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Quantum-noise reduction in a driven cavity with feedback

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We show that amplitude-squeezed states may be produced by driving a feedback-controlled cavity with a coherent input signal. The feedback controls the transmissivity of one output from the cavity and is essentially equivalent to nonlinear absorption. The cavity effectively acts as a nonlinear reflector. Hence, amplitude-squeezed states with arbitrarily strong coherent intensities can be obtained.

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I. INTRODUCTION

Feedback has long been used to stabilize the operation of optical cavities and lasers. Recently Yamamoto and co-workers [1] demonstrated that feedback could be used to reduce the intensity fluctuations in the light emitted by a semiconductor laser. In their experiment, the light from the laser illuminated a photodetector and the resulting photocurrent was fed back to the injection current of the laser. The in-loop field, that is, the light between the laser and the photodetector, does exhibit sub-shot-noise statistics, however, it is difficult to extract this field to exploit the reduced noise properties in applications. To overcome this problem Wiseman and Milburn [3] proposed a model in which the transmissivity of one output coupler from a laser cavity is controlled by a feedback circuit similar to that of Yamamoto and co-workers. The light leaving the cavity through the feedback-controlled output coupler falls on a photodetector and the resulting current used to control the transmissivity at the output. However, unlike the scheme of Yamamoto and co-workers, this scheme includes another cavity output port from which a sub-shot-noise light field may be extracted.

In both the laser feedback schemes described above the light produced has reduced intensity noise but, as is typical of lasing devices, the noise is phase independent. Squeezed states are a more general class of states of the field for which the noise is very phase dependent. Such states can exhibit either reduced intensity fluctuations or reduced phase fluctuations. It is the purpose of this paper to show that the feedback model of Wiseman and Milburn can be used to produce squeezed states. To achieve this we consider an empty, externally driven feedback-controlled cavity. The cavity thus becomes a nonlinear reflector as far as the input light is concerned. The nonlinearity, in fact, appears as nonlinear absorption.

The cavity, of course, may be designed to operate at whatever the laser input frequency is. This flexibility is one of the great attractions of producing squeezed states in this way. One does not need to rely on the frequency restrictions of a particular nonlinear optical material. Feedback enables one to engineer the nonlinearity at any frequency. The other advantage comes from the fact that the squeezing is induced on a field which has a large coherent amplitude. The theoretical description of the model is based on a model of feedback control of a travelling wave proposed by Shapiro and co-workers [2], and the general theory of photodetection from an optical cavity, given by Srinivas and Davies [4].

II. QUANTUM THEORY OF FEEDBACK-CONTROLLED CAVITY TRANSMISSIVITY

In Fig. 1 we indicate schematically the feedback system. The cavity has two outputs. The transmissivity of one output is controlled by a current derived from a photodetector illuminated by the light leaving through that port. The current controlled beam splitter may be realized in a number of ways using acousto-optic modulators and perhaps polarizing filters. A more detailed discussion of the experimental realization will be given in a future paper. The other output of the cavity is illuminated by a strong coherent field. As we now show, the input field sees a cavity containing an effective nonlinear absorber due to the feedback, where the ouput transmissivity is increased with increasing detection rate at the photodetector. The zero-feedback damping rates at the input coupler and feedback controlled coupler are γ_2 and γ_1, respectively.

The count rate at the photodetector is determined by a count superoperator $\Lambda(t)$, which depends on the counting history through

$$\Lambda(t) = \Lambda_0 \left(1 + A \int_{-\infty}^{t} e^{-\beta(t-u)} \Lambda(u)\right), \tag{2.1}$$

