# Fractional quantum revivals in the atomic gravitational cavity

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(Received 30 June 1994)

In this paper we discuss the quantum dynamics and fractional quantum revivals of an integrable nonlinear system, consisting of an atom bouncing vertically from an evanescent field, for two cases with the simplified infinite-potential and the more practical exponential potential, respectively. We study the two cases separately, then contrast and compare the results and reach the conclusion that provided the starting position of the atoms is not too close to the reflecting surface supporting the evanescent wave (this condition is always satisfied in present experiments in this field), the two cases will produce the same results. This means that the idealized infinite potential is a good approximation to the more realistic exponential potential. Because the quantum analysis of the infinite-potential case is quite simple and straightforward (since its Schrödinger equation has analytical solutions), this will greatly simplify the quantum analysis of the more complicated exponential potential case and hence has practical significance.

PACS number(s): 42.50.Vk, 33.80.Ps, 03.65.−w

## I. INTRODUCTION

Recent advances in laser cooling and trapping enable us to probe the quantum nonlinear motion of atoms in optical potentials. Indeed the isolation from fluctuations is so good that the dynamics is largely Hamiltonian and it is increasingly likely that quantum coherence effects will become experimentally accessible.

A nonlinear quantum system will typically imitate its classical counterpart on some characteristic classical time scale [1,2], after which quantum-mechanics effects will dominate. One of the quantum departures from the classical analog is seen in the appearance of collapses and revivals of the initial state, and the formation of coherent superposition states at long times. The so-called quantum fractional revivals have been observed in several systems [3–8] theoretically or experimently.

In this paper we consider an integrable nonlinear system proposed by Wallis *et al.* [9] which models an atom bouncing vertically from an optical evanescent wave. The evanescent wave provides a reflection potential step which varies rapidly with the distance from the boundary. This is a nonlinear oscillator, as the period of oscillation depends on the initial energy. The nonlinear period is, in fact, the "bounce" time, which is largely determined by the time taken for the atom to fall onto the evanescent surface from the initial position. A classical distribution of particles moving in such an oscillator shows a dispersion around curves of constant energy and the formation of whorls [10]. The quantum description is however rather different. We discuss the collapses and revivals of the initial state and the formation of coherent superposition states at long times in this system. Due to these long interaction times, the kinetic energy term, omitted in the Raman-Nath approximation of atom optics, must be included.

In the limit of large detuning, the effective potential felt by the atom is proportional to the intensity of the evanescent wave, which has an exponential decay away from the surface. The rate of fall off of this field is of the order of an optical wavelength over $2\pi$. Thus atoms bouncing from a height of the order of millimeters effectively see an infinite boundary at the surface. Of course, if they begin with energy greater than the potential energy at the surface, they hit the dielectric surface supporting the evanescent wave and are not reflected at all. The case of the infinite boundary leads to considerable simplifications in the quantum model, as there exist exact energy eigenstates in terms of Airy functions. In this paper we compare and contrast the quantum nonlinear dynamics for the infinite potential with the more realistic exponential potential. The infinite potential model breaks down if the atoms start extremely close to the surface. As this is desirable if the time scale for the formation of superposition states is to be kept experimentally reasonable, it is essential to consider the full exponential dependence of the potential if realistic predictions are to be made. In a practical system the reflecting surface is parabolic concave in order to provide stable atomic motion and well-defined cavity modes, but in order to simplify the analysis we only discuss the simpler case of flat surface in this paper.

## II. CENTER-OF-MASS MOTION OF THE ATOMS

A nonlinear oscillatory system is characterized by the fact that the period of motion is a function of the initial energy. The nonlinear frequency is defined by

$$\omega(H)=\left(\frac{\partial I(H)}{\partial H}\right)^{-1}, \tag{2.1}$$

where $I(H)$ is the action on a curve of constant energy $H$. Points on different energy surfaces in phase space rotate at different rates. This leads to a rotational shearing of a

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localized phase-space density which eventually becomes uniformly spread around the elliptic fixed point defining the periodic motion. The result for the mean position and momentum is a decay to zero on a time scale of the nonlinear period, while the variance in position and momentum saturate at a constant value determined by the radius of the annular phase-space density resulting from the shearing motion.

The quantum description, however, follows the classical description only for short times. At some time, the quantum mean values return arbitrarily closely to their initial values, as the initial state partially recurs. An even more surprising quantum effect is the appearance of coherent superpositions of copies of the initial states at intermediate times.

