

Measuring the vibrational energy of a trapped ion

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We show that an approximate quantum-nondemolition measurement of the vibrational energy of a trapped ion in a standing wave may be made by monitoring the fluorescent intensity from a probe transition. The ac component of the mean photocurrent signal is directly proportional to the average vibrational quantum number. The power spectrum of the photocurrent also contains information on the vibrational energy. We show that the phase of the vibrational motion undergoes diffusion as expected for a quantum-limited measurement of the energy of oscillation.

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I. INTRODUCTION

A trapped and laser-cooled ion, at the node of a standing wave, has recently been shown to be equivalent to the Jaynes-Cummings model, one of the paradigms of quantum optics [1]. In the Jaynes-Cummings model a simple harmonic oscillator is coupled to a two-level system in the dipole and rotating-wave approximation. If the atom is interacting with a single-mode cavity field, the harmonic oscillator is the field mode. In the trapped ion configuration, however, the harmonic oscillator corresponds to the vibrational levels of the trap. Indeed there appear to be considerable advantages in the trapped-ion realization of the Jaynes-Cummings model over that with a cavity field mode, as the harmonic oscillator representing the vibrational states of the trap is only very weakly damped and further, the strength of the coupling between the two-level atom and the oscillator can be made very strong simply by increasing the intensity of the standing wave. Cirac *et al.* [1] have suggested that an observation of the collapse and revival sequence of the mean inversion in the Jaynes-Cummings model should, in the case of a trapped ion, enable a determination of the motion of the ion in the trap. This fascinating idea raises the question of whether or not a quantum limited measurement of the center-of-mass motion of a trapped ion may be made. In this paper we give a model in which a quantum-limited measurement of the vibrational energy of the ion in the standing wave may be made, using a strong probe transition coupled to the two-level system (see Fig. 1).

The basic model of a trapped-ion realization of the Jaynes-Cummings model has been given in some detail by Cirac *et al.* [1]. We summarize their results here. If we assume that the vibrational amplitude of the ion in the trap is much less than the wavelength of the light, the Hamiltonian for an ion at the node of a standing wave is

$$H_I = \hbar \Delta \sigma_z + \hbar \nu a^\dagger a + \hbar \Omega \eta \hat{X}_1 \sigma_x, \quad (1)$$

where σ_z and σ_x are the two-level inversion and polarization operators, respectively, \hat{X}_1 is a dimensionless position operator for the position of the ion, Ω is the Rabi frequency for the transition, $\Delta = \omega_0 - \omega_L$ is the detuning between the standing-wave frequency and the atom, ν is the oscillation frequency of the ion in the trap, and a, a^\dagger are the lowering and raising operators for the vibrational states of the trap. If we scale the

position of the ion in terms of the rms position fluctuations in the ground state of the trap, the dimensionless coupling constant η is given by $\eta = k \sqrt{\hbar/2m\nu}$, where k is the wave number of the standing wave. We assume throughout this paper that on the time scales of interest the spontaneous decay rate from level two can be neglected. If we now assume that

$$\nu, |\Delta| \gg \eta \Omega, |\Delta - \nu| \quad (2)$$

with $\eta \ll 1$, the Hamiltonian in the interaction picture and rotating-wave approximation may be written

$$H = \hbar \delta \sigma_z + \frac{\hbar \eta \Omega}{2} (a \sigma_+ + a^\dagger \sigma_-), \quad (3)$$

where $\delta = \Delta - \nu$. Typical values are $\nu \approx 3$ MHz, $\eta \approx 0.07$, and $\Omega \approx 10^5$ Hz [2]. We also assume that the probe field is exactly resonant with the $1 \rightarrow 3$ transition.

As in Ref. [1], we assume that the two-level system coupled to the standing wave is very strongly coupled to a third probe level by an intense field with Rabi frequency Ω_p (see Fig. 1). Fluorescence on the $|1\rangle \leftrightarrow |3\rangle$ transition is used to determine whether or not the atom is in level $|1\rangle$. The probability of finding the atom in $|1\rangle$ is in turn determined by the position of the ion in the trap through the Jaynes-Cummings interaction with the vibrational states of the trap. This three-level scheme is the standard means of monitoring weak transitions in trapped ions [3]. It is also the basis for a quantum-limited measurement of a two-level transition leading to the Zeno effect [4]. In this latter context the scheme has been analyzed in some detail in Ref. [5]. Cirac *et al.* consider the case of a sequence of pulses applied to the $|1\rangle \leftrightarrow |3\rangle$ transition and the absence of or otherwise fluorescence on the strong transition indicates if the atom is in level $|1\rangle$. The scheme thus represents a series of (approximate) readouts of the operator $P_1 = |1\rangle\langle 1|$. As this variable is directly coupled to the vibrational states of the trap, the result-

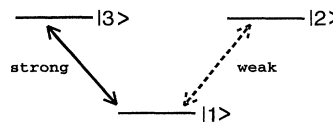


FIG. 1. Electronic energy-level scheme for the trapped ion. The $|1\rangle \leftrightarrow |2\rangle$ transition interacts with the standing wave in the trap, while the $|1\rangle \leftrightarrow |3\rangle$ transition interacts with a strong probe field.

ing statistics can be used to infer the vibrational state of the trapped ion. In this paper we consider a different scheme in which the strong probe field is always applied, in order to realize a continual quantum-limited observation of the vibrational energy of the ion.

