

Interference in a spherical phase space and asymptotic behavior of the rotation matrices

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We extend the interference in the phase-space algorithm of Wheeler and Schleich [W.P. Schleich and J.A. Wheeler, *Nature* **326**, 574 (1987)] to the case of a compact, spherical topology in order to discuss the large j limits of the angular momentum marginal probability distributions. These distributions are given in terms of the standard rotation matrices. It is shown that the asymptotic distributions are given very simply by areas of overlap in the classical spherical phase-space parametrized by the components of angular momentum. The results indicate the very general validity of the interference in phase-space concept for computing semiclassical limits in quantum mechanics.

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I. INTRODUCTION

Wheeler, Schleich, and co-workers [1-3] have given an elegant method to find the semiclassical limits of the overlap integral of quantum states in L^2 . The method is geometrical and is based on areas of overlap of suitably defined classical distribution functions, in a flat classical phase space. In a fully classical theory if the area of overlap is made up of disjoint pieces one would simply sum the areas; however, in the semiclassical theory of Schleich and Wheeler, if the areas of overlap are not connected they must be added with an appropriate phase factor to obtain the correct result. Thus interference features can arise as an interference in phase space. The technique has been applied, for example, to oscillations in the transition amplitudes for Franck-Condon transitions [3], oscillations in the photon-number distributions for squeezed states [2,4], and interference fringes exhibited by superposition states [5]. Our primary objective in this paper is to show that the technique of interference may be extended to a spherical phase space. As this is the appropriate phase space to describe angular momentum, we can thus obtain the semiclassical limits of the marginal distributions for the components of angular momentum. For example, the probability distribution for \hat{J}_x measurements on a system prepared in a \hat{J}_z eigenstate in the limit of large j is given easily by simple geometry on the sphere. Furthermore, as this marginal distribution is in fact given in terms of the standard rotation matrices the method gives a simple algorithm for computing the asymptotic form of the rotation matrices. In the remainder of this introduction we present the method of Wheeler and Schleich in a form suitable to a discussion of the angular momentum phase space.

Consider a harmonic oscillator with unit mass and frequency, oscillating around the equilibrium position $q = 0$, and prepared in an energy eigenstate $u_n(x)$ with energy $E_n = \hbar(n + \frac{1}{2})$. Suppose this system is subjected to a sudden classical force, which causes the potential to be displaced to a new equilibrium position $q = d$ but other-

wise remains unchanged in shape. What is the probability that the system will make a transition from the state $u_n(x)$ to a new energy eigenstate $\psi_m(x)$ of the displaced oscillator? This is of course given by the modulus square of the transition amplitude

$$w_{mn} = \int_{-\infty}^{\infty} dq \psi_m(q)^* u_n(q). \quad (1)$$

Wheeler and Schleich give an elegant method for determining the size of this amplitude, in the semiclassical limit, based on interference in phase space. Before presenting this method we will give an alternative interpretation of the transition amplitude, which is more appropriate for our treatment of rotation matrices in Sec. II.

Consider an oscillator prepared in the state $u_n(q)$. Let \hat{H}' be a Hermitian operator representing a physical quantity with eigenstate $\psi_m(q)$. Clearly \hat{H}' is the operator

$$\hat{H}' = \frac{\hat{p}^2}{2} + \frac{(\hat{q} - d)^2}{2} \quad (2)$$

with eigenvalues $h_m = \hbar(m + \frac{1}{2})$ and the eigenstates are related to the energy eigenstates by $|\psi_m\rangle = \hat{D}(d)|m\rangle$, where $\hat{D}(d)$ is the unitary displacement operator [6]. It is easy to see that the probability of obtaining the result h_m for a measurement of \hat{H}' for a system in the state $|n\rangle$ is given by the modulus square of the overlap in Eq. (1),

$$P(h_m) = |w_{nm}|^2. \quad (3)$$

This interpretation shows quite clearly that the interference fringes in w_{mn} are completely analogous to the interference in the probability distributions for other quantities; e.g., the position density for the two-slit experiment.

