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Wave breaking for the Stochastic Camassa–Holm equation☆

Dan Crisan, Darryl D. Holm *

Department of Mathematics, Imperial College, London SW7 2AZ, UK

HIGHLIGHTS

- The stochastic Camassa–Holm (SCH) equation is derived variationally.
- Peakon solutions and isospectrality conditions are found for the SCH equation.
- Wave breaking also survives introducing stochasticity into the SCH equation.

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ABSTRACT

We show that wave breaking occurs with positive probability for the Stochastic Camassa–Holm (SCH) equation. This means that temporal stochasticity in the diffeomorphic flow map for SCH does not prevent the wave breaking process which leads to the formation of peakon solutions. We conjecture that the time-asymptotic solutions of SCH will consist of emergent wave trains of peakons moving along stochastic space–time paths.

1. The deterministic Camassa–Holm (CH) equation

The deterministic CH equation, derived in [1], is a nonlinear shallow water wave equation for a fluid velocity solution whose profile $u(x,t)$ and its gradient both decay to zero at spatial infinity, $|x|\to\infty$, on the real line $\mathbb{R}$. Namely,

$$u_t - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}\,, \tag{1.1}$$

where subscripts t (resp. x) denote partial derivatives in time (resp. space). This nonlinear, nonlocal, completely integrable PDE may be written in *Hamiltonian form* for a momentum density $m := u - u_{xx}$ undergoing coadjoint motion, as [1]

$$m_t = \{m, h(m)\} = -(\partial_x m + m\partial_x)\frac{\delta h}{\delta m}\,, \tag{1.2}$$

which is generated by the Lie–Poisson bracket

$$\{f, h\}(m) = -\int \frac{\delta f}{\delta m}(\partial_x m + m\partial_x)\frac{\delta h}{\delta m}\,dx \tag{1.3}$$

and Hamiltonian function

$$h(m) = \frac{1}{2}\int_{\mathbb{R}} mK*m\,dx = \frac{1}{2}\int_{\mathbb{R}} u^2 + u_x^2\,dx$$

$$= \frac{1}{2}\|u\|^2_{H^1} = const. \tag{1.4}$$

Here, $K*m := \int K(x,y)m(y,t)dy$ denotes convolution of the momentum density m with Green's function of the Helmholtz operator $L = 1-\partial_x^2$, so that

$$\frac{\delta h}{\delta m} = K*m = u \quad\text{with}\quad K(x-y) = \frac{1}{2}\exp(-|x-y|)\,. \tag{1.5}$$

Alternatively, the CH equation (1.1) may be written in advective form as

$$\begin{aligned} u_t + uu_x &= -\,\partial_x\Big(K*\big(u^2+\frac{1}{2}u_x^2\big)\Big) \\ &= -\,\partial_x\int_{\mathbb{R}}\frac{1}{2}\exp(-|x-y|)\Big(u^2(y,t)+\frac{1}{2}u_y^2(y,t)\Big)\,dy\,. \end{aligned} \tag{1.6}$$

The deterministic CH equation admits signature solutions representing a wave train of peaked solitons, called *peakons*, given by

$$u(x,t) = \frac{1}{2}\sum_{a=1}^{M} p_a(t)e^{-|x-q_a(t)|} = K*m\,, \tag{1.7}$$

which emerge from smooth confined initial conditions for the velocity profile. Such a sum is an *exact solution* of the CH equation (1.1) provided the time-dependent parameters $\{p_a\}$ and $\{q_a\}$, $a = 1,\ldots,M$, satisfy certain canonical Hamiltonian equations, to be discussed later. In fact, the peakon velocity wave train in (1.7) is the *asymptotic solution* of the CH equation for any spatially confined C^1 initial condition, $u(x,0)$.

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* Corresponding author.
E-mail addresses: d.crisan@ic.ac.uk (D. Crisan), d.holm@ic.ac.uk (D.D. Holm).

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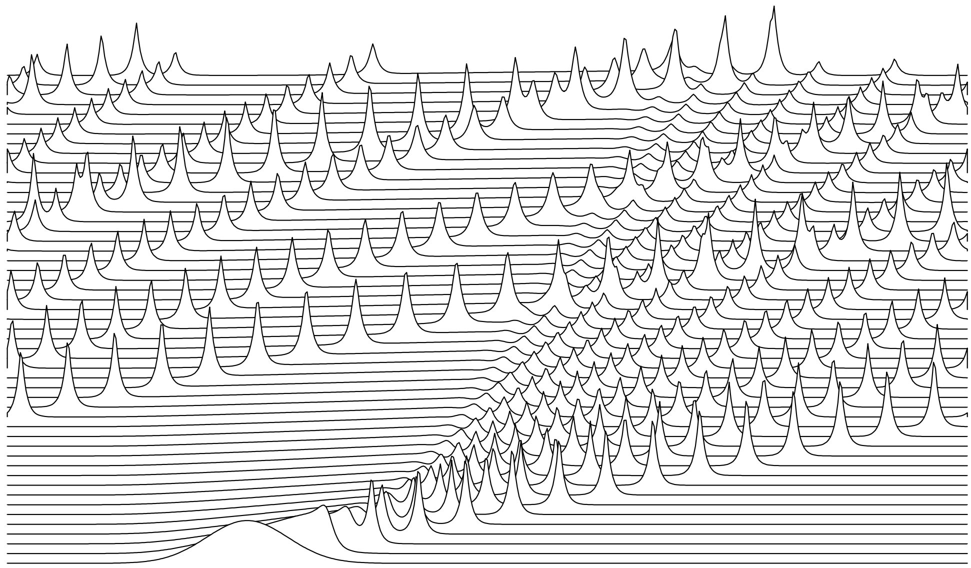

