Stability and Control of Caputo Fractional Order Systems

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

As pointed out by many researchers in the last few decades, differential equations with fractional (non-integer) order differential operators, in comparison with classical integer order ones, have apparent advantages in modelling mechanical and electrical properties of various real materials, e.g. polymers, and in some other fields. The stability and control of Caputo fractional order systems (systems of ordinary differential equations with fractional order differential operators of Caputo type) will be focused in this thesis. Our studies begin with Caputo fractional order linear systems, for which, three frequency-domain designs: pole placement, internal model principle and model matching, are developed to make the controlled systems bounded-input bounded-output stable, disturbance rejective and implementable, respectively. For these designs, fractional order polynomials are systematically defined and their root distribution, coprimeness, properness and $\rho - \kappa$ polynomials are well explored. We next move to Caputo fractional order nonlinear systems, of which the fundamental theory including the continuation and smoothness of solutions is developed; the diffusive realizations are shown to be equivalent with the systems; and the Lyapunov-like functions based on the realizations prove to be well-defined. This paves the way to stability analysis. The smoothness property of solutions suffices to yield a simple estimation for the Caputo fractional order derivative of any quadratic Lyapunov function, which together with the continuation leads to our results on Lyapunov stability, while the Lyapunov-like function contributes to our results on external stability. These stability results are then applied to H_∞ control, and finally extended to Caputo fractional order hybrid systems.

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Finally, I thank my family, especially my wife Jiaojiao, who make this possible.

Dedication

To my grandparents, parents, brother, wife and daughter.

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Chapter 1

Introduction

1.1 Motivation

In the last few decades, engineers and scientists have developed new models based on fractional order differential equations, which have been applied successfully, e.g. in material science (modelling of the behaviour of viscoelastic materials [32], relaxation and reaction kinetics of polymers [33]), fractal theory (modelling of the dynamical processes in self-similar and porous structures [34], advection and dispersion of solutes in natural porous or fractured media [35]), and psychology (modelling of the behavior of human beings based on memories [36, 37]). As explained in Remark 6.4 [1], fractional order operators are a very natural tool to model memory-dependent phenomena. They provide an excellent instrument for the description of memory and hereditary properties of various materials and processes, which endows fractional order models, in comparison with classical integer order ones, apparent advantages in modelling mechanical and electrical properties of real materials, e.g. polymers, and in many other fields [2].

There are mainly two types of fractional order derivatives (generalizations of $d^n f(t)/dt^n$ to the case $n \notin \{1, 2, ...\}$): the Riemann-Liouville fractional order derivative and the Caputo fractional order derivative. The former concept is the historically first. However, the initial conditions of ordinary differential equations with Riemann-Liouville fractional order derivatives involve the limit values of the Riemann-Liouville fractional order derivatives at the lower terminals [2]. In practical applications, these values are frequently not available, and it may not even be clear what their physical meaning is [38]. In other words, there is no known physical interpretation for such types of initial conditions [2]. Thus, the solutions of these initial value problems are practically useless [2]. To avoid this practical difficulty, Caputo proposed the so-called Caputo fractional order derivative in 1967, so that the initial conditions for ordinary differential equations with Caputo fractional order derivatives take on the same form as for integer order differential equations, i.e. have known physical interpretations [39]. Nevertheless, the mathematical theory in this area seems to be still lagging behind the needs of those practical applications. In the thesis, we intend to complement the theory centering on Caputo fractional order systems (systems of ordinary differential equations with Caputo fractional order derivatives), and their stability and control.

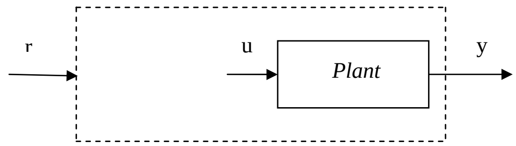


Figure 1.1: Design of control systems.

As the simplest type of systems, Caputo fractional order linear systems have aroused much interest in the past two decades, especially in the area of control. D. Matignon studied the bounded-input bounded-output (BIBO) stability in 1998 [5]. One year later, I. Podlubny proposed the fractional order ($PI^{\lambda}D^{\mu}$) controller [40], and I. Petras and L. Dorak introduced the frequency method [41]. In the new century, M. Karimi-Ghartemani, F. Merrikh-Bayat and H. Rasouli made some new theoretical progresses, see [6, 42, 43]. Except for those results in theory, Caputo fractional order linear control systems (Caputo fractional order linear systems with control inputs) also have had various practical applications: control of magnetic fluxes [43], DC motors [12, 44], and electrical radiators [45, 46, 47]. As appearing in these mentioned references, Caputo fractional order linear control systems, just like most of control systems, can be formulated as shown in Figure 1.1 referred from [9], in which the plant and the reference signal $r(t)$ are given, and the control $u(t)$ (or an overall system) is to be designed so that the plant output $y(t)$ will follow as closely as possible the reference signal. In order to avoid outputs blowing up, to reject disturbance, and to well implement systems by hardware, e.g. circuits, designed control systems are required to be stable (at least BIBO stable), disturbance rejective and implementable in practice. For these basic requirements of different aspects, the corresponding frequency methods (or designs in frequency domain): pole placement, internal model principle and model matching have been introduced, respectively. The pole placement is a method employed in feedback control system theory to place the closed-loop poles of a plant in pre-determined (desired) locations in the s-plane [48]. For disturbance rejection, the internal model principle, in which the internal model supplies closed-loop transmission zeros which cancel the unstable poles of the disturbance and reference signals, was introduced by B.A. Francis and W.M. Wonham in 1976 [49]. In 1987, C.T. Chen proposed the concept of implementable transfer function: an overall transfer function that can be implemented under four constrains - properness of compensators, well posedness, total stability and no plant leakage, and introduced the two-degrees-of-freedom configuration model matching to realize the implementable transfer functions [10].

As discussed above, the pole placement, internal model principle and model matching for integer order linear control systems have been well investigated [48, 49, 10]. However, these design problems still remain open for Caputo fractional order linear control systems. A Caputo fractional order linear control system, in the frequency domain, is given by a transfer function - a quotient of two fractional order polynomials. For example, consider a well-known fractional order control system [45], $0.8\,{}_0^CD_t^{2.2}y(t)+0.5\,{}_0^CD_t^{0.9}y(t)+y(t) = u(t)$ with zero initial condition, where u, y denotes the control input, output, and ${}^CD$ denotes the Caputo fractional order derivative. The Laplace transform of the derivative with zero initial condition has the form: $\mathscr{L}[{}_0^CD_t^{\alpha}x(t)] = s^{\alpha}X(s)$, where s denotes the variable in the frequency domain and the capital letter X denotes the Laplace transform function of x, see page 106 in [2]. It follows that the transfer function of the system is $G(s) = Y(s)/U(s) = 1/(0.8s^{2.2} + 0.5s^{0.9} + 1)$, which is a ratio of fractional order polynomials. The fractional order polynomials are defined on the Riemann surface (for more about the Riemann surface, see page 171 in [7]), unlike regular complex polynomials forming transfer functions of integer order linear control systems, only defined on the complex plane. This difference essentially complicates the fractional-version of those three frequency-domain designs. As far as we know, M. Karimi-Ghartemani and F. Merrikh-Bayat, in 2008, proposed the definition of fractional order polynomial with fractional degree and explored the distribution of roots of fractional order polynomials on the Riemann surface. Based on this, they further investigated the internal model principle [6]. But it is not constructive, i.e. they did not give an algorithm to design the internal model for a specific control system. There may be two main reasons for this limitation. First, advanced definitions related to fractional order polynomials, such as fractional order basis, *kth* corresponding polynomial and relative fractional degree, and their related properties including root distribution, coprimeness, properness and $\rho-\kappa$ polynomial, which are necessary to realize pole placement and design internal model constructively, had not been proposed and explored. Second, the pole placement for Caputo fractional order linear control systems, which is necessary for the internal model principle (because the internal model principle requires not only that the internal model includes the least common denominator of all unstable poles but also that the poles of the whole system are placed at desired locations), had not been investigated. In 2013, F. Merrikh-Bayat studied isolatedly (without pole placement, internal model principle and model matching) fractional order unstable pole-zero cancellation [42]. In 2015, H. Rasouli etc. proposed an algorithm for the fractional-version pole placement under an implicit assumption that their Diophantine equations were solvable (i.e. without sufficient condition) [43]. As for the fractional-version model matching, we can not find related results.

The Lyapunov stability of Caputo fractional order nonlinear systems has not been fully investigated yet as well, even though there have been some results. Some criteria of Lyapunov stability for Caputo fractional order nonlinear nonautonomous scalar systems were proposed in [1]. As the most important tool to investigate Lyapunov stability, the Lyapunov direct method for fractional order nonautonomous systems was presented in [21, 22]. However, it is not very easy to apply this method in practice. Since the fractional order derivative of any quadratic Lyapunov function is an infinite series even if the solution (or the state variable) involved in this function is analytic in t [1], it seems difficult to prove this fractional order derivative to be negative definite as required by the method. In order to apply the Lyapunov direct method or investigate the Lyapunov stability, some authors proposed special functionals [50, 51], while some others assumed the boundedness condition of the fractional order derivative [52, 53]. However, the problem has not been really solved. Up to 2014, Aguila-Camacho etc., assuming that the solution is differentiable, proposed an inequality to estimate the Caputo fractional derivative of a quadratic Lyapunov function, which provided a new thought to overcome the difficulty, see [16]. But, in general, even if the vector field function is infinitely many times differentiable, the derivative of the solution of a Caputo fractional order system may go to infinity at the initial time, i.e. the solution is not certainly differentiable. For example, the non-differentiable function x given by $x(t) = t^{1/2}$ is the unique solution of the Caputo fractional order differential equation ${}_0^C D_t^{1/2} x(t) = \Gamma(3/2)$ with $x(0) = 0$, in which the vector field function $f = \Gamma(3/2)$ is analytic, see page 116 in [1]. Unfortunately, this imperfect point has been always overlooked and the proposed inequality has been directly used in over one hundred research articles, e.g. [54, 55, 56]. Except for the difficulty in applying, there is another problem in the fractional Lyapunov direct method. Without any results available on the continuation of solutions to Caputo fractional order systems, the method implicitly assumes a direct consequence of the continuation: boundedness suffices global existence. Specifically, in [21, 22], only local Lipschitz conditions are assumed but Laplace transforms of Lyapunov functions involving solutions are taken. From this discussion, we may have noticed that those two problems of the Lyapunov direct method for Caputo fractional order systems originally arise from the lack of results on the continuation and smoothness of solutions.

As well known, the continuation of solutions is an indispensable part of the fundamental theory of differential equations, because it tells us the tendency of solutions - a solution of a given differential equation with continuous vector field function will tend to the boundary of the function's domain. As such, it is important for investigating the global existence and Lyapunov stability of solutions. Without the knowledge of continuation, we would have to first assume the global Lipschitz condition to guarantee the global existence, when studying the Lyapunov stability by the Lyapunov direct method. Naturally, all these are the same for Caputo fractional order differential equations. There have been many local existence and uniqueness results for Caputo fractional order differential equations, of which some were systematically presented, see Section 6.1, 6.2 in [1] and Section 3.5.1 in [4]. Compared to local ones, fewer global results have appeared, see Corollary 6.4, 6.7, 6.9 in [1], and [57], where comparison, global Lipschitz, and global Lipschitz like conditions are assumed. As far as we know, no global existence result has been derived from the general continuation of solutions yet. Up to 2016, Li and Sarwar proposed a continuation result for a scalar Caputo fractional order differential equation whose vector field function is define on $(0, \infty) \times \mathbb{R}$ [19]. As what we have read, all these results on the existence, uniqueness and continuation of solutions are only for scalar differential equations with zero initial time. Those for systems are all omitted and left to readers. Moreover, no general continuation result like the well-known continuation theorem for ordinary differential equations is developed.

It is common sense that the solutions to ordinary differential equations are differentiable (or smooth) if the vector field functions are continuous. However, this is not the case for the solutions to Caputo fractional order differential equations. In the example mentioned at the bottom of the last page, the solution $x(t) = t^{1/2}$ is not differentiable (at $t = 0$), even though the vector field function $f = \Gamma(3/2)$ is analytic. Here an interesting question for the smoothness of the solution to a Caputo fractional order differential equation with a sufficiently smooth vector field function (on the right-hand side of the equation) arises. This is a core question for the convergence analysis of numerical solutions to weakly singular integral equations [14], and for the Lyapunov stability analysis of Caputo fractional order systems since the smoothness property could be used to derive an estimation for the Caputo fractional order derivative of any quadratic Lyapunov function. However, only few papers delved into this question. In 1971, Miller and Feldstein in [58] studied the differential properties of solutions to nonlinear integral equations of the form: $x(t) = f(t) + \int_0^t a(t-s)g(s, x(s))ds$, $0 \le t \le T$, where $f(t)$ and $g(t, x)$ are smooth functions, $a(t) \in C(0, T] \cap L^1(0, T)$ but $a(t)$ may become unbounded as $t \to 0$. Let $a(t) = t^{\alpha-1}$, $\alpha > 0$, then it satisfies the smoothness condition and $x(t)$ may be considered as a solution on $[0, T]$ to a Caputo fractional order differential equation. With some additional assumptions on g and a, and an assumption of global existence of solution on $[0, T]$, it was proven that $x(t) \in C[0, T] \cap C^1(0, T]$. In 1999, Brunner, Pedas and Vainikko in [59] investigated the smoothness of solutions to nonlinear weakly singular Volterra equations that may be considered as Caputo fractional order differential equations when the weakly singular kernels are taken to be the convolution kernels appearing in the equivalent integral equations of Caputo fractional order differential equations. They applied the smoothness results from [15] to prove that the solutions belong to a space of special smooth functions. In 2014, Pedas and Tamme in [14] worked on the smoothness properties of solutions to nonlinear fractional differential equations. Following a similar idea (of the fixed point theorem) to [15], they concluded that the solutions are in the same space of special smooth functions as the one mentioned above. We find that these results are only for scalar equations with fixed initial time 0, in which the existence of solutions on some intervals is assumed.

There is no result available on the differential properties of local and global solutions to systems of Caputo fractional order differential equations with arbitrary initial time.

Similar to Lyapunov stability (called internal stability for linear systems) that describes the insensitivity of the solution x of a dynamical system to small changes in its initial condition x_0, external stability measures the reflection on the output y of a control system with zero initial condition from its input u by an L_2 norm inequality $\|y\|_{L_2} \leq \gamma\|u\|_{L_2}$, where γ is called the L_2 gain [23]. In other words, if every input $u \in L_2$ of a control system generates a zero-state output $y \in L_2$, then the control system is externally stable. External stability plays a special role in system analysis because it is natural to work with square integrable signals which can be viewed as finite energy signals [60]. As well known, if one thinks of $u(t)$ as current or voltage, then $u^T(t)u(t)$ is proportional to the instantaneous power of the signal, and its integral over all time is a measure of the energy content of the signal. There are many control problems related to the L_2 norm inequality (external stability) for integer order control systems such as H_∞ control [61], disturbance attenuation [62] and L_2 gain analysis [63]. For fractional order linear control systems, there are already some results in this area such as H_∞ control [64], L_p norm finiteness property [65] and H_2 norm computation [66]. However, it is rare to see related results for fractional order nonlinear control systems. As far as we consider, this is mainly because it is not easy to involve the vector field function into the proof of the L_2 norm inequality, through the Caputo fractional order derivative of an usual Lyapunov function V. For integer order nonlinear control systems, we can just integrate the both sides of $y^T(t)y(t) - \gamma^2 u^T(t)u(t) + \dot{V}(t) \leq 0$ from 0 to ∞, e.g. see (26) in [67]. But after replacing $\dot{V}$ by ${}_0^C D_t^\alpha V(t)$, we cannot easily do it any more. One the one hand, to derive an estimation of ${}_0^C D_t^\alpha V(t)$, assumptions more than continuity should have been imposed on $u(t)$. On the other hand, it is hard to deal with the improper integral of ${}_0^C D_t^\alpha V(t)$. Fortunately, the diffusive realization presented in [24] and the Lyapunov-like function proposed in [25] provide us an inspiration to solve this problem, even though the equivalence between Caputo fractional order control systems and their diffusive realizations, and the existence of the Lyapunov-like function both need to be proven.

This thesis is motivated by all those mentioned theory deficiencies in our research area, and organized according to the order that they have been introduced. In Chapter 2, after necessary preliminaries, we first systematically define fractional order polynomials and investigate their root distribution, coprimeness, properness and $\rho - \kappa$ polynomials, then develop the fractional-version of three frequency-domain designs: pole placement, internal model principle and model matching. In Chapter 3, the fundamental theory of Caputo fractional order systems with arbitrary initial time including the continuation and smoothness of solutions is first developed. Then the smoothness part is used to derive an estimation for the Caputo fractional order derivative of a general quadratic Lyapunov function, which together with the continuation part solves the Lyapunov stability problem for Caputo fractional order nonlinear systems. On the other hand, the equivalence between Caputo fractional order nonlinear control systems and their diffusive realizations, and the existence of the Lyapunov-like functions based on the realizations, are both proven, which solve the external stability problem for Caputo fractional order nonlinear control systems. Finally, these results of two aspects are applied to H_∞. In Chapter 4, the results on the Lyapunov and external stability for Caputo fractional order nonlinear systems are extended for Caputo fractional order hybrid systems. In Chapter 5, conclusions and future work are summarized.

1.2 An Example Application

To illustrate the application of fractional order differential equations, we introduce a realistic example of modelling mechanical property of real materials.

The traditional way to describe the behaviour of certain materials under the influence of external forces, specifically, the relation between stress (tension) σ and strain (deformation) ε, uses the laws of Hooke and Newton. If we are dealing with elastic solid, Hooke's law

$$\sigma(t) = E\varepsilon(t), \tag{1.1}$$

where E is the modulus of elasticity of the material, is the method of our choice. If the object is viscous fluid, then Newton's law

$$\sigma(t) = \eta D^1 \varepsilon(t), \tag{1.2}$$

where η is the viscosity of the material and D^1 denotes the first order differential operator, can be applied. There are various ordinary linear models such as Maxwell, Voigt, Zener and Kelvin, representing different series or parallel combinations of Hooke and Newton type units, of which a general form can be given by [4],

$$\sum_{k=0}^{n} a_k D^k \sigma(t) = \sum_{k=0}^{m} b_k D^k \varepsilon(t).$$

However, as summarized in [4], these classical models did not adjust themselves well to the behaviour demonstrated by many viscoelastic materials. The so-called viscoelastic materials, e.g. polymers, exhibit a behaviour somewhere between the pure viscous fluid and the pure elastic solid [1].

In 1971, Caputo and Mainardi proposed the following model based on four parameters [68],

$$\sigma(t) + b\, {}_0^C D_t^{\alpha} \sigma(t) = E_0 \varepsilon(t) + E_1\, {}_0^C D_t^{\alpha} \varepsilon(t), \tag{1.3}$$

where $0 < \alpha < 1$ and ${}^C D$ denotes the Caputo fractional order derivative. Fifteen years later, Bagley and Torvik consolidated this as the definitive model with the following restrictions

$$E_0 \geq 0, E_1 > 0, b \geq 0; E_1 \geq bE_0.$$

and also showed experimentally that this model is in close agreement with the behaviour of over one hundred and fifty viscoelastic materials [69]. Let $b = 0$ and $E_0 = 0$, then (1.3) becomes

$$\sigma(t) = E_1\, {}_0^C D_t^{\alpha} \varepsilon(t). \tag{1.4}$$

As we observe, the equation (1.4) "interpolates" between (1.1) (that may be considered as $\sigma(t) = ED^0\varepsilon(t)$) and (1.2).

In fact, (1.4) with $E_1 = \sqrt{E\eta}$ and $\alpha = 1/2$ is equivalent to the self-similar (fractal) tree model for viscoelastic materials shown in Figure 1.2, where springs and pistons represent the elastic and viscous properties of the materials

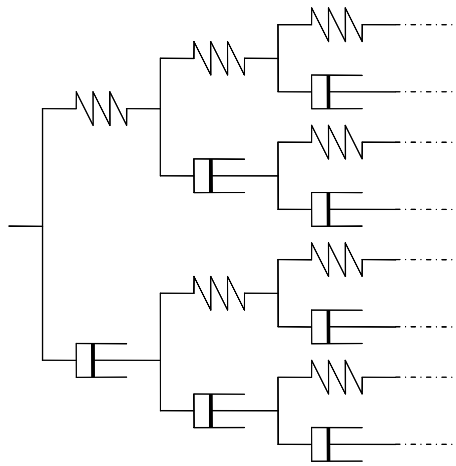


Figure 1.2: Self-similar tree model for viscoelastic materials

respectively, and the modulus of each spring and the viscosity of fluid in each piston are E and η, respectively. Now we begin to show this equivalence. Taking the Laplace transform of (1.1), and (1.2) with $\varepsilon(0) = 0$ yields,

$$G_E(s) = \frac{\mathscr{L}[\varepsilon(t)]}{\mathscr{L}[\sigma(t)]} = \frac{1}{E}, G_\eta(s) = \frac{\mathscr{L}[\varepsilon(t)]}{\mathscr{L}[\sigma(t)]} = \frac{1}{s\eta}.$$

Then the transfer function of each spring G_E and piston G_η is $1/E$ and $1/(s\eta)$, respectively. There are two facts useful for the derivation of a total transfer function of a series or parallel combination of these elastic and viscous units. In any series branch, the stress on each component of the branch is equal; in any two parallel branches with the same terminals, the strain of each branch is equal. Let G_n denote the total transfer function of the tree that has n levels of branches, then

$$G_1 = \frac{1}{\frac{1}{G_E} + \frac{1}{G_\eta}}, \; G_2 = \frac{1}{\frac{1}{G_E+G_1} + \frac{1}{G_\eta+G_1}}, \; G_3 = \frac{1}{\frac{1}{G_E+G_2} + \frac{1}{G_\eta+G_2}}, \; \ldots .$$

Thus, the total transfer function of the tree shown in Figure 1.2, $G = \lim_{n\to\infty} G_n$. On the other hand, due to the self-similar nature of the tree: any part of the tree from any branching point to "infinity" is identical to the whole tree (for a definition of self-similarity, see page 34-41, 349-350 in [34]),

$$G = \frac{1}{\frac{1}{G_E+G} + \frac{1}{G_\eta+G}}.$$

Therefore,

$$G = \sqrt{G_E G_\eta} = \frac{1}{\sqrt{E\eta}} s^{-\frac{1}{2}}.$$

In the time domain,

$$\sigma(t) = \sqrt{E\eta}\, {}_0^C D_t^{\frac{1}{2}} \varepsilon(t).$$

Chapter 2

Linear System

To design Caputo fractional order linear control systems in the frequency domain, as required to be BIBO stable, disturbance rejective and implementable, the fractional-version pole placement, internal model principle and model matching are developed in this chapter.

As introduced in Chapter 1, the pole placement is a method employed in feedback control system theory to place the closed-loop poles of a plant in pre-determined locations in the s-plane. Referring to Figure 2.1, it is, for a given plant $G(s)$, to design the compensator $C(s)$ such that all poles of the closed-loop control system are placed in desired locations out of the closed right half plane of the principal sheet of the Riemann surface, i.e. the overall system is BIBO stable.

As we know, in the internal model principle, the internal model supplies closed-loop transmission zeros which cancel the unstable poles of the disturbance and reference signals. In Figure 2.2, the internal model is a block $1/\phi(s)$ inserted inside the loop, between the input terminal of the reference r (and feedback) and that of the plant $G(s)$, of which the denominator $\phi(s)$ includes the least common denominator of the unstable poles of the disturbance w and reference r such that the output component excited by the disturbance $y_w(t) \to 0$ and the output component excited by the reference $y_r(t) \to r$ then the output $y(t) \to r$, as $t \to \infty$, i.e. the effect of the disturbance $w(t)$ will be rejected and the output $y(t)$ will approach the reference r asymptotically, as desired.

We consider the two-degrees-of-freedom configuration model matching to realize implementable transfer functions. That is: given a plant with transfer function $G(s)$ and given a desired overall transfer function $G_o(s)$, to find a proper compensator with two inputs and one output $\mathbf{C}(s)$ such that the closed-loop transfer function of every possible input-output pair is proper and BIBO stable, see Figure 2.3.

Before coming to these three cores, it is necessary to first give a definition for fractional integral, differential operators and Mittag-Leffler functions, and an introduction to BIBO stability. As a prerequisite for the pole placement, internal model principle and model matching, fractional order polynomials and their properties including root distribution, coprimeness and properness are well defined and explored. Especially, for the internal model principle, $\rho - \kappa$ polynomials are proposed with a constructive algorithm to formulate the fractional-version internal model.

2.1 Preliminaries

Now we are in a position to give a first definition for fractional integral and differential operators, and an introduction to Caputo fractional order linear systems and BIBO stability.

2.1.1 Fractional Calculus

The uniform formula of a fractional integral with $\gamma > 0$ is defined on $L_1[a,b]$ by

$$ {}_a\mathscr{D}_t^{-\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\gamma}} \,\mathrm{d}\tau, \; a \le t \le b, $$

where $\Gamma(\cdot)$ denotes the Gamma function, which converges in the right half of the complex plane [1].

For an arbitrary positive real fractional (non-integer) number q, the Riemann-Liouville and Caputo fractional derivatives are defined respectively as

$$ {}_a^R D_t^q f(t) = D^{[q]+1}[{}_a\mathscr{D}_t^{-([q]-q+1)} f(t)]; $$

$$ {}_a^C D_t^q f(t) = {}_a\mathscr{D}_t^{-([q]-q+1)}[D^{[q]+1} f(t)], $$

where $[q]$ stands for the integer part of q; D, ${}^R D$ and ${}^C D$ denote the first-order derivative, Riemann-Liouville and Caputo fractional derivatives respectively [1].