To make this point more directly let us represent the states $|\psi_m\rangle$ and $|n\rangle$ by classical distributions in phase space. Each state is represented by an annulus of width unity centered on a Kramers trajectory defined through the semiclassical quantization condition for the action

$$J_m = \oint dq p_m(q) \quad (4)$$

$$= 2\pi\hbar(m + \frac{1}{2}). \quad (5)$$

Thus the Kramers trajectory is defined by

$$\frac{1}{2}(p^2 + q^2) = \hbar(m + \frac{1}{2}). \quad (6)$$

The annulus representing $|n\rangle$ is centered on the origin, while the annulus representing $|\psi_m\rangle$ is centered on $q = d, p = 0$ (Fig. 1). These annuli are referred to as Planck-Bohr-Sommerfeld (PBS) bands. The area of a PBS band is $2\pi\hbar$. We now regard the PBS bands as classical phase-space densities. Let $P(p, q|n)$ be the conditional probability density for the point (p, q) , given that the system is in the PBS band labeled n , while $P(m|p, q)$ is the conditional probability of obtaining the result h_m for a measurement of the classical physical quantity corresponding to \hat{H}' , given the result (p, q) . These distributions are chosen as

$$P(p, q|n) = \begin{cases} \frac{1}{2\pi\hbar}, & 2n\hbar \leq p^2 + q^2 \leq 2(n+1)\hbar \\ 0, & \text{otherwise,} \end{cases} \quad (7)$$

$$P(m|p, q) = \begin{cases} 1, & p^2 + (q-d)^2 \leq 2m\hbar \\ & \leq p^2 + (q-d)^2 + 1 \\ 0, & \text{otherwise.} \end{cases} \quad (8)$$

Then it is clear that

$$P(m|n) = \int_{-\infty}^{\infty} dp dq P(m|p, q)P(p, q|n) \quad (9)$$

$$= \frac{1}{2\pi\hbar} \times (\text{area of overlap}) \quad (10)$$

$$= \frac{2\mathcal{A}_{mn}}{2\pi\hbar}, \quad (11)$$

where the area of overlap is represented by the shaded regions in Fig. 1 and the area of each diamond shaped shaded region is \mathcal{A}_{mn} . In Dowling et al. [3] this area, for

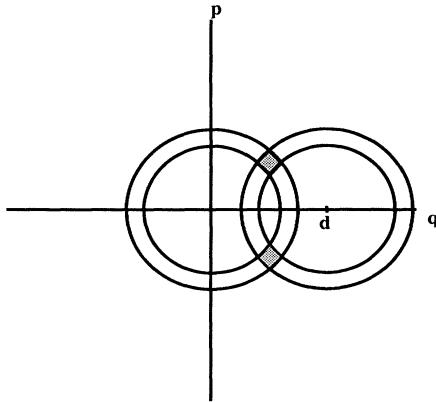


FIG. 1. Phase-space representations of harmonic oscillator states as Planck-Bohr-Sommerfeld bands.

unit frequency, is shown to be

$$\mathcal{A}_{mn} = \hbar^2 \{p_m(q_c)p_n(q_c)|p'_m(q_c) - p'_n(q_c)|\}^{-1}, \quad (12)$$

where q_c is the position at which the two Kramers trajectories intersect, $p_m(q_c)$ is the value of the momentum at the intersection, while $p'_m(q_c)$ is the slope of the trajectory at the point of intersection. In Fig. 2 we plot $P(m|n)$ calculated from the area of overlap for $n = 4$ versus m for different displacement parameters d .

Of course one can directly compute $P(m|n)$ directly from the quantum-mechanical formalism:

$$P(m|n) = |\langle \psi_m | n \rangle|^2. \quad (13)$$

This is shown in Fig. 3. Clearly, interference fringes distinguish the quantum result from the classical result in Fig. 2. We now demonstrate how these interference fringes may be derived by assigning amplitudes to areas of overlap in phase space.