II. THE MEASUREMENT MODEL

Following the method in Ref. [5] we assume that the Rabi frequency of the probe field Ω_p is much greater than that for the standing wave Ω . Furthermore, we assume that the coupling rates are such that we are in the *good measurement* limit

$$\frac{\Omega_p}{\gamma} \ll 1 \ll \frac{\Omega_p^2}{\gamma \Omega \eta}, \quad (4)$$

where γ is the spontaneous emission rate on the $|1\rangle \leftrightarrow |3\rangle$ (strong) transition. As one expects that the transition $|1\rangle \leftrightarrow |2\rangle$ will be a weak, electric-quadrupole transition [1], the right inequality will not be hard to satisfy. In this case the probe system may be adiabatically eliminated, giving the following master equation for the two-level system of interest:

$$\frac{d\rho}{dt} = -i\delta[\sigma_z, \rho] - \frac{i\eta\Omega}{2}[a\sigma_+ + a^\dagger\sigma_-, \rho] - \Gamma[P_1, [P_1, \rho]], \quad (5)$$

where the measurement parameter Γ is

$$\Gamma = \frac{\Omega_p^2}{2\gamma}. \quad (6)$$

The measured signal is the photocurrent produced by direct detection of the fluorescent field from the $|1\rangle \leftrightarrow |3\rangle$ transition. The scattered field, due to this transition, may be written in terms of the dipole moment operator for that transition as [6]

$$\hat{E}_s^{(+)}(t) = \sqrt{\gamma}\sigma_-^{13}(t), \quad (7)$$

where $\sigma_-^{13} = |1\rangle\langle 3|$. The mean photocurrent (in units of s^{-1}) is then

$$E(i(t)) = \epsilon \langle \hat{E}_s^{(-)}(t) \hat{E}_s^{(+)}(t) \rangle \quad (8)$$

$$= \epsilon \gamma p_3(t), \quad (9)$$

where ϵ is the effective quantum efficiency and $p_3(t)$ is the probability to find the atom in state $|3\rangle$. We shall return to a more detailed analysis of the signal and signal noise in Sec. IV. In the remainder of this section we analyze the motion of the trapped ion in the good measurement regime.

The effect of continuous observation on the ion is contained in the last term in Eq. (5). This term tends to destroy coherence between states $|1\rangle$ and $|2\rangle$, as one would expect for a readout of the population in level $|1\rangle$. This is easily seen by calculating the partial matrix elements of ρ in the atomic basis $\rho_{ij} = \langle i|\rho|j\rangle$, where $i, j = 1, 2$.

$$\dot{\rho}_{11} = \frac{i\eta\Omega}{2}(\rho_{12}a - a^\dagger\rho_{21}), \quad (10)$$

$$\dot{\rho}_{22} = \frac{i\eta\Omega}{2}(\rho_{21}a^\dagger - a\rho_{12}), \quad (11)$$

$$\dot{\rho}_{12} = (i\delta - \Gamma)\rho_{12} + \frac{i\eta\Omega}{2}(\rho_{11}a^\dagger - a^\dagger\rho_{22}), \quad (12)$$

and $\rho_{21} = \rho_{12}^\dagger$. The overdot denotes differentiation with respect to time. These matrix elements are actually operators on the vibrational states. The total center-of-mass state of the ion is given by $\rho_{c.m.} = \rho_{11} + \rho_{22}$. Similar equations for the two-level transition were given in [1]; however, the mechanism for coherence destruction in that paper is different from that given here and was attributed to phase noise on the probe laser. Thus the coherence decay rate Γ (in our notation) is different.

Equations (10)–(12) show that the measurement has no effect on the diagonal matrix elements, but causes the off-diagonal elements to decay at the rate Γ . For a good measurement we expect this decay to be rapid. We proceed by adiabatically eliminating the coherence operators ρ_{12} and ρ_{21} , to obtain an approximate master equation that describes the center-of-mass motion. In contrast, Cirac *et al.* [1] consider the underdamped case in which the coherence decay rate is less than the effective Rabi frequency on the two-level transition.

Solving for the steady-state coherences and substituting into the population equations we find that

$$\dot{\rho}_{11} = (i\kappa\delta[a^\dagger a, \rho_{11}] + \chi(2a^\dagger\rho_{22}a - \{\rho_{11}, a^\dagger a\})), \quad (13)$$

$$\dot{\rho}_{22} = (-i\kappa\delta[aa^\dagger, \rho_{22}] + \chi(2a\rho_{11}a^\dagger - \{\rho_{22}, aa^\dagger\})), \quad (14)$$

where

$$\kappa = \frac{\eta^2\Omega^2}{4(\delta^2 + \Gamma^2)}, \quad (15)$$

$$\chi = \kappa\Gamma, \quad (16)$$

and $\{A, B\} = AB + BA$. We can cast this in a Lindblad-form master equation for the density operator by defining the operators $A = \sigma_+ a$ and $A^\dagger = \sigma_- a^\dagger$. The result is

$$\dot{\rho} = \chi(2A\rho A^\dagger - A^\dagger A\rho - \rho A^\dagger A) + \chi(2A^\dagger\rho A - AA^\dagger\rho - \rho AA^\dagger) - i\kappa\delta[[A, A^\dagger], \rho]. \quad (17)$$