If $f \in AC^{[q]+1}[a,b]$ (the set of functions with absolutely continuous derivative of order $[q]$), then the fractional derivatives ${}_a^R D_t^q f$ and ${}_a^C D_t^q f$ exist almost everywhere on $[a,b]$ [1]. In particular, for $0 < q < 1$, ${}_a^R D_t^q f$ and ${}_a^C D_t^q f$ exist almost everywhere on $[a,b]$ if $f \in AC[a,b]$.

One important property of integer order integral operators is preserved as follows, see Theorem 2.2 in [1].

Theorem 2.1.1. *[1] Let $p, q \ge 0$ and $f \in L_1[a,b]$. Then ${}_a\mathscr{D}_t^{-p}\, {}_a\mathscr{D}_t^{-q} f = {}_a\mathscr{D}_t^{-(p+q)} f$ holds almost everywhere on $[a,b]$. If additionally $f \in C[a,b]$ or $p + q \ge 1$, then the identity holds everywhere on $[a,b]$.*

When it comes to the composition of fractional integrals and Caputo fractional derivatives, we find that the Caputo derivative is a left inverse of the fractional integral as stated in the following theorem, see Theorem 3.7 in [1].

Theorem 2.1.2. *[1] If f is continuous and $q \ge 0$, then ${}_a^C D_t^q\, {}_a\mathscr{D}_t^{-q} f = f$.*

In general, it turns out that the Laplace transform is an extremely useful tool for the analysis of linear differential equations. More precisely, that is the foundation of frequency analysis. Here we introduce the Laplace transform of the Caputo fractional order derivative, see the following theorem originated from Theorem 7.1 in [1].

Theorem 2.1.3. *[1] Assume that $f : [0,\infty) \to \mathbb{R}$ is such that the Laplace transform $\mathscr{L} f = F(s)$ exists on $[s_0, \infty)$ with some $s_0 \in \mathbb{R}$. Then for $s > \max\{0, s_0\}$, $\mathscr{L}[{}_0^C D_t^q f(t)] = s^q F(s) - \sum_{k=1}^{[q]} s^{q-k} f^{(k-1)}(0)$.*

2.1.2 System Solution

As the exponential function, e^z, in the theory of integer order linear differential equations, the Mittag-Leffler function plays a similar and important role in fractional order linear systems, see Theorem 2.1.4. The Mittag-Leffler function is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)},$$

where $\alpha > 0$ and $z \in \mathbb{C}$ is a complex number. The Mittag-Leffler function with two parameters is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)},$$

where $\beta > 0$. For all $z \in \mathbb{C}$, $E_{\alpha,\beta}$ is convergent, i.e. $E_{\alpha,\beta}$ is an entire function [1]. For $\beta = 1$, $E_{\alpha,1}(z) = E_\alpha(z)$. In particular, $E_{1,1}(z) = e^z$.

The Laplace transform of the Mittag-Leffler function (multiplied by the power function) with two parameters is

$$\mathscr{L}[t^{\beta-1}E_{\alpha,\beta}(-zt^\alpha)] = \frac{s^{\alpha-\beta}}{s^\alpha + z},\ R_e(s) > |z|^{\frac{1}{\alpha}},$$

where t, s are the variables in the time domain and frequency domain, respectively, and $R_e(s)$ denotes the real part of s [2].

Now we are ready to introduce the Caputo fractional order linear control system and its solution as follows. In fact, there are already some results on solutions to linear systems, e.g. [1] pp.135, [3] pp.43 and [4] pp.323. However, the first one here is only for scalar differential equations with zero initial time, while the latter two are derived by using the Laplace transform without an assumption of continuity imposed on inputs. As we can see below, the continuity assumption is necessary.

Theorem 2.1.4. *Assume $u \in (C[t_0, t_0 + c], \mathbb{R}^l)$, then the Caputo fractional order linear control system*

$$\begin{cases} {}_{t_0}^{C}D_t^\alpha x = Ax + Bu \\ y = Cx + Du \\ x^{(k)}(t)|_{t=t_0} = x_{0,k},\ k = 0, 1, 2, ..., m, \end{cases} \tag{2.1}$$

where $m < \alpha < m + 1$, $m \in \{0, 1, 2, ...\}$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$, has a unique solution $x(t) \in C^m[t_0, t_0 + c]$ and

$$x(t) = \sum_{k=0}^{m} \{{}_{t_0}\mathscr{D}_t^{-k} E_\alpha[A(t - t_0)^\alpha]\} x_{0,k} + \int_{t_0}^{t} (t - \tau)^{\alpha-1} E_{\alpha,\alpha}[A(t - \tau)^\alpha] Bu(\tau) d\tau.$$

Proof. Clearly, the uniqueness is trivial. We then show that the given solution satisfies the system equation. According to the definitions of the fractional integral and Mittag-Leffler function,

$${}_{t_0}\mathscr{D}_t^{-k} E_\alpha[A(t - t_0)^\alpha] = {}_{t_0}\mathscr{D}_t^{-k} \sum_{j=0}^{\infty} \frac{A^j (t - t_0)^{j\alpha}}{\Gamma(j\alpha + 1)} = \sum_{j=0}^{\infty} \frac{A^j (t - t_0)^{j\alpha+k}}{\Gamma(j\alpha + k + 1)}.$$

Then

$$
\begin{aligned}
{}^{C}_{t_0}D^{\alpha}_{t}\{{}_{t_0}\mathscr{D}_t^{-k}E_\alpha[A(t-t_0)^\alpha]\} &= {}^{C}_{t_0}D^{\alpha}_{t}[\sum_{j=0}^{\infty}\frac{A^j(t-t_0)^{j\alpha+k}}{\Gamma(j\alpha+k+1)}] = {}_{t_0}\mathscr{D}_t^{-(m+1-\alpha)}D^{m+1}[\sum_{j=0}^{\infty}\frac{A^j(t-t_0)^{j\alpha+k}}{\Gamma(j\alpha+k+1)}] \\
&= {}_{t_0}\mathscr{D}_t^{-(m+1-\alpha)}[\sum_{j=1}^{\infty}\frac{A^j(t-t_0)^{j\alpha+k-m-1}}{\Gamma(j\alpha+k-m)}] = \sum_{j=1}^{\infty}\frac{A^j(t-t_0)^{(j-1)\alpha+k}}{\Gamma[(j-1)\alpha+k+1]} \\
&= A\sum_{j=0}^{\infty}\frac{A^j(t-t_0)^{j\alpha+k}}{\Gamma(j\alpha+k+1)} = A\,{}_{t_0}\mathscr{D}_t^{-k}E_\alpha[A(t-t_0)^\alpha].
\end{aligned}
$$

The convolution term of the solution can be rewritten as

$$
\begin{aligned}
\int_{t_0}^{t}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}[A(t-\tau)^\alpha]Bu(\tau)d\tau &= \int_{t_0}^{t}(t-\tau)^{\alpha-1}[\sum_{j=0}^{\infty}\frac{A^j(t-\tau)^{j\alpha}}{\Gamma(j\alpha+\alpha)}]Bu(\tau)d\tau \\
&= \sum_{j=0}^{\infty}A^j\frac{1}{\Gamma[(j+1)\alpha]}\int_{t_0}^{t}\frac{Bu(\tau)}{(t-\tau)^{1-(j+1)\alpha}}d\tau \\
&= \sum_{j=0}^{\infty}A^j\,{}_{t_0}\mathscr{D}_t^{-(j+1)\alpha}Bu(t).
\end{aligned}
$$

Due to $u(t)\in C[t_0,t_0+c]$, according to Theorem 2.1.1 and 2.1.2,

$$
\begin{aligned}
{}^{C}_{t_0}D^{\alpha}_{t}\{\int_{t_0}^{t}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}[A(t-\tau)^\alpha]Bu(\tau)d\tau\} &= \sum_{j=0}^{\infty}A^j\,{}^{C}_{t_0}D^{\alpha}_{t}[{}_{t_0}\mathscr{D}_t^{-(j+1)\alpha}Bu(t)] = \sum_{j=0}^{\infty}A^j\,{}_{t_0}\mathscr{D}_t^{-j\alpha}Bu(t) \\
&= Bu(t) + A\sum_{j=1}^{\infty}A^{j-1}\,{}_{t_0}\mathscr{D}_t^{-[(j-1)+1]\alpha}Bu(t) \\
&= Bu(t) + A\sum_{j=0}^{\infty}A^{j}\,{}_{t_0}\mathscr{D}_t^{-(j+1)\alpha}Bu(t) \\
&= A\int_{t_0}^{t}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}[A(t-\tau)^\alpha]Bu(\tau)d\tau + Bu.
\end{aligned}
$$

Next we shall show that the solution satisfies the initial condition. Trivially,

$$
D^i_{t_0}\mathscr{D}_t^{-k}E_\alpha[A(t-t_0)^\alpha]|_{t=t_0} = \begin{cases} {}_{t_0}\mathscr{D}_t^{-(k-i)}E_\alpha[A(t-t_0)^\alpha]|_{t=t_0} = 0, & i<k; \\ E_\alpha[A(t-t_0)^\alpha]|_{t=t_0} = 1, & i=k. \end{cases}
$$

For $i>k$, $1\le i-k\le m$ and

$$
D^i_{t_0}\mathscr{D}_t^{-k}E_\alpha[A(t-t_0)^\alpha] = D^{i-k}E_\alpha[A(t-t_0)^\alpha] = D^{i-k}\sum_{j=0}^{\infty}\frac{A^j(t-t_0)^{j\alpha}}{\Gamma(j\alpha+1)} = \sum_{j=1}^{\infty}\frac{A^j(t-t_0)^{j\alpha-i+k}}{\Gamma(j\alpha+1-i+k)}.
$$

As $t = t_0$, it equals 0. Thus, for $i = 0, 1, ..., m$, we have $D^i \sum_{k=0}^{m}\{_{t_0}\mathscr{D}_t^{-k} E_\alpha[A(t-t_0)^\alpha]\}x_{0,k} = x_{0,i}$ as $t = t_0$, and it is continuous on $[t_0, t_0 + c]$. Moreover,

$$D^i \int_{t_0}^{t} (t-\tau)^{\alpha-1} E_{\alpha,\alpha}[A(t-\tau)^\alpha] Bu(\tau) d\tau = D^i \sum_{j=0}^{\infty} A^j \, {}_{t_0}\mathscr{D}_t^{-(j+1)\alpha} Bu(t) = \sum_{j=0}^{\infty} A^j \, {}_{t_0}\mathscr{D}_t^{-[(j+1)\alpha - i]} Bu(t).$$

Since $u(t)$ is continuous on $[t_0, t_0 + c]$, $\mathscr{D}_t^{-[(j+1)\alpha-i]} Bu(t)$ is continuous on $[t_0, t_0 + c]$, and

$${}_{t_0}\mathscr{D}_t^{-[(j+1)\alpha-i]} Bu(t) = \frac{1}{\Gamma[(j+1)\alpha - i]} \int_{t_0}^{t} \frac{Bu(\tau)}{(t-\tau)^{1-[(j+1)\alpha-i]}} d\tau \leq \frac{\|B\|_1 \max_{t_0 \leq t \leq t_0 + c} \|u(t)\|_1}{\Gamma[(j+1)\alpha - i][(j+1)\alpha - i]} (t - t_0).$$

Thus, $D^i \int_{t_0}^{t} (t-\tau)^{\alpha-1} E_{\alpha,\alpha}[A(t-\tau)^\alpha] Bu(\tau) d\tau = 0$, as $t = t_0$. Therefore, $x(t) \in C^m[t_0, t_0 + c]$ and $x^{(k)}(t)|_{t=t_0} = x_{0,k}$, $k = 0, 1, 2, ..., m$. □

The time-domain linear control system (2.1) can be also represented by its transfer function in frequency domain. Assume that the Laplace transform of $u(t)$ exists. Taking the Laplace transform in (2.1) with $t_0 = 0$ and $x_{0,k} = 0$ for each k, we derive

$$\mathscr{L}[y(t)] = G(s)\mathscr{L}[u(t)],$$

where $G(s) = C(s^\alpha I - A)^{-1}B + D$ is called the system transfer function. By Cramer's Rule,

$$G(s) = \frac{C \, \mathrm{adj}(s^\alpha I - A) B}{\det(s^\alpha I - A)} + D.$$

Moreover, it follows from the Laplace transform equation that

$$y(t) = g(t) * u(t) = \int_0^t g(t-\tau) u(\tau) d\tau,$$

where $g(t) = \mathscr{L}^{-1}[G(s)]$. On the other hand, according to Theorem 2.1.4,

$$y(t) = C \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}[A(t-\tau)^\alpha] Bu(\tau) d\tau + Du(t).$$

Thus,

$$g(t) = Ct^{\alpha-1} E_{\alpha,\alpha}(At^\alpha)B + D\delta(t),$$

where δ denotes the Dirac delta function. As we see, $g(t)$ is also the impulse response of the control system (2.1), i.e. the output of (2.1) as $u(t) = \delta(t)$ and $x_{0,k} = 0$, for $k = 0, 1, 2, ..., m$.

Let $l = p = 1$, then (2.1) becomes a single-input-single-output (SISO) system that will be the object of investigation for the pole placement, internal model principle and model matching later in this chapter. In this case, G is a ratio of two fractional order polynomials.

2.1.3 BIBO Stability

Based on the presented preliminaries, we are in the position to introduce the BIBO stability, and a theorem referred from [5] for SISO systems.

Definition 2.1.1. *A system is said to be BIBO stable if every bounded input excites a bounded zero-state output.*

Note that an input $u(t)$ is said to be bounded if there exists a constant u_c such that $|u(t)| \leq u_c$ for all $t \geq 0$. Moreover, as well known, a system is BIBO stable if and only if its impulse response is absolutely integrable on $[0, \infty)$.

Theorem 2.1.5. *[5] A transfer function $G(s) = Q(s^{\alpha})/P(s^{\alpha})$, where P, Q are coprime polynomials and $0 < \alpha < 1$ is the fractional order, is BIBO stable, if and only if $|arg\{\sigma\}| > \alpha\pi/2$, where σ is any complex number such that $P(\sigma) = 0$.*

As immediate consequences of the BIBO stability theorem, the following two corollaries will be employed in our frequency-domain designs later. The terminologies: proper, coprime and Riemmann surface, appearing in the corollaries, will be introduced in the next section.

Corollary 2.1.1. *A proper transfer function $G(s) = Q(s^{\frac{1}{q}})/P(s^{\frac{1}{q}})$, where P, Q are coprime polynomials and $q \in \mathbb{Z}^+$ ($\{1, 2, ...\}$), is BIBO stable, if and only if $|arg\{\omega\}| > \pi/(2q)$, where ω is any complex number such that $P(\omega) = 0$.*

Proof. The proof is similar to that of Theorem 2.1.5 in [5]. According to the partial fraction decomposition, the transfer function has the form:

$$G(s) = d + \sum_{i=1}^{r}\sum_{j=1}^{v_i} \frac{c_{ij}}{(\omega - \omega_i)^j} = d + \sum_{i=1}^{r}\sum_{j=1}^{v_i} \frac{c_{ij}}{(s^{\frac{1}{q}} - \omega_i)^j}, \tag{2.2}$$

where $d \in \mathbb{R}$, $c_{ij} \in \mathbb{C}$ and $\omega_i \in \mathbb{C}$. Then the impulse response can be expressed by the generalized Mittag-Leffler function [5] as $g(t) = d\delta(t) + \sum_{i=1}^{r}\sum_{j=1}^{v_i} c_{ij} E^{*j}_{\frac{1}{q}}(\omega_i, t)$. According to Theorem 2.17 in [5], it follows from $|arg\{\omega_i\}| > \pi/(2q)$ that as $t \to \infty$, the generalized function is equivalent to the following

$$E^{*j}_{\frac{1}{q}}(\omega_i, t) \sim \frac{\frac{1}{q}}{\Gamma(1 - \frac{1}{q})} j(-\omega_i)^{-1-j} t^{-1-\frac{1}{q}}.$$

Thus, $\int_0^{\infty} |g(t)| dt < \infty$. This completes the proof. □

Corollary 2.1.2. *A proper transfer function $G(s) = Q(s^{\frac{1}{q}})/P(s^{\frac{1}{q}})$, where P, Q are coprime polynomials and $q \in \mathbb{Z}^+$, is BIBO stable, if and only if $P(s^{\frac{1}{q}}) = 0$ has no roots in the closed right half plane of the principal sheet of the Riemann surface.*

Proof. The roots of $P(s^{\frac{1}{q}}) = 0$ are those $s \in \mathbb{C}$ such that $(s^{\frac{1}{q}} - \omega_i) = 0$, where ω_i denotes the same one appearing in (2.2). According to Lemma 2.2.1 and Remark 2.2.3, $(s^{\frac{1}{q}} - \omega_i) = 0$ has one root $|\omega_i|^q e^{jq\angle\omega_i}$, where $\angle\omega$ is defined as the angle of any $\omega \in \mathbb{C}$, $-\pi/q < \angle\omega \leq 2\pi - \pi/q$, in the closed right half plane of the principal sheet, if and only if $-\pi/(2q) \leq \angle\omega_i \leq \pi/(2q)$, i.e. $|arg\{\omega_i\}| \leq \pi/(2q)$. As we see, the condition here is equivalent to that of Corollary 2.1.1. Therefore, the conclusion follows. □

2.2 Fractional Order Polynomial

A fractional order linear control system, in frequency domain, can be given by its transfer function consisting of fractional order polynomials. For the frequency analyses: pole placement, internal model principle and model matching, it is necessary to first focus on the fractional order polynomials and their root distribution, coprimeness, properness and $\rho - \kappa$ polynomials. We now begin to define the so-called fractional order polynomials.

Definition 2.2.1. *The function*

$$P(s) = a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + ... + a_1 s^{\alpha_1} + a_0$$

is defined as a fractional order polynomial, if $\alpha_i \in \mathbb{Q}^+$ (the set of positive rational numbers), for $i = 1, 2, ..., n$ and $a_i \in \mathbb{R}$, for $i = 0, 1, ..., n$ [6].

Let $p_i/q_i = \alpha_i$, where p_i, $q_i \in \mathbb{Z}^+$ are coprime, for $i = 1, 2, ..., n$ (if $\alpha_i \in \mathbb{Z}^+$ for some i, $q_i := 1$) and let q be the least common multiple (lcm) of q_1, q_2,...,q_n, then

$$P(s) = a_n(s^{\frac{1}{q}})^{u_n} + a_{n-1}(s^{\frac{1}{q}})^{u_{n-1}} + ... + a_1(s^{\frac{1}{q}})^{u_1} + a_0.$$

The fractional degree (fdeg) of $P(s)$ with respect to $1/q$ is defined as [6]

$$fdeg_{\frac{1}{q}}\{P(s)\} := max\{u_1, u_2, ..., u_n\},$$

where $1/q$ is defined as the fractional order basis of $P(s)$.

The kth corresponding polynomial of $P(s)$ with respect to $s^{\frac{1}{kq}}$ is defined as

$$P_k(s^{\frac{1}{kq}}) := a_n(s^{\frac{1}{kq}})^{ku_n} + a_{n-1}(s^{\frac{1}{kq}})^{ku_{n-1}} + ... + a_1(s^{\frac{1}{kq}})^{ku_1} + a_0,$$

where $k \in \mathbb{Z}^+$ and the domain of s is the Riemann surface (where the origin is a branch point and the branch cut is assumed at $\mathbb{R}^-$ (the negative half of the real axis), see page 171 in [7]) of kq sheets.

The relative fractional degree (rfdeg) of $P(s)$ with respect to $1/(kq)$ is defined as

$$rfdeg_{\frac{1}{kq}}\{P(s)\} := k\, max\{u_1, u_2, ..., u_n\},$$

where $1/(kq)$ is defined as the kth corresponding fractional order basis of $P(s)$.

Remark 2.2.1. *The kq sheets of the Riemann surface are determined by*

$$s = |s|e^{j\phi},$$

where $(2m + 1)\pi < \phi \leq (2m + 3)\pi, m = -1, 0, ..., kq - 2$. In particular, the case of $m = -1$ is the principal (first) sheet. For the mapping $\lambda = s^{\frac{1}{kq}}$, these sheets become the regions of the plane λ defined by

$$\lambda = |\lambda|e^{j\theta},$$

where $(2m + 1)\pi/(kq) < \theta \leq (2m + 3)\pi/(kq)$, also see page 19 in [3].

Remark 2.2.2. *Let $\lambda = s^{\frac{1}{kq}}$, then $P_k(\lambda)$ is a regular polynomial of degree $rfdeg_{\frac{1}{kq}}\{P(s)\}$ in terms of variable λ and $P_k(\lambda) = P(s)$. In particular, $P_1(s^{\frac{1}{q}})$ has the same form as $P(s)$. Moreover, $P(s)$ is even a regular polynomial in terms of s (defined on the complex plane), if $\alpha_i \in \mathbb{Z}^+$ for all i.*

2.2.1 Root Distribution

As one of the most important properties of the just defined fractional order polynomials, root distribution determines the BIBO stability as discussed in Subsection 2.1.3. Now a useful lemma is first stated for the introduction to the root distribution.

Lemma 2.2.1. *For any $\omega \in \mathbb{C}$, the equation $(s^{\frac{1}{kq}})^k - \omega = 0$ has k roots in kq sheets of the Riemann surface, $|\omega|^q e^{jq\angle\omega}$, $|\omega|^q e^{jq\angle\omega} e^{j2q\pi}$, ..., $|\omega|^q e^{jq\angle\omega} e^{j2q(k-1)\pi}$, where $\angle\omega$ is defined as the angle of ω, $-\pi/q < \angle\omega \leq 2\pi - \pi/q$.*

Proof. For any $\omega \in \mathbb{C}$, $-\pi < arg\{\omega\} \leq \pi$. First consider the case: $-\pi/q < arg\{\omega\} \leq \pi$. It follows from the equation that

$$(s^{\frac{1}{kq}} - |\omega|^{\frac{1}{k}} e^{j\frac{arg\{\omega\}}{k}})(s^{\frac{1}{kq}} - |\omega|^{\frac{1}{k}} e^{j\frac{arg\{\omega\}+2\pi}{k}})...(s^{\frac{1}{kq}} - |\omega|^{\frac{1}{k}} e^{j\frac{arg\{\omega\}+2(k-1)\pi}{k}}) = 0.$$

Then the k roots are $|\omega|^q e^{jq\,arg\{\omega\}}$, $|\omega|^q e^{jq\,arg\{\omega\}} e^{j2q\pi}$, ..., $|\omega|^q e^{jq\,arg\{\omega\}} e^{j2q(k-1)\pi}$. Since $q\,arg\{\omega\} + 2q(k-1)\pi \in (-\pi + 2q(k-1)\pi, (2kq - q)\pi] \subseteq (-\pi, (2kq - 1)\pi]$, all roots are, as defined, in the kq sheets of the Riemann surface. Thus, in this case, let $\angle\omega = arg\{\omega\}$, then the conclusion follows.

Then consider the case: $-\pi < arg\{\omega\} \leq -\pi/q$. Following the proof for the previous case, we derive the angle of roots as $q\,arg\{\omega\} + 2q(m-1)\pi \in (-q\pi + 2q(m-1)\pi, -\pi + 2q(m-1)\pi]$, $m = 1, 2, ..., k$. When $m = 1$, the angle belongs to $(-q\pi, -\pi]$. The corresponding root is not in the Riemann surface. Thus, in this case, we can not directly use $arg\{\omega\}$. Let $\angle\omega = arg\{\omega\} + 2\pi$, then $\pi < \angle\omega \leq 2\pi - \pi/q$ and the k roots are $|\omega|^q e^{jq\angle\omega}$, $|\omega|^q e^{jq\angle\omega} e^{j2q\pi}$, ..., $|\omega|^q e^{jq\angle\omega} e^{j2q(k-1)\pi}$. Since $q\angle\omega + 2q(k-1)\pi \in (q\pi + 2q(k-1)\pi, (2kq-1)\pi] \subseteq (-\pi, (2kq-1)\pi]$, all roots are in the kq sheets of the Riemann surface. ◻

Remark 2.2.3. *The root $|\omega|^q e^{jq\angle\omega} e^{j2q(m-1)\pi}$, $m = 1, 2, ..., k$, is in the $[(m-1)q + i]th$ sheet, where $i \in \mathbb{Z}^+$ and $1 \leq i \leq q$, if and only if $-\pi/q + (i-1)2\pi/q < \angle\omega \leq \pi/q + (i-1)2\pi/q$. In particular, the root $|\omega|^q e^{jq\angle\omega}$ is in the closed right half plane of the principal sheet if and only if $-\pi/(2q) \leq \angle\omega \leq \pi/(2q)$.*

Based on the lemma above, we can deduce a further result about the root distribution as follows.