Averbukh and Perelman [11] have discussed the formation of such superposition states in nonlinear oscillators and we use their results here. In dimensionless scale, if the commutation relations are

$$[q,p] = iK, \tag{2.2}$$

where $q$ is the scaled position variable, $p$ is the scaled momentum variable, and $K$ plays the role of a dimensionless Planck constant. Then the time scale for revivals is determined by

$$T_{\rm rev} = \frac{4\pi}{\omega(H)}\left(K\left|\frac{\partial\omega(H)}{\partial H}\right|\right)^{-1}. \tag{2.3}$$

The above expression is valid only when the condition $K\left|\partial\omega(H)/\partial H\right| \ll 1$ is satisfied.

For strongly nonlinear systems the variation of $\omega(H)$ with frequency is large and the time scale for revivals is short, likewise if the system is strongly quantum (large $K$). In the following sections we will discuss the infinite potential case and the exponential potential case, respectively.

### A. The infinite-potential case

The infinite potential corresponds to an instantaneous 100% reflection off a plane atomic mirror so the Hamiltonian is

$$H_0 = \frac{p_z^2}{2m} + V(z) \tag{2.4}$$

with

$$V(z) = \begin{cases} mgz & \text{if } z \geq 0 \\ \infty & \text{if } z < 0, \end{cases} \tag{2.5}$$

where $m$ is the mass of the atom, $p_z$ is the vertical momentum component along the $z$ axis, $g$ is the acceleration of gravity, and $z$ is the distance of the atom from the boundary surface. By setting the potential at infinity at $z = 0$ we idealize the actual reflecting potential step which may be realized by an evanescent light wave whose extension is typically in the micron range.

Scaling variables by means of the following transformations:

$$x = z/\beta, \tag{2.6}$$

$$y = p_z/m\beta\omega, \tag{2.7}$$

$$H = H_0/m\beta^2\omega^2, \tag{2.8}$$

with the position scale $\beta = \sqrt[3]{\hbar^2/2m^2g}$, the frequency scale $\omega = 2\pi mg\beta/\hbar$, and the scaled acceleration $\mu = g/\beta\omega^2$. We have

$$H = \frac{y^2}{2} + \mu x. \tag{2.9}$$

The canonical commutation relations are

$$[x,y] = iK, \tag{2.10}$$

where $K = \hbar/m\beta^2\omega$.

According to Wallis *et al.* [9] by neglecting the internal evolution due to the interaction with the reflecting mirror the corresponding stationary Schrödinger equation is given by

$$\psi''(x) - (x - x_E)\psi(x) = 0, \tag{2.11}$$

where $x_E = z_E/\beta$, $z_E = E/mg$, and $E$ is the initial energy. The energy eigenstates are in the form of Airy functions, that is

$$\psi_n(x) = \mathrm{Ai}(x - x_n), \tag{2.12}$$

where $x_n$ are defined as the solutions to the equation $\mathrm{Ai}(-x_n) = 0$. The energy eigenvalues are

$$E_n = mgz_n \tag{2.13}$$

with $z_n = \beta x_n$.

The action variable is

$$I(H) = \frac{1}{\pi}\int_0^{H/\mu}\sqrt{2(H-\mu x)}dx = \frac{2H\sqrt{2H}}{3\mu\pi}. \tag{2.14}$$

By means of (2.1), the nonlinear frequency is

$$\omega(H) = \frac{\mu\pi}{\sqrt{2H}}. \tag{2.15}$$

The classical period is

$$t_b = \frac{2\pi}{\omega(H)} = \frac{2\sqrt{2H}}{\mu}. \tag{2.16}$$

By means of (2.3), the quantum revival time is

$$T_{\rm rev} = \frac{16H^2}{\pi K\mu^2}. \tag{2.17}$$

### B. The exponential potential case

In this case, the Hamiltonian for the center-of-mass motion is

$$\tilde{H}_0 = p_z^2/2m + mgz + \varepsilon e^{-\alpha z}, \tag{2.18}$$

where $p_z, m, z,$ and $\omega$ are the same as those in (2.4), and $\varepsilon$ and $\alpha$ are the amplitude and decay rate of the evanescent wave, respectively.