Fig. 1.1. Under the evolution of the CH equation (1.1), an ordered *wave train of peakons* emerges from a smooth localized initial condition (a Gaussian). The speeds are proportional to the heights of the peaks. The spatial profiles of the velocity at successive times are offset in the vertical to show the evolution. The peakon wave train eventually wraps around the periodic domain, thereby allowing the faster peakons which emerge earlier to overtake slower peakons emerging later from behind in collisions that conserve momentum and preserve the peakon shape but cause phase shifts in the positions of the peaks, as discussed in [1].

Remark 1. The peakon-train solutions of CH represent an *emergent phenomenon*. A wave train of peakons emerges in solving the initial-value problem for the CH equation (1.1) for any smooth spatially confined initial condition. An example of the emergence of a wave train of peakons from a Gaussian initial condition is shown in Fig. 1.1.

Remark 2. By Eq. (1.5), the momentum density corresponding to the peakon wave train (1.7) in velocity is given by a sum over delta functions in momentum density, representing the *singular solution*,

$$m(x,t) = \sum_{a=1}^{M} p_a(t)\,\delta(x - q_a(t))\,, \tag{1.8}$$

in which the *delta function* $\delta(x-q)$ is defined by

$$f(q) = \int f(x)\delta(x-q)\,\mathrm{d}x\,, \tag{1.9}$$

for an arbitrary smooth function f. Physically, the relationship (1.8) represents the dynamical coalescence of the CH momentum density into particle-like coherent structures (Young measures) which undergo elastic collisions as a result of their nonlinear interactions. Mathematically, the singular solutions of CH are captured by recognizing that the singular solution ansatz (1.8) itself is an equivariant momentum map from the canonical phase space of M points embedded on the real line, to the dual of the vector fields on the real line. Namely,

$$m : T^*\mathrm{Emb}(\mathbb{Z},\mathbb{R}) \to \mathfrak{X}(\mathbb{R})^*. \tag{1.10}$$

This momentum map property explains, for example, why the singular solutions (1.8) form an invariant manifold for any value of M and why their dynamics form a canonical Hamiltonian system [2].

The complete integrability of the CH equation as a Hamiltonian system follows from its isospectral problem.

Theorem 3 (*Isospectral Problem for CH [1]*)**.** *The CH equation in (1.1) follows from the compatibility conditions for the following CH isospectral eigenvalue problem and evolution equation for the real eigenfunction $\psi(x,t)$,*

$$\psi_{xx} = \left(\frac{1}{4} - \frac{m}{2\lambda}\right)\psi\,, \tag{1.11}$$

$$\partial_t\psi = -(\lambda+u)\psi_x + \frac{1}{2}u_x\psi\,, \tag{1.12}$$

with real isospectral parameter, λ.

Proof. By direct calculation, equating cross derivatives $\partial_t\psi_{xx} = \partial_x^2\partial_t\psi$ using Eqs. (1.11) and (1.12) implies the CH equation in (1.1), provided $d\lambda/dt = 0$. □

Remark 4. The complete integrability of the CH equation as a Hamiltonian system and its soliton paradigm explain the emergence of peakons in the CH dynamics. Namely, their emergence reveals the initial condition's soliton (peakon) content.

1.1. Steepening lemma: the mechanism for peakon formation

In the following we will continue working on the entire real line $\mathbb{R}$, although similar results are also available for a periodic domain with only minimal effort. We use the notation $\|u\|_2$, $\|u\|_{1,2}$ and $\|u\|_\infty$ to denote, respectively,

$$\|u\|_2^2 := \int_{-\infty}^{\infty}(u^2)\,\mathrm{d}y\,,\quad \|u\|_{1,2}^2 := \int_{-\infty}^{\infty}\left(u^2 + \frac{1}{2}u_y^2\right)\mathrm{d}y\,,\quad \text{and}$$
$$\|u\|_\infty := \sup_{x\in\mathbb{R}}\|u(x)\|.$$

Remark 5 (*Local Well-Posedness of CH*)**.** As reviewed in [2], the deterministic CH equation (1.1) is locally well posed on $\mathbb{R}$, for initial conditions in H^s with $s > 3/2$. In particular, with such initial data, CH solutions are C^∞ in time and the Hamiltonian $h(m)$ in (1.4) is bounded for all time,

$$h := \|u(\cdot,t)\|_{1,2} < \infty.$$

In fact, CH solutions preserve the Hamiltonian in (1.4) given by the $\|u(\cdot,t)\|_{1,2}$ norm

$$\|u(\cdot,t)\|_{1,2} = h = constant\,,\quad \text{for all } x\in\mathbb{R}. \tag{1.13}$$