Property 2.2.1. *If $P(s)$ is a fractional order polynomial of $fdeg_{\frac{1}{q}}\{P(s)\} = n$, then the equation $P_k(s^{\frac{1}{kq}}) = 0$ has kn roots in kq sheets of the Riemann surface. Moreover, the locations of the roots in the $[(m-1)q+1]th, [(m-1)q+2]th, ..., mqth$ sheets for each $m = 2, 3, ..., k$, are the same as the locations of those in the $1st, 2rd, ..., qth$ sheets, respectively.*

Proof. Since $\text{fdeg}_{\frac{1}{q}}\{P(s)\} = n$, the general form of $P_1(s^{\frac{1}{q}})$ is as

$$P_1(s^{\frac{1}{q}}) = a_n(s^{\frac{1}{q}})^n + a_{n-1}(s^{\frac{1}{q}})^{n-1} + ... + a_1(s^{\frac{1}{q}}) + a_0,$$

where $a_n \neq 0$. Let $\varpi = s^{\frac{1}{q}}$, then

$$P_1(\varpi) = a_n \varpi^n + a_{n-1}\varpi^{n-1} + ... + a_1 \varpi + a_0,$$

which is a regular polynomial of degree n. Thus, the equation $P_1(\varpi) = 0$ has n roots: $\varpi_1, \varpi_2, ..., \varpi_n$, i.e.

$$(s^{\frac{1}{q}} - \varpi_1)(s^{\frac{1}{q}} - \varpi_2)...(s^{\frac{1}{q}} - \varpi_n) = 0.$$

Consider $-\pi/q < \angle\varpi_i \leq 2\pi - \pi/q$, for $i = 1, 2, ..., n$, then according to Lemma 2.2.1, the equation above $P_1(s^{\frac{1}{q}}) = 0$ has n roots: $|\varpi_1|^q e^{jq\angle\varpi_1}, |\varpi_2|^q e^{jq\angle\varpi_2}, ..., |\varpi_n|^q e^{jq\angle\varpi_n}$ (or $(\varpi_1)^q, (\varpi_2)^q, ..., (\varpi_n)^q$) in the Riemann surface of q sheets. This conclusion can be seen in [6]. There is also a similar statement for the second part of the property but no proof available. It follows from the relation between $P_k(s^{\frac{1}{kq}})$ and $P_1(s^{\frac{1}{q}})$ that $P_k(s^{\frac{1}{kq}}) = 0$ is the same as

$$[(s^{\frac{1}{kq}})^k - \varpi_1][(s^{\frac{1}{kq}})^k - \varpi_2]...[(s^{\frac{1}{kq}})^k - \varpi_n] = 0.$$

According to Lemma 2.2.1, the equation above has kn roots in kq sheets of the Riemann surface.

Specifically, the $(im)th$ ($i = 1, 2, ..., n, m = 1, 2, ..., k$) root of $P_k(s^{\frac{1}{kq}}) = 0$ is

$$s_{im} = |\varpi_i|^q e^{jq\angle\varpi_i} e^{2q(m-1)\pi}. \tag{2.3}$$

It follows that $s_{11}, s_{21}, ..., s_{n1}$ are in the first q sheets ($1st, 2rd, ..., qth$ sheets) since $\angle s_{i1} = q\angle\varpi_i \in (-\pi, (2q-1)\pi]$, while $s_{1m}, s_{2m}, ..., s_{nm}$ are in the mth q sheets ($[(m-1)q+1]th, [(m-1)q+2]th, ..., mqth$ sheets) since $\angle s_{im} = q\angle\varpi_i \in (-\pi + 2q(m-1)\pi, (2qm-1)\pi]$. And the locations of $s_{1m}, s_{2m}, ..., s_{nm}$ in the mth q sheets are the same as those of $s_{11}, s_{21}, ..., s_{n1}$ in the first q sheets, because the angle difference between s_{i1} and s_{im} is $2q(m-1)\pi$. □

It follows from (2.3) that $s_{i1} = (\varpi_i)^q$ for each i=1, 2, ..., n. This implies that the roots of $P_k(s^{\frac{1}{kq}}) = 0$ in the first q sheets of P_k's Riemann surface are the same as those of $P_1(s^{\frac{1}{q}}) = 0$ in P_1's Riemann surface of q sheets so that the roots of $P_k(s^{\frac{1}{kq}}) = 0$ in the principal sheet are the same as those of $P_1(s^{\frac{1}{q}}) = 0$ in the principal sheet. Thus, we have the following remark.

Remark 2.2.4. *s_0 is a root of $P_1(s^{\frac{1}{q}}) = 0$ in the closed right half plane of the principal sheet if and only if s_0 is a root of $P_k(s^{\frac{1}{kq}}) = 0$ in the closed right half plane of the principal sheet.*

2.2.2 Coprimeness

In frequency-domain design, we usually need to consider the cancellations of zeros and poles, which makes it necessary to discuss coprimeness of the fractional order polynomials. We first give its definition.

Definition 2.2.2. *Fractional order polynomials $P(s)$, $Q(s)$ with fractional order bases $1/q_1$, $1/q_2$, respectively, are defined to be coprime, if polynomials $P_{k_1}(\lambda)$, $Q_{k_2}(\lambda)$, where $\lambda = s^{\frac{1}{q}}$, $q = lcm\{q_1, q_2\}$, $k_1 = q/q_1$ and $k_2 = q/q_2$, are coprime.*

The following lemma provides the necessary and sufficient conditions for two fractional order polynomials to be coprime.

Lemma 2.2.2. *Fractional order polynomials $P(s)$, $Q(s)$ with fractional order bases $1/q_1$, $1/q_2$, respectively, are coprime, if and only if for any $l \in \mathbb{Z}^+$, $P_{k_1}(\lambda)$, $Q_{k_2}(\lambda)$, where $\lambda = s^{\frac{1}{lq}}$, $q = lcm\{q_1, q_2\}$, $k_1 = lq/q_1$ and $k_2 = lq/q_2$, are coprime.*

Proof. (Necessity) Let $\omega = s^{\frac{1}{q}}$, then

$$\frac{Q(s)}{P(s)} = \frac{(\omega - b_1)(\omega - b_2)...(\omega - b_m)}{(\omega - a_1)(\omega - a_2)...(\omega - a_n)}, \tag{2.4}$$

where m, n are the relative fractional degrees of $Q(s)$, $P(s)$, respectively, with respect to $1/q$. And $a_i \neq b_j$ for all $i = 1, 2, ..., n$, $j = 1, 2, ..., m$, since $P(s)$, $Q(s)$ are coprime. It follows from $\omega = \lambda^l$ and Remark 2.2.2 that

$$\frac{Q_{k_2}(\lambda)}{P_{k_1}(\lambda)} = \frac{Q(s)}{P(s)} = \frac{(\lambda^l - b_1)(\lambda^l - b_2)...(\lambda^l - b_m)}{(\lambda^l - a_1)(\lambda^l - a_2)...(\lambda^l - a_n)}.$$

For the sake of contradiction, assume that $P_{k_1}(\lambda)$, $Q_{k_2}(\lambda)$ are not coprime, i.e. for some i, j, there exists $\lambda_0 \in \mathbb{C}$ such that

$$\frac{(\lambda^l - b_j)}{(\lambda^l - a_i)} = \frac{(\lambda - \lambda_0)(\lambda^{l-1} + ...)}{(\lambda - \lambda_0)(\lambda^{l-1} + ...)}.$$

Then it follows that $a_i = (\lambda_0)^l = b_j$, which contradicts $a_i \neq b_j$ for all i, j. Therefore, $P_{k_1}(\lambda)$, $Q_{k_2}(\lambda)$ are coprime.

(Sufficiency) Assume $P(s)$, $Q(s)$ are not coprime, then in (2.4), a_i=b_j for some i, j, so that $P_{k_1}(\lambda)$ and $Q_{k_2}(\lambda)$ have common term $\lambda^l - a_i$ (or $\lambda^l - b_j$). Therefore, $P_{k_1}(\lambda)$, $Q_{k_2}(\lambda)$ are not coprime. By the law of contrapositive, if $P_{k_1}(\lambda)$, $Q_{k_2}(\lambda)$ are coprime, then $P(s)$, $Q(s)$ are coprime. □

Remark 2.2.5. *If the fractional order polynomial $R(s)$ is coprime to fractional order polynomials $P(s)$ and $Q(s)$, then $R(s)$ is coprime to $P(s)Q(s)$. Let $1/q_1$, $1/q_2$, $1/q_3$ denote the fractional order bases of $P(s), Q(s), R(s)$, $q = lcm\{q_1, q_2, q_3\}$, $k_i = q/q_i$ ($i = 1, 2, 3$) and $\lambda = s^{\frac{1}{q}}$, then according to Lemma 2.2.2, $R_{k_3}(\lambda)$ is coprime to $P_{k_1}(\lambda)$ and $Q_{k_2}(\lambda)$. Thus, $R_{k_3}(\lambda)$ is coprime to $P_{k_1}(\lambda)Q_{k_2}(\lambda)$. Let $T(s) = P(s)Q(s)$, then $P_{k_1}(\lambda)Q_{k_2}(\lambda) = T_k(\lambda)$ for some $k \in \mathbb{Z}^+$. According to Lemma 2.2.2, $R(s)$ is coprime to $T(s)$.*

2.2.3 Properness

Properness is an important property of transfer functions comprised by fractional order polynomials. For why, we shall discuss after giving its definition.

Definition 2.2.3. *The fractional order transfer function $G(s) = Q(s)/P(s)$, where $P(s)$, $Q(s)$ are fractional order polynomials with fractional order bases $1/q_1$, $1/q_2$, respectively, is defined to be proper (or strictly proper), if $rfdeg_{\frac{1}{q}}\{Q(s)\} \leq rfdeg_{\frac{1}{q}}\{P(s)\}$ (or $rfdeg_{\frac{1}{q}}\{Q(s)\} < rfdeg_{\frac{1}{q}}\{P(s)\}$), where $q = lcm\{q_1, q_2\}$.*

Remark 2.2.6. *The implementation of improper fractional order transfer functions requires pure differentiators. Let* $\lambda = s^{\frac{1}{q}}$*, then according to the polynomial long division,*

$$G(s) = \frac{Q_{\frac{q}{q_2}}(\lambda)}{P_{\frac{q}{q_1}}(\lambda)} = T(\lambda) + \frac{R(\lambda)}{P_{\frac{q}{q_1}}(\lambda)},$$

where $T(\lambda)$, $R(\lambda)$ *are polynomials of* λ *and* $deg\{T(\lambda)\} = rfdeg_{\frac{1}{q}}\{Q(s)\} - rfdeg_{\frac{1}{q}}\{P(s)\} \geq 1$. *It follows from Table 1 in [8] that all approximations (approximate rational transfer functions) of* $H(s) = 1/s^m$ *for different m (in increments of* 0.1*) are strictly proper. Thus, the approximation of* $G(s)$ *is improper. According to the discussion in page 283 of [9], the implementation of improper rational transfer functions involves pure differentiators.*

Remark 2.2.7. *Improper fractional order transfer functions amplify high-frequency noise. Let* $s = j\omega$*, then as* $\omega \to \infty$, $|T[(j\omega)^{\frac{1}{q}}]| \to \infty$, $|R[(j\omega)^{\frac{1}{q}}]/P_{\frac{1}{q}}[(j\omega)^{\frac{1}{q}}]| \to 0$ *or some other constants, such that* $|G(j\omega)| \to \infty$.

Since the pure differentiator is generally unstable and is not available in practice, see [10], and high-frequency noise often exists in real world, see page 15 of [9], improper fractional order transfer functions, according to the two remarks above, need to be practically avoided.

2.2.4 $\rho - \kappa$ Polynomial

Now we propose the definition of $\rho - \kappa$ polynomial that will be used to formulate the fractional-version internal model.

Definition 2.2.4. *Given a fractional order polynomial* $P(s)$ *with* $fdeg_{\frac{1}{q_1}}\{P(s)\} = n$ *and k roots in the closed right half plane of the principal sheet, as following*

$$P(s) = P_1(s^{\frac{1}{q_1}}) = (s^{\frac{1}{q_1}} - \omega_1)(s^{\frac{1}{q_1}} - \omega_2)...(s^{\frac{1}{q_1}} - \omega_{k-l_1})(s^{\frac{1}{q_1}})^{l_1}(s^{\frac{1}{q_1}} - \omega_{k+1})(s^{\frac{1}{q_1}} - \omega_{k+2})...(s^{\frac{1}{q_1}} - \omega_n),$$

where $\omega_i \neq 0$ *and* $|arg\{\omega_i\}| \leq \pi/(2q_1)$ *for* $i = 1, 2, ..., k - l_1$; $|arg\{\omega_i\}| > \pi/(2q_1)$ *for* $i = k+1, k+2, ..., n$. *The fractional order polynomial* $Q(s)$*, as following*

$$Q(s) = (s^{\frac{1}{q_2}} - \varpi_1)(s^{\frac{1}{q_2}} - \varpi_2)...(s^{\frac{1}{q_2}} - \varpi_{k-l_1})(s^{\frac{1}{q_2}})^{l_2},$$

where $|arg\{\varpi_i\}| \leq \pi/(2q_2)$ *for* $i = 1, 2, ..., k - l_1$, *is defined as the* $\rho - \kappa$ ($\rho = l_1/q_1 - l_2/q_2, \kappa = q_1/q_2$) *polynomial of the roots, in the closed right half plane of the principal sheet, of* $P(s)$*, if* $\rho < 1$ *and* $\varpi_i = |\omega_i|^{\kappa} e^{j\kappa\, arg\{\omega_i\}}$ *for* $i = 1, 2, ..., k - l_1$.

Remark 2.2.8. *All coefficients of* $Q(s)$ *are real such that* $Q(s)$ *is naturally a fractional order polynomial. Assume* $\omega_1 \in \mathbb{C}$*, then its conjugate* $\omega_1^* \in \{\omega_2, \omega_3, ..., \omega_{k-l_1}\}$ *(named as* ω_2*), because complex roots appear in conjugate pairs, and* $|arg\{\omega_i\}| \leq \pi/(2q_1)$. *Due to* $\varpi_i = |\omega_i|^{\kappa} e^{j\kappa\, arg\{\omega_i\}}$, $\varpi_1 = |\omega_1|^{\kappa} e^{j\kappa\, arg\{\omega_1\}}$, $\varpi_2 = |\omega_2|^{\kappa} e^{j\kappa\, arg\{\omega_2\}} = |\omega_1|^{\kappa} e^{-j\kappa\, arg\{\omega_1\}}$. *It follows,* $\varpi_2 = \varpi_1^*$. *Thus, complex numbers in* $\{\varpi_1, \varpi_2, ..., \varpi_{k-l_1}\}$ *also appear in conjugate pairs. This suffices.*

The following lemma shows the root distribution of the denominator in the fraction simplified from $Q(s)/P(s)$.

Lemma 2.2.3. *If the fractional order polynomial $Q(s)$ is a $\rho-\kappa$ polynomial of the roots, in the closed right half plane of the principal sheet, of a fractional order polynomial $P(s)$, then*

$$\frac{Q(s)}{P(s)} = \frac{N(s)}{s^{\rho} D(s)},$$

where $N(s)$, $D(s)$ are fractional order polynomials and $D(s)$ has no roots in the closed right half plane of the principal sheet.

Proof. For convenience, the same parameters as in Definition 2.2.4, for $P(s)$ and $Q(s)$, are considered. Let q be a multiple of q_1, q_2, and $k_1 = q/q_1$, $k_2 = q/q_2$, then for $i = 1, 2, ..., k - l_1$,

$$s^{\frac{1}{q_1}} - \omega_i = (s^{\frac{1}{q}})^{k_1} - \omega_i = (s^{\frac{1}{q}} - |\omega_i|^{\frac{1}{k_1}} e^{j\frac{arg\{\omega_i\}}{k_1}})(s^{\frac{1}{q}} - |\omega_i|^{\frac{1}{k_1}} e^{j\frac{arg\{\omega_i\}+2\pi}{k_1}})...(s^{\frac{1}{q}} - |\omega_i|^{\frac{1}{k_1}} e^{j\frac{arg\{\omega_i\}+(k_1-1)2\pi}{k_1}}),$$

$$s^{\frac{1}{q_2}} - \varpi_i = (s^{\frac{1}{q}})^{k_2} - \varpi_i = (s^{\frac{1}{q}} - |\varpi_i|^{\frac{1}{k_2}} e^{j\frac{arg\{\varpi_i\}}{k_2}})(s^{\frac{1}{q}} - |\varpi_i|^{\frac{1}{k_2}} e^{j\frac{arg\{\varpi_i\}+2\pi}{k_2}})...(s^{\frac{1}{q}} - |\varpi_i|^{\frac{1}{k_2}} e^{j\frac{arg\{\varpi_i\}+(k_2-1)2\pi}{k_2}}).$$

Since $\varpi_i = |\omega_i|^{\kappa} e^{j\kappa\, arg\{\omega_i\}}$, $\kappa = q_1/q_2 = k_2/k_1$, then

$$|\omega_i|^{\frac{1}{k_1}} e^{j\frac{arg\{\omega_i\}}{k_1}} = |\varpi_i|^{\frac{1}{k_2}} e^{j\frac{arg\{\varpi_i\}}{k_2}}.$$

For simplicity, let $\hat{\omega}_{im} = |\omega_i|^{\frac{1}{k_1}} e^{j\frac{arg\{\omega_i\}+(m-1)2\pi}{k_1}}$ and $\hat{\varpi}_{im} = |\varpi_i|^{\frac{1}{k_2}} e^{j\frac{arg\{\varpi_i\}+(m-1)2\pi}{k_2}}$, then

$$\frac{s^{\frac{1}{q_2}} - \varpi_i}{s^{\frac{1}{q_1}} - \omega_i} = \frac{(s^{\frac{1}{q}} - \hat{\varpi}_{i2})(s^{\frac{1}{q}} - \hat{\varpi}_{i3})...(s^{\frac{1}{q}} - \hat{\varpi}_{ik_2})}{(s^{\frac{1}{q}} - \hat{\omega}_{i2})(s^{\frac{1}{q}} - \hat{\omega}_{i3})...(s^{\frac{1}{q}} - \hat{\omega}_{ik_1})}.$$

According to Lemma 2.2.1 and Remark 2.2.3, the roots remaining in the denominator of $s^{\frac{1}{q_2}} - \varpi_i / s^{\frac{1}{q_1}} - \omega_i$ are $|\omega_i|^{q_1} e^{jq_1\, arg\{\omega_i\}} e^{j2q_1\pi}, |\omega_i|^{q_1} e^{jq_1\, arg\{\omega_i\}} e^{j2q_1 2\pi}, ..., |\omega_i|^{q_1} e^{jq_1\, arg\{\omega_i\}} e^{j2q_1(k_1-1))\pi}$, in the $(q_1+1)th$, $(2q_1+1)th$, ..., $[(k_1-1)q_1+1]th$ sheet, respectively, because $|arg\{\omega_i\}| \le \pi/(2q_1)$ and $\angle\omega_i = arg\{\omega_i\}$. Thus, the roots remaining in the denominator of $s^{\frac{1}{q_2}} - \varpi_i / s^{\frac{1}{q_1}} - \omega_i$, $i = 1, 2, ..., k - l_1$, are not in the closed right half plane of the principal sheet.

According to Lemma 2.2.1 and Remark 2.2.3, the roots of $s^{\frac{1}{q_1}} - \omega_i = 0$, $i = k + 1, k + 2, ..., n$, are also not in the closed right half plane of the principle sheet, due to $|arg\{\omega_i\}| > \pi/(2q_1)$.

Let $D(s) = [\prod_{i=1}^{k-l_1} \prod_{m=2}^{k_1} (s^{\frac{1}{q}} - \hat{\omega}_{im})] \prod_{i=k+1}^{n} (s^{\frac{1}{q_1}} - \omega_i)$ and $N(s) = \prod_{i=1}^{k-l_1} \prod_{m=2}^{k_2} (s^{\frac{1}{q}} - \hat{\varpi}_{im})$, then $Q(s)/P(s) = N(s) /[s^{\rho} D(s)]$, where $\rho = l_1/q_1 - l_2/q_2 < 1$, and $D(s)$ has no roots in the closed right half plane of the principal sheet.

In the following, we show that all coefficients of $D(s)$ and $N(s)$ are real such that $D(s)$, $N(s)$ are fractional order polynomials as defined.

If $\omega_i \in \mathbb{R}$, for some $i = 1, 2, ..., k - l_1$, then $\hat{\omega}_{i1} \in \mathbb{R}$ (and $\varpi_i \in \mathbb{R}$). Since complex roots appear in conjugate pairs, complex elements in $\{\hat{\omega}_{i2}, \hat{\omega}_{i3}, ..., \hat{\omega}_{ik_1}\}$ are in conjugate pairs. Thus, the coefficients of $\prod_{m=2}^{k_1} (s^{\frac{1}{q}} - \hat{\omega}_{im})$ are real. In the same way, coefficients of $\prod_{m=2}^{k_2} (s^{\frac{1}{q}} - \hat{\varpi}_{im})$ are real.

If $\omega_i \in \mathbb{C}$, for some $i = 1, 2, ..., k - l_1$, according to Remark 2.2.8, then $\omega_i^* \in \{\omega_1, \omega_2, ..., \omega_{k-l_1}\}$ (named as ω_j), and ϖ_i, ϖ_j are conjugate. Thus, $\hat{\omega}_{i1}$, $\hat{\omega}_{j1}$ are conjugate as well. In $(s^{\frac{1}{q_2}} - \varpi_i)(s^{\frac{1}{q_2}} - \varpi_j)/\ [(s^{\frac{1}{q_1}} - \omega_i)(s^{\frac{1}{q_1}} - \omega_j)]$ (its all coefficients are real), $(s^{\frac{1}{q}} - \hat{\omega}_{i1})\ (s^{\frac{1}{q}} - \hat{\omega}_{j1})$ (its coefficients are also real) will be canceled. Therefore, all coefficients of $\prod_{m=2}^{k_1}(s^{\frac{1}{q}} - \hat{\omega}_{im})\ \prod_{m=2}^{k_1}(s^{\frac{1}{q}} - \hat{\omega}_{jm})$ and $\prod_{m=2}^{k_2}(s^{\frac{1}{q}} - \hat{\varpi}_{im})\ \prod_{m=2}^{k_2}(s^{\frac{1}{q}} - \hat{\varpi}_{jm})$ are real.

Consequently, all coefficients of $N(s)$ are real. Since complex numbers in $\{\omega_1, \omega_2, ..., \omega_{k-l_1}\}$ appear in conjugate pairs, all coefficients of $\prod_{i=1}^{k-l_1}(s^{\frac{1}{q_1}} - \omega_i)$ are real, so are coefficients of $\prod_{i=k+1}^{n}(s^{\frac{1}{q_1}} - \omega_i)$. Therefore, all coefficients of $D(s)$ are also real. □

Remark 2.2.9. *If the fractional order polynomial $Q(s)$ is a $\rho - \kappa$ polynomial of the roots, in the closed right half plane of the principal sheet, of $P(s)R(s)$, where $P(s)$ and $R(s)$ are fractional order polynomials, then for some certain constants ρ_p, $\rho_r \le \rho$, $Q(s)/P(s) = N_p(s)/[s^{\rho_p} D_p(s)]$ and $Q(s)/R(s) = N_r(s)/[s^{\rho_r} D_r(s)]$, where $N_p(s)$, $D_p(s)$, $N_r(s)$ and $D_r(s)$ are fractional order polynomials, and $D_p(s)$ and $D_r(s)$ have no roots in the closed right half plane of the principal sheet. This is straightforward from the lemma.*

Here we introduce a lemma cited from [9] that will be applied to solve equations involving the fractional order polynomials in our designs later.

Lemma 2.2.4. *[9], pp.273-275. Given coprime polynomials $D(s)$ and $N(s)$ with $deg\{N(s)\} < deg\{D(s)\} = n$. Let $m \ge n - 1$, then for any polynomial $F(s)$ of degree $(n + m)$, there exist polynomial solutions $A(s)$ and $B(s)$ with $deg\{B(s)\} \le deg\{A(s)\} = m$ for the equation $A(s)D(s) + B(s)N(s) = F(s)$.*

2.3 Frequency-Domain Designs

In this section, we shall develop the designs in frequency domain: pole placement, internal model principle and model matching.

2.3.1 Pole Placement

We first state the following theorem to demonstrate how to arbitrarily assign an overall fractional order polynomial with constraints in terms of coprimeness and properness.

Theorem 2.3.1. *Given coprime fractional order polynomials $D(s) = D_1(s^{\frac{1}{q_D}})$ and $N(s) = N_1(s^{\frac{1}{q_N}})$ with*

$$rfdeg_{\frac{1}{q_G}}\{N(s)\} < rfdeg_{\frac{1}{q_G}}\{D(s)\} = n,$$

where $q_G = lcm\{q_D, q_N\}$. For any fractional order polynomial $F(s) = F_*(s^{\frac{1}{q_{F_*}}})$, where q_{F_*} denotes a corresponding fractional order basis of $F(s)$, with

$$rfdeg_{\frac{1}{q_{F_*}}}\{F(s)\} \ge 2n\frac{q_{F_*}}{q_G} - \frac{q_{F_*}}{\sigma},$$

where σ is a certain constant, $\sigma \geq lcm\{q_G, q_{F_*}\}$, there exist fractional order polynomial solutions $A(s)$ and $B(s)$ with

$$rfdeg_{\frac{1}{q}}\{B(s)\} \leq rfdeg_{\frac{1}{q}}\{A(s)\},$$

where q denotes $lcm\{q_G, q_{F_*}\}$, for the equation $A(s)D(s) + B(s)N(s) = F(s)$.