Using the following scaled variables and relations:

$$\tilde{x} = \alpha z, \tag{2.19}$$
$$\tilde{y} = \alpha p_z/m\omega, \tag{2.20}$$
$$T(\tilde{y}) = \tilde{y}^2/2, \tag{2.21}$$
$$V(\tilde{x}) = \kappa e^{-\tilde{x}} + \lambda\tilde{x}, \tag{2.22}$$
$$\tilde{H} = \tilde{H}_0\alpha^2/m\omega^2, \tag{2.23}$$

with $\kappa = \varepsilon\alpha^2/m\omega^2$, $\lambda = \alpha g/\omega^2$, and $\omega = 2\pi mg/\alpha\hbar$. Then the effective Hamiltonian in this case is

$$\tilde{H} = \frac{\tilde{y}^2}{2} + \kappa e^{-\tilde{x}} + \lambda\tilde{x} = T(\tilde{y}) + V(\tilde{x}) \tag{2.24}$$

with the canonical commutation relations

$$[\tilde{x}, \tilde{y}] = i\tilde{K}, \tag{2.25}$$

where $\tilde{K} = \hbar\alpha^2/m\omega$. In order to ensure that the atoms bounce in the gravitational field if they start from rest, there exists the restriction $\lambda[1 + \ln(\kappa/\lambda)] < \tilde{H} < \kappa$.

Obviously the scaling scheme here is equivalent to that for the infinite-potential case if we choose $\alpha = 1/\beta$, which means we have the following correspondence: $x \sim \tilde{x}, y \sim \tilde{y}, H \sim \tilde{H}, K \sim \tilde{K}$, etc., and for convenience we can use the same variable symbols for the two cases, thus making the direct constrast and comparision of the two cases easier. From now on we will use the same variables—$x, y, H, K$, etc., for both cases instead of distinguishing them by means of different symbols. In this case analytical formulas are not available so we have to resort to numerical methods.

First we have to find the turning points on the $x$ axis $(x_1, 0)$ and $(x_2, 0)$ in the classical phase space in order to calculate the action variable, nonlinear frequency, etc. Thus we need to solve the equation $\kappa e^{-x} + \lambda x = H$ around $(\kappa - H)/(\kappa - \lambda)$ for $x_1$ and $H/\lambda$ for $x_2$. We then calculate the action, nonlinear frequency, etc., as follows. The action variable is

$$I(H) = \frac{1}{\pi}\int_{x_1}^{x_2} \sqrt{2(H - \kappa e^{-x} - \lambda x)}dx\ . \tag{2.26}$$

The nonlinear frequency is

$$\omega(H) = \pi\left(\int_{x_1}^{x_2} \frac{dx}{\sqrt{2(H - \kappa e^{-x} - \lambda x)}}\right)^{-1}. \tag{2.27}$$

The classical period is

$$t_b(\exp) = 2\pi/\omega(H), \tag{2.28}$$

where "exp" denotes exponential. The quantum revival time is

$$T_{\rm rev}(\exp) = 2t_b(\exp)\left(K\left|\frac{\partial\omega(H)}{\partial H}\right|\right)^{-1}. \tag{2.29}$$

## III. FRACTIONAL QUANTUM REVIVAL

In order to analyze the fractional quantum revivals of the two cases mentioned above, we choose the minimum uncertainty state as the initial state which has the wave function

$$\phi(x, 0) = (1/\sqrt[4]{2\pi})e^{-(x-x_0)^2/4} \tag{3.1}$$

with means $\langle x\rangle = x_0, \langle y\rangle = 0$ and position variance $\sigma_x = 1$ and momentum variance $\sigma_y = K^2/4\sigma_x$. We choose the following parameters $x(0) = x_0, y(0) = 0, K = 2$, and $\mu = \lambda = 2$. In all cases we choose $\kappa = 100$.

### A. The infinite-potential case

Under the conditions given above, the bounce time and the revival time are given by

$$t_b = 2\sqrt{x_0}, \tag{3.2}$$
$$T_{\rm rev} = \frac{8x_0^2}{\pi}. \tag{3.3}$$

The above formulas are used in our numerical examples.

The quantum dynamics is easily determined by expanding the initial state in terms of the energy eigenstates given in Sec. II. Thus

$$\phi(x, t) = \sum_{n=1}^{\infty} c_n\ \mathrm{Ai}(x - x_n)e^{-ix_n t} \tag{3.4}$$

with $c_n = \int_{-\infty}^{\infty}\phi(x, 0)\mathrm{Ai}(x - x_n)dx$ and the time scaled in units of $\hbar/mg\beta$. The momentum representation $\phi(p, t)$ of the state at any time is then found by Fourier transforming $\phi(x, t)$. We then calculate the position and momentum probability densities at a quarter of the revival time $t = (1/4)T_{\rm rev}$.