By a standard Sobolev embedding theorem, (1.13) also implies the useful relation that

$$M := \sup_{t\in[0,\infty)}\|u(\cdot,t)\|_\infty < \infty. \tag{1.14}$$

The mechanism for the emergent formation of the peakons seen in Fig. 1.1 may also be understood as a variant of classical formations of weak solutions in fluid dynamics by showing that initial conditions exist for which the solution of the CH equation (1.1) can develop a vertical slope in its velocity $u(t, x)$, in finite time. The mechanism turns out to be associated with *inflection points of negative slope*, such as take place on the leading edge of a rightward-propagating, spatially-confined velocity profile. In particular,

Lemma 6 (*Steepening Lemma [1]*). *Suppose the initial profile of velocity $u(x, 0)$ has an inflection point at $x = \bar{x}$ to the right of its maximum, and otherwise it decays to zero in each direction and that $\|u(\cdot, 0)\|_{1,2} < \infty$. Moreover we assume that $u_x(\bar{x}, 0) < -\sqrt{2M}$, where M is the constant defined in (1.14). Then, the negative slope at the inflection point will become vertical in finite time.*

Proof. Consider the evolution of the slope at the inflection point $t \mapsto \bar{x}(t)$ that starts at time 0 from an inflection point $x = \bar{x}$ of $u(x, 0)$ to the right of its maximum so that

$$s_0 := u_x(\bar{x}(0), 0) < \infty.$$

Define $s_t := u_x(\bar{x}(t), t), \quad t \geq 0$. From the spatial derivative of the advective form of the CH equation (1.6) one obtains

$$\partial_x(u_t + uu_x) = -\partial_x^2 K * (u^2 + \frac{1}{2}u_x^2) = u^2 + \frac{1}{2}u_x^2 - K * (u^2 + \frac{1}{2}u_x^2),$$

which leads to

$$\partial_t u_x = -uu_{xx} + u^2 - \frac{1}{2}u_x^2 - K * (u^2 + \frac{1}{2}u_x^2).$$

This, in turn, yields an equation for the evolution of $t \mapsto s_t$. Namely, by using $u_{xx}(\bar{x}(t), t) = 0$ and (1.14) one finds

$$\begin{aligned} \frac{\mathrm{d}s}{\mathrm{d}t} &= -\frac{1}{2}s^2 + u^2(\bar{x}(t), t) - \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-|\bar{x}(t)-y|}\left(u^2 + \frac{1}{2}u_y^2\right)\mathrm{d}y \\ &\leq -\frac{1}{2}s^2 + M\,. \end{aligned} \tag{1.15}$$

Let $\tilde{s}$ be the solution of the equation

$$\frac{\mathrm{d}\tilde{s}}{\mathrm{d}t} = -\frac{1}{2}\tilde{s}^2 + M, \quad \tilde{s}_0 = s_0\,. \tag{1.16}$$

Observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}((s_t - \tilde{s}_t)\mathrm{e}^{\frac{1}{2}\int_0^t (s_p + \tilde{s}_p)dp}) \leq 0, \quad s_0 - \tilde{s}_0 = 0,$$

therefore, $s_t \leq \tilde{s}_t$ for all $t > 0$ (as long as both are well defined). However, for $s_0 := u_x(\bar{x}(0), 0) < -\sqrt{2M}$, Eq. (1.16) admits the (unique) explicit solution

$$\bar{s} = \sqrt{2M}\coth\left(\sigma + \frac{t}{2}\sqrt{2M}\right), \quad \sigma = \coth^{-1}\left(\frac{s_0}{\sqrt{2M}}\right) < 0.$$

Since $\lim_{t \mapsto -2\sigma/\sqrt{2M}} \bar{s}_t = -\infty$ it follows that there exists a time $\tau \leq -2\sigma/\sqrt{2M}$ by which the slope $s_t = u_x(\bar{x}(t), t)$ becomes negative and vertical, i.e. $\lim_{t \mapsto \tau} \bar{s}_t = -\infty$. □

Remark 7. Suppose the initial condition is anti-symmetric, so the inflection point at $u = 0$ is *fixed* and $\mathrm{d}\bar{x}/\mathrm{d}t = 0$, due to the symmetry $(u, x) \to (-u, -x)$ admitted by Eq. (1.1). In this case, the total momentum vanishes, i.e. $M = 0$, and no matter how small $|s(0)|$ (with $s(0) < 0$), the verticality $s \to -\infty$ develops at $\bar{x}$ in finite time.

Remark 8. The Steepening Lemma of [1] proves that in one dimension any initial velocity distribution whose spatial profile has an inflection point with negative slope (for example, any antisymmetric smooth initial distribution of velocity on the real line) will develop a vertical slope in finite time. Note that the peakon solution (1.7) has no inflection points, so it is not subject to the steepening lemma.

The Steepening Lemma underlies the mechanism for forming these singular solutions, which are continuous but have discontinuous spatial derivatives. Indeed, the numerical simulations in Fig. 1.1 show that the presence of an inflection point of negative slope in any confined initial velocity distribution triggers the steepening lemma as the *mechanism* for the formation of the peakons. Namely, according to Fig. 1.1, the initial (positive) velocity profile "leans" to the right and steepens, then produces a peakon that is taller than the initial profile, so it propagates away to the right, since the peakon moves at a speed equal to its height. This process leaves a profile behind with an inflection point of negative slope; so it repeats, thereby producing a wave train of peakons with the tallest and fastest ones moving rightward in order of height. In fact, Fig. 1.1 shows that this recurrent process produces only peakon solutions, as in (1.7). This is a result of the isospectral property of CH as a completely integrable Hamiltonian system [1]. Namely, the eigenvalues of the initial profile $u(x, 0)$ for the associated CH isospectral problem are equal to the asymptotic speeds of the peakons in the wave train (1.7).