Proof. It is given that

$$\text{rfdeg}_{\frac{1}{q}}\{N(s)\} < \text{rfdeg}_{\frac{1}{q}}\{D(s)\} = n\frac{q}{q_G} \tag{2.5}$$

and

$$\text{rfdeg}_{\frac{1}{q}}\{F(s)\} \geq 2n\frac{q}{q_G} - \frac{q}{\sigma}. \tag{2.6}$$

Since $D(s), N(s)$ are coprime, according to Lemma 2.2.2, $D_{\frac{q}{q_D}}(\lambda)$, $N_{\frac{q}{q_N}}(\lambda)$, where $\lambda = s^{\frac{1}{q}}$, are coprime. It follows from (2.5) and (2.6) that

$$\deg\{N_{\frac{q}{q_N}}(\lambda)\} < \deg\{D_{\frac{q}{q_D}}(\lambda)\} = n\frac{q}{q_G},\ \deg\{F_{\frac{q}{q_{F_*}}}(\lambda)\} \geq 2n\frac{q}{q_G} - \frac{q}{\sigma}.$$

According to Lemma 2.2.4, there exist polynomial solutions $\hat{A}(\lambda)$, $\hat{B}(\lambda)$ with

$$\deg\{\hat{B}(\lambda)\} \leq \deg\{\hat{A}(\lambda)\} = \deg\{F_{\frac{q}{q_{F_*}}}(\lambda)\} - \deg\{D_{\frac{q}{q_D}}(\lambda)\}$$

such that $\hat{A}(\lambda)D_{\frac{q}{q_D}}(\lambda) + \hat{B}(\lambda)N_{\frac{q}{q_N}}(\lambda) = F_{\frac{q}{q_{F_*}}}(\lambda)$, because $\sigma \geq q$, $\deg\{\hat{A}(\lambda)\} \geq \deg\{D_{\frac{q}{q_D}}(\lambda)\} - 1$. Let $A(s) = \hat{A}(\lambda)$ and $B(s) = \hat{B}(\lambda)$, then $A(s)$, $B(s)$ are the fractional order polynomial solutions. □

Remark 2.3.1. *The fractional order bases of $A(s)$, $B(s)$ may be not $1/q$. In fact, it is not easy to preassign the fractional order bases as $A(s) = A_1(s^{\frac{1}{q_A}})$, $B(s) = B_1(s^{\frac{1}{q_B}})$. Let q denote $lcm\{q_A, q_B, q_G, q_{F_*}\}$, if $\sigma \geq q$, it follows (from the same proof) that the polynomial solutions (exist) are $\hat{A}(s^{\frac{1}{q}})$, $\hat{B}(s^{\frac{1}{q}})$, where q is only a common multiple of q_A and q_B.*

Remark 2.3.2. *It follows from [9] pp.273-275 that the solutions $A(s)$ (or $\hat{A}(\lambda)$), $B(s)$ (or $\hat{B}(\lambda)$) can be derived. Let $n_D = deg\{D_{\frac{q}{q_D}}(\lambda)\}$, $n_A = deg\{F_{\frac{q}{q_{F_*}}}(\lambda)\} - n_D$, then*

$$\begin{aligned}
D_{\frac{q}{q_D}}(\lambda) &= D_0 + D_1\lambda + ... + D_{n_D}\lambda^{n_D}, D_{n_D} \neq 0,\\
N_{\frac{q}{q_N}}(\lambda) &= N_0 + N_1\lambda + ... + N_{n_D}\lambda^{n_D},\\
\hat{A}(\lambda) &= A_0 + A_1\lambda + ... + A_{n_A}\lambda^{n_A},\\
\hat{B}(\lambda) &= B_0 + B_1\lambda + ... + B_{n_A}\lambda^{n_A},\\
F_{\frac{q}{q_{F_*}}}(\lambda) &= F_0 + F_1\lambda + ... + F_{n_D+n_A}\lambda^{n_D+n_A},
\end{aligned}$$

where all coefficients are real constants, not necessarily nonzero.

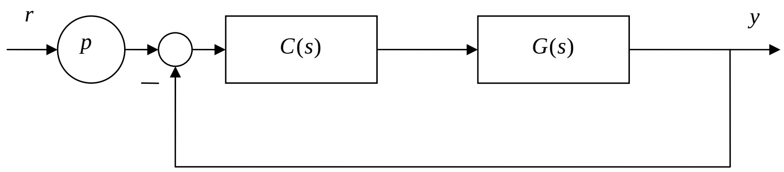


Figure 2.1: Unity-feedback configuration for pole placement.

Substitute into $\hat{A}(\lambda)D_{\frac{q}{q_D}}(\lambda) + \hat{B}(\lambda)N_{\frac{q}{q_N}}(\lambda) = F_{\frac{q}{q_{F_*}}}(\lambda)$*, then* $[A_0\ B_0\ A_1\ B_1\ ...\ A_{n_A}\ B_{n_A}]S = [F_0\ F_1\ ...\ F_{n_D+n_A}]$*, where* S *is* $2(n_A + 1) \times (n_D + n_A + 1)$ *as following*

$$S := \begin{bmatrix} D_0 & D_1 & ... & D_{n_D} & 0 & ... & 0 \\ N_0 & N_1 & ... & N_{n_D} & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & D_0 & ... & D_{n_D-1} & D_{n_D} & ... & 0 \\ 0 & N_0 & ... & N_{n_D-1} & N_{n_D} & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & D_0 & ... & D_{n_D} \\ 0 & 0 & ... & 0 & N_0 & ... & N_{n_D} \end{bmatrix}.$$

Since $n_A \geq n_D - 1$ *and* $D_{\frac{q}{q_D}}(\lambda)$, $N_{\frac{q}{q_N}}(\lambda)$ *are coprime, as discussed in [9],* S *has full column rank. Thus,* $\hat{A}(\lambda)$ *and* $\hat{B}(\lambda)$ *(are unique if* $n_A = n_D - 1$*; are not unique if* $n_A > n_D - 1$*) always exist, and are derivable from the algebraic equation.*

It is now ready to present the theorem for the design of the fractional-version pole placement.

Theorem 2.3.2. *Consider the unity-feedback configuration shown in Figure 2.1. The plant is described by a strictly proper transfer function* $G(s) = N(s)/D(s)$*, where* $D(s)$*,* $N(s)$ *are coprime, and* $D(s) = D_1(s^{\frac{1}{q_D}})$ *and* $N(s) = N_1(s^{\frac{1}{q_N}})$ *with*

$$rfdeg_{\frac{1}{q_G}}\{N(s)\} < rfdeg_{\frac{1}{q_G}}\{D(s)\} = n,$$

where $q_G = lcm\{q_D, q_N\}$*. For any desired fractional order polynomial* $F(s) = F_*(s^{\frac{1}{q_{F_*}}})$ *with*

$$fdeg\{F(s)\} \geq 2n\frac{q_{F_*}}{q_G} - \frac{q_{F_*}}{\sigma},$$

where σ *is a certain constant,* $\sigma \geq lcm\{q_G, q_{F_*}\}$*, there exists a proper compensator* $C(s) = B(s)/A(s)$ *with*

$$rfdeg_{\frac{1}{q}}\{B(s)\} \leq rfdeg_{\frac{1}{q}}\{A(s)\},$$

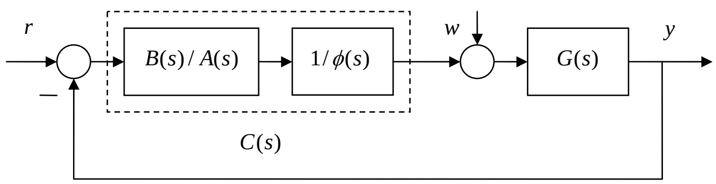


Figure 2.2: Unity-feedback configuration for internal model principle.

where q denotes $lcm\{q_G, q_{F_*}\}$, such that the overall transfer function equals

$$G_{r\to y}(s) = \frac{pN(s)B(s)}{A(s)D(s) + B(s)N(s)} = \frac{pN(s)B(s)}{F(s)}.$$

Proof. It follows straightforwardly from Theorem 2.3.1. Note that $C(s)$ is derivable from Remark 2.3.2. □

2.3.2 Internal Model Principle

Based on the results on the fractional-version pole placement and $\rho-\kappa$ polynomial, we then state the fractional-version internal model principle.

Theorem 2.3.3. *Consider the unity-feedback configuration shown in Figure 2.2. The plant is described by a strictly proper transfer function $G(s) = N(s)/D(s)$, where $D(s)$, $N(s)$ are coprime fractional order polynomials. The reference signal $r(t)$ and disturbance $w(t)$ are modeled as $R(s) = Nr(s)/Dr(s)$ and $W(s) = Nw(s)/Dw(s)$. Let $\phi(s)$ be a $\rho-\kappa$ polynomial of the roots, in the closed right half plane of the principal sheet, of the fractional order polynomial $Dr(s)Dw(s)$. If $\phi(s)$ and $N(s)$ are coprime, then there exists a proper compensator such that the output $y(t)$ will track $r(t)$ and reject $w(t)$ both asymptotically and robustly.*

Proof. Let $\bar{D}(s) = D(s)\phi(s)$, then

$$\mathrm{rfdeg}_{\frac{1}{q_{\bar{G}}}}\{N(s)\} < \mathrm{rfdeg}_{\frac{1}{q_{\bar{G}}}}\{\bar{D}(s)\} = \bar{n}$$

where $q_{\bar{G}} = lcm\{q_{\bar{D}}, q_N\}$, because $G(s)$ is strictly proper. Since $N(s)$ is coprime to $D(s)$ and $\phi(s)$, according to Remark 2.2.5, $\bar{D}(s)$, $N(s)$ are coprime. According to Theorem 2.3.2, for any desired fractional order polynomial $F(s)$ (has no roots in the closed right half plane of the principal sheet) with

$$\mathrm{rfdeg}_{\frac{1}{q_{F_*}}}\{F(s)\} \geq 2\bar{n}\frac{q_{F_*}}{q_{\bar{G}}} - \frac{q_{F_*}}{\bar{\sigma}},$$

where $\bar{\sigma} \geq lcm\{q_{\bar{G}}, q_{F_*}\}$, there exist fractional order polynomial solutions $A(s)$ and $B(s)$ with

$$\mathrm{rfdeg}_{\frac{1}{\bar{q}}}\{B(s)\} \leq \mathrm{rfdeg}_{\frac{1}{\bar{q}}}\{A(s)\}, \tag{2.7}$$

where $\bar{q}$ denotes $lcm\{q_{\bar{G}}, q_{F_*}\}$, for the equation $A(s)\bar{D}(s) + B(s)N(s) = F(s)$. Claim that the compensator, as shown in Figure 2.2,

$$C(s) = \frac{B(s)}{A(s)\phi(s)},$$

then the compensator is proper due to (2.7). Compute the transfer function from w to y,

$$G_{w\to y}(s) = \frac{G(s)}{1 + C(s)G(s)} = \frac{N(s)A(s)\phi(s)}{A(s)D(s)\phi(s) + B(s)N(s)} = \frac{N(s)A(s)\phi(s)}{F(s)}.$$

According to Lemma 2.2.3 and Remark 2.2.9, then the Laplace transform of $y_w(t)$ (the output excited by $w(t)$)

$$Y_w(s) = G_{w\to y}(s)W(s) = \frac{N(s)A(s)\phi(s)}{F(s)}\frac{Nw(s)}{Dw(s)} = \frac{N(s)A(s)Nw(s)}{F(s)}\frac{\hat{N}w(s)}{s^{\rho_w}\hat{D}w(s)}, \tag{2.8}$$

where $\rho_w \le \rho < 1$, $\hat{D}w(s), \hat{N}w(s)$ are fractional order polynomials and $\hat{D}w(s)$ has no roots in the closed right half plane of the principal sheet, because $\phi(s)$ is a $\rho - \kappa$ polynomial of the roots, in the closed right half plane of the principal sheet, of the fractional order polynomial $D_r(s)Dw(s)$.

According to the final value theorem proposed in [6], it follows that

$$\lim_{t\to\infty} y_w(t) = \lim_{s\to 0} sY_w(s) = \lim_{s\to 0} \frac{N(s)A(s)Nw(s)}{F(s)}\frac{\hat{N}w(s)}{\hat{D}w(s)}s^{1-\rho_w} = 0. \tag{2.9}$$

Alternatively, it follows from (2.8) that

$$Y_w(s) = \frac{N(s)A(s)Nw(s)\hat{N}w(s)}{(s^{\frac{1}{q}} - \omega_1)(s^{\frac{1}{q}} - \omega_2)...(s^{\frac{1}{q}} - \omega_{k-q\rho_w})(s^{\frac{1}{q}})^{q\rho_w}} \tag{2.10}$$

where $1/q, k$ are the fractional order basis, fractional degree of $F(s)s^{\rho_w}\hat{D}w(s)$ and $|arg\{\omega_i\}| > \pi/(2q)$ for $i = 1, 2, ...,$ $k - q\rho_w$. Taking the inverse Laplace transform of (2.10), then we can conclude that $y_w(t)$ consists of the generalized Mittag-Leffler functions $E^{*j}_{\frac{1}{q}}(\omega_i, t)$, $1 \le j \le k - q\rho_w$ and $t^{\rho_w - 1}$. According to Theorem 2.17 in [5],

$$\lim_{t\to\infty} |E^{*j}_{\frac{1}{q}}(\omega_i, t))| = \lim_{t\to\infty} \frac{\frac{1}{q}}{\Gamma(1-\frac{1}{q})} j\,|(-\omega_i)^{-1-j}\,|t^{-1-\frac{1}{q}} = 0. \tag{2.11}$$

Thus, $\lim_{t\to\infty} y_w(t) = 0$, i.e. the output excited by $w(t)$ is asymptotically suppressed.

Next we compute the Laplace transforms of $y_r(t)$ (the output excited by $r(t)$) and $e(t) := r(t) - y_r(t)$ as following

$$Y_r(s) = G_{r\to y}(s)R(s) = \frac{B(s)N(s)}{F(s)}\frac{Nr(s)}{Dr(s)},\ E(s) = R(s) - Y_r(s) = \frac{A(s)D(s)\phi(s)}{F(s)}\frac{Nr(s)}{Dr(s)}.$$

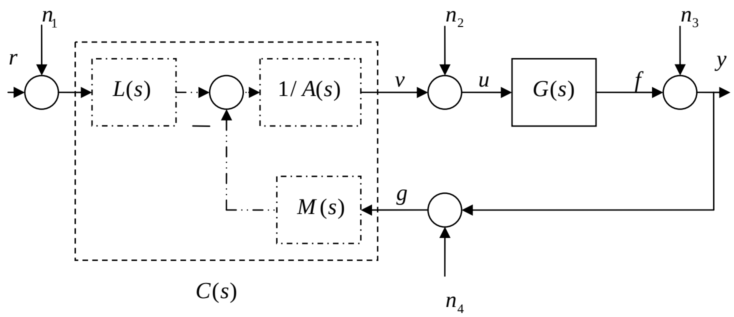


Figure 2.3: Two-degrees-of-freedom configuration for model matching.

Following the same processes as (2.8)-(2.9) or (2.8), (2.10)-(2.11), we can conclude, $\lim_{t\to\infty} e(t) = \lim_{t\to\infty} r(t) - y_r(t) = 0$. Since $y(t) = y_r(t) + y_w(t)$, $y(t) \to r(t)$ as $t \to \infty$. This shows the asymptotic tracking and disturbance rejection.

As we see, even if the parameters of $D(s), N(s), A(s)$ and $B(s)$ change, as long as all the roots of $F(s)$ remain outside of the closed right half plane of the principal sheet, and the roots in the closed right half plane of the principal sheet of $Dr(s)$ and $Dw(s)$ are canceled by $\phi(s)$ with $\rho_w, \rho_r < 1$, the output $y(t)$ will still track $r(t)$ and reject $w(t)$ asymptotically. This shows the robustness. □

2.3.3 Model Matching

Now we are in the right position to work on our last frequency-domain design: the fractional-version model matching.

Theorem 2.3.4. *Consider the two-degrees-of-freedom configuration shown in Figure 2.3. Given a plant described by a strictly proper transfer function $G(s) = N(s)/D(s)$, where $D(s)$, $N(s)$ are coprime fractional order polynomials. For a given overall transfer function (from r to y) $G_o(s) = E(s)/F(s)$, where $E(s)$, $F(s)$ are coprime fractional order polynomials, there exists a proper compensator with two inputs and one output $\mathbf{C}(s) = [1/A(s)][L(s) \; - M(s)]$, where $A(s)$, $L(s)$ and $M(s)$ are fractional order polynomials, such that the closed-loop transfer function of every possible input (r, n_1, n_2, n_3, n_4)-output (v, u, f, y, g) pair is proper and BIBO stable, if and only if*

1. *$F(s)$ has no roots in the closed right half plane of the principal sheet;*

2. *$rfdeg_{\frac{1}{q}}\{F(s)\} - rfdeg_{\frac{1}{q}}\{E(s)\} \geq rfdeg_{\frac{1}{q}}\{D(s)\} - rfdeg_{\frac{1}{q}}\{N(s)\}$, where $q = lcm\{q_D, q_N, q_E, q_F\}$, and $1/q_D$, $1/q_N$, $1/q_E$ and $1/q_F$ denote the fractional order bases of $D(s)$, $N(s)$, $E(s)$ and $F(s)$, respectively;*

3. *$E(s) = Q(s)R(s)$, where $Q(s)$ is a $\rho - \kappa$, $\rho \leq 0$, polynomial of the roots, in the closed right half plane of the principal sheet, of $N(s)$, and $R(s)$ is the rest of $E(s)$.*

Proof. (Sufficiency) First (referring to Procedure 9.1 in [9]) show that there exist $A(s)$, $L(s)$ and $M(s)$ such that $L(s)/A(s)$, $-M(s)/A(s)$ are proper and

$$\frac{E(s)}{F(s)} = \frac{L(s)N(s)}{A(s)D(s) + M(s)N(s)}.$$

To find $A(s)$, $L(s)$ and $M(s)$, consider

$$\frac{\hat{L}(\lambda)N_{\frac{q}{q_N}}(\lambda)}{\hat{A}(\lambda)D_{\frac{q}{q_D}}(\lambda) + \hat{M}(\lambda)N_{\frac{q}{q_N}}(\lambda)} = \frac{E_{\frac{q}{q_E}}(\lambda)N_{\frac{q}{q_N}}(\lambda)}{F_{\frac{q}{q_F}}(\lambda)N_{\frac{q}{q_N}}(\lambda)} = \frac{\bar{E}(\lambda)N_{\frac{q}{q_N}}(\lambda)}{\bar{F}(\lambda)}, \tag{2.12}$$

where $\lambda = s^{\frac{1}{q}}$, $\bar{E}(\lambda)/\bar{F}(\lambda) = E_{\frac{q}{q_E}}(\lambda)/[F_{\frac{q}{q_F}}(\lambda)N_{\frac{q}{q_N}}(\lambda)]$, and $\bar{E}(\lambda)$, $\bar{F}(\lambda)$ are coprime. Here, we may just consider to set $\hat{L}(\lambda) = \bar{E}(\lambda)$ and solve $\hat{A}(\lambda)$, $\hat{M}(\lambda)$ from $\bar{F}(\lambda) = \hat{A}(\lambda)D_{\frac{q}{q_D}}(\lambda) + \hat{M}(\lambda)N_{\frac{q}{q_N}}(\lambda)$. However, proper solutions may not exist due to the possibility that $\deg\{\bar{F}(\lambda)\}$ is not sufficiently high. To avoid this possibility, we introduce an arbitrary fractional order polynomial (without roots in the closed right half plane of the principal sheet) $\tilde{F}(\lambda)$ such that $\deg\{\bar{F}(\lambda)\tilde{F}(\lambda)\} \geq 2\deg\{D_{\frac{q}{q_D}}(\lambda)\} - 1$. Now rewrite (2.12) as

$$\frac{\hat{L}(\lambda)N_{\frac{q}{q_N}}(\lambda)}{\hat{A}(\lambda)D_{\frac{q}{q_D}}(\lambda) + \hat{M}(\lambda)N_{\frac{q}{q_N}}(\lambda)} = \frac{\bar{E}(\lambda)\tilde{F}(\lambda)N_{\frac{q}{q_N}}(\lambda)}{\bar{F}(\lambda)\tilde{F}(\lambda)}. \tag{2.13}$$

Set $\hat{L}(\lambda) = \bar{E}(\lambda)\tilde{F}(\lambda)$ and $\bar{F}(\lambda)\tilde{F}(\lambda) = \hat{A}(\lambda)D_{\frac{q}{q_D}}(\lambda) + \hat{M}(\lambda)N_{\frac{q}{q_N}}(\lambda)$, according to Lemma 2.2.4 and Remark 2.3.2, we then derive $\hat{A}(\lambda)$, $\hat{M}(\lambda)$ with $\deg\{\hat{M}(\lambda)\} \leq \deg\{\hat{A}(\lambda)\}$. It follows from condition 2) and (2.13) that

$$\deg\{\bar{F}(\lambda)\tilde{F}(\lambda)\} - \deg\{\hat{L}(\lambda)\} - \deg\{N_{\frac{q}{q_N}}(\lambda)\} \geq \deg\{D_{\frac{q}{q_D}}(\lambda)\} - \deg\{N_{\frac{q}{q_N}}(\lambda)\}.$$

This implies

$$\deg\{\hat{L}(\lambda)\} \leq \deg\{\bar{F}(\lambda)\tilde{F}(\lambda)\} - \deg\{D_{\frac{q}{q_D}}(\lambda)\} = \deg\{\hat{A}(\lambda)\}.$$

Let $A(s) = \hat{A}(\lambda)$, $L(s) = \hat{L}(\lambda)$ and $M(s) = \hat{M}(\lambda)$, then the existence of the proper compensator is shown.

Second show that the closed-loop transfer function of every possible input-output pair is proper. According to the Mason's Gain Formula [11], transfer functions (the signs of which are ignored) of input-output pairs are

$$\frac{1}{1 + \frac{M(s)}{A(s)}G(s)}, \frac{\frac{L(s)}{A(s)}}{1 + \frac{M(s)}{A(s)}G(s)}, \frac{\frac{M(s)}{A(s)}}{1 + \frac{M(s)}{A(s)}G(s)}, \frac{G(s)}{1 + \frac{M(s)}{A(s)}G(s)}, \frac{\frac{L(s)}{A(s)}G(s)}{1 + \frac{M(s)}{A(s)}G(s)}, \frac{\frac{M(s)}{A(s)}G(s)}{1 + \frac{M(s)}{A(s)}G(s)}. \tag{2.14}$$

Since $M(s)/A(s)$ is proper and $G(s)$ is strictly proper, then $\lim_{s\to\infty}[M(s)/A(s)]G(s) = 0$. Because $L(s)/A(s)$ is also proper, it is easy to see that all the transfer functions approach constants as s goes to ∞. This implies the properness.

Third show that the closed-loop transfer function of every possible input-output pair is BIBO stable. According to Lemma 2.2.3, it follows from condition 3) that

$$\frac{E(s)N(s)}{F(s)N(s)} = \frac{s^{-\rho}\hat{Q}(s)R(s)}{\hat{N}(s)}\frac{N(s)}{F(s)},$$

where $\hat{Q}(s)$, $\hat{N}(s)$ are fractional order polynomials and $\hat{N}(s)$ has no roots in the closed right half plane of the principal sheet. It follows from condition 1) that $\hat{N}(s)F(s)$ has no roots in the closed right half plane of the principal sheet so that $\bar{F}(s^{\frac{1}{q}})$ has no roots in the closed right half plane of the principal sheet. Thus, $A(s)D(s)+M(s)N(s) = \bar{F}(s^{\frac{1}{q}})\tilde{F}(s^{\frac{1}{q}})$ has no roots in the closed right half plane of the principal sheet. Substitute $G(s) = N(s)/D(s)$ into those transfer functions, then according to Corollary 2.1.2, the BIBO stability follows.

(Necessity) Condition 1) follows from the BIBO stability of $G_o(s)$. The properness and BIBO stability of $G_{r\rightarrow u}(s) = G_o(s)/G(s) = E(s)D(s)/[F(s)N(s)]$ imply condition 2) and 3), respectively. □

Corollary 2.3.1. *The conclusion in Theorem 2.3.4 holds if and only if $G_o(s)$ is BIBO stable, and $G_o(s)/G(s)$ is proper and BIBO stable.*

Proof. It is straightforward that three conditions in Theorem 2.3.4 are equivalent to the conditions in this corollary. □

Remark 2.3.3. *There is no need to concern the properness of the three blocks contained in dash in Figure 2.3. They are implemented as an integrated block-$\boldsymbol{C}(s)$ which has two inputs (r, g) and one output (v), see Figure 9.5 in [9]. These blocks are only used to indicate the relation between r, g and v but not independently implemented. In fact, $\boldsymbol{C}(s)$'s two components $L(s)/A(s)$, $-M(s)/A(s)$ are proper. Moreover, as discussed in [9], pp.291-292, the configuration shown in Figure 2.3 is better than other two-degrees-of-freedom configurations for model matching.*

2.4 Illustrative Examples

In this section, two examples for the applications of the pole placement, internal model principle and model matching are provided.

Example 2.4.1. *Consider the unity-feedback configuration in Figure 2.2 with $G(s) = 1/(s^{1/3}+1)$. Design a proper compensator $C(s) = B(s)/[A(s)\phi(s)]$ such that the output $y(t)$ will track any step reference input $r(t) = a$ and reject the generalized Mittag-Leffler type disturbance $w(t) = bE^{*1}_{\frac{1}{3}}(1,t)$, with unknown constants a and b, both asymptotically and robustly.*

First of all, we investigate the existence of $C(s)$. The Laplace transforms of $r(t)$ and $w(t)$ are a/s and $b/(s^{1/3}-1)$, respectively. It follows that $D_r(s) = s$, $D_w(s) = s^{1/3}-1$ and $D_r(s)D_w(s) = (s^{1/3}-1)s$. According to Definition 2.2.4, we can select $\phi(s) = (s^{1/3}-1)s^{1/3}$ such that $\phi(s)$ is a $\rho-\kappa$ (where $\rho = 2/3 < 1$ and $\kappa = 1$) polynomial of the roots, in the closed right half plane of the principal sheet, of $D_r(s)D_w(s)$. Since $\phi(s)$ and $N(s) = 1$ are coprime, according to Theorem 2.3.3, there exists $A(s)$, $B(s)$ such that $C(s)$ will be proper and $y(t)$ will track r(t) and reject $w(t)$ asymptotically and robustly.

