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Noise reduction in a laser by nonlinear damping

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We consider the reduction of the intensity fluctuations in a laser by intracavity nonlinear absorption. The optimum operating conditions for reducing the intracavity intensity fluctuations are not the same as the conditions for reducing the intensity fluctuations in the output field. For a quite general class of models, we show that at the optimum operating point for reducing intensity fluctuations in the output field reduction in the intracavity intensity fluctuations is half of the maximum level that can be achieved in the model. We also show that as the laser intensity fluctuations are reduced the phase fluctuations, as measured by the laser linewidth, are correspondingly increased.

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I. INTRODUCTION

Since the demonstration by Yamamoto and co-workers [1] that the intensity fluctuations in the output of a semiconductor laser can be reduced below the shot-noise limit by feedback, there has been considerable effort given to similar schemes to reduce quantum noise in general laser systems [2–11]. In the work of Ritsch [6] and Walls, Collett, and Lane [7], the possibility of reducing intensity noise by introducing intracavity nonlinear absorbers was discussed. The examples discussed in these papers suggested that the optimum conditions for reducing intensity fluctuations inside the cavity and outside the cavity do not coincide. In this paper, we take up this issue and find for a general class of nonlinear absorbers, the conditions under which the intensity fluctuations in the intracavity field or the external field may be reduced below the shot-noise limit.

A convenient measure of the intensity fluctuations inside the cavity is given by the Mandel Q factor, defined by

$$Q=\frac{V(n)-\bar{n}}{\bar{n}} . \tag{1.1}$$

In terms of this parameter, the shot-noise limit is defined as $Q=0$. Sub-shot-noise intensity fluctuations inside the cavity thus correspond to $-1<Q<0$. Outside the cavity a linearized intensity fluctuation spectrum characterizes the intensity fluctuations. This spectrum is the Fourier transform of the photocurrent two-time correlation function [4] (normalized to have units of s^{-1}) and takes the form

$$S(\omega)=\kappa\bar{n}\left[1+R\frac{k^2}{k^2+\omega^2}\right] , \tag{1.2}$$

where κ is the cavity decay rate, $\bar{n}$ is the steady-state mean photon number inside the cavity, and k is a model-dependent parameter. In this paper we show that if the best Q factor obtained inside the cavity is Q_∞, the optimum reduction in output intensity fluctuations is given by

$$R_{\text{opt}}=\tfrac{1}{2}Q_\infty , \tag{1.3}$$

which agrees with the results of Walls, Collett, and Lane [7]. Explicit expressions for these quantities are given in Eqs. (2.22) and (2.26).

II. LASER WITH NONLINEAR DAMPING

The density operator for the state of the field inside a cavity with a nonlinear absorber evolves according to the master equation

$$\dot{\rho}=\frac{\kappa}{2}(2a\rho a^\dagger-a^\dagger a\rho-\rho a^\dagger a)+\frac{\gamma}{2}(2A\rho A^\dagger-A^\dagger A\rho-\rho A^\dagger A) . \tag{2.1}$$

The linear loss term proportional to κ represents coupling to the modes external to the cavity. The nonlinear term is assumed to be of the form

$$A=\{(a^\dagger)^P a^Q\} , \tag{2.2}$$

where P,Q are integers and the curly brackets indicate that the order of the operators is unspecified. Following Ritsch [6], we define two integers M,N such that $N\ge M\ge 1$, and $(N-M)$ mod $2=0$ with the interpretation that N is the number of photons involved in the process ($N\ge 2$ implies a nonlinear process), while M is the number of photons absorbed in the process ($M<N$ indicates a Raman-type process). Thus we chose

$$P=\frac{N-M}{2} , \tag{2.3}$$

$$Q=\frac{N+M}{2} . \tag{2.4}$$

Note that we are implicitly assuming that the absorption is unsaturated. This gives the best noise reduction. For the same reason we will assume later that the gain is saturated. As examples we have the following.

Linear loss,

$$M=N=1,\ A=a\ ;$$

two-photon absorption,

$$M=N=2,\ A=a^2\ ;$$

three-photon Raman process,

$$M=1,\ N=3,\ A=a^\dagger a^2\ .$$

For specificity we will assume that

$$A=(a^\dagger a)^P a^M\ . \tag{2.5}$$

This includes the examples given above. The results of this paper do not depend on this choice, providing that N is small compared to the mean photon number $\bar{n}$, as it will be in realistic cases.