We can obtain an asymptotic expression for the transition amplitude in Eq. (1) by using semiclassical wave functions in the integrand. These functions may be constructed by standard WKB techniques for bound systems or more generally by the Maslov construction, which associates a wave function to an arbitrary curve in phase space [7]. To each PBS band we associate a wave function $u_n^{\text{sc}}(q)$ given by

$$u_n^{\text{sc}}(q) = 2 [T_n p_n(q)]^{-1/2} \cos[S_n(q) - \pi/4], \quad (14)$$

where T_n is the period of the associated orbit, $p_n(q)$ defines the corresponding Kramers trajectory Eq. (6), and $S_n(q)$ is the action function along the orbit. The phase shift $\pi/4$ is a topological phase determined by the number of turning points in the position variable.

The probability may then be written

$$P(m|n) = \left| \left(\frac{\mathcal{A}_{mn}}{2\pi\hbar} \right)^{1/2} e^{-i\phi_{mn}} + \left(\frac{\mathcal{A}_{mn}}{2\pi\hbar} \right)^{1/2} e^{i\phi_{mn}} \right|^2, \quad (15)$$

where \mathcal{A}_{mn} is, as before, the area of the diamond shaped regions of overlap of the PBS bands. The phase ϕ_{mn} is

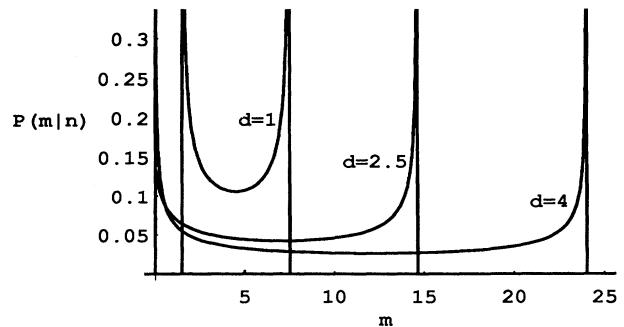


FIG. 2. Classical probability distribution $P(m|n)$ for $n = 4$ with various displacements (units such that $\hbar = 1$).

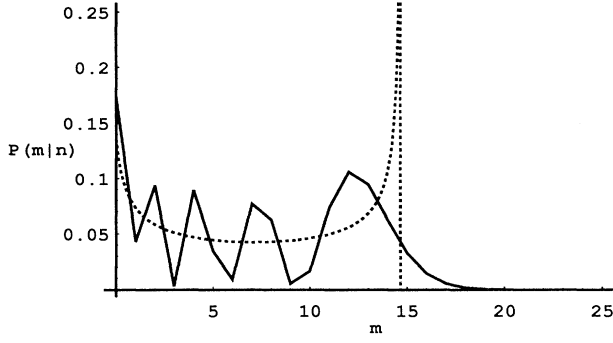


FIG. 3. Quantum-mechanical probability distribution $P(m|n)$ with $n = 4$ and $d = 2.5$ (units such that $\hbar = 1$). The dashed line is the classical result, as in Fig.2

the area in common between the areas bounded by the Kramers trajectories (Fig. 1). The essential difference between the semiclassical and the classical result is the usual one: to determine a probability we first determine the total amplitude of the overlap, then take the modulus square, whereas in the classical result we simply sum the areas of overlap. The method thus assigns to each area of overlap an amplitude and a phase. To see this even more clearly, we write the semiclassical result as

$$P(m|n) = 2 \frac{\mathcal{A}_{mn}}{2\pi\hbar} [1 + \cos(2\phi_{mn})]. \quad (16)$$

The classical result is obtained when we neglect the interference term.

The interference in the phase-space method presented above is justified by associating a semiclassical (WKB) wave function to phase-space representations of the state in terms of PBS bands. However, one strongly suspects that the method is of sufficiently general validity to enable asymptotic expressions for overlap integrals to be worked out purely from the geometry of overlapping PBS bands, without knowing the underlying WKB wave functions. We will answer this question in the affirmative by considering the case of a spherical (S^2) phase space. The method then enables asymptotic results for the rotation matrices to be obtained by simple geometry on the surface of a sphere.