Next we try to derive $A(s)$, $B(s)$. Let $\bar{D}(s) = D(s)\phi(s) = (s^{1/3}+1)(s^{1/3}-1)s^{1/3}$, then $\mathrm{rdeg}_{\frac{1}{3}}\{\bar{D}(s)\} = 3$. Since $\bar{D}(s)$ and $N(s)$ are coprime, according to Theorem 2.3.1, for any $F(s)$ with corresponding fractional order basis 1/3 and $\mathrm{rfdeg}_{\frac{1}{3}}\{F(s)\} \geq 6 - 3/\sigma$, where σ is a certain constant and $\sigma \geq 3$, there exists a proper compensator $B(s)/A(s)$

such that $A(s)\bar{D}(s) + B(s)N(s) = F(s)$. For the existence, we can select $\sigma = 3$ and then $\text{rfdeg}_{\frac{1}{3}}\{F(s)\} = 5$. Now we are ready to select $F(s)$. Choose $F(s) = (s^{1/3} + 1)^5$, then $1/q_F = 1/3$, and according to Remark 2.2.3 and Property 2.2.1, $F(s)$ has no roots in the closed right half plane of the principle sheet, since $\angle - 1 = \pi$.

Now we follow Remark 2.3.2 to derive $A(s)$ and $B(s)$. Let $\lambda = s^{1/3}$, then $\bar{D}_1(\lambda) = \lambda^3 - \lambda$, $N_3(\lambda) = 1$, $F_1(\lambda) = \lambda^5 + 5\lambda^4 + 10\lambda^3 + 10\lambda^2 + 5\lambda + 1$. Clearly, $n_A = \deg\{F_1(\lambda)\} - n_{\bar{D}} = 2$. Let $\hat{A}(\lambda) = A_0 + A_1\lambda + A_2\lambda^2$ and $\hat{B}(\lambda) = B_0 + B_1\lambda + B_2\lambda^2$, then $[A_0\ B_0\ A_1\ B_1\ A_2\ B_2]S = [1\ 5\ 10\ 10\ 5\ 1]$, where

$$S = \begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Since $\bar{D}(s)$ and $N(s)$ are coprime, S has full column rank then has inverse. The solution is [11 1 5 16 1 15]. Thus, $A(s) = s^{2/3} + 5s^{1/3} + 11$ and $B(s) = 15s^{2/3} + 16s^{1/3} + 1$. Obviously, $C(s)$ is proper, as desired.

Then confirm that $y(t)$ will track r(t) and reject $w(t)$, i.e. $y(t) \to r(t)$ and $y_w(t) \to 0$. It follows from the block diagram Figure 2.2,

$$Y_w(s) = \frac{b(s^{\frac{2}{3}} + 5s^{\frac{1}{3}} + 11)s^{\frac{1}{3}}}{(s^{\frac{1}{3}} + 1)^5},\ E(s) = R(s) - Y_r(s) = \frac{a(s^{\frac{2}{3}} + 5s^{\frac{1}{3}} + 11)(s^{\frac{1}{3}} - 1)}{(s^{\frac{1}{3}} + 1)^4 s^{\frac{2}{3}}}.$$

It is easy to see, $\rho_w = -1/3 < 1$ and $\rho_r = 2/3 < 1$. According to the final value theorem in [6], $y_w(t) \to 0$ and $e(t) = r(t) - y_r(t) \to 0$. Thus, $y(t) \to r(t)$.

As we see, even if the parameters of $D(s), N(s), A(s)$ and $B(s)$ change, as long as all the roots of $F(s)$ remain outside of the closed right half plane of the principal sheet, and the roots in the closed right half plane of the principal sheet of $Dr(s)$ and $Dw(s)$ are cancelled by $\phi(s)$ with ρ_w, $\rho_r < 1$, the output $y(t)$ will still track $r(t)$ and reject $w(t)$ asymptotically. This guarantees the robustness. So far, the design has been completed.

Example 2.4.2. *Consider the two-degrees-of-freedom configuration in Figure 2.3 with* $G(s) = 0.08/[s(0.05s + 1)]$ *(DC motor). Match* $G_o(s) = (0.05s + 1)/(0.05s^{2.5} + s^{1.5} + 0.05s + 1)$*. This stable overall transfer function (system) can asymptotically track the unit step reference with desired overshoot and settling time, and has desired phase margin* 45° *and infinite gain margin, see [12].*

We first apply Theorem 2.3.4 to check the feasibility of matching, i.e. the existence of $\mathbf{C}(s)$. $G_o(s)$ can be simplified to $G_o(s) = 1/(s^{3/2} + 1)$. Let $E(s) = 1$ and $F(s) = s^{3/2} + 1 = (s^{1/2} + 1)[s^{1/2} - (1/2 + j\sqrt{3}/2)][s^{1/2} - (1/2 - j\sqrt{3}/2)]$, then $E(s)$, $F(s)$ are coprime. Since $\angle - 1 = \pi$, $\angle(1/2 + j\sqrt{3}/2) = \pi/3$ and $\angle(1/2 - j\sqrt{3}/2) = -\pi/3$, it follows from Remark 2.2.3 and Property 2.2.1 that $F(s)$ has no roots in the closed right half plane of the principal sheet. Thus, condition 1) in the theorem is satisfied. However, condition 2) is not satisfied, since $\text{rfdeg}_{\frac{1}{2}}\{F(s)\} - \text{rfdeg}_{\frac{1}{2}}\{E(s)\} = 3 < 4 = \text{rfdeg}_{\frac{1}{2}}\{D(s)\} - \text{rfdeg}_{\frac{1}{2}}\{N(s)\}$. Therefore, $G_o(s)$ can not be matched.

The failure of condition 2) leads to the fact that the transfer function from the reference (the constant block) to the input of the DC motor, see Fig.12 in [12],

$$G_{r\to u}(s) = \frac{0.03125s^3 + 1.25s^2 + 12.5s}{0.05s^{2.5} + s^{1.5} + 0.05s + 1}$$

is improper. As analyzed in Remark 2.2.7, once some high-frequency noise appears in the constant block, the input of the DC motor would blow up then damage the motor.

In order to illustrate the model matching, we may consider to modify the plant transfer function in Example 2.4.2 as $G(s) = 0.08/[s^{0.5}(0.05s + 1)]$, while keep $G_o(s)$ the same. Then condition 2) is satisfied. As for condition 3), it is also satisfied with $\rho = 0$. Therefore, for this $G(s)$, there exists a proper compensator with two inputs and one output $\mathbf{C}(s) = (1/A(s))[L(s)\ \ -M(s)]$ such that the overall transfer function is $G_o(s)$, and the closed-loop transfer function of every possible input (r, n_1, n_2, n_3, n_4)-output (v, u, f, y, g) pair is proper and BIBO stable.

In the following, we follow the proof of Theorem 2.3.4 to find $A(s)$, $L(s)$ and $M(s)$. Let $\lambda = s^{\frac{1}{q}}, q = 2$, then $E_2(\lambda) = 1$, $F_1(\lambda) = \lambda^3+1$, $N_2(\lambda) = 2/25$ and $D_1(\lambda) = \lambda(1/20\lambda^2+1)$. It follows that $\bar{E}(\lambda) = 1$ and $\bar{F}(\lambda) = 2/25(\lambda^3+1)$. Since $\deg\{\bar{F}(\lambda)\} < 2\deg\{D_1(\lambda)\} - 1$, we need to introduce $\tilde{F}(\lambda)$ (without roots in the closed right half plane of the principal sheet) such that $\deg\{\bar{F}(\lambda)\tilde{F}(\lambda)\} \geq 2\deg\{D_1(\lambda)\} - 1$. We can select $\tilde{F}(\lambda) = 1/4(\lambda + 1)^2$, then set $\hat{L}(\lambda) = \bar{E}(\lambda)\tilde{F}(\lambda)$ (i.e. $L(s) = 1/4s + 1/2s^{1/2} + 1/4$) and $\bar{F}(\lambda)\tilde{F}(\lambda) = \hat{A}(\lambda)D_1(\lambda) + \hat{M}(\lambda)N_2(\lambda)$, where $\bar{F}(\lambda)\tilde{F}(\lambda) = 1/50(\lambda^5 + 2\lambda^4 + \lambda^3 + \lambda^2 + 2\lambda + 1)$.

Now we follow Remark 2.3.2 to derive $\hat{A}(\lambda)$ and $\hat{M}(\lambda)$. As we see, $n_A = \deg\{\bar{F}(\lambda)\tilde{F}(\lambda)\} - \deg\{D_1(\lambda)\} = 2$. Let $\hat{A}(\lambda) = A_0 + A_1\lambda + A_2\lambda^2$ and $\hat{M}(\lambda) = M_0 + M_1\lambda + M_2\lambda^2$, then $[A_0\ M_0\ A_1\ M_1\ A_2\ M_2]S = 1/50[1\ 2\ 1\ 1\ 2\ 1]$, where

$$S = \begin{bmatrix} 0 & 1 & 0 & \frac{1}{20} & 0 & 0 \\ \frac{2}{25} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{20} & 0 \\ 0 & \frac{2}{25} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{20} \\ 0 & 0 & \frac{2}{25} & 0 & 0 & 0 \end{bmatrix}.$$

The solution is $[-38/5\ 1/4\ 4/5\ 191/2\ 2/5\ \ -39/4]$. Thus, $A(s) = 2/5s + 4/5s^{1/2} - 38/5$, and $M(s) = -39/4s + 191/2s^{1/2} + 1/4$. Since both $L(s)/A(s)$ and $-M(s)/A(s)$ are proper, $\mathbf{C}(s)$ is proper.

Finally, we confirm that the overall transfer function is $1/(s^{3/2} + 1)$, and the closed-loop transfer function of every possible input-output pair is proper and BIBO stable. It follows from the block diagram Figure 2.3, the overall transfer function

$$G_o(s) = \frac{L(s)N(s)}{A(s)D(s) + M(s)N(s)}.$$

Substitute the derived $A(s)$, $L(s)$ and $M(s)$ into the equation above, we can derive $G_o(s) = 1/(s^{3/2} + 1)$. Since $\mathbf{C}(s)$ is really proper, all transfer functions, see (2.14), of possible input-output pairs are proper. And since all those transfer functions have the same denominator $A(s)D(s)+M(s)N(s) = \bar{F}(s^{1/2})\tilde{F}(s^{1/2}) = 1/50(s^{5/2}+2s^2+s^{3/2}+s+2s^{1/2}+1)$, of which no root is in the closed right half plane of the principal sheet, they are all BIBO stable. This completes the model matching.

Chapter 3

Nonlinear System

This chapter focuses on the Lyapunov and external stability of Caputo fractional order nonlinear systems. As we know, Lyapunov stability describes the behavior of system solutions for $t \to \infty$. It seems necessary to specially study the global existence of solutions before analyzing Lyapunov stability. However, this is not the case. We may consider a direct consequence of the continuation of solutions: a bounded Lyapunov function already incidentally implies the global existence. Another prerequisite for the analysis of fractional Lyapunov stability is the smoothness of solutions. The differential property of solutions to Caputo fractional order systems suffices to yield a simple estimation for the Caputo fractional order derivative of any quadratic Lyapunov function. Thus, for the fractional Lyapunov stability, we shall review the existence and uniqueness, and develop the continuation and smoothness, of the solution to the following general system of Caputo fractional order differential equations

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha} x = f(t, x) \\ x^{(k)}(t)|_{t=t_0} = x_{0,k},\ k = 0, ..., m, \end{cases} \tag{3.1}$$

where ${}^{C}D$ denotes the Caputo fractional derivative; $m < \alpha < m + 1$, $m \in \{0, 1, 2, ...\}$; f defined on some open set $D \subseteq \mathbb{R} \times \mathbb{R}^n$ is the given vector field function; and $x_{0,k} \in \mathbb{R}^n$ is the initial value vector.

For the external stability, we shall consider to use the Lyapunov-like function, instead of the quadratic Lyapunov function as usual. Without certain assumptions imposed on the inputs $u(t)$ of Caputo fractional order nonlinear control systems, we cannot derive an estimation for the Caputo fractional order derivative of a quadratic Lyapunov function so that it is difficult to involve the vector field function into the proof of the L_2 norm inequality (for external stability), through the Caputo fractional order derivative of the usual quadratic Lyapunov function V, i.e. by immediately integrating the both sides of $y^T(t)y(t) - \gamma^2 u^T(t)u(t) + {}_0^{C}D_t^{\alpha} V \leq 0$ from 0 to ∞. Fortunately, the Lyapunov-like function, as we shall see, works well. We shall first prove the equivalence between the control systems and their diffusive realizations, and then demonstrate the Lyapunov-like functions based on the realizations well-defined.

These stability results will be applied to H_∞ control that requires the controlled system without disturbance input to be (Lyapunov) asymptotically stable and the controlled system with disturbance to be externally stable from the disturbance to the output.

3.1 Existence and Uniqueness

Theorem 3.1.1. *(**Existence**). Assume that f is continuous on the closed set $\bar{S} = \{(t, x) : t \in [t_0, t_0 + a], \|x - \sum_{k=0}^{m}(t - t_0)^k x_{0,k}/k!\|_1 \leq b\}$, for some $a > 0$, $b > 0$ such that $\bar{S} \subset D$. Then (3.1) has a solution $x(t) \in C[t_0, t_0 + h]$, where $h = min\{a, [b\Gamma(\alpha + 1)/M]^{1/\alpha}\}$ and $M = \max_{(t,x)\in\bar{S}} \|f(t, x)\|_1$.*

Theorem 3.1.2. *(**Uniqueness**). Assume that f is continuous in t and Lipschitz in x on the closed set $\bar{S}$. Then (3.1) has a unique solution $x(t) \in C[t_0, t_0 + h]$.*

Remark 3.1.1. *Assume the hypothesis of Theorem 3.1.1, except that the set $\bar{S}$ is taken to be $\bar{S}_g = \{(t, x) : t \in [t_0, t_0 + a], x \in \mathbb{R}^n\}$. Moreover, assume that there exist constants $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ and $0 \leq \mu < 1$ such that $\|f(t, x)\|_1 \leq \lambda_1 + \lambda_2\|x\|_1^{\mu}$ for any $(t, x) \in \bar{S}_g$, then (3.1) has a solution $x(t) \in C[t_0, t_0 + a]$. Or assume that there exists a constant $L > 0$ such that $\|f(t, x) - f(t, y)\|_1 \leq L\|x - y\|_1$, for any $(t, x), (t, y) \in \bar{S}_g$, then (3.1) has a unique solution $x(t) \in C[t_0, t_0 + a]$. Here the parameter a may be taken as $+\infty$.*

Lemma 3.1.1. *Assume the hypothesis of Theorem 3.1.1. Then $x(t) \in C[t_0, t_0 + h]$ is a solution of (3.1) if and only if it is a solution of the Volterra integral equation of the second kind*

$$x(t) = \sum_{k=0}^{m} \frac{(t - t_0)^k}{k!} x_{0,k} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - \tau)^{\alpha-1} f(\tau, x(\tau))d\tau.$$

Theorem 3.1.1, Theorem 3.1.2 and Lemma 3.1.1 are respectively extended from Theorem 6.1, Theorem 6.5 and Lemma 6.2 in [1], according to Remark 6.1 in [1]. In fact, the proofs for these extensions are the same as the corresponding ones in [1] with the following replacements: the scalar real space $\mathbb{R}$, the initial time 0 and the absolute norm $|\cdot|$ are taken to be $\mathbb{R}^n$, t_0 and $\|\cdot\|_1$, respectively. The adaption of $\|\cdot\|_1$ here is to be consistent with Section 3.3.

3.2 Continuation

Let $x(t)$ be a solution of (3.1) on an interval $J = [t_0, t_0 + h]$ (or $[t_0, t_0 + h)$). By a continuation of $x(t)$, we mean an extension $\tilde{x}(t)$ of $x(t)$ to a larger interval $\tilde{J} = [t_0, t_0 + \tilde{h}]$ (or $[t_0, t_0 + \tilde{h})$), where $\tilde{h} > h$, such that $\tilde{x}(t)$ is a solution of (3.1) on $\tilde{J}$ and $\tilde{x}(t) = x(t)$ on J. If it is not possible to extend J, then $x(t)$ is called non-continuable. In this case, the interval J is called a maximal interval of existence for $x(t)$. At first, we introduce the following preliminaries.

Definition 3.2.1. *Let $F \subseteq (C[a, b], \mathbb{R}^n)$, then F is called*

i. *uniformly bounded, if there exists an $M > 0$ such that for all $f \in F$ and all $t \in [a, b]$, $\|f(t)\|_1 \leq M$;*

ii. *equicontinuous, if for any $\epsilon > 0$, there exists a $\delta > 0$ such that for all $f \in F$ and all $t_1, t_2 \in [a, b]$ with $|t_1 - t_2| < \delta$, $\|f(t_1) - f(t_2)\|_1 < \epsilon$.*

Definition 3.2.2. *Let (X,d) be a metric space. The set $U \subseteq X$ is called relatively compact in X, if the closure of U is a compact subset of X.*

Theorem 3.2.1. *(**Arzelà-Ascoli**). Let $V \subseteq (C[a,b],\mathbb{R}^n)$, be equipped with norm $\|\cdot\|_{1,\infty}$, where $\|x\|_{1,\infty} := \max_{t\in[a,b]}\|x(t)\|_1$. If V is uniformly bounded and equicontinuous, then V is relatively compact in $C[a,b]$.*

Theorem 3.2.2. *(**Schauder's Fixed Point Theorem**). Let (X,d) be a complete metric space. If W is a closed convex subset of X and $T: W \to W$ is a mapping such that $\{Tx : x \in W\}$ is relatively compact in X, then T has a fixed point in W.*

The theorems above may be found in many books, e.g. [1], pp.230. Now we can state our continuation theorem.

Theorem 3.2.3. *Assume $f \in C(D,\mathbb{R}^n)$. If $x(t)$ is a solution of (3.1) on some interval, then it can be extended over a maximal interval of existence. Moreover, if $[t_0,\beta)$ is a maximal interval of existence, then $(t,x(t))$ tends to the boundary of D as $t \to \beta^-$.*

Proof. According to Theorem 3.1.1, the continuity of f on $\bar{S} \subset D$ suffices that (3.1) has a solution $x(t) \in C[t_0,t_0+h]$, and $(t,x(t)) \in \bar{S}$ for $t \in [t_0,t_0+h]$. Choose a sequence D_n of open sets in D such that $\cup_{n=1}^{\infty} D_n = D$, $\bar{D}_n$ is bounded and $\bar{D}_n \subset D_{n+1}$ for $n = 1,2...$, then there exists $N > 0$ such that $n > N$ implies $\bar{S} \subset D_n$.

We shall start to show that there is an extension of $x(t)$ to an interval $[t_0,t_0+h+h_e]$ for some $h_e > 0$, by using Schauder's Fixed Point Theorem. Define the following operator

$$(T\hat{x})(t) = z(t) + \frac{1}{\Gamma(\alpha)}\int_{t_0+h}^{t}(t-\tau)^{\alpha-1}f(\tau,\hat{x}(\tau))d\tau, t \in [t_0+h,t_0+h+h_e],$$

where $z(t) = \sum_{k=0}^{m}(t-t_0)^k x_{0,k}/k! + [1/\Gamma(\alpha)]\int_{t_0}^{t_0+h}(t-\tau)^{\alpha-1}f(\tau,x(\tau))d\tau$, and $\hat{x} \in A$. Here the domain of the operator $A := \{y \in C[t_0+h,t_0+h+h_e] : \|y(t)-z(t)\|_{1,\infty,A} \le b_e\}$, where $\|y\|_{1,\infty,A} := \max_{t\in[t_0+h,\,t_0+h+h_e]}\|y(t)\|_1$ and $b_e = M_n h_e^{\alpha}/\Gamma(\alpha+1)$, $M_n := \max_{(t,x)\in\bar{D}_n}\|f(t,x)\|_1$. It can be shown as follows that A is closed and convex. Suppose $\{y_n\} \subset A$ and $\lim_{n\to\infty} y_n = y^*$, then $y^* \in C[t_0+h,t_0+h+h_e]$, due to the completeness of the space of continuous functions on $[t_0+h,t_0+h+h_e]$, equipped with the norm defined above. Moreover,

$$\|y^*(t)-z(t)\|_{1,\infty,A} \le \|y^*(t)-y_n(t)\|_{1,\infty,A} + \|y_n(t)-z(t)\|_{1,\infty,A} = \|y^*(t)-y_n(t)\|_{1,\infty,A} + b_e.$$

Take the limit as $n \to \infty$, then $\|y^*(t)-z(t)\|_{1,\infty,A} \le b_e$. Thus, $y^* \in A$. This proves that A is closed. Let $y_3(t) = \theta y_1(t) + (1-\theta)y_2(t)$, where $y_1, y_2 \in A$ and $\theta \in [0,1]$. Then

$$\|y_3(t)-z(t)\|_{1,\infty,A} = \|\theta[y_1(t)-z(t)] + (1-\theta)[y_2(t)-z(t)]\|_{1,\infty,A} \le \theta b_e + (1-\theta)b_e = b_e.$$

Thus, $y_3 \in A$, then A is convex. Moreover, for any $(t,y(t))$, where $t \in [t_0+h,t_0+h+h_e]$ and $y \in A$, $|t-t_0| \le h+h_e$ and

$$\begin{aligned}\|y(t) - \sum_{k=0}^{m}\frac{(t-t_0)^k}{k!}x_{0,k}\|_1 &= \|y(t)-z(t)+z(t) - \sum_{k=0}^{m}\frac{(t-t_0)^k}{k!}x_{0,k}\|_1 \\ &\le b_e + \frac{M}{\Gamma(\alpha)}\int_{t_0}^{t_0+h}(t-\tau)^{\alpha-1}d\tau \\ &\le b_e + \frac{M}{\Gamma(\alpha+1)}(h+h_e)^{\alpha}.\end{aligned}$$

As $h_e \to 0$, $|t-t_0| \le h \le a$ and $\|y(t)-\sum_{k=0}^{m}(t-t_0)^k x_{0,k}/k!\|_1 \le b$. Thus, for a sufficiently small h_e, $(t, y(t)) \in D_n \subset \bar{D}_n$, $n > N$, where $t \in [t_0+h, t_0+h+h_e]$ and $y \in A$.

We first show that for any $\hat{x} \in A$, $T\hat{x} \in A$. For any $t_0+h \le t_1 \le t_2 \le t_0+h+h_e$,

$$\begin{aligned}
&\|\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_0+h}(t_1-\tau)^{\alpha-1}f(\tau,x(\tau))d\tau - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_0+h}(t_2-\tau)^{\alpha-1}f(\tau,x(\tau))d\tau\|_1 \\
&= \frac{1}{\Gamma(\alpha)}\|\int_{t_0}^{t_0+h}[(t_1-\tau)^{\alpha-1}-(t_2-\tau)^{\alpha-1}]f(\tau,x(\tau))d\tau\|_1 \\
&\le \frac{M}{\Gamma(\alpha+1)}\begin{cases} (t_2-t_0-h)^\alpha-(t_1-t_0-h)^\alpha+(t_1-t_0)^\alpha-(t_2-t_0)^\alpha & \text{if } \alpha<1 \\ (t_1-t_0-h)^\alpha-(t_2-t_0-h)^\alpha+(t_2-t_0)^\alpha-(t_1-t_0)^\alpha & \text{if } \alpha>1 \end{cases},
\end{aligned} \tag{3.2}$$

and for all $\hat{x} \in A$,

$$\begin{aligned}
&\|\frac{1}{\Gamma(\alpha)}\int_{t_0+h}^{t_1}(t_1-\tau)^{\alpha-1}f(\tau,\hat{x}(\tau))d\tau - \frac{1}{\Gamma(\alpha)}\int_{t_0+h}^{t_2}(t_2-\tau)^{\alpha-1}f(\tau,\hat{x}(\tau))d\tau\|_1 \\
&= \frac{1}{\Gamma(\alpha)}\|\int_{t_0+h}^{t_1}[(t_1-\tau)^{\alpha-1}-(t_2-\tau)^{\alpha-1}]f(\tau,\hat{x}(\tau))d\tau - \int_{t_1}^{t_2}(t_2-\tau)^{\alpha-1}f(\tau,\hat{x}(\tau))d\tau\|_1 \\
&\le \frac{M_n}{\Gamma(\alpha)}[\int_{t_0+h}^{t_1}|(t_1-\tau)^{\alpha-1}-(t_2-\tau)^{\alpha-1}|d\tau + \int_{t_1}^{t_2}(t_2-\tau)^{\alpha-1}d\tau] \\
&= \frac{M_n}{\Gamma(\alpha+1)}\begin{cases} 2(t_2-t_1)^\alpha+(t_1-t_0-h)^\alpha-(t_2-t_0-h)^\alpha & \text{if } \alpha<1 \\ (t_2-t_0-h)^\alpha-(t_1-t_0-h)^\alpha & \text{if } \alpha>1 \end{cases}.
\end{aligned} \tag{3.3}$$

As we see, the two differences both converge to zero, as $t_1 \to t_2$. Thus, $z(t) \in C[t_0+h, t_0+h+h_e]$, and $(T\hat{x})(t) \in C[t_0+h, t_0+h+h_e]$. For all $\hat{x} \in A$,

$$\|(T\hat{x})(t)-z(t)\|_1 = \frac{1}{\Gamma(\alpha)}\|\int_{t_0+h}^{t}(t-\tau)^{\alpha-1}f(\tau,\hat{x}(\tau))d\tau\|_1 \le \frac{M_n}{\Gamma(\alpha)}\int_{t_0+h}^{t}(t-\tau)^{\alpha-1}d\tau \le \frac{M_n}{\Gamma(\alpha+1)}h_e^\alpha = b_e.$$

Thus, $(T\hat{x}) \in A$, for all $\hat{x} \in A$.