In Figs. 1 and 2 we show the position and momentum distribution at one quarter of the revival time for various initial states. In each case the dashed line corresponds to the infinite-potential case. The bounce times and revival times for these cases are given in Table I. Note that the momentum distribution is dominated by two symmetrically placed peaks. This corresponds to the formation of a superposition of two localized states at the same position of oppositely directed momentum. The absolute value of this momentum is roughly what would be expected for a classical particle at this time for this initial condition. In position space this superposition is manifest as a single peak modulated by interference fringes. As the initial state moves further from the potential, the momentum scale for the dynamics increases. Thus the superposed states have a larger momentum separation and, as expected, the interference fringes in the position probability distribution become more closely spaced.

### B. The exponential potential case

In this case we must calculate the bounce time numerically using Eq. (2.28). The revival time is not so easy to determine however. While it can be determined numerically using Eq. (2.29), this will only be an approximation. For initial states far from the classical fixed point the particle moves quite fast and it is very easy to miss a fractional revival. We can use the results of the infinite-potential case as a guide. Noting that a superposition of momentum states must correspond to zero mean momentum with a large variance, we can examine the dynamics of these moments to look for possible superposition states. The revival time obtained by these methods is shown in Table I.

The time evolution of the wave function is done by means of the so-called split-operator method [12] by separating the Hamiltonian into two parts, that is the kinetic energy part and the potential part. In our case we have

$$H = T(y) + V(x), \tag{3.5}$$

where $T(y) = y^2/2, V(x) = \kappa e^{-x} + \lambda x,$ and $[x, y] = iK$.

Evolution over one time step $\Delta t$ is given by

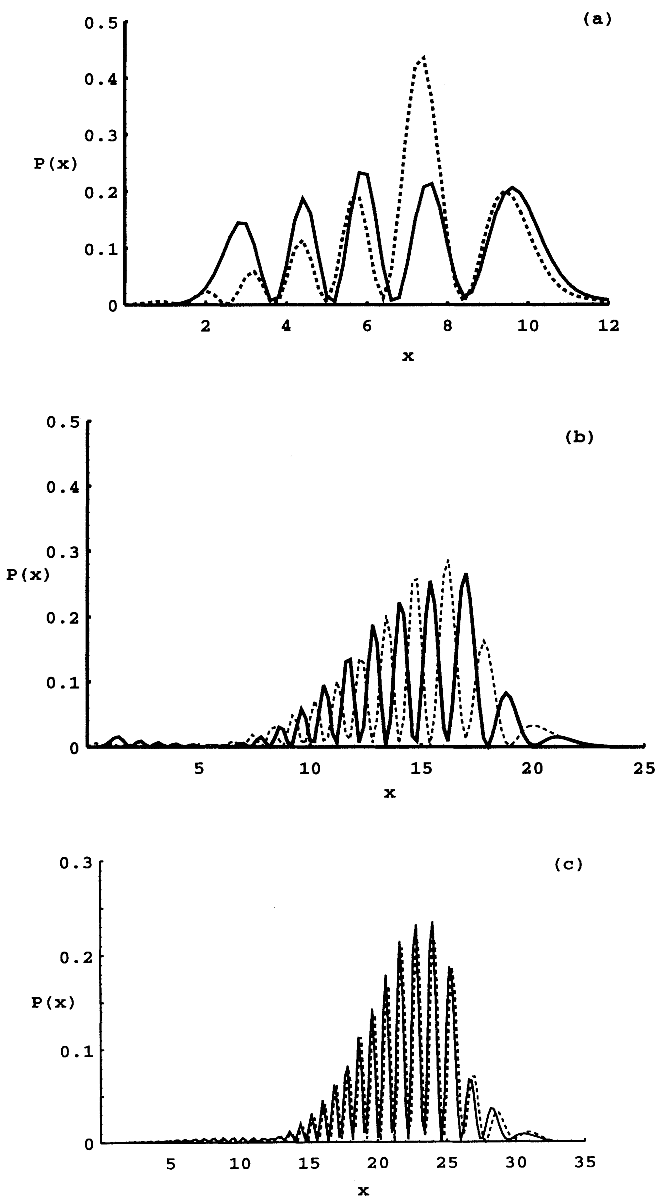


FIG. 1. The position probability distribution at one quarter of the revival time for various initial conditions. (a) $x_0$=10; (b) $x_0$=20; and (c) $x_0$=30. Dashed line: infinite potential; solid line: exponential potential.