We now proceed to extract the physical content of Eqs. (13) and (14). In the eigenstates of vibrational quanta $|n\rangle$, where $a^\dagger a|n\rangle = n|n\rangle$, we define the joint probability $P_i(n) = \langle n|\rho_{ii}|n\rangle$ to find n vibrational quanta and the atom in state i ($i = 1, 2$). Then

$$\frac{dP_1(n)}{dt} = \chi((2nP_2(n-1) - 2nP_1(n))), \quad (18)$$

$$\frac{dP_2(n)}{dt} = \chi(2(n+1)P_1(n+1) - 2(n+1)P_2(n)). \quad (19)$$

Equations (18) and (19) represent a vector valued birth-death Markov process, in which atoms in state $|2\rangle$ make a transition to state $|1\rangle$, increasing the phonon number by one, while

atoms in state $|1\rangle$ absorb a phonon, thus moving to state $|2\rangle$. It is now possible to see how information about the atomic state gives information on the phonon number. To begin, we note an interesting constant of motion

$$\frac{d}{dt}[P_1(n,t) + P_2(n-1,t)] = 0. \quad (20)$$

The probability to find the atom in state i , at time t , is given by

$$p_i(t) = \sum_{n=0}^{\infty} P_i(n,t). \quad (21)$$

Using Eqs. (20) and (21) we find that

$$p_1(t) = \langle \hat{n}(t) \rangle - \langle \hat{n}(0) \rangle + p_1(0), \quad (22)$$

where $\hat{n} = a^\dagger a$. We thus see that a determination of the population of the ground state as a function of time gives direct information on the vibrational center-of-mass energy. We show in Sec. IV how this information may be extracted from the fluorescence of the probe transition. Another important consequence of Eq. (22) is

$$|\langle \hat{n}(t) \rangle - \langle \hat{n}(0) \rangle| \leq 1, \quad (23)$$

which indicates that, for large initial vibrational excitation, the phonon number is very nearly a constant of motion. At the other extreme, if the system starts in the vibrational ground state [$\langle \hat{n}(0) \rangle = 0$] and in the electronic ground state [$p_1(0) = 1$], then it must remain in this state.

Using Laplace transforms, the solutions to Eqs. (18) and (19) are easily found to be

$$P_1(n,t) = \begin{cases} \frac{1}{2}P_1(n,0)(1 + e^{-4\chi n t}) + \frac{1}{2}P_2(n-1,0) \\ \quad \times (1 - e^{-4\chi n t}), & n > 0 \\ P_1(0,0), & n = 0, \end{cases} \quad (24)$$

$$P_2(n,t) = \frac{1}{2}P_1(n+1,0)(1 - e^{-4\chi(n+1)t}) + \frac{1}{2}P_2(n,0) \\ \times (1 + e^{-4\chi(n+1)t}). \quad (25)$$

The probability to find the atom in the ground state at time t is given by Eq. (21), with $i=1$, and is a rapidly decaying function of time if the initial distribution of vibrational energy is sharply peaked at a large value of n . For short times $\chi t \langle n \rangle \ll 1$ and the atom initially in the ground state, we find that to a good approximation

$$p_1(t) = 1 - 2\chi t \langle \hat{n}(0) \rangle. \quad (26)$$

In this case we have a measurement of the initial mean vibrational quantum number and the information can be extracted from the initial response of the signal. The time scale for a possible experiment is discussed in Sec. V.

If the vibrational state of the ion is initially thermal with mean number \bar{n} and the atom starts in the ground state, we find the probability to find the atom in the ground state at time t is

$$p_1(t) = \frac{1}{2} + \frac{1}{2[1 + \bar{n}(1 - e^{-4\chi t})]}. \quad (27)$$

If the vibrational state of the ion is coherent, with mean quantum number \bar{n} , the initial phonon-number distribution is Poissonian. If the atom starts in the ground state then

$$p_1(t) = \frac{1}{2} + \frac{1}{2}\exp[-\bar{n}(1 - e^{-4\chi t})]. \quad (28)$$

Thus the steady-state probability, in both cases, is approximately 0.5 for \bar{n} large. This corresponds to saturating the probe transition.

We can use Eq. (22) together with Eqs. (27) and (28) to obtain the mean vibrational quantum number. In the case of an initial thermal state

$$\langle \hat{n}(t) \rangle = \bar{n} + \frac{1}{2}\{[1 + \bar{n}(1 - e^{-4\chi t})]^{-1} - 1\}. \quad (29)$$

In the case of an initial Poisson state,

$$\langle \hat{n}(t) \rangle = \bar{n} + \frac{1}{2}\{\exp[-\bar{n}(1 - e^{-4\chi t})] - 1\}. \quad (30)$$

It is then easy to see that the mean vibrational quantum number changes very little. In both cases the steady state, for large \bar{n} , is

$$\bar{n}_\infty \approx \bar{n} - \frac{1}{2}. \quad (31)$$

III. STATE REDUCTION AND QUANTUM BACK-ACTION NOISE

The measurement model thus realizes a measurement of the vibrational quantum number $a^\dagger a$. If this measurement is quantum limited, we expect the density operator for the system to become diagonalized in the phonon-number basis. This dynamical state reduction is typical for continuous quantum-limited measurements [8]. Complementary to this decay of off-diagonal coherence in the number basis, the theory of continuous quantum-limited measurement suggests that the quantity conjugate to the measured quantity should undergo a diffusion process. We now show that this is indeed the case here. For simplicity we set the detuning to zero ($\delta=0$) in what follows.