II. ANGULAR MOMENTUM PHASE SPACE

We first must decide on the appropriate phase space for angular momentum states. To be specific, consider a particle of unit mass constrained to move on the surface of a sphere of unit radius. The position of the particle is most easily described in terms of the spherical polar angles (θ, ϕ) . The Lagrangian for the system is then

$$L = \frac{1}{2} \left\{ \sin^2 \theta (\dot{\phi})^2 + (\dot{\theta})^2 \right\}. \quad (17)$$

The canonical momenta are

$$p_\theta = \dot{\theta}, \quad (18)$$

$$p_\phi = \dot{\phi} \sin^2 \theta, \quad (19)$$

and the Hamiltonian is

$$H = \frac{p_\theta^2}{2} + \frac{p_\phi^2}{2 \sin^2 \theta}. \quad (20)$$

Clearly, p_ϕ is a constant of the motion. There are thus two constants of the motion (including the energy), and the four-dimensional phase-space must reduce to two dimensions.

A more standard choice of canonical variables, however, is given by the components of angular momentum (which may be considered as a canonical transformation of the original variables) defined by

$$J_x = -p_\theta \sin \phi - p_\phi \cot \theta \cos \phi, \quad (21)$$

$$J_y = p_\theta \cos \phi - p_\phi \cot \theta \sin \phi, \quad (22)$$

$$J_z = p_\phi. \quad (23)$$

The total angular momentum is then

$$|\mathbf{J}|^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta}, \quad (24)$$

making the Hamiltonian

$$H = \frac{1}{2} |\mathbf{J}|^2. \quad (25)$$

The classical phase space is then the sphere with Cartesian coordinates (J_x, J_y, J_z) and radius $J^2 = |\mathbf{J}|^2$. The trajectory corresponding to say the J_x eigenstate is then simply a circle defined by $J_x = \text{const}$; likewise for other components.

In quantum mechanics, however, angular momentum is represented by the operators $(\hat{J}_x, \hat{J}_y, \hat{J}_z)$, which obey the commutation relations

$$[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z, \quad (26)$$

$$[\hat{J}_y, \hat{J}_z] = i\hbar \hat{J}_x, \quad (27)$$

$$[\hat{J}_z, \hat{J}_x] = i\hbar \hat{J}_y. \quad (28)$$

The simultaneous eigenstates of \hat{J}_z, \hat{J}^2 are

$$\hat{J}^2 |j, m\rangle = \hbar^2 j(j+1) |j, m\rangle, \quad (29)$$

$$\hat{J}_z |j, m\rangle = \hbar m |j, m\rangle, \quad (30)$$

where as usual $j = 0, 1, \frac{1}{2}, \dots$ and $m = -j, \dots, j$. Eigenstates for the other components can be obtained by rotation of the z eigenstates.

Any rotation in three dimensions can be expressed in terms of the standard rotation operator

$$D(\alpha, \beta, \gamma) = \exp\left(-\frac{i\alpha}{\hbar} J_z\right) \times \exp\left(-\frac{i\beta}{\hbar} J_y\right) \exp\left(-\frac{i\gamma}{\hbar} J_z\right). \quad (31)$$

The matrix elements of this operator are

$$D_{m'm}^j(\alpha, \beta, \gamma) = \langle j, m' | D(\alpha, \beta, \gamma) | j, m \rangle. \quad (32)$$

A rotation purely around the y axis has the matrix elements defined by

$$d_{m'm}^j(\beta) = D_{m'm}^j(0, \beta, 0). \quad (33)$$

Suppose now the system is prepared in an eigenstate of \hat{J}_z , which we denote $|j, m\rangle_z$. Let \hat{J}_n be the component in the direction \mathbf{n} . What is the probability that a measurement of \hat{J}_n will give the result $n\hbar$? This is of course given by

$$P(n|m) = |\mathbf{n}\langle j, n|j, m\rangle_z|^2, \quad (34)$$

where $|j, n\rangle_n$ is an eigenstate of \hat{J}_n . For the sake of simplicity we will assume that \mathbf{n} lies in the x - z plane; thus

$$|j, m\rangle_n = \exp(-i\hat{J}_y\beta) |j, m\rangle_z, \quad (35)$$

and the overlap is given directly in terms of the rotation matrix by

$$\mathbf{n}\langle j, n|j, m\rangle_z = d_{nm}^j(\beta), \quad (36)$$

where β is the angle between \mathbf{n} and the z axis.