The peakon solutions lie in H^1 and have finite energy. We conclude that solutions with initial conditions in H^s with $s > 3/2$ go to infinity in the H^s norm in finite time, but they remain in H^1 and presumably continue to exist in a weak sense for all time in H^1.

2. Advective form of the Stochastic Camassa–Holm (SCH) equation

Following [3], we derive the SCH equation by introducing the stochastic Hamiltonian function,

$$\begin{aligned} \widetilde{h}(m) &= \frac{1}{2}\int_{\mathbb{R}} m(x, t)K * m(x, t)\,dx\,dt \\ &+ \int_{\mathbb{R}} m(x, t)\sum_{i=1}^{N}\xi^i(x) \circ dW_t^i\,dx\,. \end{aligned} \tag{2.1}$$

The second term generates spatially correlated random displacements, by pairing the momentum density with the Stratonovich noise in (2.1) via a set of time-independent prescribed functions $\xi^i(x)$, $i = 1, 2, \ldots, N$, representing the spatial correlations. Thus, the resulting SCH equation is given by

$$0 = \mathrm{d}m + (\partial_x m + m\partial_x)\frac{\delta\widetilde{h}(m)}{\delta m} = \mathrm{d}m + (\partial_x m + m\partial_x)\upsilon\,, \tag{2.2}$$

where $m := u - u_{xx}$ and the stochastic vector field υ, defined by

$$\upsilon(x, t) := u(x, t)\,dt + \sum_{i=1}^{N}\xi^i(x) \circ dW_t^i\,, \tag{2.3}$$

represents random spatially correlated shifts in the velocity [3–6]. Thus, the noise introduced in (2.2) and (2.3) represents an additional stochastic perturbation in the momentum transport velocity.

2.1. Peakon solutions and isospectrality for the SCH equation

Theorem 9. *The SCH equation (2.2) with the stochastic vector field υ in (2.3) admits the singular momentum solution for CH in (1.9) for peakon wave trains.*

Proof. Substituting the singular momentum relation (1.9) into the stochastic Hamiltonian $\widetilde{h}(m)$ in (2.1) and performing the integrals yield the Hamiltonian for the stochastic peakon trajectories as

$$\widetilde{h}(q,p) := \frac{1}{4}\sum_{a,b=1}^{M} p_a(t)p_b(t)e^{-|q_a(t)-q_b(t)|} + \sum_{a,b=1}^{M} p_a(t)\sum_{i=1}^{N}\xi^i(q_a(t))\circ dW_t^i .$$

The canonical Hamiltonian equations for the stochastic peakon trajectories and momenta are thus given by

$$\mathrm{d}q_a = \frac{\partial\widetilde{h}}{\partial p_a} = \frac{1}{2}\sum_{b=1}^{M} p_b(t)e^{-|q_a(t)-q_b(t)|}\,dt + \sum_{i=1}^{N}\xi^i(q_a(t))\circ dW_t^i = u(q_a(t))\,dt + \sum_{i=1}^{N}\xi^i(q_a(t))\circ dW_t^i = v(q_a(t)) ,$$

and

$$\mathrm{d}p_a = -\frac{\partial\widetilde{h}}{\partial q_a} = -p_a(t)\frac{\partial u}{\partial q_a}\,dt - p_a(t)\sum_{i=1}^{N}\frac{\partial \xi^i}{\partial q_a}\circ dW_t^i = -p_a(t)\frac{\partial v(q_a(t))}{\partial q_a} .$$

Substituting these stochastic canonical Hamiltonian equations for $q_a(t)$ and $p_a(t)$ into the singular momentum solution for CH in (1.9) recovers the SCH equation (2.2) and the stochastic vector field v in (2.3). □

Thus, the SCH equation (2.2) admits peakon wave train solutions whose peaks in velocity follow the stochastic trajectories given by the stochastic vector field v in (2.3) and satisfy stochastic canonical Hamiltonian equations. The corresponding canonical Hamiltonian equations in the absence of noise describe the trajectories and momenta of CH wave trains. For numerical studies of the interactions of stochastic peakon solutions, see [5,6].

Remarkably, a certain amount of the isospectral structure for the deterministic CH equation is preserved by the addition of the stochastic transport perturbation we have introduced in (2.2) and (2.3).