The general solution to the master equation (5) in the number basis is

$$\langle n | \rho_{11}(t) | m \rangle = \frac{1}{2}e^{-\chi(\sqrt{n} - \sqrt{m})^2 t} (q_{n,m} + r_{n-1,m-1}) \\ + \frac{1}{2}e^{-\chi(\sqrt{n} + \sqrt{m})^2 t} (q_{n,m} - r_{n-1,m-1}), \quad (32)$$

$$\begin{aligned} \langle n|\rho_{22}(t)|m\rangle &= \frac{1}{2}e^{-\chi(\sqrt{n+1}-\sqrt{m+1})^2 t}(q_{n+1,m+1}+r_{n,m}) \\ &\quad - \frac{1}{2}e^{-\chi(\sqrt{n+1}+\sqrt{m+1})^2 t}(q_{n+1,m+1}-r_{n,m}), \end{aligned} \quad (33)$$

where $n, m > 0$, $q_{nm} = \langle n|\rho_{11}(0)|m\rangle$, and $r_{nm} = \langle n|\rho_{22}(0)|m\rangle$. Off-diagonal matrix elements decay at two rates. The fast rate is given by $-\chi(\sqrt{n+1}-\sqrt{m+1})^2$ and is due to the birth-death process previously identified for the phonon-number distribution. The slow rate $-\chi(\sqrt{n+1}+\sqrt{m+1})^2$ is an additional coherence decay term that becomes greater the larger the number difference between the superposed states that the coherence describes. This term ensures that the steady-state distribution is diagonal in the number basis. The steady-state density operator for the system, given that the atom starts in the ground state, is easily seen to be

$$\begin{aligned} \rho_\infty &= \left(P(0,0)|0\rangle\langle 0| + \frac{1}{2} \sum_{n=1}^{\infty} P(n,0)|n\rangle\langle n| \right) \\ &\quad \otimes |1\rangle\langle 1| + \frac{1}{2} \left(\sum_{n=0}^{\infty} P(n+1,0)|n\rangle\langle n| \right) \otimes |2\rangle\langle 2|. \end{aligned} \quad (34)$$

If the initial phonon-number distribution has a large mean, this equation may be approximated by

$$\rho_\infty \approx \sum_{n=0}^{\infty} P(n,0)|n\rangle\langle n| \otimes \frac{I}{2}, \quad (35)$$

where I is the two-dimensional identity matrix. The approximate steady-state solution is diagonal in the phonon number, with the diagonal matrix elements unchanged from the initial phonon-number distribution. This is characteristic of a quantum-nondemolition (QND) measurement of the number operator for a simple harmonic oscillator [9] and thus the measurement scheme analyzed here is approximately a QND number measurement. The electronic state is an equal mixture of the ground and excited states. A QND measurement of the quantum number of an oscillator must necessarily add noise to the oscillator phase. We now show that the center-of-mass oscillations undergo a phase diffusion as expected.

The Q functions for the operators ρ_{ii} are defined by coherent state matrix elements as

$$Q_i(\alpha, t) = \langle \alpha | \rho_{ii}(t) | \alpha \rangle, \quad (36)$$

where $|\alpha\rangle$ is a coherent state of the oscillatory motion in the trap. The equations of motion for each of these functions may be obtained directly from Eqs. (13) and (14) using standard operator correspondence rules [9]. The results are

$$\dot{Q}_1(\alpha) = 2\chi|\alpha|^2(Q_2 - Q_1) - \chi \left(\alpha^* \frac{\partial Q_1}{\partial \alpha^*} + \alpha \frac{\partial Q_1}{\partial \alpha} \right), \quad (37)$$

$$\begin{aligned} \dot{Q}_2(\alpha) &= 2\chi \frac{\partial^2 Q_1}{\partial \alpha^* \partial \alpha} - 2\chi(1+|\alpha|^2)(Q_2 - Q_1) \\ &\quad + 2\chi \left(\alpha^* \frac{\partial Q_1}{\partial \alpha^*} + \alpha \frac{\partial Q_1}{\partial \alpha} \right) - \chi \left(\alpha^* \frac{\partial Q_2}{\partial \alpha^*} + \alpha \frac{\partial Q_2}{\partial \alpha} \right). \end{aligned} \quad (38)$$

The Q function for the vibrational motion is given by $Q(\alpha, t) = Q_1(\alpha, t) + Q_2(\alpha, t)$ and satisfies the equation

$$\dot{Q} = \chi \left(\frac{\partial}{\partial \alpha^*} \alpha^* + \frac{\partial}{\partial \alpha} \alpha \right) (Q_1 - Q_2) + 2\chi \frac{\partial^2 Q_1}{\partial \alpha \partial \alpha^*}. \quad (39)$$

The last term in Eq. (39), involving second-order partial derivatives, leads to phase diffusion. This may be seen by writing the equations of motion in terms of polar coordinates (r, ϕ) , where $\alpha = r e^{i\phi}$. This gives a term of the form

$$\frac{\chi}{2r^2} \frac{\partial^2 Q_1}{\partial \phi^2}.$$

We have previously noted that for an initial state with large mean phonon number, the average $\langle a^\dagger a \rangle$ is very nearly constant. Thus if we start in an initial state of large amplitude α_0 , we can replace r^2 by $|\alpha_0|^2$. If we assume the atom starts in the ground state, then for a short time $Q = Q_1$ and $Q_2 = 0$. The initial evolution then corresponds to a simple harmonic oscillator decaying into a zero-temperature heat bath, with the amplitude decay rate of χ . If the initial mean vibrational energy is large, then for long times $Q_1 \approx Q_2$. This can be seen by noting that the off-diagonal elements of ρ_{ii} decay rapidly [see Eqs. (32) and (33)], thus the Q functions for long times can be approximated by

$$Q_i(\alpha, t) \approx \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2} P_i(n, t). \quad (40)$$