Second show that $T(A) := \{T\hat{x} : \hat{x} \in A\}$ is precompact. According to Arzelà-Ascoli Theorem, we need to prove that $T(A)$ is uniformly bounded and equicontinuous. For any $\hat{x} \in A$,

$$\|T\hat{x}(t)\|_1 \le \|z\|_{1,\infty,A} + \frac{1}{\Gamma(\alpha)}\int_{t_0+h}^{t}(t-\tau)^{\alpha-1}\|f(\tau,\hat{x}(\tau))\|_1 d\tau \le \|z\|_{1,\infty,A} + \frac{M_n}{\Gamma(\alpha+1)}h_e^\alpha.$$

Thus, the uniform boundedness is proven. It follows from (3.2) and (3.3) that for all $\hat{x} \in A$ and any $t_0+h \le t_1 \le t_2 \le t_0+h+h_e$,

$$\|(T\hat{x})(t_1)-(T\hat{x})(t_2)\|_1 \le \sum_{k=0}^{m}\frac{\|x_{0,k}\|_1}{k!}[(t_2-t_0)^k-(t_1-t_0)^k] + \frac{M_{0,n}}{\Gamma(\alpha+1)}\begin{cases} 2(t_2-t_1)^\alpha+(t_1-t_0)^\alpha-(t_2-t_0)^\alpha & \text{if } \alpha<1 \\ (t_2-t_0)^\alpha-(t_1-t_0)^\alpha & \text{if } \alpha>1 \end{cases}$$

$$\leq \|x_{0,1}\|_1(t_2-t_1)+\sum_{k=2}^{m}\frac{\|x_{0,k}\|_1}{k!}[(t_2-t_0)^k-(t_1-t_0)^k]+\frac{M_{0,n}}{\Gamma(\alpha+1)}\begin{cases}2(t_2-t_1)^\alpha & \text{if } \alpha<1\\ (t_2-t_0)^\alpha-(t_1-t_0)^\alpha & \text{if } \alpha>1\end{cases}$$

$$=\|x_{0,1}\|_1(t_2-t_1)+\sum_{k=2}^{m}\frac{\|x_{0,k}\|_1}{k!}k(\xi_k-t_0)^{k-1}(t_2-t_1)+\frac{M_{0,n}}{\Gamma(\alpha+1)}\begin{cases}2(t_2-t_1)^\alpha & \text{if } \alpha<1\\ \alpha(\xi_\alpha-t_0)^{\alpha-1}(t_2-t_1) & \text{if } \alpha>1\end{cases}$$

$$\leq \|x_{0,1}\|_1(t_2-t_1)+\sum_{k=2}^{m}\frac{\|x_{0,k}\|_1}{k!}k(h+h_e)^{k-1}(t_2-t_1)+\frac{M_{0,n}}{\Gamma(\alpha+1)}\begin{cases}2(t_2-t_1)^\alpha & \text{if } \alpha<1\\ \alpha(h+h_e)^{\alpha-1}(t_2-t_1) & \text{if } \alpha>1\end{cases},$$

where $M_{o,n} := \max\{M, M_n\}$, and $\xi_k, \xi_\alpha \in [t_2, t_1]$ are the "mean" points appearing in the application of the Mean Value Theorem to $(t-t_0)^k$, $(t-t_0)^\alpha$, respectively. Thus, for $|t_1-t_2|<\delta$,

$$\|(T\hat{x})(t_1)-(T\hat{x})(t_2)\|_1 \leq \|x_{0,1}\|_1\delta+\sum_{k=2}^{m}\frac{\|x_{0,k}\|_1}{k!}k(h+h_e)^{k-1}\delta+\frac{M_{0,n}}{\Gamma(\alpha+1)}\begin{cases}2\delta^\alpha & \text{if } \alpha<1\\ \alpha(h+h_e)^{\alpha-1}\delta & \text{if } \alpha>1\end{cases}.$$

This proves the equicontinuity.

According to the Schauder's Fixed Point Theorem, T has a fixed point $\hat{x}_* \in A$. Thus, for $t \in [t_0+h, t_0+h+h_e]$,

$$\hat{x}_*(t)=(T\hat{x}_*)(t)=z(t)+\frac{1}{\Gamma(\alpha)}\int_{t_0+h}^{t}(t-\tau)^{\alpha-1}f(\tau,\hat{x}_*(\tau))d\tau=\sum_{k=0}^{m}\frac{(t-t_0)^k}{k!}x_{0,k}+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_0+h}(t-\tau)^{\alpha-1}f(\tau,x(\tau))d\tau$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_0+h}^{t}(t-\tau)^{\alpha-1}f(\tau,\hat{x}_*(\tau))d\tau.$$

Since $|\int_{t_0+h}^{t}(t-\tau)^{\alpha-1}f(\tau,\hat{x}_*(\tau))d\tau| \leq M_n(t-t_0-h)^\alpha/\alpha$, $\int_{t_0+h}^{t}(t-\tau)^{\alpha-1}f(\tau,\hat{x}_*(\tau))d\tau=0$, as $t=t_0+h$. Thus, $\hat{x}_*(t_0+h)=x(t_0+h)$. Let $\tilde{x}(t)=\begin{cases}x(t) & \text{if } t\in[t_0,t_0+h]\\ \hat{x}_*(t) & \text{if } t\in[t_0+h,t_0+h+h_e]\end{cases}$, then $\tilde{x}(t)\in C[t_0,t_0+h+h_e]$ and

$$\tilde{x}(t)=\sum_{k=0}^{m}\frac{(t-t_0)^k}{k!}x_{0,k}+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}f(\tau,\tilde{x}(\tau))d\tau, t\in[t_0+h,t_0+h+h_e].$$

Thus,

$$\tilde{x}(t)=\sum_{k=0}^{m}\frac{(t-t_0)^k}{k!}x_{0,k}+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}f(\tau,\tilde{x}(\tau))d\tau, t\in[t_0,t_0+h+h_e].$$

According to Lemma 3.1.1, $\tilde{x}(t)$ is a solution to (3.1) on $[t_0, t_0+h+h_e]$. Therefore, $x(t)$ can be extended to $[t_0, t_0+h+h_e]$.

Since $\bar{D}_n$ is compact, we can continue this extension process finite times to get an extension of $x(t)$ to $[t_0, h_n]$ such that $(h_n, x(h_n)) \notin \bar{D}_n$. Similarly, for D_{n+1}, $n>N$, there exists h_{n+1} such that the solution has an extension to $[t_0, h_{n+1}]$ and $(h_{n+1}, x(h_{n+1})) \notin \bar{D}_{n+1}$. As we see, $\{h_n\}$ is a monotone increasing sequence. Let $\beta=\lim_{n\to\infty} h_n$, then $\beta \leq \infty$. Thus, $x(t)$ has been extended to $[t_0,\beta)$ and cannot be extended further, since the sequence $\{(h_n, x(h_n))\}$ is either unbounded or has a limit point on the boundary of D. Therefore, the solution can be extended over its maximal interval of existence $[t_0,\beta)$.

If $\beta = \infty$, then $(t, x(t))$ tends to the boundary of D, due to $t \to \infty$, as $t \to \beta^-$.

If $\beta < \infty$, we shall prove the theorem by contradiction. Suppose that $(t, x(t))$ does not tend to the boundary of D as $t \to \beta$. Then there exists an open bounded set U with $\bar{U} \subset D$ and a constant $\gamma \in [t_0, \beta)$ such that $(t, x(t)) \in U$, for all $t \in [\gamma, \beta)$. Moreover, there exists a closed and bounded set $\bar{V} \subset D$ such that $(t, x(t)) \in \bar{V}$, for all $t \in [t_0, \gamma]$. We shall first show that $\lim_{t\to\beta^-} x(t)$ exists, i.e. $\lim_{t\to\beta^-} x(t) = x(\beta^-)$. Let $M_{\bar{U},\bar{V}} = \max_{(t,x)\in\bar{U}\cup\bar{V}} \|f(t, x)\|_1$, then for any $t_0 \le t_1 \le t_2 < \beta$,

$$
\begin{aligned}
\|x(t_1) - x(t_2)\|_1 &= \|\sum_{k=0}^{m} \frac{(t_1 - t_0)^k}{k!} x_{0,k} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t - \tau)^{\alpha-1} f(\tau, x(\tau)) d\tau - \sum_{k=0}^{m} \frac{(t_2 - t_0)^k}{k!} x_{0,k} \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_2} (t - \tau)^{\alpha-1} f(\tau, x(\tau)) d\tau\|_1 \\
&\le \|x_{0,1}\|_1 (t_2 - t_1) + \sum_{k=2}^{m} \frac{\|x_{0,k}\|_1}{k!} k\beta^{k-1} (t_2 - t_1) + \frac{M_{\bar{U},\bar{V}}}{\Gamma(\alpha+1)} \begin{cases} 2(t_2 - t_1)^\alpha & \text{if } \alpha < 1 \\ \alpha\beta^{\alpha-1}(t_2 - t_1) & \text{if } \alpha > 1 \end{cases}.
\end{aligned}
$$

Thus, $x(t)$ is uniformly continuous on $[t_0, \beta)$. Thus, $\lim_{t\to\beta^-} x(t) = x(\beta^-)$ exists. Since $\bar{U}$ is closed, $(\beta, x(\beta^-)) \in \bar{U}$. Let $x(\beta) = x(\beta^-)$, then $x(t) \in C[t_0, \beta]$ and $f(t, x(t)) \in C[t_0, \beta]$ so that the integral $\int_{t_0}^{t} (t - \tau)^{\alpha-1} f(\tau, x(\tau)) d\tau \in C[t_0, \beta]$. For $t \in [t_0, \beta)$,

$$
x(t) = \sum_{k=0}^{m} \frac{(t - t_0)^k}{k!} x_{0,k} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - \tau)^{\alpha-1} f(\tau, x(\tau)) d\tau.
$$

Taking the limit as $t \to \beta^-$, then we know that $x(t)$ satisfies (3.1) at $t = \beta$. Thus, (3.1) has a solution $x(t) \in C[t_0, \beta]$. Let $\bar{W} = \bar{U} \cup \bar{V}$, then $\bar{W} \subset D$, and $(t, x(t)) \in \bar{W}$ for $t \in [t_0, \beta]$. As we did before, we can extend the solution to $[t_0, \beta + h_\epsilon]$ for some $h_\epsilon > 0$. This contradicts with the claim that $[t_0, \beta)$ is the maximal interval of existence. Therefore, in this case, $(t, x(t))$ also tends to the boundary of D as $t \to \beta^-$. The proof is complete. □

The following corollary gives some useful consequences of the continuation theorem.

Corollary 3.2.1. *Assume $f \in C(D, \mathbb{R}^n)$, where $D = [0, \infty) \times \mathbb{R}^n$. If $x(t)$ is a solution of (3.1) on a maximal interval of existence $J = [t_0, \beta)$, then*

i. either $\beta = \infty$ or $\lim_{t\to\beta^-} \sup \|x(t)\|_1 = \infty$;

ii. $\beta = \infty$, if for any $\gamma > t_0$, $x(t)$ is bounded on $J \cap [t_0, \gamma)$.

Proof. i. According to Theorem 3.2.3, as $t \to \beta^-$, $(t, x(t))$ tends to the boundary of $[0, \infty) \times \mathbb{R}^n$. Thus, either $\beta = \infty$ or $\lim_{t\to\beta^-} \sup \|x(t)\|_1 = \infty$.

ii. If $x(t)$ is bounded on $J \cap [t_0, \gamma)$, for any $\gamma > t_0$, then $\lim_{t\to\beta^-} \sup \|x(t)\|_1 < \infty$, for any $\beta < \infty$. It follows from i that $\beta = \infty$. □

3.3 Smoothness

The differential properties of local and global solutions to systems of Caputo fractional order differential equations are examined in this section. Suggesting that the solutions belong to a space of special smooth functions whose derivatives may not exist at initial time but grow describably nearby.

3.3.1 Preliminaries

Here the derivative formula for a composite function with a vector argument, and preliminaries about complete s-paces, nonempty and closed sets for the application of contraction mapping theorem, are introduced through three corresponding lemmas.

Lemma 3.3.1. *Assume that all the necessary derivatives are defined, then for any integer $i \geq 1$,*

$$\frac{d^i}{dt^i} f(t, x(t)) = \sum_0 \sum_1 \dots \sum_i \frac{i!}{\prod_{j=1}^{i} (j!)^{k_j} \prod_{j=1}^{i} \prod_{l=0}^{n} v_{jl}!} \begin{bmatrix} \frac{\partial^k}{\partial t^{u_0} \partial x_1{}^{u_1} \dots \partial x_n{}^{u_n}} f_1(t, x(t)) \\ \frac{\partial^k}{\partial t^{u_0} \partial x_1{}^{u_1} \dots \partial x_n{}^{u_n}} f_2(t, x(t)) \\ \vdots \\ \frac{\partial^k}{\partial t^{u_0} \partial x_1{}^{u_1} \dots \partial x_n{}^{u_n}} f_n(t, x(t)) \end{bmatrix} \prod_{j=1}^{i} [x_1^{(j)}(t)]^{v_{j1}} [x_2^{(j)}(t)]^{v_{j2}} \dots [x_n^{(j)}(t)]^{v_{jn}},$$

where $f(t, x(t)) = [f_1(t, x(t)), f_2(t, x(t)), ..., f_n(t, x(t))]^T$, $x(t) = [x_1(t), x_2(t), ..., x_n(t)]^T$; the respective sums are over all nonnegative integer solutions of the following Diophantine equations

$$\begin{aligned} &\sum_0 \to k_1 + 2k_2 + \dots + ik_i = i, \\ &\sum_1 \to v_{10} + v_{11} + \dots + v_{1n} = k_1, \\ &\sum_2 \to v_{20} + v_{21} + \dots + v_{2n} = k_2, \\ &\vdots \\ &\sum_i \to v_{i0} + v_{i1} + \dots + v_{in} = k_i; \end{aligned}$$

$v_{20} = v_{30} = \dots = v_{i0} = 0$; $u_l = v_{1l} + v_{2l} + \dots + v_{il}$, for $l = 0, 1, ..., n$; and $k = u_0 + u_1 + \dots + u_n = k_1 + k_2 + \dots + k_i$.

Proof. Consider t as the number zero element of the vector argument, then the conclusion follows straightforwardly from the unique theorem in [13]. □

Proposition 3.3.1. *In Lemma 3.3.1, $u_0, u_1, ..., u_n$ are all possible nonnegative integers such that $1 \leq u_0+u_1+\dots+u_n \leq i$, except, for $i \geq 2$, $u_0 = 1, 2, ..., i-1$, $u_1 = u_2 = \dots = u_n = 0$.*

Proof. Let $u_0+u_1+...+u_n < 1$, then $k = u_0+u_1+...+u_n = 0$ so that $k_j = 0$, $j = 1,2,...,i$, and $i = k_1+2k_2+...+ik_i = 0$, which is not true. Similarly, let $u_0 + u_1 + ... + u_n > i$, then $k_1 + k_2 + ... + k_i > i$ so that $k_1 + 2k_2 + ... + ik_i > i$, which is also not the case. Thus, $1 \leq u_0 + u_1 + ... + u_n \leq i$.

If $i = 1$, let $u_0 = \bar{u}_0$, $u_1 = \bar{u}_1$, ..., $u_n = \bar{u}_n$, where $\bar{u}_0, \bar{u}_1, ..., \bar{u}_n$ are any nonnegative integers such that $\bar{u}_0+\bar{u}_1+...+\bar{u}_n = 1$, then $k_1 = 1$ and other k's are 0. Thus, we can select a solution: $v_{10} = \bar{u}_0$, $v_{11} = \bar{u}_1$, ..., $v_{1n} = \bar{u}_n$, and other v's are 0.

If $i \geq 2$, let $u_0 = i$ and $u_1 = u_2 = ... = u_n = 0$, then we can select $v_{10} = i$ and other v's as 0. Let $u_0 = \bar{u}_0$, $u_1 = \bar{u}_1$, ..., $u_n = \bar{u}_n$, where $\bar{u}_0, \bar{u}_1, ..., \bar{u}_n$ are any nonnegative integers and not all the latter n ones are 0 such that $\bar{u}_0 + \bar{u}_1 + ... + \bar{u}_n = j$, $1 \leq j \leq i$ and $i \geq 2$. If $j = i$, then $k_1 = i$ and other k's are 0. Thus, we can select a solution: $v_{10} = \bar{u}_0$, $v_{11} = \bar{u}_1$, ..., $v_{1n} = \bar{u}_n$, and other v's are 0. If $j \leq i-1$, for $k_1 + 2k_2 + ... + ik_i = i$ and $k_1 + k_2 + ... + k_i = j$, there always exists a solution $k_1 = j-1$, $k_2 = 0$, ..., $k_{i-j} = 0$, $k_{i-(j-1)} = 1$, $k_{i-j+2} = 0$, ..., $k_i = 0$. In this solution, $k_1 + k_{i-(j-1)} = j > \bar{u}_0$ so that we can select $v_{10} = \bar{v}_{10} \leq k_1$ and $v_{[i-(j-1)]\,0} = \bar{v}_{[i-(j-1)]\,0} = 0$ such that $v_{10} + v_{[i-(j-1)]\,0} = \bar{u}_0$. Similarly, $k_1 - \bar{v}_{10} + k_{i-(j-1)} - \bar{v}_{[i-(j-1)]\,0} = j - \bar{u}_0 \geq \bar{u}_1$ so that we can select $v_{11} = \bar{v}_{11} \leq k_1 - \bar{v}_{10}$ and $v_{[i-(j-1)]\,1} = \bar{v}_{[i-(j-1)]\,1} \leq k_{i-(j-1)} - \bar{v}_{[i-(j-1)]\,0}$ such that $v_{11} + v_{[i-(j-1)]\,1} = \bar{u}_1$. Keep doing this, we know, $k_1 - \sum_{l=0}^{l=n-1} \bar{v}_{1l} + k_{i-(j-1)} - \sum_{l=0}^{l=n-1} \bar{v}_{[i-(j-1)]\,l} = j - \sum_{l=0}^{l=n-1} \bar{u}_l = \bar{u}_n$ so that we can select $v_{1n} = \bar{v}_{1n} = k_1 - \sum_{l=0}^{l=n-1} \bar{v}_{1l}$ and $v_{[i-(j-1)]\,n} = \bar{v}_{[i-(j-1)]\,n} = k_{i-(j-1)} - \sum_{l=0}^{l=n-1} \bar{v}_{[i-(j-1)]\,l}$ such that $v_{1n} + v_{[i-(j-1)]\,n} = \bar{u}_n$. Therefore, there always exist solutions, except, for $i \geq 2$, $u_0 = 1,2,...,i-1$, $u_1 = u_2 = ... = u_n = 0$.

For $i \geq 2$, let $u_0 = j$, $j = 1,2,...,i-1$, and $u_1 = u_2 = ... = u_n = 0$, then $v_{10} = j$ and other v's are 0 so that $i = k_1 + 2k_2 + ... + ik_i = k_1 = j$. This is not true. Thus, for the "except" cases, there is no solutions for the Diophantine equations. $\square$

Lemma 3.3.2. *$(C^{q,m,\nu}(t_1,t_2], \|\cdot\|_{1,q,m,\nu})$ is complete, where $C^{q,m,\nu}(t_1,t_2]$ ($q = \{m+1, m+2, ...\}$, $\nu \in [1-(\alpha-m), 1)$) is the set of functions x: $[t_1,t_2] \to \mathbb{R}^n$ which are m times continuously differentiable on $[t_1,t_2]$; q times continuously differentiable on $(t_1,t_2]$ and $\|x^{(i)}(t)\|_1 \leq c(t-t_1)^{1-\nu-(i-m)}$, $t \in (t_1,t_2]$, $i = m+1, m+2, ..., q$, and c is a positive constant; and $\|x\|_{1,q,m,\nu} = \|x\|_{1,\infty} + \sum_{i=1}^{m} \|x^{(i)}\|_{1,\infty} + \sum_{i=m+1}^{q} \sup_{t\in(t_1,t_2]} (t-t_1)^{\nu-1+(i-m)} \|x^{(i)}(t)\|_1$.*

Proof. Suppose that $\{x_u\}$ is an arbitrary Cauchy sequence in $(C^{q,m,\nu}(t_1,t_2], \|\cdot\|_{1,q,m,\nu})$, then for any $\epsilon > 0$, there exists $U > 0$ such that $u, v > U$ implies

$$\begin{aligned}\|x_u - x_v\|_{1,q,m,\nu} &= \|x_u - x_v\|_{1,\infty} + \sum_{i=1}^{m} \|x_u^{(i)} - x_v^{(i)}\|_{1,\infty} + \sum_{i=m+1}^{q} \sup_{t\in(t_1,t_2]} (t-t_1)^{\nu-1+(i-m)} \|x_u^{(i)}(t) - x_v^{(i)}(t)\|_1 \\ &= \max_{t\in[t_1,t_2]} \|x_u(t) - x_v(t)\|_1 + \sum_{i=1}^{m} \max_{t\in[t_1,t_2]} \|x_u^{(i)}(t) - x_v^{(i)}(t)\|_1 + \sum_{i=m+1}^{q} \sup_{t\in(t_1,t_2]} \|(t-t_1)^{\nu-1+(i-m)} [x_u^{(i)}(t) - x_v^{(i)}(t)]\|_1 \\ &< \epsilon.\end{aligned}$$

Now fix $t \in [t_1,t_2]$, then $\{x_u(t)\}$, $\{x_u^{(i)}(t)\}$ ($i = 1,2,...,m$) are both Cauchy sequences in $\mathbb{R}^n$. Similarly, fix $t \in (t_1,t_2]$, then $\{(t-t_1)^{\nu-1+(i-m)} x_u^{(i)}(t)\}$ ($i = m+1, m+2, ..., q$) is a Cauchy sequence in $\mathbb{R}^n$ as well. Thus, there exist $x(t)$, $x^{(i)}(t)$ and $(t-t_1)^{\nu-1+(i-m)} x^{(i)}(t)$ such that $x_u(t) \to x(t)$, $x_u^{(i)}(t) \to x^{(i)}(t)$ and $(t-t_1)^{\nu-1+(i-m)} x_u^{(i)}(t) \to (t-t_1)^{\nu-1+(i-m)} x^{(i)}(t)$ as $u \to \infty$. This well defines $x(t)$, $x^{(i)}(t)$ for $t \in [t_1,t_2]$, and $(t-t_1)^{\nu-1+(i-m)} x^{(i)}(t)$ for $t \in (t_1,t_2]$.

Let $v \to \infty$, then for all $t \in [t_1, t_2]$, $\|x_u(t)-x(t)\|_1 \le \epsilon$, $\|x_u^{(i)}(t)-x^{(i)}(t)\|_1 \le \epsilon$; for all $t \in (t_1, t_2]$, $\|(t-t_1)^{\nu-1+(i-m)}x_u^{(i)}(t)-(t-t_1)^{\nu-1+(i-m)}x^{(i)}(t)\|_1 \le \epsilon$. As we see, $\{x_u(t)\}$, $\{x_u^{(i)}(t)\}$ and $\{(t-t_1)^{\nu-1+(i-m)}x_u^{(i)}(t)\}$ converge uniformly to $x(t)$, $x^{(i)}(t)$ and $(t-t_1)^{\nu-1+(i-m)}x^{(i)}(t)$, respectively. Thus, $x(t)$, $x^{(i)}(t)$ are continuous on $[t_1, t_2]$, and $(t-t_1)^{\nu-1+(i-m)}x^{(i)}(t)$ is continuous on $(t_1, t_2]$. Moreover, for all $t \in (t_1, t_2]$,

$$\|(t-t_1)^{\nu-1+(i-m)}x^{(i)}(t)\|_1 \le \|(t-t_1)^{\nu-1+(i-m)}x^{(i)}(t)-(t-t_1)^{\nu-1+(i-m)}x_u^{(i)}(t)\|_1 + \|(t-t_1)^{\nu-1+(i-m)}x_u^{(i)}(t)\|_1 < \epsilon + c.$$

It follows from the arbitrariness of ϵ, $(t-t_1)^{\nu-1+(i-m)}\|x^{(i)}(t)\|_1 \le c$. Thus, $x(t)$ is m times continuously differentiable on $[t_1, t_2]$ and for $i = m+1, ..., q$, $\|x^{(i)}(t)\|_1 \le c(t-t_1)^{1-\nu-(i-m)}$, $t \in (t_1, t_2]$, i.e. $x(t) \in C^{q,m,\nu}(t_1, t_2]$. Obviously, for $u > U$, $\|x_u - x\|_{1,q,m,\nu} < (q+1)\epsilon$, which implies $\|x_u - x\|_{1,q,m,\nu} \to 0$ as $u \to \infty$. Therefore, $\{x_u\}$ converges to x. □

Lemma 3.3.3. *$B = \{x \in C^{q,m,\nu}(t_1, t_2] : \|x - \sum_{k=0}^{m}(t-t_0)^k x_{0,k}/k!\|_{1,\infty,B} \le b$ and $\|x\|_{1,q,m,\nu,B} \le c\}$ is nonempty and closed such that $(B, \|\cdot\|_{1,q,m,\nu,B})$ is nonempty and complete, where $\|x\|_{1,q,m,\nu,B} = (W+1)\|x\|_{1,\infty,B} + \sum_{i=1}^{m}\|x^{(i)}\|_{1,\infty,B} + \sum_{i=m+1}^{q}\sup_{t\in(t_1,t_2]}(t-t_1)^{\nu-1+(i-m)}\|x^{(i)}(t)\|_1$ (W is a positive constant), $t_0 \le t_1 < t_2 \le t_0 + h$, $c > (W+1)\sum_{k=0}^{m} h^k\|x_{0,k}\|_1/k! + \sum_{i=1}^{m}\sum_{k=i}^{m} h^{k-i}\|x_{0,k}\|_1/(k-i)!$, and $t_2 - t_1$ is sufficiently small.*

Proof. Let $x(t) = \sum_{k=0}^{m}(t-t_0)^k x_{0,k}/k! + \epsilon(t-t_1)^{1-\nu+m} + o[(t-t_1)^{1-\nu+m}]$, where ϵ is a sufficiently small constant, then $\|x(t) - \sum_{k=0}^{m}(t-t_0)^k x_{0,k}/k!\|_{1,\infty,B} \le |\epsilon|n(t_2-t_1)^{1-\nu+m} + o[(t_2-t_1)^{1-\nu+m}] \le b$ and

$$\begin{aligned}
\|x(t)\|_{1,q,m,\nu} &= (W+1)\|x(t)\|_{1,\infty,B} + \sum_{i=1}^{m}\|x^{(i)}(t)\|_{1,\infty,B} + \sum_{i=m+1}^{q}\sup_{t\in(t_1,t_2]}(t-t_1)^{\nu-1+(i-m)}\|x^{(i)}(t)\|_1 \\
&\le (W+1)\{\sum_{k=0}^{m}\frac{(t_2-t_0)^k}{k!}\|x_{0,k}\|_1 + |\epsilon|n(t_2-t_1)^{1-\nu+m} + o[(t_2-t_1)^{1-\nu+m}]\} \\
&\quad + \sum_{i=1}^{m}\sum_{k=1}^{m}\frac{(t_2-t_0)^{k-i}}{(k-i)!}\|x_{0,k}\|_1 + |\epsilon|n(1-\nu+m)...(1-\nu+m-i+1)(t_2-t_1)^{1-\nu+m-i} + o[(t_2-t_1)^{1-\nu+m-i}] \\
&\quad + \sum_{i=m+1}^{q}\sup_{t\in(t_1,t_2]}(t-t_1)^{\nu-1+(i-m)}\{|\epsilon|n|(1-\nu+m)...(1-\nu+m-i+1)|(t-t_1)^{1-\nu+m-i} + o[(t-t_1)^{1-\nu+m-i}]\} \\
&\le (W+1)\{\sum_{k=0}^{m}\frac{h^k}{k!}\|x_{0,k}\|_1 + |\epsilon|n(t_2-t_1)^{1-\nu+m} + o[(t_2-t_1)^{1-\nu+m}]\} \\
&\quad + \sum_{i=1}^{m}\sum_{k=1}^{m}\frac{h^{k-i}}{(k-i)!}\|x_{0,k}\|_1 + |\epsilon|n(1-\nu+m)...(1-\nu+m-i+1)(t_2-t_1)^{1-\nu+m-i} + o[(t_2-t_1)^{1-\nu+m-i}] \\
&\quad + \sum_{i=m+1}^{q}|\epsilon|n|(1-\nu+m)...(1-\nu+m-i+1)| + \sum_{i=m+1}^{q}\sup_{t\in(t_1,t_2]}(t-t_1)^{\nu-1+(i-m)}o[(t-t_1)^{1-\nu+m-i}] \\
&\le c.
\end{aligned}$$

Thus, $x(t) \in B$ and B is nonempty.