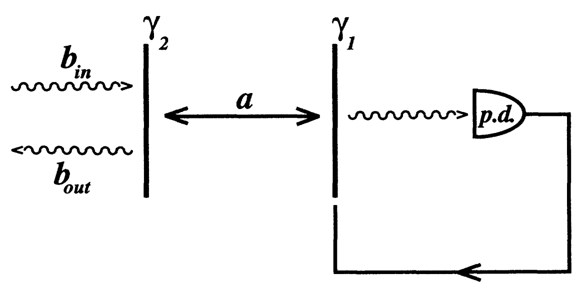

FIG. 1. A schematic representation of the cavity with feedback model where p.d. denotes the photodetector. The modes $b_{\rm in}$ and $b_{\rm out}$ denote the input and output fields, respectively.

where

$$\Lambda_0 = \gamma_1 \mathcal{J} \tag{2.2}$$

and

$$\mathcal{J}\rho = a\rho a^\dagger \tag{2.3}$$

is the count superoperator in the absence of the feedback circuit. The quantity A is a dimensionless parameter determining the strength of the feedback, and $\frac{1}{\beta}$ is the memory parameter . This feedback model was motivated by a classical Markovian self-exciting point process [3]. If we iterate Eq. (2.1) we find that the count superoperator defines a stationary count process. The count superoperator is given by

$$\Lambda = \gamma_1 \mathcal{J} \sum_{n=0}^{\infty} \chi^n \mathcal{J}^n, \tag{2.4}$$

where

$$\chi = \frac{\gamma_1 A}{\beta} \tag{2.5}$$

is the feedback parameter. This equation converges provided

$$\mathrm{Tr}(|\chi|\mathcal{J}\rho) \; < \; 1. \tag{2.6}$$

This is effectively a restriction on the allowed photon number for physically acceptable solutions. Equation (2.4) is the series expansion for

$$\Lambda = \gamma_1 \mathcal{J}\left(1 + \frac{A}{\beta}\Lambda\right). \tag{2.7}$$

The mean count rate is determined by a rate operator (not a superoperator) R, defined by

$$\frac{d\bar{m}}{dt} = \mathrm{Tr}(\rho R) = \mathrm{Tr}(\Lambda\rho), \tag{2.8}$$

hence R is given by

$$R = \gamma_1 \sum_{n=0}^{\infty} \chi^n (a^\dagger)^{n+1} a^{n+1}, \tag{2.9}$$

under the assumption that every photon that leaves the cavity is counted.

Using the Srinivas-Davies result for the time dependence of ρ,

$$\frac{d\rho}{dt} = \Lambda\rho - \frac{R}{2}\rho - \rho\frac{R}{2}, \tag{2.10}$$

we get the following master equation for a cavity with feedback:

$$\frac{d\rho_{\mathrm{fb}}}{dt} = \frac{\gamma_1}{2}\sum_{n=0}^{\infty}\chi^n \Big[2a^{n+1}\rho(a^\dagger)^{n+1} - (a^\dagger)^{n+1}a^{n+1}\rho - \rho(a^\dagger)^{n+1}a^{n+1}\Big]. \tag{2.11}$$

Note that we are using a cavity driven by a coherent input at the other mirror. This mirror has a damping rate γ_2. We must therefore include driving and damping terms in the master equation. The complete master equation is therefore given by

$$\frac{d\rho}{dt} = [\epsilon a^\dagger - \epsilon^* a, \rho] + \frac{\gamma_2}{2}\left(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a\right) + \frac{d\rho_{\mathrm{fb}}}{dt}, \tag{2.12}$$

where we assumed that the bath is at zero temperature. We also note that ϵ is amplitude of the driving field *inside* the cavity.