We now turn to a phase-space representation of the angular momentum states. To do this we define the PBS bands in the following way. A z eigenstate $|j, m\rangle_z$ is represented by a Kramers trajectory with $J_z = m$ on the surface of a sphere of radius $R = [j(j+1)]^{1/2}$. The PBS band is then centered on this trajectory and lies between $J_z = m - \frac{1}{2}$ and $J_z = m + \frac{1}{2}$. The same idea is applied to eigenstates of \hat{J}_n . In Fig. 4 we illustrate two PBS bands corresponding to \hat{J}_x and \hat{J}_z eigenstates. Clearly evident are two areas of overlap, and thus we expect to see interference fringes in the probability distribution in Eq. (34). We now discuss this case in some detail.

The overlap between the J_z and J_x band is made up of the two small areas $A_{m,n}^j$ shown as shaded in Fig. 4.

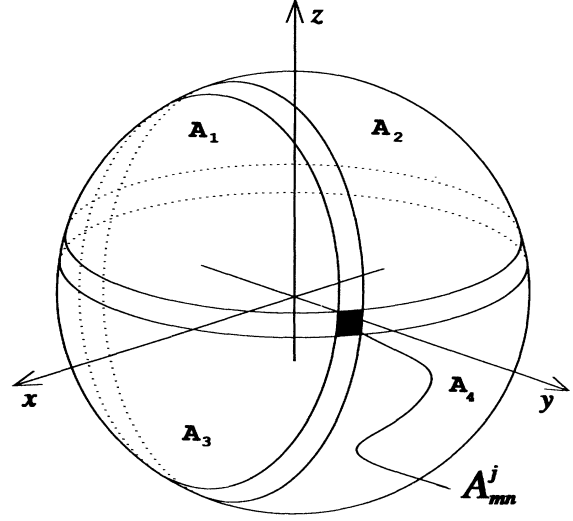


FIG. 4. Phase-space representation of eigenstates of J_x and J_y . The area of overlap is designated $A_{m,n}^j$.

We can now directly apply the area of the overlap algorithm to compute the amplitude ${}_z\langle j, m|j, n\rangle_x$ and corresponding rotation matrix element $d_{mn}^j(\pi/2)$. However, it is not at all clear how the area of overlap determines the amplitude and phase in the case of a spherical phase space. The area of overlap is simple enough. As the area of each of the PBS bands is of width unity, the area of overlap is 1. However, as we show below, this area must be weighted by the inverse of the radius R . The next problem is to determine which of four possible areas (see Fig. 4) contained between the Kramers trajectories will determine the phase.

Define $a^j(z, x)$ as the area bounded by the circles $J_x = x$, $J_z = z$. This is given by

$$a^j(z, x) = 2R^2 \arctan\left(\frac{R}{xz} \sqrt{R^2 - x^2 - z^2}\right) - 2Rz \arctan\left(\frac{\sqrt{R^2 - x^2 - z^2}}{x}\right) - 2Rx \arcsin\left(\sqrt{\frac{R^2 - x^2 - z^2}{R^2 - x^2}}\right) \quad \text{for } x, z \geq 0. \quad (37)$$

The areas for x or z less than 0 are obtained by the relations

$$a^j(-z, x) = 2\pi R(R - z) - a^j(z, x), \quad (38)$$

$$a^j(z, -x) = 2\pi R(R - x) - a^j(z, x). \quad (39)$$

There are four possible areas to consider in Fig. 4:

$$A_1 = a^j(m, n), \quad (40a)$$

$$A_2 = 2\pi R(R - m) - a^j(m, n), \quad (40b)$$

$$A_3 = 2\pi R(R - n) - a^j(m, n), \quad (40c)$$

$$A_4 = 2\pi R(m + n) + a^j(m, n). \quad (40d)$$

To distinguish which area should contribute to the phase, we consider the case $m = 0, n = 1$.