Using Eqs. (24) and (25) and assuming the atom starts in the ground state we find

$$Q_1(\alpha, t) \approx \frac{1}{2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2} P_1(n, 0) (1 + e^{-4\chi n t}), \quad (41)$$

$$Q_2(\alpha, t) \approx \frac{1}{2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2} P_1(n+1, 0) (1 - e^{-4\chi(n+1)t}). \quad (42)$$

If the mean photon number is large, there is very little difference between $P_1(n, 0)$ and $P_1(n+1, 0)$. Thus for long times these two Q functions are very nearly equal. This result is also confirmed by numerical solution indicated in Fig. 2. In this case the long time evolution can be approximated by

$$\dot{Q} \approx \chi \frac{\partial^2 Q}{\partial \alpha^* \partial \alpha}, \quad (43)$$

leading to phase diffusion at the rate $D_\phi = \chi/4|\alpha_0|^2$. The phase diffuses at a slower rate for larger initial mean vibrational energies.

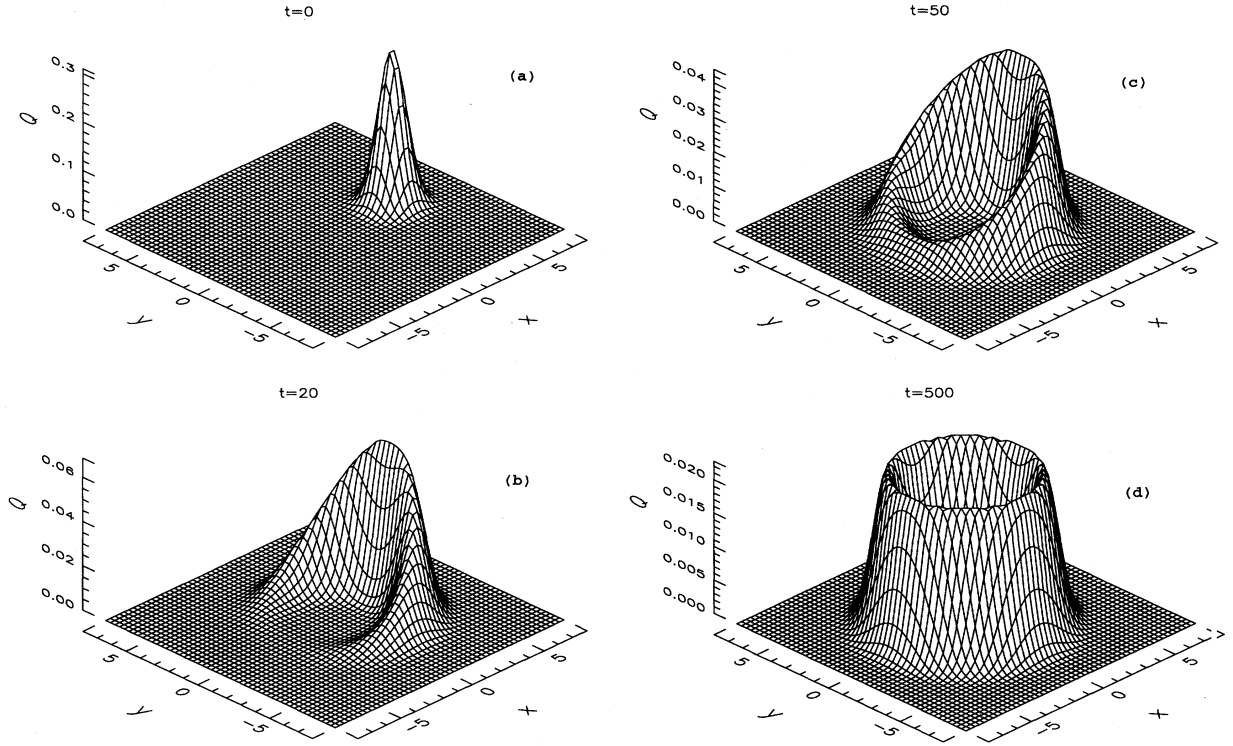


FIG. 2. Plot of the Q function, versus real (x) and imaginary (y) parts of α , at various times for an initial coherent state with coherent amplitude $\alpha_0 = 4$. (a) $\chi t = 0$, (b) $\chi t = 20$, (c) $\chi t = 50$, and (d) $\chi t = 500$.

Rather than solve Eqs. (37) and (38) directly, we solve Eqs. (13) and (14) in the photon-number basis and then compute the Q functions via the general results in Eqs. (32) and (33). In order to show the phase diffusion we take the initial center-of-mass state to be a coherent state $|\alpha_0\rangle$ and assume that the atom is initially in the ground state. The Q function for the initial coherent state is a bivariate Gaussian centered on α_0 . The results are shown in Fig. 2, where we plot the Q function versus real and imaginary α at increasing times. Note that as time proceeds the initial coherent state undergoes a phase diffusion, eventually producing an annular distribution at approximately the same radius as the initial coherent state. As expected, a measurement of the number operator $a^\dagger a$ causes the phase variable to undergo diffusion by quantum back-action noise, while the initial mean phonon number remains very nearly constant. This is consistent with our identification of the measurement scheme as an approximate QND measurement of the vibrational energy of the trapped ion.

