

which vanishes for $j-\mu$ odd.

We now proceed to a determination of the fundamental limit to phase shift measurements in the interferometer [6,7,11]. A positive operator-valued measure (POVM), or effect, is an operator $\hat{\mathsf{E}}(x)$ [16,17], which satisfies the criteria (i) $\int dx\, \hat{\mathsf{E}}(x) = \hat{1}$, and (ii) the spectrum of $\hat{\mathsf{E}}(x)$ is positive for all x. The probability density for the corresponding measurement results is

$$P(x) = \mathrm{tr}\Big[\hat{\rho}\hat{\mathsf{E}}(x)\Big] \quad (9)$$

for $\hat{\rho}$ the density matrix for the system.

To construct the effect for optimal estimation of the phase shift parameter ϕ in Eq. (5), we must first determine the representation in which the generator $\hat{J}_y$ is a pure differential operator. In this representation the unitary transformation in Eq. (5) is a pure translation. This ensures the resulting effect produces a shift-invariant probability density. This representation is constructed as follows. By introducing the $\hat{J}_y$ eigenstates $\{|j\mu\rangle_y\}$, which are analogous to the $\hat{J}_z$ eigenstates of Eq. (2), the normalized phase state can be written as [18]

$$|j\theta\rangle = (2j+1)^{-1/2}\sum_{\mu=-j}^{j} e^{i\mu\theta}|j\mu\rangle_y. \quad (10)$$

The action of the unitary transformation Eq. (5) on these states is easily seen to be a pure translation.

The overlap of two distinct phase states is given by

$$(2j+1)\langle j\theta|j\theta+2\phi\rangle = U_{2j}(\cos\phi) = \mathrm{Tr}\Big[\hat{I}(2\phi)\Big], \quad (11)$$

when $U_n(x)$ is the Chebyshev polynomial of the second kind [19]. The phase state (10) has been normalized such that

$$\lim_{j\to\infty}[(2j+1)\langle j\theta'|j\theta\rangle] = 2\pi\delta(\theta-\theta') \quad (12)$$

and

$$\lim_{\theta\to\theta'}\langle j\theta'|j\theta\rangle = 1. \quad (13)$$

The effect

$$\hat{\mathsf{E}}(\theta)d\theta = (2j+1)|j\theta\rangle\langle j\theta|d\theta/2\pi \quad (14)$$

satisfies

$$\hat{I}^\dagger(\phi)\hat{\mathsf{E}}(\theta)\hat{I}(\phi) = \hat{\mathsf{E}}(\theta+\phi), \quad (15)$$

and, for an arbitrary input state $\hat{\rho}$, the probability distribution over θ is given by

$$P_j(\theta|\phi)d\theta = (2j+1)\langle j\theta+\phi|\hat{\rho}|j\theta+\phi\rangle d\theta/2\pi. \quad (16)$$

As an example, consider the distribution for the phase state $|j,\theta=0\rangle$. This is

$$P_j(\theta|\phi=0)d\theta = \frac{[U_{2j}(\cos\theta/2)]^2}{(2j+1)}\frac{d\theta}{2\pi}, \quad (17)$$

which is normalized such that [19]

$$\int_{-\pi}^{\pi} P_j(\theta|\phi=0)\,d\theta = 1. \quad (18)$$

The odd moments of $\sin(\theta/2)$ for the phase state distribution (17) are zero as $\sin^m(\theta/2)P_j(\theta)$ is an odd function with respect to $\theta/2$ for m odd. The variance with respect to $\sin(\theta/2)$ is thus

$$\left[\Delta\sin\left(\frac{\theta}{2}\right)\right]^2 = \int_{-\pi}^{\pi}\sin^2\left(\frac{\theta}{2}\right)P_j(\theta)\,d\theta = \frac{1}{2(2j+1)}, \quad (19)$$

which, for small θ, is approximated by

$$\Delta\theta \approx (j+1/2)^{-1/2}. \quad (20)$$

The phase uncertainty decreases as $1/\sqrt{j}$, and the phase state is thus not the best input state given phase measurements of the form (14).