The asymptotic result for the rotation matrix for this case is

$$d_{10}^j\left(\frac{\pi}{2}\right) \approx -(-1)^{j-1} \left(\frac{2}{\pi(j + \frac{1}{2})}\right)^{1/2}. \quad (41)$$

Given that the area of overlap for A_{10}^j is unity, it is possible to write

$$\left|d_{10}^j\left(\frac{\pi}{2}\right)\right| = 2 \left(\frac{A_{10}^j}{2\pi R}\right)^{1/2}, \quad (42)$$

where the radius of the sphere R for large j is given by $R = \sqrt{j + 1/2}$. This factor is the same form as that for the flat phase space, except that the area of overlap is weighted by $(2\pi R)^{-1}$. With this in mind we conjecture that the area of overlap formula for the amplitude should be

$$d_{mn}^j\left(\frac{\pi}{2}\right) = 2e^{i\theta} \left(\frac{A_{mn}^j}{2\pi R}\right)^{1/2} \cos\left(\frac{a^j(m, n)}{2\pi R} - \frac{\pi}{4}\right), \quad (43)$$

where θ is a phase factor yet to be determined.

Using Eq. (43), the phase contributions from each of the areas in Eq. (40) for j large are

$$\cos\left(\frac{A_1}{2R} - \frac{\pi}{4}\right) = \cos\left(\frac{a^j(m, n)}{2R} - \frac{\pi}{4}\right), \quad (44a)$$

$$\cos\left(\frac{A_2}{2R} - \frac{\pi}{4}\right) = (-1)^{j-m} \cos\left(\frac{a^j(m, n)}{2R} - \frac{\pi}{4}\right), \quad (44b)$$

$$\cos\left(\frac{A_3}{2R} - \frac{\pi}{4}\right) = (-1)^{j-n} \cos\left(\frac{a^j(m, n)}{2R} - \frac{\pi}{4}\right), \quad (44c)$$

$$\cos\left(\frac{A_4}{2R} - \frac{\pi}{4}\right) = (-1)^{m+n} \cos\left(\frac{a^j(m, n)}{2R} - \frac{\pi}{4}\right). \quad (44d)$$

With $n = 1, m = 0$, Eq. (44) together with Eq. (41) imply that either A_2 or A_4 is the correct choice. Now

$$d_{nm}^j(\beta) = (-1)^{n-m} d_{mn}^j(\beta). \quad (45)$$

So $d_{01}^j(\pi/2)$ has a phase of $(-1)^{(j-1)/2}$. Thus the correct area is A_2 . Thus we postulate that the area of the overlap rule for spherical phase space is

$$d_{mn}^j\left(\frac{\pi}{2}\right) = (-1)^{j-m} 2 \frac{A_{mn}^j}{2\pi R} \cos\left(\frac{a^j(m, n)}{2R} - \frac{\pi}{4}\right). \quad (46)$$

The case for which the area-of-overlap result is easiest to verify is $d_{m0}^j(\pi/2)$. Using the relation [8]

$$d_{m0}^j\left(\frac{\pi}{2}\right) = (-1)^{j-m} d_{m0}^j\left(\frac{\pi}{2}\right), \quad (47)$$

it is clear that if j is even $d_{m0}^j(\pi/2) = 0$ for m odd and if j is odd, then $d_{m0}^j(\pi/2) = 0$ for m even. Using interference in phase space we have

$$\begin{aligned} |{}_z\langle j, m | j, 0 \rangle_x|^2 &= \left| d_{m0}^j\left(\frac{\pi}{2}\right) \right|^2 \\ &= 4 \left(\frac{A_{m0}^j}{2\pi R} \right)^2 \cos^2\left(\frac{a^j(m, 0)}{2R} - \frac{\pi}{4}\right), \end{aligned} \quad (48)$$

where in this case $a^j(m, 0) = \pi R(R - m)$. Using the asymptotic expression for the radius, $R \approx j + \frac{1}{2}$, the phase in the cosine is $\phi = (j - m)\frac{\pi}{2}$. So

$$\cos^2 \phi = \begin{cases} 0 & \text{for } (j - m) \text{ odd} \\ 1 & \text{for } (j - m) \text{ even,} \end{cases} \quad (49)$$

which is consistent with the exact result.