Theorem 10 (*Isospectral Problem for SCH*)**.** *The SCH equation in (2.2) follows from the compatibility condition for the deterministic CH isospectral eigenvalue problem (1.11), and a stochastic evolution equation for the real eigenfunction* ψ*,*

$$\psi_{xx} = \left(\frac{1}{4} - \frac{m}{2\lambda}\right)\psi , \tag{2.4}$$

$$\mathrm{d}\psi = -(\lambda + v)\psi_x + \frac{1}{2}v_x\psi , \tag{2.5}$$

$$\text{with}\quad v := u\,dt + \sum_{i=1}^{N}\xi^i(x)\circ dW_t^i , \tag{2.6}$$

and real isospectral parameter, λ*, provided* $\mathrm{d}\lambda = 0$ *and* $\xi^i(x) = C^i + A^ie^x + B^ie^{-x}$*, for constants* A^i*,* B^i *and* C^i*.*

Proof. By direct calculation, equating cross derivatives $\mathrm{d}\psi_{xx} = \partial_x^2\mathrm{d}\psi$ using Eqs. (2.4) and (2.5) implies, when $\mathrm{d}\lambda = 0$, that

$$\mathrm{d}m + (\partial_x m + m\partial_x)v + \lambda\big(m_x - (v_x - v_{xxx})\big) = 0.$$

Consequently, the compatibility condition for Eqs. (2.4) and (2.5) implies the SCH equation in (2.2), provided $\mathrm{d}\lambda = 0$ and $\xi_x^i(x) - \xi_{xxx}^i(x) = 0$. The latter means that $\xi^i(x)$ is either constant, or exponential. □

Remark 11. Theorem 10 means that the SCH equation (2.2) with stochastic vector field v (2.3) with $\xi_x^i(x) - \xi_{xxx}^i(x) = 0$ has the same countably infinite set of conservation laws as for the deterministic CH equation (1.1). However, the SCH Hamiltonian $\widetilde{h}(m)$ in (2.1) is not conserved by the SCH equation (2.2), because it depends explicitly on time. Consequently, for those choices of $\xi^i(x)$, the SCH equation is equivalent to consistency of the linear equations (2.4) and (2.5). Therefore, SCH is solvable by the isospectral method for each realization of the stochastic process in (2.6). However, it remains to be seen whether SCH is integrable as a Hamiltonian system. See [7] for an example of an integrable stochastic deformation of the CH equation.

The issue now and for the remainder of the paper is to find out whether the wave breaking property which is the mechanism for the creation of peakon wave trains in the deterministic case also survives the introduction of stochasticity.

2.2. Wave breaking estimates for SCH

In the following we will assume the conditions under which the stochastic integrals appearing in Eq. (2.2) for u as well as the equation for u_x are well defined and summable. In particular, we assume that the vector fields ξ_i are smooth and bounded and that

$$\sum_{i>0}\left((\|\xi^i\|_\infty)^2 + (\|\xi_x^i\|_\infty)^2 + (\|\xi^i\|_{2,1})^2\right) < \infty.$$

Let A^i, $\partial_x A^i$, $i \in \mathbb{Z}_+$ be the following set of operators

$$A^i(u) = u_x\xi^i - K * \left(u_x\xi_{xx}^i(x) + 2u\xi_x^i(x)\right) , \tag{2.7}$$

$$\partial_x A^i(u) = u_{xx}\xi^i + u_x\xi_x^i - \partial_x K * \left(u_x\xi_{xx}^i(x) + 2u\xi_x^i(x)\right) , \tag{2.8}$$

($\partial_x A^i$ is obtained by formally differentiating A^i). In the following we will assume that there is a local solution of Eq. (2.2) such that the operators A^i, $\partial_x A^i$, $i \in \mathbb{Z}_+$ are well defined.

Remark 12. The analysis presented in [8] for the local existence and uniqueness of the solution of stochastic Euler equation can be a template for proving the existence of a local solution of the stochastic partial differential equation (2.2). Essentially, one needs to show that there is a global solution of a suitably chosen truncated version of (2.2). This is done via a relative compactness argument applied to a sequence of stochastic partial differential equations with vanishing diffusion terms.

Let us deduce first the equation for the velocity slope, u_x. We have the following lemma:

Lemma 13 (*Evolution of the Velocity Slope*)**.** *Under the above conditions, we have*

$$\mathrm{d}u_x = -\frac{1}{2}\left(u_x^2 + 2uu_{xx} - u^2\right)\,dt - K * \left(u^2 + \frac{1}{2}u_x^2\right)dt - \sum_i A_x^i(u)\circ dW_t^i . \tag{2.9}$$

Proof. Expanding out the SCH equation in terms of u and v gives

$$\begin{aligned}
0 &= \mathrm{d}m + (\partial_x m + m\partial_x)v \\
&= (1-\partial_x^2)\mathrm{d}u + 2uv_x + u_xv - 2v_xu_{xx} - vu_{xxx} \\
&= (1-\partial_x^2)(\mathrm{d}u + vu_x) + u_xv_{xx} + 2uv_x \\
&= (1-\partial_x^2)(\mathrm{d}u + vu_x) + \partial_x\left(u^2 + \frac{1}{2}u_x^2\right)\,dt \\
&\quad + \sum\left(u_x\xi_{xx}^i(x) + 2u\xi_x^i(x)\right)\circ dW_t^i .
\end{aligned}$$