According to Lemma 3.3.2, $(C^{q,m,\nu}(t_1,t_2],\|\cdot\|_{1,q,m,\nu,B})$ is complete, due to $\|x\|_{1,q,m,\nu} \le \|x\|_{1,q,m,\nu,B} \le (W+1)\|x\|_{1,q,m,\nu}$. Suppose that $\{x_u\} \subset \{x \in C^{q,m,\nu}(t_1,t_2] : \|x - \sum_{k=0}^{m}(t-t_0)^k x_{0,k}/k!\|_{1,\infty,B} \le b\}$ and $\lim_{u\to\infty} x_u = x$, then $\{x_u\} \subset C^{q,m,\nu}(t_1,t_2]$ and $\lim_{u\to\infty}\|x_u - x\|_{1,q,m,\nu,B} = 0$. Thus, $x \in C^{q,m,\nu}(t_1,t_2]$ and $\lim_{u\to\infty}\|x_u - x\|_{1,\infty,B} = 0$. For any u, $\|x - \sum_{k=0}^{m}(t-t_0)^k x_{0,k} /k!\|_{1,\infty,B} \le \|x - x_u\|_{1,\infty,B} + \|x_u - \sum_{k=0}^{m}(t-t_0)^k x_{0,k}/k!\|_{1,\infty,B}$. Then $\|x - \sum_{k=0}^{m}(t-t_0)^k x_{0,k}/k!\|_{1,\infty,B} \le \lim_{u\to\infty}\|x - x_u\|_{1,\infty,B} + \lim_{u\to\infty}\|x_u - \sum_{k=0}^{m}(t-t_0)^k x_{0,k}/k!\|_{1,\infty,B} \le b$. Thus, the limit of sequence remains in the set so that $\{x \in C^{q,m,\nu}(t_1,t_2] : \|x - \sum_{k=0}^{m}(t-t_0)^k x_{0,k}/k!\|_{1,\infty,B} \le b\}$ is closed. Since $\{x \in C^{q,m,\nu}(t_1,t_2] : \|x\|_{1,q,m,\nu,B} \le c\}$ is also closed, B is a closed set in $(C^{q,m,\nu}(t_1,t_2],\|\cdot\|_{1,q,m,\nu,B})$. Therefore, $(B,\|\cdot\|_{1,q,m,\nu,B})$ is complete. □

3.3.2 Local Smoothness

The smoothness property of local solutions is focused in this subsection. The following main theorem for local smoothness is the combination of Theorem 3.3.2 ($0 < \alpha < 1$) and Theorem 3.3.3 ($\alpha > 1$).

Theorem 3.3.1. *Assume that f is continuous in t and x on $\bar{S}$, and $q-m$ times continuously differentiable with respect to t and x on $S = \{(t,x) : t \in (t_0, t_0+a], \|x - \sum_{k=0}^{m}(t-t_0)^k x_{0,k}/k!\|_1 \le b\}$, and there exist constants $\nu \in [1-(\alpha-m), 1)$, M_d and L_d such that, for any $(t,x) \in S$ and all possible nonnegative integers $u_0, u_1, ..., u_n$ with $1 \le u_0+u_1+...+u_n \le q-m$,*

$$\|\frac{\partial^{u_0+u_1+...+u_n}}{\partial t^{u_0}\partial {x_1}^{u_1}...\partial {x_n}^{u_n}} f(t,x)\|_1 \le M_d \begin{cases} 1 & \text{if } u_0 = 0 \\ (t-t_0)^{1-\nu-u_0} & \text{if } u_0 \ge 1 \end{cases}; \tag{3.4}$$

and for any (t,x), $(t,y) \in S$ and those $u_0, u_1, ..., u_n$ with $u_0 + u_1 + ... + u_n = q-m$,

$$\|\frac{\partial^{q-m}}{\partial t^{u_0}\partial {x_1}^{u_1}...\partial {x_n}^{u_n}} f(t,x) - \frac{\partial^{q-m}}{\partial t^{u_0}\partial {y_1}^{u_1}...\partial {y_n}^{u_n}} f(t,y)\|_1 \le L_d \|x-y\|_1 \begin{cases} 1 & \text{if } u_0 = 0 \\ (t-t_0)^{1-\nu-u_0} & \text{if } u_0 \ge 1 \end{cases}. \tag{3.5}$$

Then (3.1) has a unique solution $x(t) \in C^{q,m,\nu}(t_0, t_0+h]$.

Remark 3.3.1. *If $q = m$, i.e. only the continuity of f on $\bar{S}$ is assumed, then it follows from the part of proof of Theorem 3.3.3 before (3.19) that (3.1) has a solution $x(t) \in C^m[t_0, t_0+h]$.*

Remark 3.3.2. *It follows from the part of proof below (3.13) that (3.5) is satisfied if for any (t,x), $(t,y) \in S$, $l = 1, 2, ..., n$ and those $u_0, u_1, ..., u_n$ with $u_0 + u_1 + ... + u_n = q-m$,*

$$\|\frac{\partial^{q-m+1}}{\partial t^{u_0}\partial {x_1}^{u_1}...\partial {x_l}^{u_l+1}...\partial {x_n}^{u_n}} f(t,x)\|_1 \le \frac{1}{n} L_d \begin{cases} 1 & \text{if } u_0 = 0 \\ (t-t_0)^{1-\nu-u_0} & \text{if } u_0 \ge 1 \end{cases}.$$

The "if" part in (3.4) and (3.5) can be removed as in the following corollary. Since involving parts of the proofs for Theorem 3.3.2 and 3.3.3, the proof for this corollary will be given at the end of this subsection.

Corollary 3.3.1. *Assume that f is $q-m$ times continuously differentiable on $\bar{S}$, and there exists a constant L_d such that, for any $(t,x), (t,y) \in \bar{S}$ and all possible nonnegative integers $u_0, u_1, ..., u_n$ with $u_0 + u_1 + ... + u_n = q-m$,*

$$\|\frac{\partial^{q-m}}{\partial t^{u_0}\partial {x_1}^{u_1}...\partial {x_n}^{u_n}} f(t,x) - \frac{\partial^{q-m}}{\partial t^{u_0}\partial {y_1}^{u_1}...\partial {y_n}^{u_n}} f(t,y)\|_1 \le L_d \|x-y\|_1. \tag{3.6}$$

Then (3.1) has a unique solution $x(t) \in C^{q,m,1-(\alpha-m)}(t_0, t_0+h]$.

Corollary 3.3.2. *Assume that f is $q - m + 1$ times continuously differentiable on $\bar{S}$. Then (3.1) has a unique solution $x(t) \in C^{q,m,1-(\alpha-m)}(t_0, t_0 + h]$.*

Proof. It follows straightforwardly from Corollary 3.3.1. □

Now we state the local smoothness theorem for $0 < \alpha < 1$. It will be proven by using the contraction mapping theorem.

Theorem 3.3.2. *Let $0 < \alpha < 1$. Assume that f is continuous in t and x on $\bar{S}$, and q times continuously differentiable with respect to t and x on $S = \{(t, x) : t \in (t_0, t_0 + a], \|x - x_{0,0}\|_1 \leq b\}$, and there exist constants $\nu \in [1 - \alpha, 1)$, M_d and L_d such that, for any $(t, x) \in S$ and all possible nonnegative integers $u_0, u_1, ..., u_n$ with $1 \leq u_0 + u_1 + ... + u_n \leq q$,*

$$\left\|\frac{\partial^{u_0+u_1+...+u_n}}{\partial t^{u_0}\partial x_1{}^{u_1}...\partial x_n{}^{u_n}} f(t, x)\right\|_1 \leq M_d \begin{cases} 1 & \text{if } u_0 = 0 \\ (t - t_0)^{1-\nu-u_0} & \text{if } u_0 \geq 1 \end{cases}; \tag{3.7}$$

and for any (t, x), $(t, y) \in S$ and those $u_0, u_1, ..., u_n$ with $u_0 + u_1 + ... + u_n = q$,

$$\left\|\frac{\partial^q}{\partial t^{u_0}\partial x_1{}^{u_1}...\partial x_n{}^{u_n}} f(t, x) - \frac{\partial^q}{\partial t^{u_0}\partial y_1{}^{u_1}...\partial y_n{}^{u_n}} f(t, y)\right\|_1 \leq L_d \|x - y\|_1 \begin{cases} 1 & \text{if } u_0 = 0 \\ (t - t_0)^{1-\nu-u_0} & \text{if } u_0 \geq 1 \end{cases}. \tag{3.8}$$

Then (3.1) has a unique solution $x(t) \in C^{q,\nu}(t_0, t_0 + h]$.

Proof. Since f is continuous on $\bar{S}$, according to Theorem 3.1.1, (3.1) has a continuous solution on $[t_0, t_0 + h]$. Let $x_*(t)$ denote this solution, then $x_*(t) \in C[t_0, t_0 + h]$, and according to Lemma 3.1.1,

$$x_*(t) = x_{0,0} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - \tau)^{\alpha-1} f(\tau, x_*(\tau)) d\tau, \ t \in [t_0, t_0 + h].$$

Referring to [14], we fix two arbitrary different points in $[t_0, t_0 + h]$ and let t_2 denote the larger one, t_1 the other one, then $t_0 \leq t_1 < t_2 \leq t_0 + h$. Consider the following integral equation

$$x(t) = (Tx)(t) + z(t), \ t \in (t_1, t_2], \tag{3.9}$$

where

$$(Tx)(t) = \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t - \tau)^{\alpha-1} f(\tau, x(\tau)) d\tau, \ t \in (t_1, t_2],$$

and

$$z(t) = x_{0,0} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t - \tau)^{\alpha-1} f(\tau, x_*(\tau)) d\tau, \ t \in (t_1, t_2].$$

As observed, $x_*(t)$, $t \in (t_1, t_2]$ is a solution to (3.9). We shall show that (3.9) is uniquely solvable on $(t_1, t_2]$ and the solution is in $C^{q,\nu}(t_1, t_2]$. Since t_1, t_2 are arbitrary, as we shall see, this finally proves $x_*(t) \in C^{q,\nu}(t_0, t_0 + h]$.

Another important observation is $z(t) \in C^{q,v}(t_1, t_2]$. For $t_1 \leq s_1 \leq s_2 \leq t_2$,

$$\begin{aligned}
\|z(s_1) - z(s_2)\|_1 &= \frac{1}{\Gamma(\alpha)}\| \int_{t_0}^{t_1} (s_1-\tau)^{\alpha-1} f(\tau, x_*(\tau))d\tau - \int_{t_0}^{t_1} (s_2-\tau)^{\alpha-1} f(\tau, x_*(\tau))d\tau\|_1 \\
&= \frac{1}{\Gamma(\alpha)}\| \int_{t_0}^{t_1} [(s_1-\tau)^{\alpha-1} - (s_2-\tau)^{\alpha-1}] f(\tau, x_*(\tau))d\tau\|_1 \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} |(s_1-\tau)^{\alpha-1} - (s_2-\tau)^{\alpha-1}| \, \|f(\tau, x_*(\tau))\|_1 d\tau \\
&\leq \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t_1} (s_1-\tau)^{\alpha-1} - (s_2-\tau)^{\alpha-1} d\tau \\
&= \frac{M}{\Gamma(\alpha)\alpha}[(s_2-t_1)^\alpha - (s_1-t_1)^\alpha + (s_1-t_0)^\alpha - (s_2-t_0)^\alpha].
\end{aligned}$$

As $s_1 \to s_2$, $\|z(s_1) - z(s_2)\|_1 \to 0$, which implies $z(t) \in C[t_1, t_2]$. In $z(t)$, $t \neq \tau$ and $f(\tau, x_*(\tau))$ is continuous so that $(t-\tau)^{\alpha-1} f(\tau, x_*(\tau))$ is continuous and its partial derivatives with respect to t are also continuous. Thus, we have

$$z^{(i)}(t) = \frac{(\alpha-1)...(\alpha-i)}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t-\tau)^{\alpha-1-i} f(\tau, x_*(\tau))d\tau, \; t \in (t_1, t_2], \; i = 1, 2, ..., q,$$

which is continuous. Moreover, for sufficiently small $t_2 - t_1$ such that $t - t_1 < 1$, we estimate

$$\begin{aligned}
\|z^{(i)}(t)\|_1 &\leq |\frac{(\alpha-1)...(\alpha-i)}{\Gamma(\alpha)}| \int_{t_0}^{t_1} (t-\tau)^{\alpha-1-i} \|f(\tau, x_*(\tau))\|_1 d\tau \\
&\leq |\frac{(\alpha-1)...(\alpha-i)}{\Gamma(\alpha)}| M \int_{t_0}^{t_1} (t-\tau)^{\alpha-1-i} d\tau \\
&= |\frac{(\alpha-1)...(\alpha-i)}{\Gamma(\alpha)}| M \frac{-1}{\alpha-i}[(t-t_1)^{\alpha-i} - (t-t_0)^{\alpha-i}] \\
&\leq |\frac{(\alpha-1)...(\alpha-i+1)}{\Gamma(\alpha)}| M(t-t_1)^{1-\nu-i}, \; t \in (t_1, t_2], \; i = 1, 2, ..., q.
\end{aligned}$$

Therefore, $z(t) \in C^{q,\nu}(t_1, t_2]$.

Define $(Sx)(t) = (Tx)(t) + z(t)$, $t \in (t_1, t_2]$. We shall show that S maps the closed ball $B = \{x \in C^{q,\nu}(t_1, t_2] : \|x - x_{0,0}\|_{1,\infty,B} \leq b$ and $\|x\|_{1,q,\nu,B} \leq c\}$, where, referring to [15], $\|x\|_{1,q,\nu,B} = (W+1)\|x\|_{1,\infty,B} + \sum_{i=1}^{q} \sup_{t\in(t_1,t_2]}(t-t_1)^{\nu-1+i} \times \|x^{(i)}(t)\|_1$, $W > \max\{\sum_{i=1}^{q} c_{T,i}, \sum_{i=1}^{q} c_{S,i,1,1}\}$ and $c > c_z + W$, into itself. Here c_z denotes the right hand side (constant) of the following inequality

$$\|z\|_{1,q,\nu,B} \leq (W+1)[\|x_{0,0}\|_1 + \frac{Mh^\alpha}{\Gamma(\alpha+1)}] + \sum_{i=1}^{q} \frac{|(\alpha-1)...(\alpha-i+1)|M}{\Gamma(\alpha)}.$$

According to Lemma 3.3.3, B equipped with $\|\cdot\|_{1,q,\nu,B}$ is nonempty and closed such that it is complete.

For any $x \in B$, $t_1 \le s_1 \le s_2 \le t_2$,

$$\begin{aligned}
\|(Tx)(s_1) - (Tx)(s_2)\|_1 &= \frac{1}{\Gamma(\alpha)}\| \int_{t_1}^{s_1} (s_1-\tau)^{\alpha-1} f(\tau, x(\tau))d\tau - \int_{t_1}^{s_2} (s_2-\tau)^{\alpha-1} f(\tau, x(\tau))d\tau\|_1 \\
&= \frac{1}{\Gamma(\alpha)}\| \int_{t_1}^{s_1} [(s_1-\tau)^{\alpha-1} - (s_2-\tau)^{\alpha-1}] f(\tau, x(\tau))d\tau - \int_{s_1}^{s_2} (s_2-\tau)^{\alpha-1} f(\tau, x(\tau))d\tau\|_1 \\
&\le \frac{1}{\Gamma(\alpha)}\{\| \int_{t_1}^{s_1} [(s_1-\tau)^{\alpha-1} - (s_2-\tau)^{\alpha-1}] f(\tau, x(\tau))d\tau\|_1 + \| \int_{s_1}^{s_2} (s_2-\tau)^{\alpha-1} f(\tau, x(\tau))d\tau\|_1\} \\
&\le \frac{1}{\Gamma(\alpha)}\{\int_{t_1}^{s_1} [(s_1-\tau)^{\alpha-1} - (s_2-\tau)^{\alpha-1}] \|f(\tau, x(\tau))\|_1 d\tau + \int_{s_1}^{s_2} (s_2-\tau)^{\alpha-1} \|f(\tau, x(\tau))\|_1 d\tau\} \\
&\le \frac{M}{\Gamma(\alpha)}[\int_{t_1}^{s_1} (s_1-\tau)^{\alpha-1} - (s_2-\tau)^{\alpha-1} d\tau + \int_{s_1}^{s_2} (s_2-\tau)^{\alpha-1} d\tau] \\
&= \frac{M}{\Gamma(\alpha+1)}[2(s_2-s_1)^{\alpha} + (s_1-t_1)^{\alpha} - (s_2-t_1)^{\alpha}].
\end{aligned}$$

As $s_1 \to s_2$, $\|(Tx)(s_1) - (Tx)(s_2)\|_1 \to 0$, which implies $(Tx)(t) \in C[t_1, t_2]$. This together with $z(t) \in C[t_1, t_2]$ suffices $(Sx)(t) \in C[t_1, t_2]$. Moreover, for any $x \in B$,

$$\begin{aligned}
\|(Sx)(t) - x_{0,0}\|_{1,\infty,B} &= \frac{1}{\Gamma(\alpha)} \max_{t\in[t_1,t_2]} \| \int_{t_1}^{t} (t-\tau)^{\alpha-1} f(\tau, x(\tau))d\tau + \int_{t_0}^{t_1} (t-\tau)^{\alpha-1} f(\tau, x_*(\tau))d\tau\|_1 \\
&\le \frac{1}{\Gamma(\alpha)} \max_{t\in[t_1,t_2]} [\int_{t_1}^{t} (t-\tau)^{\alpha-1} \|f(\tau, x(\tau))\|_1 d\tau + \int_{t_0}^{t_1} (t-\tau)^{\alpha-1} \|f(\tau, x_*(\tau))\|_1 d\tau] \\
&\le \frac{M}{\Gamma(\alpha)} \max_{t\in[t_1,t_2]} [\int_{t_1}^{t} (t-\tau)^{\alpha-1} d\tau + \int_{t_0}^{t_1} (t-\tau)^{\alpha-1} d\tau] \\
&= \frac{M}{\Gamma(\alpha)} \max_{t\in[t_1,t_2]} \int_{t_0}^{t} (t-\tau)^{\alpha-1} d\tau \\
&\le \frac{M}{\Gamma(\alpha)} \max_{t\in[t_1,t_2]} \frac{1}{\alpha}(t-t_0)^{\alpha} \\
&\le \frac{M}{\Gamma(\alpha+1)} h^{\alpha} \\
&\le b,\ t \in [t_1, t_2].
\end{aligned}$$

We need to further show that for any $x \in B$, $(Sx)(t) \in C^{q,\nu}(t_1, t_2]$ and $\|(Sx)(t)\|_{1,q,\nu,B} \le c$. In order to differentiate $(Tx)(t)$, we referring to [14], let $s(t) = t_1 + (t - t_1)/2$, then rewrite $(Tx)(t)$ as

$$(Tx)(t) = (T_1x)(t) + (T_2x)(t),$$

where

$$(T_1x)(t) = \frac{1}{\Gamma(\alpha)} \int_{t_1}^{s(t)} (t-\tau)^{\alpha-1} f(\tau, x(\tau))d\tau,\ t \in (t_1, t_2],$$

and

$$(T_2x)(t) = \frac{1}{\Gamma(\alpha)} \int_{s(t)}^{t} (t-\tau)^{\alpha-1} f(\tau, x(\tau)) d\tau,\ t \in (t_1, t_2].$$

In $(T_1x)(t)$, $t \neq \tau$ due to $t > s(t)$ so that $(t-\tau)^{\alpha-1} f(\tau, x(\tau))$ is continuous and its partial derivatives with respect to t are also continuous. Thus, for $t \in (t_1, t_2]$,

$$\begin{aligned}(T_1x)'(t) &= \frac{\alpha-1}{\Gamma(\alpha)} \int_{t_1}^{s(t)} (t-\tau)^{\alpha-2} f(\tau, (x(\tau)) d\tau + \frac{1}{\Gamma(\alpha)2^{\alpha}} (t-t_1)^{\alpha-1} f(\tau, x(\tau))|_{\tau=s(t)} \\ &= c_1 \int_{t_1}^{s(t)} (t-\tau)^{\alpha-1-1} f(\tau, (x(\tau)) d\tau + c_{10} (t-t_1)^{\alpha-1+0} f(\tau, x(\tau))|_{\tau=s(t)},\end{aligned}$$

and

$$\begin{aligned}(T_1x)''(t) &= c_1(\alpha-2) \int_{t_1}^{s(t)} (t-\tau)^{\alpha-3} f(\tau, (x(\tau)) d\tau + c_1 \frac{1}{2^{\alpha-1}} (t-t_1)^{\alpha-2} f(\tau, x(\tau))|_{\tau=s(t)} \\ &\quad + c_{10}(\alpha-1)(t-t_1)^{\alpha-2} f(\tau, x(\tau))|_{\tau=s(t)} + c_{10} \frac{1}{2} (t-t_1)^{\alpha-1} \left.\frac{df(\tau, x(\tau))}{d\tau}\right|_{\tau=s(t)} \\ &= c_2 \int_{t_1}^{s(t)} (t-\tau)^{\alpha-2-1} f(\tau, (x(\tau)) d\tau + c_{20} (t-t_1)^{\alpha-2+0} f(\tau, x(\tau))|_{\tau=s(t)} + c_{21} (t-t_1)^{\alpha-2+1} \left.\frac{df(\tau, x(\tau))}{d\tau}\right|_{\tau=s(t)} \\ &= c_2 \int_{t_1}^{s(t)} (t-\tau)^{\alpha-2-1} f(\tau, (x(\tau)) d\tau + \sum_{j=0}^{2-1} c_{2j} (t-t_1)^{\alpha-2+j} \frac{d^j}{d\tau^j} f(\tau, x(\tau))|_{\tau=s(t)}.\end{aligned}$$