In Fig. 5 we plot the probability distributions

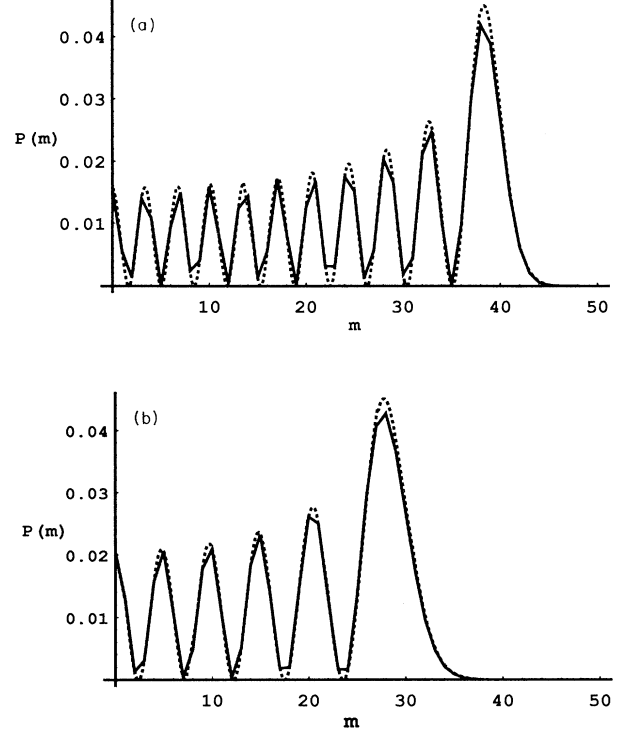


FIG. 5. Probability distributions for J_z eigenvalues, m , for two different J_x eigenstates (a) $|j = 50, m = 30\rangle_x$ and (b) $|j = 50, m = 40\rangle_x$. The solid line is the distribution obtained from a numerical evaluation of the exact result using recursion relations, and the dashed line is the result using the interference in the phase-space method.

$|d_{m,30}^{50}(\pi/2)|^2$ and $|d_{m,40}^{50}(\pi/2)|^2$, respectively. The results obtained by the interference in the phase space method (supplemented with an Airy function as described below), are compared with a direct numerical computation of the rotation matrices using appropriate recursion relations [9]. The agreement is quite good. Towards the tails

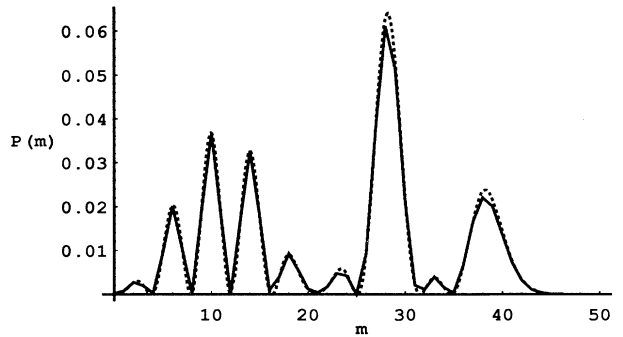


FIG. 6. Probability distribution for J_z eigenvalues for a system prepared in a superposition of J_x eigenstates. The solid line is the exact distribution calculated numerically by recursion relations, while the dashed line represents the result of the interference in the phase-space method.

of the distributions the corresponding PBS bands have only one area of overlap and approach the no-overlap region, so one would expect the approximation to break down here. The problem is to find the correct semiclassical wave functions near the classical turning points. This is an old problem. The solution in this region is given in terms of Airy functions. In the Appendix we show how this can be used to supplement the area-of-overlap method to give a better approximation near the tails of the distribution.