In the photon-number representation we use

$$A^\dagger|n\rangle=\left[\frac{(n+M)!}{n!}\right]^{1/2} n^P|n+M\rangle \tag{2.6}$$

and

$$A^\dagger A|n\rangle=(n-M)^{2P}\frac{n!}{(n-M)!}|n\rangle\ . \tag{2.7}$$

To model the laser gain we use the model of Scully and Lamb [12]. This is based on a four-level model in which inelastic collisions quickly deplete the populations of the lasing levels enabling the atomic dynamics to be adiabatically eliminated. The photon-number distribution for the field then obeys the master equation

$$\frac{d\rho_n}{dt}=-G\left[\frac{n+1}{1+(n+1)/n_s}\rho_n-\frac{n}{1+n/n_s}\rho_{n-1}\right]\ , \tag{2.8}$$

where G is the gain and n_s is the saturation number. We find that the best noise reduction occurs in the limit of large gain where $\bar{n}\gg n_s$, in which case the master equation may be approximated by

$$\frac{d\rho_n}{dt}=Gn_s(\rho_{n-1}-\rho_n)\ . \tag{2.9}$$

The diagonal matrix elements of the field density operator then obey the master equation

$$\begin{aligned}\dot{\rho}_n={}&Gn_s(\rho_{n-1}-\rho_n)+\kappa[(n+1)\rho_{n+1}-n\rho_n]\\&+\gamma\left[\frac{(n+M)!}{n!}n^{N-M}\rho_{n+M}\right.\\&\left.-\frac{n!}{(n-M)!}(n-M)^{(N-M)}\rho_n\right]\ ,\end{aligned} \tag{2.10}$$

where $\rho_n=\langle n|\rho|n\rangle$. For the initial condition $\rho_n(0)=\delta_{n,m}$, the short time solution is

$$\begin{aligned}\rho_n(t)={}&\delta_{n,m}\left[1-\kappa tm-\gamma t\frac{m!}{(m-M)!}(m-M)^{N-M}-Gn_st\right]\\&+\delta_{n,m-1}\kappa tm+\delta_{n,m-M}\gamma t\frac{m!}{(m-M)!}(m-M)^{N-M}\\&+\delta_{n,m+1}Gn_st\ .\end{aligned} \tag{2.11}$$

This equation indicates that we may replace the birth-death master equation with a nonlinear diffusion process with drift and diffusion coefficients given by

$$d(m)=\frac{1}{t}\langle n-m\rangle\approx-\kappa m-\gamma Mm^N+Gn_s\ , \tag{2.12}$$

$$D(m)=\frac{1}{t}\langle(n-m)^2\rangle\approx\kappa m+\gamma M^2m^N+Gn_s\ . \tag{2.13}$$

From Eq. (2.12) we estimate the stationary mean photon number $\bar{n}$ in the cavity as given by the solution of

$$\gamma M\bar{n}^N+\kappa\bar{n}-Gn_s=0\ . \tag{2.14}$$

We now introduce the scaled parameters

$$\mu=\frac{\bar{n}}{\nu}\ , \tag{2.15}$$

$$\chi=\nu^{N-1}\frac{\gamma}{\kappa}M\ , \tag{2.16}$$

where $\nu=(G/\kappa)n_s$ is the standard ($\gamma=0$) expression for the mean photon number well above threshold [12]. In terms of the scaled parameters the mean is given by

$$\chi\mu^N+\mu-1=0\ . \tag{2.17}$$

In Fig. 1 we plot the mean photon number versus χ for various values of N. In general $\bar{n}$ is less than ν, so that $0<\mu<1$.

To proceed we consider fluctuations around the deterministic steady state by approximating the nonlinear diffusion process by a linear one (i.e., an Ornstein-Uhlenbeck process). The drift and diffusion constant of the process are

$$k=[N-(N-1)\mu]\frac{\kappa}{\mu}\ , \tag{2.18}$$

$$D=\bar{n}[(M+1)-(M-1)\mu]\frac{\kappa}{\mu}\ . \tag{2.19}$$

The variance of the internal photon number distribution in the steady state is given by

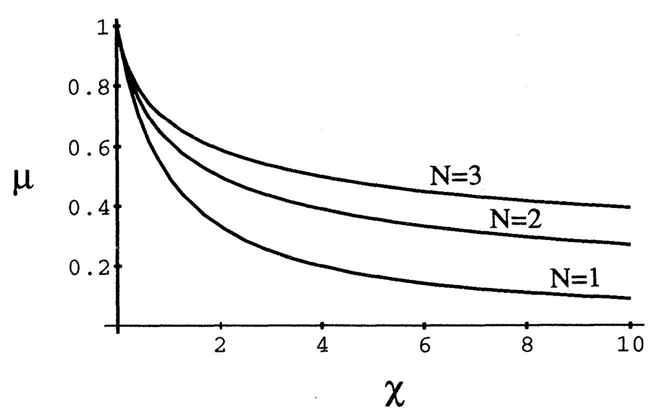

FIG. 1. A plot of the scaled intracavity mean photon number μ vs the scaled nonlinear absorption χ for various values of N.

$$\sigma^2 = \frac{D}{2k} = \bar{n}\frac{(M+1)-(M-1)\mu}{2[N-(N-1)\mu]} . \quad (2.20)$$