Therefore, applying the smoothing operator $K* := (1-\partial_x^2)^{-1}$, given by the convolution with Green's function $K(x,y)$ in (1.5) for the Helmholtz operator $(1-\partial_x^2)$, to both sides of the previous equation yields

$$\begin{aligned} \mathrm{d}u + uu_x\,dt &= -u_x\left(\sum \xi^i \circ dW_t^i\right) - \partial_x K * \left(u^2 + \frac{1}{2}u_x^2\right) dt \\ &\quad + \sum K * \left(u_x\xi^i_{xx} + 2u\xi^i_x\right) \circ dW_t^i \\ &= -\partial_x K * \left(u^2 + \frac{1}{2}u_x^2\right) dt \\ &\quad - \sum (u_x\xi^i - K * \left(u_x\xi^i_{xx}(x) + 2u\xi^i_x(x)\right)) \circ dW_t^i \\ &= -\partial_x K * \left(u^2 + \frac{1}{2}u_x^2\right) dt - \sum A^i(u) \circ dW_t^i, \quad (2.10) \end{aligned}$$

in which the derivative ∂_x is understood to act on everything standing to its right. Consequently, we have

$$\begin{aligned} \mathrm{d}u_x &= -\left(u_x^2 + uu_{xx}\right) dt - \partial_{xx}K * \left(u^2 + \frac{1}{2}u_x^2\right) dt \\ &\quad - \sum A_x^i(u) \circ dW_t^i. \end{aligned} \quad (2.11)$$

Then, since

$$\partial_{xx}K * \left(u^2 + \frac{1}{2}u_x^2\right) = -\left(u^2 + \frac{1}{2}u_x^2\right) + K * \left(u^2 + \frac{1}{2}u_x^2\right),$$

we deduce (2.9). □

Remark 14. Observe that

$$\begin{aligned} &\partial_x K * \left(u_x\xi^i_{xx}(x) + 2u\xi^i_x(x)\right) \\ &\quad = \partial_{xx}K * (u\xi^i_{xx}(x)) - \partial_x K * \left(u\xi^i_{xxx}(x) + 2u\xi^i_x(x)\right) \\ &\quad = -u\xi^i_{xx}(x) + K * (u\xi^i_{xx}(x)) - \partial_x K * \left(u\xi^i_{xxx}(x) + 2u\xi^i_x(x)\right). \quad (2.12) \end{aligned}$$

Hence, the last term in the expression of (2.8) can be controlled by the supremum norm of u.

Just as in the deterministic case, we define next the process $t \mapsto \nu_t$ as the inflection point of u to the right of its maximum so that

$$u_{xx}(\nu_t, t) = 0, \quad \text{and } s_t = u_x(\nu_t, t) < 0.$$

In what follows, we will assume, without proof, that the process $t \mapsto \nu_t$ is a semi-martingale. The argument to show that validity of this property is based on the implicit function theorem. Indeed one can show that ν satisfies the equation

$$\mathrm{d}\nu_t = -\frac{1}{u_{xxx}(\nu_t, t)}(\mathrm{d}u_{xx})(\nu_t, t),$$

provided the equation is well defined. That is, assume we have an inflection point, not an inflection interval; which means that we have assumed $u_{xxx}(\nu_t, t) \neq 0$. Using the semimartingale property of ν and the Itô-Wentzell formula (see, e.g., [9]), we deduce that

$$\mathrm{d}\left(u_x(\nu_t, t)\right) = (\mathrm{d}u_x)(\nu_t, t) + u_{xx}(\nu_t, t) \circ \mathrm{d}\nu_t = (\mathrm{d}u_x)(\nu_t, t).$$

Hence, by (2.9) and (2.12), we find that

$$\begin{aligned} \mathrm{d}s_t &= -\left(\frac{1}{2}s_t^2 - u^2(\nu_t, t)\right) dt - K * \left(u^2 + \frac{1}{2}u_x^2\right)(\nu_t)\,dt \\ &\quad - \sum_i \left(s_t\xi^i_{\nu_t} + B^i(u)|_{\nu_t}\right) \circ dW_t^i, \end{aligned} \quad (2.13)$$

where the operators B^i are given by

$$B^i(u) = -u\xi^i_{xx}(x) + K * (u\xi^i_{xx}(x)) - \partial_x K * \left(u\xi^i_{xxx}(x) + 2u\xi^i_x(x)\right),$$
$i \in \mathbb{Z}_+$.

We will henceforth consider the particular case when the vector fields ξ^i are spatially homogeneous, so that $B^i(u) = 0$. (This is also the isospectral case, which we discussed in the previous section.) In this case, just as in the deterministic case, we have

$$\|u(\cdot,t)\|_{1,2} = \|u(\cdot,0)\|_{1,2}, \quad \text{for all } x \in \mathbb{R}. \quad (2.14)$$

and, again, (2.14) implies that

$$M := \sup_{t\in[0,\infty)} \|u(\cdot,t)\|_\infty < \infty. \quad (2.15)$$

This bound arises because the stochastic term vanishes when computing $\mathrm{d}\,\|u(\cdot,t)\|^2_{1,2}$. More precisely, the stochastic term is given by the expression

$$\sum (2uu_x + u_xu_{xx})\,\xi^i \circ W_t^i,$$

whose spatial integral over the real line vanishes for constant ξ^i, for the class of solutions $u(\cdot,t)$ which vanish at infinity and whose gradient also vanishes at infinity. Note that the constant M in (2.15) is independent of the realization of the Brownian motions W^i, $i \in \mathbb{Z}_+$. By a standard Sobolev embedding theorem, (1.13) also implies the useful relation that

$$M := \sup_{t\in[0,\infty)} \|u(\cdot,t)\|_\infty < \infty. \quad (2.16)$$

Proposition 15. *As in the deterministic case, suppose the initial profile of velocity $u(x,0)$ has an inflection point at $x = \bar{x}$ to the right of its maximum, and it decays to zero in each direction; so that $\|u(\cdot,0)\|_{1,2} < \infty$. Consider the expectation of the slope at the inflection point, $\bar{s}_t = E[s_t]$. If $u_x(\bar{x},0)$ is sufficiently small, then there exists $\tau < \infty$ such that $\lim_{t\mapsto\tau}\bar{s}_t = -\infty$.*