Another interesting case occurs for the probability dis-

tribution for \hat{J}_z measurements when the system is prepared in a superposition of two distinct \hat{J}_x eigenstates. Suppose, for example, we choose

$$|\psi\rangle_x = \frac{1}{\sqrt{2}}(|50, 30\rangle_x + |50, 40\rangle_x). \quad (50)$$

In this case there are four regions of overlap as the state $|\psi\rangle_x$ is represented by two PBS bands. The area-of-overlap method then gives the probability distribution by

$$|{}_z\langle 50, m|\psi\rangle_x|^2 \approx \left| 2 \left[\frac{A_{m,30}^{50}}{2\pi R} \right]^{\frac{1}{2}} \cos \left(\frac{a^{50}(m, 30)}{2\pi R} - \frac{\pi}{4} \right) + 2 \left[\frac{A_{m,40}^{50}}{2\pi R} \right]^{\frac{1}{2}} \cos \left(\frac{a^{50}(m, 40)}{2\pi R} - \frac{\pi}{4} \right) \right|^2. \quad (51)$$

In Fig. 6 we compare this result (again supplemented with an Airy function for the tails) with that computed by recursion relations for the rotation matrices. Again the agreement is very good.

III. CONCLUSION

We have shown how the area-of-overlap method of Schleich and Wheeler [4] may be adapted for calculating the overlap of states of definite angular momentum. Whereas for one-dimensional potentials with a flat phase space, interference in phase space is not very different from other approximation methods and so is mostly useful as a conceptual guide, its application to angular momentum is in many ways easier than the alternative asymptotic techniques. It is easier for example to calculate areas on a sphere than to take the limit of Legendre functions.

The method enables one to calculate asymptotic expressions for the rotation matrices by calculating simple areas on the surface of a sphere. In addition the geometry of overlapping areas on the surface of the sphere enables one to determine at a glance the gross features of the relevant rotation matrix in terms of interference fringes. Thus the method not only provides a simple way to compute rotation matrices in the semiclassical limit but also gives valuable physical insight into the essential differences between the classical and quantum descriptions of angular momentum.

APPENDIX

In this appendix we show how to supplement the area of overlap algorithm in order to get good agreement at the tails of the distributions; that is, in the region where there is only one area of overlap. The idea used in Ref. [10] is to approximate the WKB wave function near a turning point with an Airy function solution. Consider the overlap of the states $|j, m\rangle_z$ and $|j, n\rangle_x$. The phase factor from the interference in phase space method is of the form $\cos \left(\int_M^K q(z) dz - \frac{\pi}{4} \right)$, where

$$q(z) = R \arccos \left(\frac{n}{\sqrt{1-z^2}} \right), \quad (A1)$$

where $M = m/R$, $N = n/R$, and $K = k/R$. Typically a WKB wave function for a periodic system has a phase factor of the form $\cos \left[\int_x^{x_0} p(x) dx - \frac{\pi}{4} \right]$, where $p(x)$ is the momentum as a function of position along a curve of constant energy, and x_0 is the classical turning point $p(x_0) = 0$. This satisfies the second-order differential equation

$$\frac{d^2 u}{dx^2} + p^2(x)u = 0. \quad (A2)$$

Near the turning point x_0 this becomes

$$\frac{d^2 u}{dx^2} + a(x - x_0)u(x) \approx 0, \quad (A3)$$

which has the solution

$$u(x) \approx \text{Ai}(a^{1/3}(x - x_0)), \quad (A4)$$

where Ai is the Airy function, and a is a constant that depends on the turning point and the form of the potential. This is the approximate solution in the neighborhood of the turning point. We thus expect that near the turning points $q(z)$ in Eq. (A1) should be an approximate solution to the differential equation

$$\frac{d^2 y}{dM^2} + q^2(M)y = 0. \quad (A5)$$

A Taylor expansion of q^2 to first order gives

$$q^2(M) \approx a(K - M). \quad (A6)$$

Where $a = -dq^2(z)/dz|_K$ let

$$u(z) = \frac{N}{\sqrt{1-z^2}}. \quad (A7)$$

Then

$$a = 2r^2 u'(K). \quad (\text{A8})$$

The differential equation thus becomes

$$\frac{d^2 y}{dM^2} + 2R^2 u'(K)(K - M)y = 0. \quad (\text{A9})$$

Thus the approximate solution near the turning points is

$$y(M) = \text{const} \times \text{Ai} \left(\left(\frac{2R^2 NK}{(1 - K^2)^{3/2}} \right)^{1/3} (M - K) \right). \quad (\text{A10})$$

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