The deviation of the steady-state distribution from Poisson statistics may be quantified by the Q parameter

$$Q = \frac{\sigma^2 - \bar{n}}{\bar{n}} = \frac{(\mu-1)(2N-M-1)}{2[N-(N-1)\mu]} . \quad (2.21)$$

For no nonlinearity $\mu=1$ and $Q=Q_0=0$. If the intracavity absorption (χ) is large, $\mu \to 0$ and

$$Q \to Q_\infty = -1 + \frac{M+1}{2N} , \quad (2.22)$$

which is the minimum value as found by Ritsch [6]. Note that the best results for reducing the intracavity intensity fluctuations occur with N large and M small. The sensitivity of the absorber to intensity fluctuations is determined by the degree of nonlinearity (N) of the damping process. This explains why large N gives the best noise reduction. However, it is not desirable for the number of photons absorbed in the damping process (M) to be large because this would lower the mean photon number $\bar{n}$, increasing the variance to mean ratio. For the two examples discussed above, $M=N=2 \Longrightarrow Q_\infty = -\frac{1}{4}$, as found by Walls, Collett, and Lane [7], and $M=1, N=3 \Longrightarrow Q_\infty = -\frac{2}{3}$. These results are depicted in Fig. 2.

The intensity fluctuations outside the cavity are determined by the Fourier transform of the normalized photocurrent two-time correlation function [4]

$$\langle \delta i^2(\omega) \rangle = \kappa \bar{n} \left[1 + R\frac{k^2}{k^2+\omega^2} \right] , \quad (2.23)$$

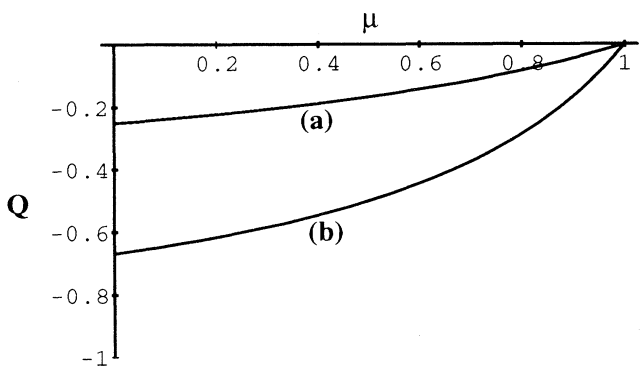

FIG. 2. A plot of the Q parameter, which measures deviation from Poisson statistics inside the cavity, vs the scaled intracavity mean photon number μ. (a) $N=M=2$; (b) $N=3, M=1$.

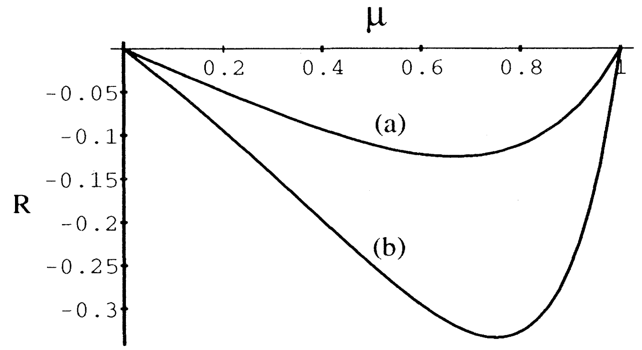

FIG. 3. A plot of the output intensity noise reduction parameter R vs the scaled intracavity mean photon number μ. (a) $N=M=2$; (b) $N=3, M=1$.

where

$$R = 2Q\frac{\kappa}{k} = \frac{(\mu-1)(2N-M-1)\mu}{[N-(N-1)\mu]^2} . \quad (2.24)$$

The parameter R in fact determines the reduction of the output intensity fluctuations below the shot-noise limit as is seen by evaluating the spectrum at zero frequency,

$$\langle \delta i^2(0) \rangle = \kappa \bar{n}(1+R) . \quad (2.25)$$

In the limit $\mu=1$, $R=R_0=0$. Also in the limit $\mu \to 0$, $R \to R_\infty = 0$ from below. In Fig. 3 we plot R versus μ for the case of two-photon absorption and a Raman process. The optimum value of R is found by solving $dR/d\mu=0$. We find

$$R_{\mathrm{opt}} = \frac{1}{2}\left[\frac{M+1}{2N} - 1 \right] \quad (2.26)$$

at

$$\mu_{\mathrm{opt}} = \frac{N}{N+1} . \quad (2.27)$$

Note that the best noise reduction obtainable by such a nonlinear process is $R=-\frac{1}{2}$, which occurs for M small and N large. At the optimum value for μ we find

$$Q_{\mathrm{opt}} = \frac{1}{2}\left[\frac{M+1}{2N} - 1 \right] . \quad (2.28)$$

Thus we have shown that

$$R_{\mathrm{opt}} = Q_{\mathrm{opt}} = \tfrac{1}{2} Q_\infty . \quad (2.29)$$

This agrees with the results found by Walls, Collett, and Lane [7] for the case of two-photon absorption.