IV. SIGNAL AND NOISE

In Sec. III we demonstrated that the probability to find the atom in the ground state at time t is proportional to the average vibrational quantum number of the trapped atom at the same time. However, we do not observe $p_1(t)$ directly. The actual signal is a photocurrent produced by monitoring the fluorescent light from the probe transition.

The mean photocurrent is proportional to the probability to find the atom in state $|3\rangle$, Eq. (9). However, in the adiabatic approximation used in this paper, we may write this

more directly in terms of the probability to find the atom in state $|1\rangle$ (see [5]),

$$E(i(t)) = 2\Gamma \epsilon p_1(t), \quad (44)$$

where, as previously, ϵ is the quantum efficiency and $\Gamma = \Omega_p^2/2\gamma$ is the measurement parameter. Using Eq. (22) and assuming the atom starts in the ground state $|1\rangle$, we have that

$$E(i(t)) = 2\Gamma \epsilon [\langle \hat{n}(t) \rangle - \langle \hat{n}(0) \rangle + 1]. \quad (45)$$

This equation indicates that, on top of a dc current, there is a fluctuating photocurrent directly proportional to the instantaneous mean vibrational quantum number. The dc background is determined by the initial mean vibrational quantum number.

To estimate the noise in the signal we can calculate the variance in the photocurrent or, more usefully, the spectrum of fluctuations in the photocurrent. The variance in the photocurrent is

$$V(i(t)) = \frac{2\Gamma \epsilon}{\tau_D} p_1(t) - 4\Gamma^2 \epsilon^2 [p_1(t)]^2, \quad (46)$$

where τ_D is the characteristic response time of the photodetector. The first term in Eq. (46) represents the shot noise of the detector, which is the dominant term.

In an experimental context a more useful quantity than the variance is the power spectrum of the current fluctuations. We are thus led to consider the stationary two-time correlation function [7]

$$G(\tau) = \langle i(t+\tau)i(t) \rangle_{SS} \quad (47)$$

$$= \epsilon^2 \langle E^{(-)}(0)E^{(-)}(\tau)E^{(+)}(\tau)E^{(+)}(0) \rangle_{SS} + \epsilon \delta(\tau) \langle E^{(-)}E^{(+)} \rangle_{SS}, \quad (48)$$

where the subscript SS reminds us that these averages are evaluated in the stationary state of the source. For the atomic source considered here this correlation function reduces to

$$G(\tau) = i_\infty [\delta(\tau) + 2\Gamma \epsilon p_{1,SS}(\tau)], \quad (49)$$

where $i_\infty = 2\Gamma \epsilon p_1(\infty)$ is the stationary current and $p_{1,SS}(\tau)$ is defined by

$$p_{1,SS}(\tau) = \frac{1}{2} \left(1 + [p_1(\infty)]^{-1} \sum_{n=0}^{\infty} e^{-4\chi n|\tau|} P_1(n, \infty) \right), \quad (50)$$

where $P_1(n, \infty)$ is the steady-state conditional probability to find the n vibrational quanta and the atom in the ground electronic state. This probability is given by

$$P_1(n, \infty) = \begin{cases} P(0,0), & n=0 \\ \frac{1}{2} P(n,0), & n \neq 0. \end{cases} \quad (51)$$

The spectrum of fluctuations $S(\omega)$ is the Fourier transform of $G(\tau)$ and is given by

$$S(\omega) = i_\infty \{ 1 + \Gamma \epsilon [2\pi \delta(\omega) + s(\omega)] \}, \quad (52)$$

where the normalized spectrum $s(\omega)$ is

$$s(\omega) = [p_1(\infty)]^{-1} \sum_{n=1}^{\infty} P(n,0) \frac{4\chi n}{\omega^2 + 16\chi^2 n^2}. \quad (53)$$

The first term in Eq. (52) is the shot-noise contribution to the noise power, while the second term is a dc term, which we neglect in what follows. If the atom starts in the ground state then the steady-state probability of finding the atom in the ground state is $p_1(\infty) = [P(0,0) + 1]/2$, which for a vibrational number distribution with a large initial mean is approximately $1/2$. This result was previously obtained in [1]. In the frequency domain the short-time approximation corresponds to large frequencies. Thus for $\omega \gg 4\chi \langle \hat{n}(0) \rangle$ we find

$$S(\omega) \approx i_\infty \left[1 + \frac{4\Gamma \epsilon \chi}{p_1(\infty) \omega^2} \langle \hat{n}(0) \rangle \right], \quad (54)$$

where $\langle \hat{n}(0) \rangle$ is the initial mean vibrational quantum number. As expected, the initial mean vibrational quantum number can be determined as a noise component above the shot-noise component at high frequencies. In Sec. V we consider what the frequency scale is likely to be in a typical experiment.