In order to solve Eq. (2.12) we transform it to the Drummond-Gardiner positive-P representation [5]. We then obtain the following equation:

$$\frac{\partial P(\boldsymbol{\alpha},t)}{\partial t} = \Bigg\{ -\left(\epsilon\frac{\partial}{\partial\alpha} + \epsilon^*\frac{\partial}{\partial\beta}\right) + \frac{\gamma_2}{2}\left(\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\beta}\beta\right) + \frac{\gamma_1}{2}\sum_{n=0}^{\infty}\chi^n\left[2(\alpha\beta)^{n+1} - \left(\beta - \frac{\partial}{\partial\alpha}\right)^{n+1}\alpha^{n+1} - \left(\alpha - \frac{\partial}{\partial\beta}\right)^{n+1}\beta^{n+1}\right]\Bigg\} P(\boldsymbol{\alpha},t), \tag{2.13}$$

where $\boldsymbol{\alpha} = (\alpha,\beta)^T$. This equation can be truncated to first order in χ if we assume that

$$\mathrm{Tr}(|\chi|\mathcal{J}\rho) \ll 1\,. \tag{2.14}$$

Hence we obtain the following Fokker-Planck equation for the positive-P function:

$$\frac{\partial P(\boldsymbol{\alpha},t)}{\partial t} = \Bigg[-\frac{\partial}{\partial\alpha}\left(\epsilon - \frac{\gamma}{2}\alpha - \gamma_1\chi\alpha^2\beta\right) - \frac{\partial}{\partial\beta}\left(\epsilon^* - \frac{\gamma}{2}\beta - \gamma_1\chi\alpha\beta^2\right) - \frac{\partial^2}{\partial\alpha^2}\frac{\gamma_1\chi}{2}\alpha^2 - \frac{\partial^2}{\partial\beta^2}\frac{\gamma_1\chi}{2}\beta^2\Bigg] P(\boldsymbol{\alpha},t), \tag{2.15}$$

where $\gamma = \gamma_1 + \gamma_2$.

This Fokker-Planck equation is identical to that used by Drummond, Gardiner, and Walls [6] and Collett and Walls [7] when $\gamma = \gamma_2$ and $\gamma_1\chi \to 2\chi$. This means that the system considered here is formally equivalent to the two-photon-absorption model. The difference lies in the extra linear damping term γ_1 which also controls the effective two-photon-loss rate $\gamma_1\chi$ in Eq. (2.15).

III. LINEARIZED ANALYSIS OF FLUCTUATIONS

The Fokker-Planck equation derived in the preceding section is equivalent to the Ito stochastic differential equations given by

$$\frac{d\boldsymbol{\alpha}}{dt} = \mathbf{A} + \tilde{B}\mathbf{E}, \tag{3.1}$$

where the drift vector is

$$\mathbf{A} = \begin{pmatrix} \epsilon - \frac{\gamma}{2}\alpha - \gamma_1\chi\alpha^2\beta \\ \epsilon^* - \frac{\gamma}{2}\beta - \gamma_1\chi\alpha\beta^2 \end{pmatrix}. \tag{3.2}$$

The matrix $\tilde{B}$ is a postive semidefinite matrix given by

$$\tilde{B}\tilde{B}^T = \tilde{D}(\boldsymbol{\alpha}) = \begin{pmatrix} -\gamma_1\chi\alpha^2 & 0 \\ 0 & -\gamma_1\chi\beta^2 \end{pmatrix}, \tag{3.3}$$

where $\tilde{D}(\boldsymbol{\alpha})$ is the diffusion matrix. The vector $\mathbf{E} = (E_1, E_2)^T$ is the noise vector where

$$\langle E_i(t)E_j(t')\rangle = \delta_{i,j}\delta(t-t'). \tag{3.4}$$

In our case we get the following stochastic equations:

$$\begin{aligned} \frac{d\alpha}{dt} &= \epsilon - \frac{\gamma}{2}\alpha - \gamma_1\chi\alpha^2\beta + i\sqrt{\gamma_1\chi}\alpha E_1, \\ \frac{d\beta}{dt} &= \epsilon^* - \frac{\gamma}{2}\beta - \gamma_1\chi\alpha\beta^2 + i\sqrt{\gamma_1\chi}\beta E_2. \end{aligned} \tag{3.5}$$