Usually we are given states written as a superposition of the eigenstates of $\hat{J}_z$. In that case we require the phase state matrix elements

$$\langle j\theta|j\nu\rangle_z = (2j+1)^{-1/2}\sum_{\mu=-j}^{j} e^{-i\mu\theta}{}_y\langle j\mu|j\nu\rangle_z \quad (21)$$

and

$$\begin{aligned} {}_y\langle j\mu|j\nu\rangle_z &= {}_z\langle j\mu|e^{-i(\pi/2)\hat{J}_x}|j\nu\rangle_z \\ &= e^{i(\pi/2)(\nu-\mu)}I^j_{\mu\nu}(\pi/2). \end{aligned} \quad (22)$$

A number of authors have suggested that the best input states for interferometer based phase measurements have equal Fock number states in each interferometer input mode [3–5] corresponding to making j an integer and $\nu = 0$. We now consider this case.

The overlap is given by

$$\langle j\theta | j0\rangle_z = (2j+1)^{-1/2} \sum_{\mu=-j}^{j} e^{-i\mu(\theta+\pi/2)} I^{j}_{\mu 0}\left(\frac{\pi}{2}\right), \tag{23}$$

and the phase distribution $P_j(\theta|\phi = 0)$ is shown in Fig. 1(a). Provided that the phase spread is small compared to the separation of the two dominant peaks, the double-peaked nature of the distribution is not detrimental, provided that the phase is measured modulo π. A predominantly single-peaked phase distribution can be obtained by assuming the input state $[|j0\rangle_z + |j1\rangle_z]/\sqrt{2}$ [3].

The large j asymptotic expansion for Eq. (23), obtained using Eq. (8), is given by

$$\langle j\theta | j0\rangle = \frac{(-1)^{j/2}}{\sqrt{2j+1}} \sqrt{\frac{2}{\pi}} \sum_{\mu=-j/2}^{j/2} \frac{e^{-2i\mu\theta}}{[j(j+1)-(2\mu)^2]^{1/4}} \tag{24}$$

and holds for j even (μ summed over integers) and for j (μ summed over half-odd integers). In the limit of large j, the sum (24) approaches the integral

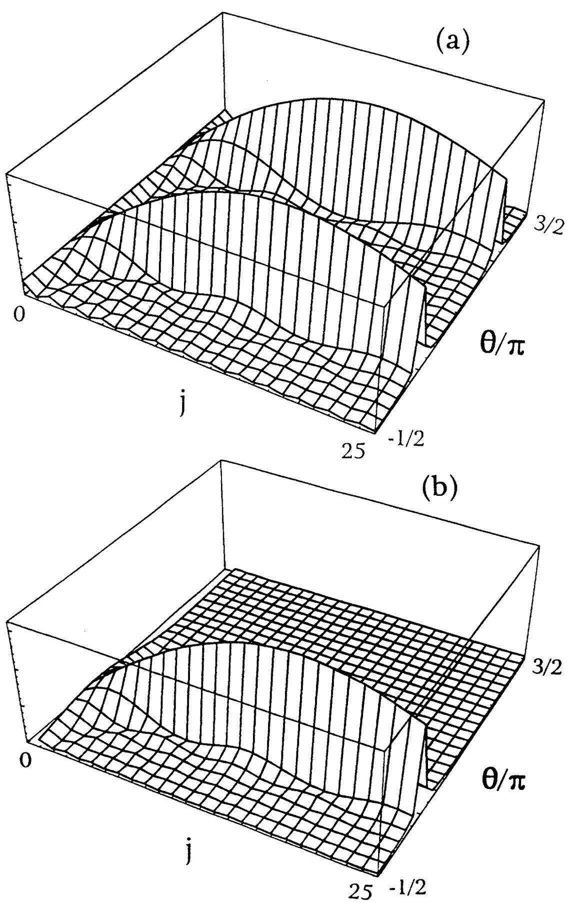

FIG. 1. Plots of $P_j(\theta|\phi = 0)$ vs θ/π for the input state $|j0\rangle_z$ for (a) the exact solution and (b) the large j asymptotic expansion.