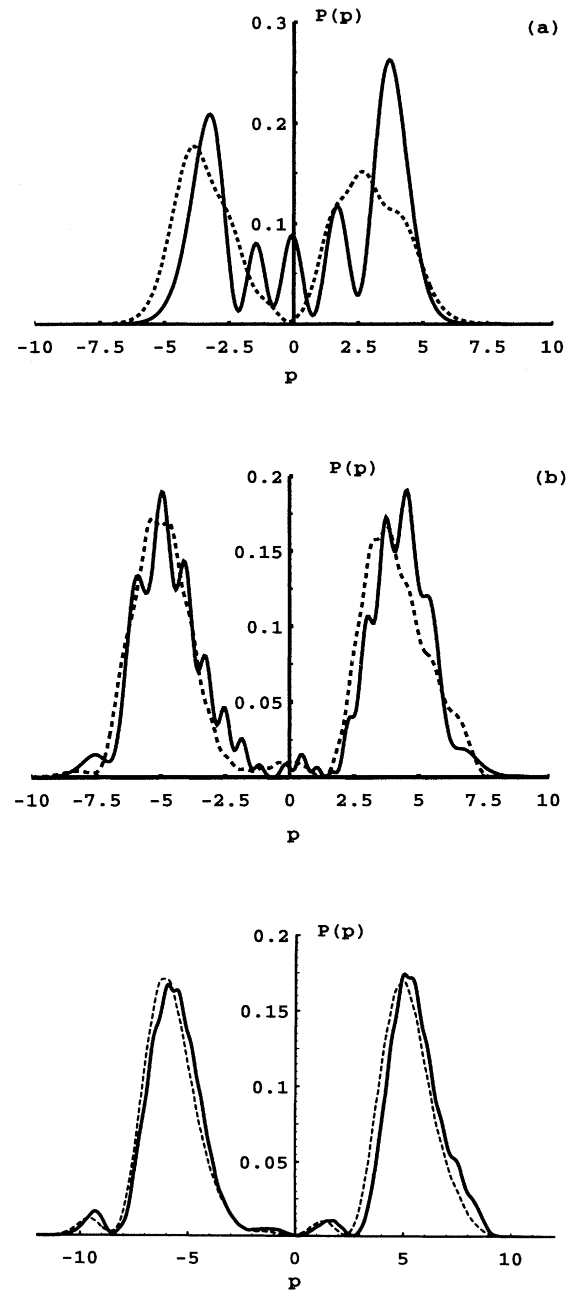


FIG. 2. The momentum probability distribution at one quarter of the revival time for various initial conditions. (a) $x_0$=10; (b) $x_0$=20; and (c) $x_0$=30. Dashed line: infinite potential; solid line: exponential potential.

TABLE I. A comparison of bounce times and revival times for the infinite (inf) and exponential (exp) cases with various initial conditions.

| $x$ | $t_b$ (inf) | $t_b$ (exp) | $T_{\rm rev}/4t_b$ (inf) | $T_{\rm rev}/4t_b$ (exp) |
|---|---|---|---|---|
| 10 | 6.3 | 6.3 | 10 | 12 |
| 20 | 8.9 | 9.0 | 28 | 28 |
| 30 | 10.9 | 11.1 | 52 | 53 |
| 50 | 14.1 | 14.3 | 113 | 115 |

$$\phi(t+\Delta t) = e^{-\frac{iH\Delta t}{K}}\phi(t) \simeq e^{-\frac{iT\Delta t}{2K}} e^{-\frac{iV\Delta t}{K}} e^{-\frac{iT\Delta t}{2K}}\phi(t) \tag{3.6}$$

which is accurate up to second order in $\Delta t$.

We have to work in position space and momentum space as well. The fast Fourier transform is used to convert the wave function between the two spaces. By means of various numbers of iteration of $\Delta t$ we can get the wave function any time we want. For efficiency we adopted the adaptive time stepsize method developed by Tan and Walls [13] in our programming.

In Figs. 1 and 2 we show the position and momentum probablity distributions at one quarter of the revival time. In each case the solid line corresponds to the exponential case. As in the infinite case one sees the formation of a superposition of two states of oppositely directed momentum. Comparing Figs. 1 and 2 it is clear that as the initial state is located further from the surface, the infinite and exponential cases become increasingly similar. Note that the interference fringes in the position probability distributions for the infinite potential appear to be shifted with respect to those for the exponential case. This is due to two facts. First, the classical turning points in the two cases are not the same. The turning point for the infinite case is always at zero, while the exponential case has turning points different from zero by a small amount. The difference in the turning points leads to a small phase shift between the wave functions at the time of interest, which leads in turn to a shift in the interference fringes. The second reason why the interference fringes do not overlap is that it is not easy to choose the revivial time the same in the two cases, due to slight differences in the the nonlinear oscillation frequency of the two models. However the case $x_0 = 30$ does show good agreement.