Proof. By changing from Stratonovich to Itô integration, we obtain from (2.13) that

$$\begin{aligned} \mathrm{d}s_t &= -\left(\frac{1}{2}s_t^2 - u^2(\nu_t, t)\right) dt - K * \left(u^2 + \frac{1}{2}u_x^2\right)(\nu_t)\,dt \\ &\quad - \sum_i s_t\xi^i dW_t^i + \frac{1}{2}\sum_i s_t\left(\xi^i\right)^2 dt \end{aligned} \quad (2.17)$$

and, by taking expectation, we deduce that

$$\frac{d}{dt}E[s_t] \le -\frac{1}{2}\left(E\left[s_t^2\right] - E[s_t]^2\right) - \frac{1}{2}\left(E[s_t]^2\right) + \frac{\|\xi\|}{2}E[s_t] + M.$$

Consequently, for arbitrary $\varepsilon \in [0,1)$, we have

$$\frac{d}{dt}\bar{s}_t \le -\frac{1}{2}(\bar{s}_t)^2 + \frac{\|\xi\|}{2}\bar{s}_t + M \le -\frac{1-\varepsilon}{2}(\bar{s}_t)^2 + \left(M + \frac{\|\xi\|^2}{2\varepsilon}\right)$$

from which we deduce that the magnitude $|\bar{s}_t|$ blows up in finite time, just as in the deterministic case. □

Note that Proposition 15 does not guarantee pathwise blow up of the process for the magnitude of the negative slope at the inflection point $|s_t|$, only the blow up of its mean, $|\bar{s}_t|$. The following theorem shows that, indeed the pathwise negative slope s_t blows up in finite time with positive probability, albeit not with probability 1.

Theorem 16 (*Wave Breaking for the Stochastic Camassa–Holm Equation*). *Under the same assumptions as those introduced in Proposition 15, with positive probability, the negative slope at the inflection point $s_t = u_x(\nu_t, t)$ will become vertical in finite time.*

Proof. We define a new Brownian motion W, as follows:

$$W_t = \frac{-\sum_i \xi^i W_t^i}{\|\xi\|}, \quad t > 0.$$

Then the equation for s_t becomes

$$ds_t = -\left(\frac{1}{2}s_t^2 - \frac{\|\xi\|^2}{2}s_t - u^2(v_t, t)\right)dt$$
$$- K * \left(u^2 + \frac{1}{2}u_x^2\right)(v_t)\,dt + \|\xi\|\, s_t dW_t.$$

We introduce a Brownian motion B such that the stochastic integral $\int_0^t s_p dW_p$ can be represented as

$$\int_0^t s_p dW_p = B_{\int_0^t s_p^2 dp}.$$

Then, as above, for arbitrary $\varepsilon \in [0, 1/2)$

$$s_t \le s_0 + \int_0^t \left(\left(M + \frac{\|\xi\|^2}{2\varepsilon}\right) - \frac{\varepsilon s_p^2}{2}\right)dp$$
$$+ \left(-\frac{1-2\varepsilon}{2}\int_0^t s_p^2 dp + \|\xi\|\, B_{\int_0^t s_p^2 dp}\right)$$
$$\le s_0 + \int_0^t \left(M + \frac{\|\xi\|^2}{2\varepsilon} - \frac{\varepsilon s_p^2}{2}\right)dp + X_{\int_0^t s_p^2 dp},$$

where X is the Brownian motion with negative drift.[1]

$$X(t) = -\frac{1-2\varepsilon}{2}t + \|\xi\|\, B_t.$$

With positive probability (though not 1!), the process X remains smaller than, say, $-s_0/2 > 0$ for all $t > 0$ and therefore so does the process $t \mapsto X_{\int_0^t s_p^2 dp}$ which is just a time-change of X. It follows that, with the same probability, we have that

$$s_t \le \frac{1}{2}s_0 + \int_0^t \left(M + \frac{\|\xi\|^2}{2\varepsilon} - \frac{\varepsilon s_p^2}{2}\right)dp.$$