A number of special cases can be considered. If the atom starts in the electronic ground state and also in the ground state of the trap,

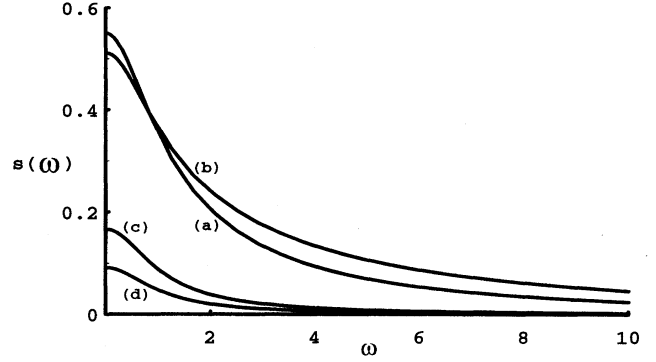


FIG. 3. Plot of the normalized photocurrent power spectrum $s(\omega)$ for an initial thermal distribution of vibrational energy. In all cases $4\chi = 1$. (a) $\bar{n} = 5$, (b) $\bar{n} = 2$, (c) $\bar{n} = 0.2$, and (d) $\bar{n} = 0.1$

$$P_1(n, \infty) = \delta_{n,0}, \quad (55)$$

then the noise spectrum is purely a flat shot-noise spectrum. This could be used as a diagnostic to determine the effectiveness of laser cooling in localizing the ion in the ground state of the trap.

If the atom starts in a thermal state, the initial phonon-number distribution is

$$P(n,0) = \frac{1}{1 + \bar{n}} \left(\frac{\bar{n}}{1 + \bar{n}} \right)^n. \quad (56)$$

In the low-temperature limit $\bar{n} \ll 1$ the spectrum is

$$S(\omega) = i_\infty \left(1 + \frac{4\Gamma \epsilon \chi \bar{n}}{p_1(\infty) (\omega^2 + 16\chi^2)} \right). \quad (57)$$

This is a Lorentzian, above the shot-noise background, centered at $\omega = 0$ with height proportional to \bar{n} and width proportional to χ .

Finally, we consider the case of an initial coherent state of vibrational motion in which case the phonon-number distribution is Poissonian and is given by

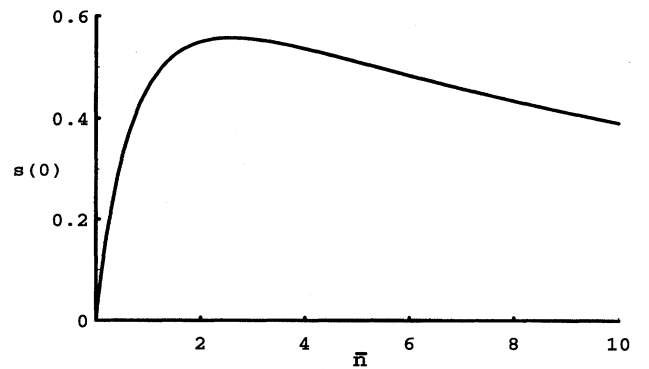


FIG. 4. Plot of the normalized photocurrent power spectrum evaluated at zero frequency $s(0)$ versus \bar{n} for an initial thermal distribution.

$$P(n,0) = \frac{\bar{n}^n}{n!} e^{-\bar{n}}. \quad (58)$$

If the initial mean quantum number is large we find

$$S(\omega) = i_\infty \left(1 + \frac{8\Gamma\epsilon\chi\bar{n}}{\omega^2 + 16\chi^2\bar{n}^2} \right). \quad (59)$$

This is a low, broad Lorentzian, above the shot-noise background and centered at zero frequency.

In Fig. 3 we plot the normalized spectrum defined by

$$s(\omega) = \frac{S(\omega) - i_\infty}{i_\infty 4\chi\Gamma\epsilon}, \quad (60)$$

for an initial thermal distribution of vibrational energy, and $4\chi = 1$. As the ion is cooled, the spectrum approaches the flat shot-noise characteristic spectrum. In Fig. 4 we plot $s(0)$ versus \bar{n} for an initial thermal distribution. This shows a crossover from the low-temperature limit Eq. (57), in which $s(0)$ is linear in \bar{n} , to the high-temperature limit in which $s(0)$ decreases as \bar{n} increases.

V. DISCUSSION AND CONCLUSION

We first estimate some likely experimental values for the parameters of our model. To begin we note that the good measurement limit requires that the Rabi frequency of the probe transition be less than the decay rate on that transition. This transition will be a typical electric-dipole cooling transition for which a typical spontaneous emission rate is $\gamma = 10$ MHz. We thus choose $\Omega_p = 2$ MHz and thus the measurement parameter is $\Gamma = 0.2$ MHz. The second part of the good measurement limit requires that $\Omega_p^2/\gamma > \eta\Omega$. The scale parameter η is typically of the order of 0.01; hence if we choose $\Omega = 0.1$ MHz, this condition is satisfied. The measurement coupling parameter is then $\chi \approx 1.25$ Hz. This parameter determines the time scale of the birth-death process of our measurement, which we see will be slow compared to the period of oscillation in the trap (which is typically of the order of microseconds).

The time scale for the short-time result in Eq. (26) is $t < 1/\chi\bar{n}$. For Doppler cooling of the ion the mean vibrational quantum number is $\bar{n} = \gamma/2\nu$. For a typical trap frequency of $\nu = 1$ MHz this gives $\bar{n} = 5$. Thus the short-time limit means time less than about 1 s, which for this kind of experiment is not very short at all. In the context of the photocurrent power spectrum we see that the widths of the distribution, for an initial thermal distribution with $\bar{n} = 5$, will also be rather small. Thus the high-frequency limit is easily achieved in an experiment.