In the semiclassical steady state these equations have a steady-state solution given by

$$\alpha_0 = \frac{\epsilon}{\frac{\gamma}{2} + \gamma_1\chi|\alpha_0|^2}. \tag{3.6}$$

If we now set $\bar{n} = |\alpha_0|^2$ to be the mean steady-state photon number inside the cavity, we obtain the following cubic in $\chi\bar{n}$:

$$\chi\bar{n}\left(\frac{D}{2} + \chi\bar{n}\right)^2 = \frac{\chi|\epsilon|^2}{\gamma_1^2}, \tag{3.7}$$

where $D = \gamma/\gamma_1 = 1 + \gamma_2/\gamma_1$ with $\gamma = \gamma_1 + \gamma_2$. Note that the value of D can never go below 1.

In order to find the stability conditions and noise characteristics of the system we linearize these equations about the deterministic steady state given by Eq. (3.6) by setting

$$\begin{aligned} \alpha(t) &= \alpha_0 + \delta\alpha(t), \\ \beta(t) &= \alpha_0^* + \delta\beta(t), \end{aligned} \tag{3.8}$$

where $\delta\alpha$ and $\delta\beta$ are small, time-dependent fluctuations about the steady states α_0 and α_0^*. We then get the following equation for the fluctuations:

$$\frac{d}{dt}(\delta\boldsymbol{\alpha}) = -\tilde{A}\delta\boldsymbol{\alpha} + \mathbf{F}, \tag{3.9}$$

where $\delta\boldsymbol{\alpha} = (\delta\alpha, \delta\beta)^T$ and $\mathbf{F} = (F_1, F_2)^T$. The noise terms are given by $F_j = i\sqrt{\gamma_1\chi\bar{n}}$ and

$$\tilde{A} = \begin{pmatrix} \frac{\gamma}{2} + 2\gamma_1\chi\bar{n} & \gamma_1\chi\bar{n} \\ \gamma_1\chi\bar{n} & \frac{\gamma}{2} + 2\gamma_1\chi\bar{n} \end{pmatrix}. \tag{3.10}$$

We note that

$$\langle F_i(t)F_j(t')\rangle = -\gamma_1\chi\bar{n}\delta_{i,j}\delta(t-t'). \tag{3.11}$$

In order for this system of equations to be stable the real part of the eigenvalues of the matrix $\tilde{A}$ must be greater than zero. These eigenvalues are

$$\begin{aligned} \lambda_1 &= \frac{\gamma}{2} + \gamma_1\chi\bar{n}, \\ \lambda_2 &= \frac{\gamma}{2} + 3\gamma_1\chi\bar{n}. \end{aligned} \tag{3.12}$$

As both eigenvalues are always positive, the system is always stable. This is not unreasonable since an increase in the number of photons inside the cavity leads to an increased damping rate out of the cavity which reduces the photon number inside it. Hence the system is always stable.

A. Noise characteristics of the intracavity radiation field

In order to find the noise characteristics of the light inside the cavity we need to calculate the covariance matrix from our fluctuation equations (3.5). This matrix can be found, by the method of Gardiner [8], to be

$$\begin{aligned} \sigma &= \begin{pmatrix} \langle\delta\alpha^2\rangle & \langle\delta\alpha\delta\beta\rangle \\ \langle\delta\alpha\delta\beta\rangle & \langle\delta\beta^2\rangle \end{pmatrix} \\ &= \frac{-\chi\bar{n}}{(D+4\chi\bar{n})^2 - (2\chi\bar{n})^2}\begin{pmatrix} D+4\chi\bar{n} & -2\chi\bar{n} \\ -2\chi\bar{n} & D+4\chi\bar{n} \end{pmatrix}. \end{aligned} \tag{3.13}$$