$$|\langle j\theta | j0\rangle| = \sqrt{\frac{2}{\pi}}\, j^{-1} \sum_{\mu=-j/2}^{j/2} e^{-2i\mu\theta}[1 - 4\mu^2/j^2]^{-1/4}$$
$$\approx \frac{1}{\sqrt{\pi}} \int_0^1 dx \cos(j\theta x)(1-x^2)^{-1/4}, \tag{25}$$

which reduces to [19]

$$|\langle j\theta | j0\rangle| = 2^{-3/4}\Gamma(3/4)(j\theta)^{-1/4} J_{1/4}(j\theta), \tag{26}$$

and the phase distribution for the input state $|j0\rangle_z$ is the bounded function

$$P_j(\theta|\phi = 0) = \frac{2j+1}{2\pi} \frac{[\Gamma(3/4)]^2}{2^{3/2}} \frac{[J_{1/4}(j\theta)]^2}{\sqrt{j\theta}}, \tag{27}$$

which is plotted in Fig. 1(b).

The Bessel function in Eq. (26) has an infinite number of zeros and all are real and symmetric about $j\theta = 0$. The width of the overlap is estimated by the width

$$\Delta(j\theta) \approx j\theta_0$$

for $j\theta_0$ the first zero which occurs near $j\theta_0 \approx 2.781$, and, therefore, the phase uncertainty is approximated by

$$\Delta\theta \approx 2.781/j, \tag{28}$$

which shows that the phase uncertainty varies inversely with j. For the optimal POVM scheme, which corresponds to phase measurements, the input state $|j0\rangle_z$ gives a phase uncertainty which varies as $1/j$ in the large j limit. A $1/j$ (or $1/n$) asymptotic limit for $\Delta\theta$ has been determined previously for (a) photon number difference ($\hat{J}_z$) measurements but only as a minimum for particular values of the phase shift parameter [3] and for (b) the same input state but a different choice of POVM based on harmonic oscillator phase state projectors [4]. Here we have determined a $1/j$ asymptotic limit for phase sensitivity which is based on using *optimal* phase measurements for interferometers using su(2) phase state projectors.

A two-mode squeezed vacuum input state provides a way to realize a superposition of $\{|n\rangle_a |n\rangle_b\}$ states which can be inserted into the interferometer [4]. The two-mode squeezed vacuum state is expressed as [20]

$$|r\rangle_{\mathrm{sv}} = \sum_{n=0}^{\infty} \sqrt{\mathcal{P}_{\bar{j}}(j)}\, |n\rangle_a |n\rangle_b = \sum_{j=0}^{\infty} \sqrt{\mathcal{P}_{\bar{j}}(j)}\, |j0\rangle_z \tag{29}$$

for $\bar{j} = \sinh^2 r$ the mean photon number and

$$\mathcal{P}_{\bar{j}}(j) = \mathrm{sech}^2 r (\tanh r)^{2j} = (i + \bar{j})^{-1} \left(\frac{\bar{j}}{1+\bar{j}}\right)^{j}, \tag{30}$$

a "thermal distribution" of j's with effective "temperature"

$$T_r = \frac{\hbar\omega}{k \ln \coth^2 r}. \tag{31}$$

The phase distribution for the squeezed vacuum state is

$$P_j(\theta|\phi = 0) = \pi^{-1}(j + 1/2)\mathcal{P}_{\bar{j}}(j)\,|\langle j\theta|j0\rangle_z|^2; \quad (32)$$

therefore, the two-mode squeezed vacuum serves as a source of product states $\{|n\rangle_a |n\rangle_b\}$ with a thermal distribution of n values. Unfortunately, the thermal distribution (30) is heavily weighted with undesirable low n input states making the squeezed vacuum source an "energy-expensive" procedure for realizing a $1/n$ limit for reducing phase uncertainty $\Delta\theta$.

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