From here, the argument in the deterministic case applies: Let $\hat{s}$ be the solution of the equation

$$\frac{d\hat{s}}{dt} = -\frac{\varepsilon}{2}\hat{s}^2 + \hat{M}, \quad \hat{s}_0 = \frac{1}{2}s_0, \tag{2.18}$$

where $\hat{M} = M + \frac{\|\xi\|^2}{2\varepsilon}$. Then, $s_t \le \hat{s}_t$ for all $t > 0$ (as long as both are well defined). However, for $s_0 := u_x(\bar{x}(0), 0) < -\sqrt{2\hat{M}/\varepsilon}$, Eq. (2.18) admits the (unique) explicit solution

$$\hat{s} = \sqrt{2\hat{M}/\varepsilon}\coth\left(\sigma + \frac{\varepsilon t}{2}\sqrt{2\hat{M}/\varepsilon}\right),$$

$$\sigma = \coth^{-1}\left(\frac{s_0}{\sqrt{2\hat{M}/\varepsilon}}\right) < 0,$$

which implies, as in the deterministic case, the existence of a finite time by which the slope $s_t = u_x(\bar{x}(t), t)$ becomes vertical. □

Remark 17. Using the properties of the Brownian motion with negative drift, a lower bound for the probability that the initially negative slope at the inflection point $s_t = u_x(v_t, t)$ will become vertical in finite time is given by $1 - \exp\left(s_0(1-2\varepsilon)/(4\|\xi\|^2)\right)$, where s_0 must satisfy the constraint $s_0 < -\sqrt{2\hat{M}/\varepsilon}$ with $\hat{M} = M + \frac{\|\xi\|^2}{2\varepsilon}$.

Remark 18. In a similar manner, one can show that $s_t \ge \tilde{s}_t$, $t \ge 0$, where

$$\tilde{s}_t = -\left(\frac{1}{2}\tilde{s}_t^2 - \frac{\|\xi\|^2}{2}\tilde{s}_t + M\right)dt + \|\xi\|\,\tilde{s}_t dW_t, \quad \tilde{s}_0 = s_0.$$

To show this, one proceeds as in the proof of the steepening lemma by first justifying the inequality

$$\frac{d}{dt}\left((s_t - \tilde{s}_t)e^{\frac{1}{2}\int_0^t (s_p + \tilde{s}_p)dp + \frac{\|\xi\|^2 t}{2} + \|\xi\| W_t}\right) \ge 0, \quad s_0 - \tilde{s}_0 = 0,$$

so that $s_t \ge \tilde{s}_t$ for all $t > 0$ (as long as both are well defined). In turn, $\tilde{s}_t$, and therefore s_t, may achieve positive values with positive probability. This could, in principle, lead to a violation of the conditions under which an initially negative slope at an inflection point may become vertical. Future work is planned by the authors to further investigate the emergence of peakons, as well as the local well-posedness of the stochastic CH equation.

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References

[1] Roberto Camassa, Darryl D. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett. 71 (1993) 1661–1664.

[2] Darryl D. Holm, Jerrold E. Marsden, Momentum maps and measure valued solutions (peakons, filaments, and sheets) of the Euler-Poincaré equations for the diffeomorphism group, in: J.E. Marsden, T.S. Ratiu (Eds.), The Breadth of Symplectic and Poisson Geometry, A Festshrift for Alan Weinstein, in: Progr. Math., vol. 232, Birkhäuser Boston, Boston, MA, 2005, pp. 203–235. Preprint at arxiv.org/abs/nlin.CD/0312048.

[3] Darryl D. Holm, Variational principles for stochastic fluid dynamics, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 471 (2015) 20140963.

[4] Darryl D. Holm, Jerrold E. Marsden, Tudor S. Ratiu, The Euler–Poincaré equations and semidirect products with applications to continuum theories, Adv. Math. 137 (1) (1998) 1–81.

[5] Darryl D. Holm, Tomasz M. Tyranowski, Variational principles for stochastic soliton dynamics, Proc. R. Soc. A 472 (2016) 20150827 2016, Preprint available at arXiv:1503.03127, http://arxiv.org/pdf/1503.03127.pdf.

[6] Darryl D. Holm, Tomasz M. Tyranowski, Stochastic Discrete Hamiltonian Variational Integrators, 2017. Preprint available at arXiv:1609.00463, http://arxiv.org/pdf/1609.00463.pdf.

[7] Alexis Arnaudon, The stochastic integrable AKNS hierarchy, 2015. arXiv:1511.07080 [nlin.SI].

[8] Dan Crisan, Franco Flandoli, Darryl D. Holm, Solution properties of a 3D stochastic Euler fluid equation, https://arxiv.org/abs/1704.06989.

[9] Daniel Ocone, Etienne Pardoux, A generalized Itô-Wentzell formula. Application to a class of anticipating stochastic differential equations, Ann. Inst. H. Poincaré Probab. Statist. 25 (1) (1989) 39–71.

[10] Sigman Karl, Notes on Brownian Motion, http://www.columbia.edu/ks20/FE-Notes/4700-07-Notes-BM.pdf.

[1] Let X be a Brownian motion with negative drift, $X(t) = \sigma B(t) + \mu t$, $\mu < 0$, $\lim_{t\mapsto\infty} X(t) = -\infty$. Let $M = \max_{s\ge 0} X(t)$. Then $P(M \ge a) = \exp\left(-a\left(\frac{2|\mu|}{\sigma^2}\right)\right)$ and we conclude that M has an exponential distribution with mean $\frac{2|\mu|}{\sigma^2}$. To put it differently, no matter where we start the Brownian motion with drift there is a positive probability that it will reach any level, before it drifts off to $-\infty$. Vice versa, it never hits level a with positive probability, see e.g. [10].