These correlations can be directly related to the expectation values of products of a and $a^\dagger$ via

$$\begin{aligned} \langle a, a\rangle &= \langle\delta\alpha^2\rangle, \\ \langle a^\dagger, a\rangle &= \langle\delta\alpha\delta\beta,\rangle \\ \langle a^\dagger, a^\dagger\rangle &= \langle\delta\beta^2\rangle, \end{aligned} \tag{3.14}$$

where

$$\langle A, B\rangle = \langle AB\rangle - \langle A\rangle\langle B\rangle. \tag{3.15}$$

Using these we can find the variances in the quadrature phase observables $X_1 = a + a^\dagger$ and $X_2 = -i(a - a^\dagger)$:

$$V(X_1) = 1 - \frac{2\chi\bar{n}}{(D + 6\chi\bar{n})},$$
$$V(X_2) = 1 + \frac{2\chi\bar{n}}{(D + 2\chi\bar{n})}. \tag{3.16}$$

As we can see from Eq. (3.7) any required value of $\chi\bar{n}$ can be obtained independently of D by correctly selecting the driving power $|\epsilon|^2$. We can therefore set D and $\chi\bar{n}$ independently in Eq. (3.16). Thus we see that the maximum squeezing approaches $\frac{5}{7}$ as $D \to 1$ and $\chi\bar{n} \to 1$.

B. Photon statistics

The Q factor which measures the deviation from Poisson photon statistics is given by

$$Q(0) = \frac{V(a^\dagger a) - \langle a^\dagger a\rangle}{\langle a^\dagger a\rangle}. \tag{3.17}$$

To first order in $\chi\bar{n}$ this becomes

$$Q(0) \approx -2\chi\bar{n}. \tag{3.18}$$

As expected from the results of Sec. III A we observe sub-Poissonian statistics corresponding to amplitude squeezing.

C. Squeezing spectrum of the output field

In order to calculate the squeezing in the output field we need to find the spectral matrix for the fluctuations in the intracavity field. This matrix is defined by [8]

$$\tilde{S}(\omega) = (\tilde{A} + i\omega\tilde{I})^{-1}\tilde{D}(\tilde{A} - i\omega\tilde{I})^{-1} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}, \tag{3.19}$$

where $\tilde{A}$ is the drift matrix given by Eq. (3.10), $\tilde{D}$ is the diffusion matrix given by Eq. (3.3), and $\tilde{I}$ is the identity matrix. The normally ordered squeezing spectrum of the output field is related to the spectral matrix via [7]:

$$: S_{\pm}^{\text{out}}(\omega) := \gamma_2 \left[\pm e^{-2i\theta} S_{11} \pm e^{2i\theta} S_{22} + S_{12} + S_{21}\right], \tag{3.20}$$

where θ is the relative phase of the quadrature being measured. If we set $\theta = 0$ then : S_{+}^{out} : is the squeezing in the X_1 quadrature, or amplitude, and : S_{-}^{out} : is the squeezing in the X_2 quadrature, or phase. For this system these are

$$S_{+}(\Omega) =: S_{+}^{\text{out}}(\omega) := \frac{-2\nu\delta}{\left(\frac{D}{2} + 3\delta\right)^2 + \Omega^2},$$
$$S_{-}(\Omega) =: S_{-}^{\text{out}}(\omega) := \frac{2\nu\delta}{\left(\frac{D}{2} + \delta\right)^2 + \Omega^2}, \tag{3.21}$$

where $\Omega = \omega/\gamma_1$, $\nu = \gamma_2/\gamma_1$ and $\delta = \chi\bar{n}$. These spectra are both Lorentzian with the maximum squeezing given by

$$S_{+}(0) = \frac{-2\nu\delta}{\left(\frac{D}{2} + 3\delta\right)^2} = \frac{-2(D-1)\delta}{\left(\frac{D}{2} + 3\delta\right)^2} = \frac{-2\frac{\gamma_2}{\gamma_1}\chi\bar{n}}{\left(\frac{1}{2} + \frac{\gamma_2}{\gamma_1} + 3\chi\bar{n}\right)^2},$$
$$S_{-}(0) = \frac{2\nu\delta}{\left(\frac{D}{2} + \delta\right)^2} = \frac{2(D-1)\delta}{\left(\frac{D}{2} + \delta\right)^2} = \frac{-2\frac{\gamma_2}{\gamma_1}\chi\bar{n}}{\left(\frac{1}{2} + \frac{\gamma_2}{\gamma_1} + \chi\bar{n}\right)^2}. \tag{3.22}$$