A more serious practical limitation to this measurement scheme is posed by the heating of the ion due to the probe field. In the Appendix we show that the effect of heating is to add to the equation for $\dot{\rho}_{11}$ [Eq. (10)] a term of the form

$$(\dot{\rho}_{11})_{\text{heat}} = \gamma_h (2a^\dagger \rho_{11} a + 2a \rho_{11} a^\dagger - \{a^\dagger a, \rho_{11}\} - \{a a^\dagger, \rho_{11}\}), \quad (61)$$

where $\{ , \}$ is an anticommutator and the heating rate is given by

$$\gamma_h = \frac{\eta_p^2 \Omega_p^2}{2\gamma}, \quad (62)$$

where η_p is the Lamb-Dicke parameter for the probe field. This term may be neglected provided $\gamma_h \ll \chi$, which requires

$$\left(\frac{\eta_p}{\eta} \right)^2 \left(\frac{\Omega_p^2}{\Omega\gamma} \right)^2 \ll 1. \quad (63)$$

If this is combined with Eq. (4), it indicates that $\eta_p \ll 1$. For the parameters used above this would require $\eta_p \ll \eta$, which may be difficult to achieve in practice. The difficulty may be overcome by only switching on the probe field for a short time. This suggests a pulsed probe field might be a better way to extract the information. A pulsed probe field would require a very different treatment to that presented in this paper.

In this paper we have proposed an approximate quantum-nondemolition measurement scheme for measuring the vibrational energy of a trapped ion in a standing wave in the Lamb-Dicke limit. By measuring the fluorescent intensity in a strong probe transition coupled to one of the levels, a direct determination of the number of vibrational quanta can be made. Current experiments on laser-cooled trapped ions suggest that such a QND measurement is possible with available technology.

APPENDIX

In this appendix we show how the effects of probe heating of the ion may be included. The probe field induces fluorescence on the $|1\rangle \leftrightarrow |3\rangle$ transition and the atom recoils at each spontaneous emission event. The effect of spontaneous emission and recoil is described by a master equation for the total system, which includes the term

$$\dot{\rho}_{\text{heat}} = \frac{\gamma}{2} (2\sigma \mathcal{N} \rho \sigma^\dagger - \sigma^\dagger \sigma \rho - \rho \sigma^\dagger \sigma), \quad (A1)$$

where $\sigma = |1\rangle\langle 3|$ and

$$\mathcal{N} \rho = \int \phi(\vec{n}) e^{i\eta_p X_1 n_1} \rho e^{-i\eta_p X_1 n_1} d\vec{n}, \quad (A2)$$

where $\phi(\vec{n})$ is the probability for a photon to be scattered in direction \vec{n} , $n_1 = \vec{k} \cdot \vec{n}/k$ is the projection of the recoil direction on the standing-wave axis, $X_1 = a + a^\dagger$ is the dimensionless position of the ion, and η_p is the Lamb-Dicke parameter for the probe transition. Expanding to second order in η_p , the effect of recoil on the center-of-mass motion is described by the term

$$\dot{\rho}_{\text{heat}} = -\frac{\gamma\eta_p^2}{2} [X_1, [X_1, \sigma \rho \sigma^\dagger]]. \quad (A3)$$

If we now transform to the interaction picture for the vibrational motion of the ion in the trap and assume that the

vibrational frequency is larger than the response bandwidth of the measurement, the effect of the heating is well described by the equation

$$\dot{\rho}_{\text{heat}} = \frac{\gamma \eta_p^2}{2} (2a^\dagger \rho_{33} a + 2a \rho_{33} a^\dagger - \{a^\dagger a, \rho_{33}\} - \{a a^\dagger, \rho_{33}\}) |1\rangle\langle 1|. \quad (\text{A4})$$

In the adiabatic approximation, level $|3\rangle$ is eliminated and

replacing $\rho_{33} \rightarrow (\Omega_p^2 / \gamma^2) \rho_{11}$, the final equation to describe the heating of the ion is

$$\dot{\rho}_{\text{heat}} = \gamma_h [(2a^\dagger \rho_{11} a + 2a \rho_{11} a^\dagger - \{a^\dagger a, \rho_{11}\} - \{a a^\dagger, \rho_{11}\}) |1\rangle\langle 1|], \quad (\text{A5})$$

where

$$\gamma_h = \frac{\Omega_p^2 \eta_p^2}{2\gamma} = \Gamma \eta_p^2. \quad (\text{A6})$$

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- [1] J. I. Cirac, R. Blatt, A. S. Parkins, and P. Zoller, *Phys. Rev. A* **49**, 1202 (1994).
 [2] C. Blockley and D. F. Walls, *Phys. Rev. A* **47**, 2115 (1993).
 [3] R. C. Thompson, K. Dholakia, G. Zs. K. Horvath, W. Power, and D. M. Segal, *J. Mod. Opt.* **41**, 1087 (1994).
 [4] W. M. Itano, D. J. Heinzen, J. J. Bollinger, and D. J. Wineland, *Phys. Rev. A* **41**, 2295 (1990).
 [5] M. J. Gagen and G. J. Milburn, *Phys. Rev. A* **47**, 1467 (1993).
 [6] H. J. Carmichael, in *An Open Systems Approach to Quantum Optics*, Lecture Notes in Physics Vol. 18 (Springer, Berlin, 1993).
 [7] H. J. Carmichael, *J. Opt. Soc. Am. B* **4**, 1588 (1987).
 [8] A. Barchielli, *Phys. Rev. A* **34**, 1642 (1986); L. Diosi, *Phys. Lett. A* **129**, 31 (1988); V. P. Belavkin, *J. Phys. A* **22**, L1109 (1989); G. J. Milburn, K. Jacobs, and D. F. Walls, *Phys. Rev. A* **50**, 5256 (1994).
 [9] D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer, Berlin, 1994).