In Fig. 3 we plot the mean and variance of the momentum for the initial position $x_0 = 50$. Note the decrease of the momentum variance at 115 and 230 kicks, corresponding to one-quarter and one-half of the revival time. The drop in the variance at 230 kicks corresponds to a near complete revival of the initial state.

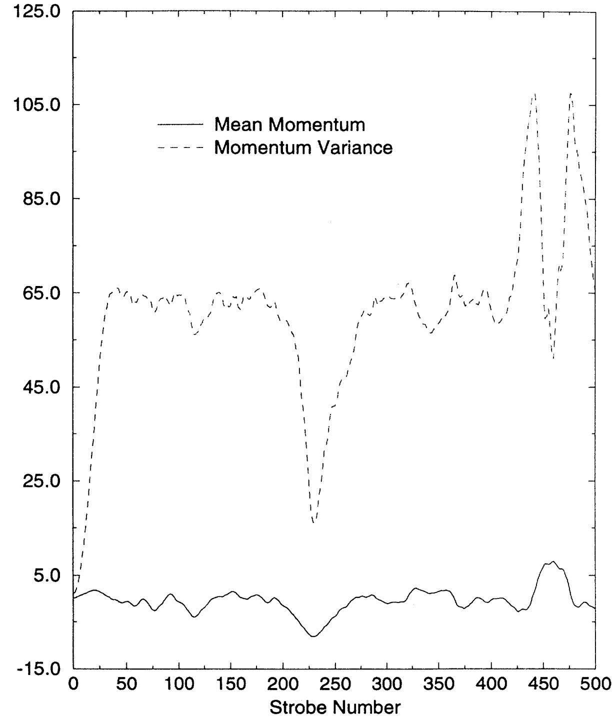


FIG. 3. The momentum mean and variance versus number of kicks, $x_0 = 50$.

## IV. CONCLUSION

The above results show that the quantum nonlinear motion of a particle bouncing vertically from a horizontal evanescent wave optical potential will lead to the formation of superposition states and corresponding interference fringes. The superposed states are localized and consist of equal but oppositely directed mean momentum. Such a state could possibly be detected by looking for Doppler-shifted resonances with a weak probe, or perhaps the position density fringes could be probed directly.

For an initial Gaussian state located sufficiently far from the surface, but still with low enough energy to be reflected, we can treat the potential as an infinite discontinuous step. In fact, there exist some differences among the distributions for the two cases when $x_0 = 10, 20$ but for $x_0 \geq 30$ the distributions are almost the same, which means that provided $x_0$ is not too small the idealized infinite potential will be a good approximation to the realistic exponential potential.

In current experiments the atoms start several millimeters above the surface [14], while the values corresponding to the dimensionless parameters used in our numerical calculations are just in the range of $z = 0.2$–$18\ \mu$m. Thus the analytical formulas of the infinite-potential case are definitely valid for these experiments in the atomic bouncer systems.

Now we will briefly consider the possibility of implementing the fractional quantum revivals in practical systems, there are two options. First the practical parameters for our numerical examples are the following $z$ : 0.2–18 $\mu$m; $t_b$ : 0.6–18 ms; $T_{\rm rev}$ : 25 ms – 1.6 s. We will need 10–115 bounces to see the revivals. The presently available maximum number of bounces is about 10 and seems hopeful. But the problem is how to start the atoms extremely close to the surface or even on the surface. One possible solution is to use grazing incidence. Second, one can start the atoms at a height of about 3 mm just as

Aminoff *et al.* in [14]. Then we will have a value of $x_0$ at the order of magnitude of $10^4$, the practical values for the bounce time and revival time at about 20 ms and $4 \times 10^4$ ms, respectively, and the system will take about $10^5$ bounces to manifest the revivals. It is still much beyond experiments that are presently accessible. In particular, losses such as spontaneous emission would have long since scattered most atoms from the system. We still have a long way to go for practical experiments for the fractional revivals in the atomic bouncer system to be performed.

## ACKNOWLEDGMENTS

Wen-Yu Chen would like to thank S. Dyrting for his penetrating analysis and stimulating discussions.

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