III. LASER LINEWIDTH

One might expect that any attempt to reduce the laser intensity fluctuations below the shot-noise limit will be at the expense of increasing the phase fluctuations and thus of increasing the laser linewidth. This is indeed the case as we now verify. We will adopt the simple procedure of Sargent III, Scully, and Lamb [12], which consists in finding the decay rate for the average field amplitude. This decay is then attributed to phase diffusion since the intensity is constant. Outside the cavity the mean electric field is proportional to $\langle a^\dagger(t)\rangle$. This quantity obeys the equation

$$\frac{d\langle a^\dagger(t)\rangle}{dt}=\sum_{n=0}^{\infty}\langle n|\dot\rho(t)|n+1\rangle\sqrt{n+1}\ . \tag{3.1}$$

From Eq. (2.1) we find the contribution due to the nonlinear terms is

$$\langle \dot a^\dagger(t)\rangle=\frac{\gamma}{2}\sum_{n=0}^{\infty}\sqrt{n+1}\left[2[n(n+1)]^P\left[\frac{(n+M)!(n+M+1)!}{n!(n+1)!}\right]^{1/2}\rho_{n+M,n+M+1}\right.$$
$$\left.-\left[(n-M)^{2P}\frac{n!}{(n-M)!}+(n+1-M)^{2P}\frac{(n+1)!}{(n+1-M)!}\right]\rho_{n,n+1}\right]\ . \tag{3.2}$$

Rearranging sums and expanding to first order in $1/n$ and using (2.14) we get

$$\langle \dot a^\dagger(t)\rangle=\frac{1}{2\bar n}(\kappa\bar n-Gn_s)\langle \dot a^\dagger(t)\rangle\ . \tag{3.3}$$

The standard Scully-Lamb terms for the phase diffusion are

$$\langle \dot a^\dagger(t)\rangle=\frac{1}{2\bar n}(Gn_s-GA-\kappa\bar n)\langle a^\dagger(t)\rangle\ , \tag{3.4}$$

where A is a dimensionless parameter determined by the decay rates of the particular atomic transition model used. When there terms are added to those arising from the nonlinearity, we find

$$\langle \dot a^\dagger(t)\rangle=-\Gamma\langle \dot a^\dagger(t)\rangle\ , \tag{3.5}$$

where the decay rate is

$$\Gamma=\frac{1}{2\bar n}GA=\kappa A\frac{1}{2\mu n_s}\ . \tag{3.6}$$

Thus the laser linewidth is

$$2\Gamma=\kappa A\frac{1}{\mu n_s}\ . \tag{3.7}$$

At the optimum operation point we find

$$2\Gamma=\frac{\kappa A}{n_s}\left[\frac{N+1}{N}\right]\ , \tag{3.8}$$

which should be compared with the standard result

$$2\Gamma=\frac{\kappa A}{n_s}\ . \tag{3.9}$$

In order to compare the result in Eq. (3.7) with the standard laser, we wish to keep the output power constant. This means we keep κ constant and, for a given nonlinearity, adjust G, the gain, to make $\bar n$ the same as it would be without nonlinear absorption. Thus μ now varies with G ($\mu\propto 1/G$) rather than $\bar n$. We are thus led to define a new dimensionless linewidth

$$\mathcal{D}=\frac{2\Gamma}{\kappa A/n_s}=\frac{1}{\mu}\ . \tag{3.10}$$

Evidently the linewidth is broadened by increasing the nonlinearity as expected.

For small nonlinearity we can write

$$\mu=1-\delta\ , \tag{3.11}$$

where δ is small ($\ll 1/N$), so that $\mathcal{D}=1+\delta$. For $\kappa,\bar n$ fixed, we can use the dimensionless noise parameter

$$\mathcal{N}=1+R=1+\frac{(\mu-1)(2N-M-1)\mu}{[N-(N-1)\mu]^2} \tag{3.12}$$

to compare the two models. This parameter does not have a simple dependence on μ. However, not far from the standard laser we have

$$\mathcal{N}=1-\delta(2N-M-1)\ , \tag{3.13}$$

which demonstrates the effect of intensity noise reduction. In particular, for the simplest kind of nonlinear absorption, two-photon absorption ($N=M=2$), we find

$$\mathcal{N}=1-\delta \tag{3.14}$$

so that

$$\mathcal{D}\approx\frac{1}{\mathcal{N}}\ , \tag{3.15}$$

which clearly indicates that decreasing the intensity fluctuations is at the expense of increased phase fluctuations.

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