These equations show that we get a reduction of fluctuations in the X_1 quadrature outside the cavity and a corresponding increase in the fluctuations in the X_2 quadrature. The optimum squeezing in the X_1 quadrature occurs at $D = 6\delta + 2$ and is given by

$$S_{+}^{\text{out}}(0) = \frac{-2\delta}{1 + 6\delta} = \frac{-2\chi\bar{n}}{1 + 6\chi\bar{n}}. \tag{3.23}$$

Since $\chi\bar{n}$ must be less than 1, the best obtainable squeezing is $-\frac{2}{7}$ when $\chi\bar{n} \to 1$ and $D \to 8$, or $\gamma_2/\gamma_1 \to 7$. This suggests that the reflectivity of the input-output coupler at the pump end of the cavity must be much less than that of the feedback end; in fact, it must be seven times smaller. It should be noted that the absolute maximum possible theoretical squeezing is -1 because of our definitions of X_1 and X_2.

Figure 2 shows the dependence of the squeezing outside the cavity on $D = (\gamma_1 + \gamma_2)/\gamma_1 = 1 + \gamma_2/\gamma_1$ and the scaled intracavity mean photon number $\chi\bar{n}$.

The best squeezing described here occurs at a value of $\chi\bar{n}$ close to 1. In Eq. (2.14), however, we assumed that $\chi\bar{n}$ is small for the purposes of the truncation of the equation for the positive-P function. Hence the best squeezing parameter regime is outside the region of validity of the model.

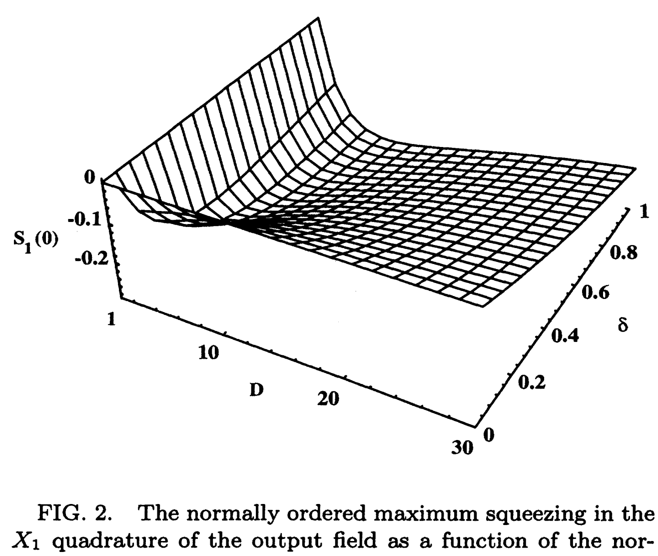

FIG. 2. The normally ordered maximum squeezing in the X_1 quadrature of the output field as a function of the normalized linear damping D and the scaled intracavity mean photon number $\chi\bar{n}$.

IV. CONCLUSION

In this paper we have shown that using a feedback scheme where the photodetector current outside a driven cavity is used to modify the damping rate can lead to squeezing in the field inside and outside the cavity. That is, it can produce squeezed states. The essential difference between this scheme and conventional laser with feedback schemes described in the introduction is the presence of a strong driving field and no intracavity amplification. The light field exiting the cavity through the same port through which the driving field enters can be extracted as a source of squeezed light. The cavity can be thought to be acting as a nonlinear reflector. This nonlinear reflectivity produces the reduced noise properties on a strong coherent light field.

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