Let $(T_1x)^{(i-1)}(t) = c_{i-1} \int_{t_1}^{s(t)} (t-\tau)^{\alpha-(i-1)-1} f(\tau, (x(\tau)) d\tau + \sum_{j=0}^{(i-1)-1} c_{(i-1)j} (t-t_1)^{\alpha-(i-1)+j} \frac{d^j}{d\tau^j} f(\tau, x(\tau))|_{\tau=s(t)}$, $i \geq 2$, then

$$\begin{aligned}(T_1x)^{(i)}(t) &= c_{i-1}(\alpha-i) \int_{t_1}^{s(t)} (t-\tau)^{\alpha-i-1} f(\tau, (x(\tau)) d\tau + c_{i-1} \frac{1}{2^{\alpha-(i-1)}} (t-t_1)^{\alpha-i} f(\tau, x(\tau))|_{\tau=s(t)} \\ &\quad + \sum_{j=0}^{(i-1)-1} c_{(i-1)j} [\alpha-(i-1)+j] (t-t_1)^{\alpha-i+j} \frac{d^j}{d\tau^j} f(\tau, x(\tau))|_{\tau=s(t)} \\ &\quad + \sum_{j=0}^{(i-1)-1} c_{(i-1)j} \frac{1}{2} (t-t_1)^{\alpha-i+(j+1)} \frac{d^{j+1}}{d\tau^{j+1}} f(\tau, x(\tau))|_{\tau=s(t)} \\ &= c_{i-1}(\alpha-i) \int_{t_1}^{s(t)} (t-\tau)^{\alpha-i-1} f(\tau, (x(\tau)) d\tau + c_{i-1} \frac{1}{2^{\alpha-(i-1)}} (t-t_1)^{\alpha-i} f(\tau, x(\tau))|_{\tau=s(t)} \\ &\quad + \sum_{j=0}^{(i-1)-1} c_{(i-1)j} [\alpha-(i-1)+j] (t-t_1)^{\alpha-i+j} \frac{d^j}{d\tau^j} f(\tau, x(\tau))|_{\tau=s(t)} \\ &\quad + c_{(i-1)(i-2)} \frac{1}{2} (t-t_1)^{\alpha-i+(i-1)} \frac{d^{i-1}}{d\tau^{i-1}} f(\tau, x(\tau))|_{\tau=s(t)} + \sum_{j=0}^{(i-1)-2} c_{(i-1)j} \frac{1}{2} (t-t_1)^{\alpha-i+(j+1)} \frac{d^{j+1}}{d\tau^{j+1}} f(\tau, x(\tau))|_{\tau=s(t)}\end{aligned}$$

$$= c_{i-1}(\alpha - i)\int_{t_1}^{s(t)} (t-\tau)^{\alpha-i-1} f(\tau,(x(\tau))d\tau + \sum_{j=0}^{(i-1)-1} c_{(i-1)j}[\alpha-(i-1)+j](t-t_1)^{\alpha-i+j}\frac{d^j}{d\tau^j}f(\tau,x(\tau))|_{\tau=s(t)}$$

$$+\sum_{k=0}^{(i-1)-1}\left\{\begin{array}{l} c_{i-1}\frac{1}{2^{\alpha-(i-1)}},\ \ k=0 \\ c_{(i-1)(k-1)}\frac{1}{2},\ k>0 \end{array}\right. \times (t-t_1)^{\alpha-i+k}\frac{d^k}{d\tau^k}f(\tau,x(\tau))|_{\tau=s(t)} + c_{(i-1)(i-2)}\frac{1}{2}(t-t_1)^{\alpha-i+(i-1)}\frac{d^{i-1}}{d\tau^{i-1}}f(\tau,x(\tau))|_{\tau=s(t)}$$

$$= c_i\int_{t_1}^{s(t)}(t-\tau)^{\alpha-i-1}f(\tau,(x(\tau))d\tau + \sum_{j=0}^{i-1}c_{ij}(t-t_1)^{\alpha-i+j}\frac{d^j}{d\tau^j}f(\tau,x(\tau))|_{\tau=s(t)}.$$

Therefore, we derive

$$(T_1x)^{(i)}(t) = c_i\int_{t_1}^{s(t)}(t-\tau)^{\alpha-i-1}f(\tau,(x(\tau))d\tau + \sum_{j=0}^{i-1}c_{ij}(t-t_1)^{\alpha-i+j}\frac{d^j}{d\tau^j}f(\tau,x(\tau))|_{\tau=s(t)}, t\in(t_1,t_2], i=1,2,...,q. \quad (3.10)$$

Since $x \in B$ (so that $(\tau, x(\tau)) \in S$,$\tau \in (t_1, t_2]$) and $f(t, x)$ is q times continuously differentiable on S, according to Lemma 3.3.1, $d^j f(\tau, x(\tau))/d\tau^j|_{\tau=s(t)}$, $t \in (t_1, t_2]$, $j = 1, 2, ..., q-1$, is always continuous. Moreover, $(t-\tau)^{\alpha-i-1} f(\tau, (x(\tau))$ is continuous due to $t > s(t)$ implied by $t > t_1$. Thus, $(T_1x)^{(i)}(t) \in C(t_1, t_2]$, $i = 1, 2, ..., q$.

As we see, $(T_2x)(t)$ is an improper integral so that we can not directly differentiate it. By using the smoothness of f, we first rewrite it as

$$\begin{aligned}(T_2x)(t) &= \frac{1}{\Gamma(\alpha)}\int_{s(t)}^{t}(t-\tau)^{\alpha-1}f(\tau,x(\tau))d\tau = \frac{-1}{\Gamma(\alpha)\alpha}\int_{s(t)}^{t}f(\tau,x(\tau))d(t-\tau)^{\alpha}\\ &= \frac{-1}{\Gamma(\alpha)\alpha}[f(\tau,x(\tau))(t-\tau)^{\alpha}\Big|_{\tau=s(t)}^{\tau=t} - \int_{s(t)}^{t}(t-\tau)^{\alpha}\frac{df(\tau,x(\tau))}{d\tau}d\tau]\\ &= \frac{1}{\Gamma(\alpha)\alpha}\int_{s(t)}^{t}(t-\tau)^{\alpha}\frac{df(\tau,x(\tau))}{d\tau}d\tau + \frac{1}{\Gamma(\alpha)\alpha 2^{\alpha}}(t-t_1)^{\alpha}f(\tau,x(\tau))|_{\tau=s(t)},\ t\in(t_1,t_2].\end{aligned}$$

For any $t \in (t_1, t_2]$, there exists t_* such that $t_1 < t_* < s(t) < t$, then

$$\begin{aligned}\int_{s(t)}^{t}(t-\tau)^{\alpha}\frac{df(\tau,x(\tau))}{d\tau}d\tau &= \int_{t_*}^{t}(t-\tau)^{\alpha}\frac{df(\tau,x(\tau))}{d\tau}d\tau - \int_{t_*}^{s(t)}(t-\tau)^{\alpha}\frac{df(\tau,x(\tau))}{d\tau}d\tau\\ &= \Gamma(\alpha+1)\,{}_{t_*}\mathscr{D}_t^{-(\alpha+1)}[\frac{df(t,x(t))}{dt}] - \int_{t_*}^{s(t)}(t-\tau)^{\alpha}\frac{df(\tau,x(\tau))}{d\tau}d\tau.\end{aligned}$$

Since $x \in B$ and $f(t, x)$ is q times continosly differentiable on S, $d^if(t, x(t))/dt^i$ is continuous on $[t_*, t_2]$. According to Theorem 2.1.1 and (1.1) in [1],

$$\frac{d}{dt}\,{}_{t_*}\mathscr{D}_t^{-(\alpha+1)}[\frac{df(t,x(t))}{dt}] = \frac{d}{dt}\,{}_{t_*}\mathscr{D}_t^{-1}\,{}_{t_*}\mathscr{D}_t^{-\alpha}[\frac{df(t,x(t))}{dt}] = {}_{t_*}\mathscr{D}_t^{-\alpha}[\frac{df(t,x(t))}{dt}] = \frac{1}{\Gamma(\alpha)}\int_{t_*}^{t}(t-\tau)^{\alpha-1}\frac{df(\tau,x(\tau))}{d\tau}d\tau.$$

It is clear that $(t-\tau)^{\alpha} d^i f(\tau, x(\tau))/d\tau^i$ is continuous on $[t_*, s(t)]$ and its partial derivative with respect to t, $\alpha(t-\tau)^{\alpha-1} d^i f(\tau, x(\tau))/d\tau^i$, is also continuous on $[t_*, s(t)]$ due to $t > s(t) > t_*$. By the Leibniz integral rule,

$$\frac{d}{dt}\int_{t_*}^{s(t)}(t-\tau)^{\alpha}\frac{df(\tau,x(\tau))}{d\tau}d\tau = \alpha\int_{t_*}^{s(t)}(t-\tau)^{\alpha-1}\frac{df(\tau,x(\tau))}{d\tau}d\tau + \frac{1}{2^{\alpha+1}}(t-t_1)^{\alpha}\frac{df(\tau,x(\tau))}{d\tau}|_{\tau=s(t)}.$$

Thus, for $t \in (t_1, t_2]$,

$$\begin{aligned}
(T_2x)'(t) &= \frac{1}{\Gamma(\alpha)}\int_{s(t)}^{t}(t-\tau)^{\alpha-1}\frac{df(\tau,x(\tau))}{d\tau}d\tau + \frac{-1}{\Gamma(\alpha)\alpha 2^{\alpha+1}}(t-t_1)^{\alpha}\frac{df(\tau,x(\tau))}{d\tau}\Big|_{\tau=s(t)}\\
&\quad + \frac{1}{\Gamma(\alpha)2^{\alpha}}(t-t_1)^{\alpha-1}f(\tau,x(\tau))|_{\tau=s(t)} + \frac{1}{\Gamma(\alpha)\alpha 2^{\alpha+1}}(t-t_1)^{\alpha}\frac{df(\tau,x(\tau))}{d\tau}\Big|_{\tau=s(t)}\\
&= d_1\int_{s(t)}^{t}(t-\tau)^{\alpha-1}\frac{df(\tau,x(\tau))}{d\tau}d\tau + d_{10}(t-t_1)^{\alpha-1+0}f(\tau,x(\tau))|_{\tau=s(t)}\\
&= d_1\frac{-1}{\alpha}\int_{s(t)}^{t}\frac{df(\tau,x(\tau))}{d\tau}d(t-\tau)^{\alpha} + d_{10}(t-t_1)^{\alpha-1}f(\tau,x(\tau))|_{\tau=s(t)}\\
&= d_1\frac{-1}{\alpha}[\frac{df(\tau,x(\tau))}{d\tau}(t-\tau)^{\alpha}\Big|_{\tau=s(t)}^{\tau=t} - \int_{s(t)}^{t}(t-\tau)^{\alpha}\frac{d^2f(\tau,x(\tau))}{d\tau^2}d\tau] + d_{10}(t-t_1)^{\alpha-1}f(\tau,x(\tau))|_{\tau=s(t)}\\
&= d_1\frac{1}{\alpha}[\frac{1}{2^{\alpha}}(t-t_1)^{\alpha}\frac{df(\tau,x(\tau))}{d\tau}\Big|_{\tau=s(t)} + \int_{s(t)}^{t}(t-\tau)^{\alpha}\frac{d^2f(\tau,x(\tau))}{d\tau^2}d\tau] + d_{10}(t-t_1)^{\alpha-1}f(\tau,x(\tau))|_{\tau=s(t)}\\
&= d_1\frac{1}{\alpha}\int_{s(t)}^{t}(t-\tau)^{\alpha}\frac{d^2f(\tau,x(\tau))}{d\tau^2}d\tau + d_1\frac{1}{\alpha 2^{\alpha}}(t-t_1)^{\alpha}\frac{df(\tau,x(\tau))}{d\tau}\Big|_{\tau=s(t)} + d_{10}(t-t_1)^{\alpha-1}f(\tau,x(\tau))|_{\tau=s(t)},
\end{aligned}$$

$$\begin{aligned}
(T_2x)''(t) &= d_1\int_{s(t)}^{t}(t-\tau)^{\alpha-1}\frac{d^2f(\tau,x(\tau))}{d\tau^2}d\tau + d_1\frac{-1}{\alpha 2^{\alpha+1}}(t-t_1)^{\alpha}\frac{d^2f(\tau,x(\tau))}{d\tau^2}\Big|_{\tau=s(t)}\\
&\quad + d_1\frac{1}{2^{\alpha}}(t-t_1)^{\alpha-1}\frac{df(\tau,x(\tau))}{d\tau}\Big|_{\tau=s(t)} + d_1\frac{1}{\alpha 2^{\alpha+1}}(t-t_1)^{\alpha}\frac{d^2f(\tau,x(\tau))}{d\tau^2}\Big|_{\tau=s(t)}\\
&\quad + d_{10}\frac{1}{\alpha-1}(t-t_1)^{\alpha-2}f(\tau,x(\tau))|_{\tau=s(t)} + d_{10}\frac{1}{2}(t-t_1)^{\alpha-1}\frac{df(\tau,x(\tau))}{d\tau}\Big|_{\tau=s(t)}\\
&= d_1\int_{s(t)}^{t}(t-\tau)^{\alpha-1}\frac{d^2f(\tau,x(\tau))}{d\tau^2}d\tau + d_{20}(t-t_1)^{\alpha-2+0}f(\tau,x(\tau))|_{\tau=s(t)} + d_{21}(t-t_1)^{\alpha-2+1}\frac{df(\tau,x(\tau))}{d\tau}\Big|_{\tau=s(t)}.
\end{aligned}$$

Let $(T_2x)^{(i-1)}(t) = d_1\int_{s(t)}^{t}(t-\tau)^{\alpha-1}\frac{d^{i-1}f(\tau,x(\tau))}{d\tau^{i-1}}d\tau + \sum_{j=0}^{(i-1)-1} d_{(i-1)j}(t-t_1)^{\alpha-(i-1)+j}\frac{d^j}{d\tau^j}f(\tau,x(\tau))|_{\tau=s(t)}, i \geq 2$, then

$$(T_2x)^{(i-1)}(t) = d_1\frac{-1}{\alpha}\int_{s(t)}^{t}\frac{d^{i-1}}{d\tau^{i-1}}f(\tau,x(\tau))d(t-\tau)^{\alpha} + \sum_{j=0}^{(i-1)-1} d_{(i-1)j}(t-t_1)^{\alpha-(i-1)+j}\frac{d^j}{d\tau^j}f(\tau,x(\tau))|_{\tau=s(t)}$$

$$= d_1\frac{1}{\alpha}\int_{s(t)}^{t}(t-\tau)^{\alpha}\frac{d^i}{d\tau^i}f(\tau,x(\tau))d\tau + d_1\frac{1}{\alpha 2^{\alpha}}(t-t_1)^{\alpha}\frac{d^{i-1}}{d\tau^{i-1}}f(\tau,x(\tau))|_{\tau=s(t)} + \sum_{j=0}^{(i-1)-1}d_{(i-1)j}(t-t_1)^{\alpha-(i-1)+j}\frac{d^j}{d\tau^j}f(\tau,x(\tau))|_{\tau=s(t)},$$

$$
\begin{aligned}
(T_2x)^{(i)}(t) &= d_1 \int_{s(t)}^{t} (t-\tau)^{\alpha-1} \frac{d^i}{d\tau^i} f(\tau, x(\tau)) d\tau - d_1 \frac{1}{\alpha 2^{\alpha+1}} (t-t_1)^{\alpha} \frac{d^i}{d\tau^i} f(\tau, x(\tau))|_{\tau=s(t)} \\
&\quad + d_1 \frac{1}{2^{\alpha}} (t-t_1)^{\alpha-1} \frac{d^{i-1}}{d\tau^{i-1}} f(\tau, x(\tau))|_{\tau=s(t)} + d_1 \frac{1}{\alpha 2^{\alpha+1}} (t-t_1)^{\alpha} \frac{d^i}{d\tau^i} f(\tau, x(\tau))|_{\tau=s(t)} \\
&\quad + \sum_{j=0}^{(i-1)-1} d_{(i-1)j} [\alpha - (i-1) + j] (t-t_1)^{\alpha-i+j} \frac{d^j}{d\tau^j} f(\tau, x(\tau))|_{\tau=s(t)} \\
&\quad + \sum_{j=0}^{(i-1)-1} d_{(i-1)j} \frac{1}{2} (t-t_1)^{\alpha-(i-1)+j} \frac{d^{j+1}}{d\tau^{j+1}} f(\tau, x(\tau))|_{\tau=s(t)} \\
&= d_1 \int_{s(t)}^{t} (t-\tau)^{\alpha-1} \frac{d^i}{d\tau^i} f(\tau, x(\tau)) d\tau + d_1 \frac{1}{2^{\alpha}} (t-t_1)^{\alpha-1} \frac{d^{i-1}}{d\tau^{i-1}} f(\tau, x(\tau))|_{\tau=s(t)} \\
&\quad + \sum_{j=0}^{(i-1)-1} d_{(i-1)j} [\alpha - (i-1) + j] (t-t_1)^{\alpha-i+j} \frac{d^j}{d\tau^j} f(\tau, x(\tau))|_{\tau=s(t)} \\
&\quad + d_{(i-1)(i-2)} \frac{1}{2} (t-t_1)^{\alpha-1} \frac{d^{i-1}}{d\tau^{i-1}} f(\tau, x(\tau))|_{\tau=s(t)} + \sum_{j=0}^{(i-1)-2} d_{(i-1)j} \frac{1}{2} (t-t_1)^{\alpha-(i-1)+j} \frac{d^{j+1}}{d\tau^{j+1}} f(\tau, x(\tau))|_{\tau=s(t)} \\
&= d_1 \int_{s(t)}^{t} (t-\tau)^{\alpha-1} \frac{d^i}{d\tau^i} f(\tau, x(\tau)) d\tau + \sum_{j=0}^{(i-1)-1} d_{(i-1)j} [\alpha - (i-1) + j] (t-t_1)^{\alpha-i+j} \frac{d^j}{d\tau^j} f(\tau, x(\tau))|_{\tau=s(t)} \\
&\quad + \sum_{k=1}^{(i-1)-1} d_{(i-1)(k-1)} \frac{1}{2} (t-t_1)^{\alpha-i+k} \frac{d^k}{d\tau^k} f(\tau, x(\tau))|_{\tau=s(t)} + d_{i(i-1)} (t-t_1)^{\alpha-i+(i-1)} \frac{d^{i-1}}{d\tau^{i-1}} f(\tau, x(\tau))|_{\tau=s(t)} \\
&= d_1 \int_{s(t)}^{t} (t-\tau)^{\alpha-1} \frac{d^i}{d\tau^i} f(\tau, x(\tau)) d\tau + \sum_{j=0}^{i-1} d_{ij} (t-t_1)^{\alpha-i+j} \frac{d^j}{d\tau^j} f(\tau, x(\tau))|_{\tau=s(t)}.
\end{aligned}
$$

Therefore,

$$
(T_2x)^{(i)}(t) = d_1 \int_{s(t)}^{t} (t-\tau)^{\alpha-1} \frac{d^i}{d\tau^i} f(\tau, x(\tau)) d\tau + \sum_{j=0}^{i-1} d_{ij} (t-t_1)^{\alpha-i+j} \frac{d^j}{d\tau^j} f(\tau, x(\tau))|_{\tau=s(t)}, t \in (t_1, t_2], i = 1, 2, ..., q. \quad (3.11)
$$

For $i = 1, 2, ..., q-1$, we can use the partial integration, like we do for $(T_2x)^{(i-1)}(t)$, to easily see $(T_2x)^{(i)}(t) \in C(t_1, t_2]$. But we can not do that again for $i = q$, since $f(\tau, x(\tau))$ is not $q+1$ times continuously differentiable.

Nevertheless, we can still show $(T_2x)^{(q)}(t) \in C(t_1, t_2]$ as follows. As we see, the sum term of $(T_2x)^{(q)}(t)$ is obviously continuous on $(t_1, t_2]$. We only need to show the continuity of its integral term. According to Lemma 3.3.1 and Proposition 3.3.1,

$$
\sum_{j=1}^{i} (1-\nu-j)(v_{j1} + v_{j2} + ... + v_{jn}) = \sum_{j=1}^{i} (1-\nu-j)(k_j - v_{j0}) \geq \sum_{j=1}^{i} -j(k_j - v_{j0}) = -i + \sum_{j=1}^{i} j v_{j0} = -i + u_0,
$$

especially, $\sum_{j=1}^{i}(1-\nu-j)(v_{j1}+v_{j2}+...+v_{jn}) = \sum_{j=1}^{i}(1-\nu-j)k_j = -i+(1-\nu)\sum_{j=1}^{i}k_j = -i+(1-\nu)\sum_{l=0}^{n}u_l \geq 1-\nu-i$, if $u_0 = 0$. Moreover, for a single $i \geq 2$, $\partial^k f(t,x(t))/\partial t^{u_0}\partial x_1^{u_1}...\partial x_n^{u_n}$ with $u_0 = 1,2,...,i-1$, $u_1 = u_2 = ... = u_n = 0$, will not appear in the expression of $d^i f(t,x(t))/dt^i$. However, as we shall see, $d^i f(t,x(t))/dt^i$ for each $i = 1,2,...q$, will be involved in the proof, which makes $\partial^k f(t,x(t))/\partial t^{u_0}\partial x_1^{u_1}...\partial x_n^{u_n}$ with all possible nonnegative integers $u_0,u_1,...,u_n$ such that $1 \leq u_0+u_1+...+u_n \leq q$ appear. Thus, we need that (3.7) holds for all these possible cases. It follows from (3.7), for sufficiently small t_2-t_1 such that $\tau-t_1<1$,

$$\begin{aligned}
\|\frac{d^i}{d\tau^i}f(\tau,x(\tau))\|_1 &= \|\sum_0\sum_1...\sum_i\frac{i!}{\prod_{j=1}^{i}(j!)^{k_j}\prod_{j=1}^{i}\prod_{l=0}^{n}v_{jl}!}\frac{\partial^k}{\partial\tau^{u_0}\partial x_1{}^{u_1}...\partial x_n{}^{u_n}}f(\tau,x(\tau))\prod_{j=1}^{i}[x_1^{(j)}(\tau)]^{v_{j1}}[x_2^{(j)}(\tau)]^{v_{j2}}...[x_n^{(j)}(\tau)]^{v_{jn}}\|_1\\
&\leq \sum_0\sum_1...\sum_i c_{i,k,n,v}\,M_d\begin{cases}1 & \text{if } u_0=0\\ (\tau-t_0)^{1-\nu-u_0} & \text{if } u_0\geq 1\end{cases}\times c^{\sum_{j=1}^{i}v_{j1}+v_{j2}+...+v_{jn}}(\tau-t_1)^{\sum_{j=1}^{i}(1-\nu-j)(v_{j1}+v_{j2}+...+v_{jn})}\\
&\leq \sum_0\sum_1...\sum_i \tilde{c}_{i,k,n,v}\,M_d\begin{cases}(\tau-t_1)^{1-\nu-i} & \text{if } u_0=0\\ (\tau-t_0)^{1-\nu-u_0}(\tau-t_1)^{-i+u_0} & \text{if } u_0\geq 1\end{cases}\\
&= \sum_0\sum_1...\sum_i \tilde{c}_{i,k,n,v}\,M_d\begin{cases}(\tau-t_1)^{1-\nu-i} & \text{if } u_0=0\\ (\tau-t_1)^{1-\nu-i} & \text{if } u_0\geq 1\end{cases}\\
&\leq c_{f,i}(\tau-t_1)^{1-\nu-i}, \tau\in(t_1,t_2], i=1,2,...,q,
\end{aligned}$$

where $c_{f,i}$ is independent of t_1 and t_2. For $t_1 < s_1 \leq s_2 \leq t_2$, $s_1 - s(s_2) = (s_1-s_2+s_1-t_1)/2 > 0$ if s_2 is sufficiently close to s_1. In this situation,

$$\begin{aligned}
&\|\int_{s(s_1)}^{s_1}(s_1-\tau)^{\alpha-1}\frac{d^i}{d\tau^i}f(\tau,x(\tau))d\tau - \int_{s(s_2)}^{s_2}(s_2-\tau)^{\alpha-1}\frac{d^i}{d\tau^i}f(\tau,x(\tau))d\tau\|_1\\
&= \|\int_{s(s_1)}^{s(s_2)}(s_1-\tau)^{\alpha-1}\frac{d^i}{d\tau^i}f(\tau,x(\tau))d\tau+\int_{s(s_2)}^{s_1}[(s_1-\tau)^{\alpha-1}-(s_2-\tau)^{\alpha-1}]\frac{d^i}{d\tau^i}f(\tau,x(\tau))d\tau-\int_{s_1}^{s_2}(s_2-\tau)^{\alpha-1}\frac{d^i}{d\tau^i}f(\tau,x(\tau))d\tau\|_1\\
&\leq \int_{s(s_1)}^{s(s_2)}(s_1-\tau)^{\alpha-1}\|\frac{d^i}{d\tau^i}f(\tau,x(\tau))\|_1 d\tau+\int_{s(s_2)}^{s_1}[(s_1-\tau)^{\alpha-1}-(s_2-\tau)^{\alpha-1}]\|\frac{d^i}{d\tau^i}f(\tau,x(\tau))\|_1 d\tau+\int_{s_1}^{s_2}(s_2-\tau)^{\alpha-1}\|\frac{d^i}{d\tau^i}f(\tau,x(\tau))\|_1 d\tau\\
&\leq c_{f,i}\Big\{\int_{s(s_1)}^{s(s_2)}(s_1-\tau)^{\alpha-1}(\tau-t_1)^{1-\nu-i}d\tau+\int_{s(s_2)}^{s_1}[(s_1-\tau)^{\alpha-1}-(s_2-\tau)^{\alpha-1}](\tau-t_1)^{1-\nu-i}d\tau+\int_{s_1}^{s_2}(s_2-\tau)^{\alpha-1}(\tau-t_1)^{1-\nu-i}d\tau\Big\}\\
&\leq c_{f,i}\Big\{\int_{s(s_1)}^{s(s_2)}(s_1-\tau)^{\alpha-1}d\tau\,[s(s_1)-t_1]^{1-\nu-i}+\int_{s(s_2)}^{s_1}[(s_1-\tau)^{\alpha-1}-(s_2-\tau)^{\alpha-1}]\,d\tau\,[s(s_2)-t_1]^{1-\nu-i}\\
&\quad+\int_{s_1}^{s_2}(s_2-\tau)^{\alpha-1}d\tau\,(s_1-t_1)^{1-\nu-i}\Big\}\\
&\leq c_{f,i}\Big\{\int_{s(s_1)}^{s(s_2)}(s_1-\tau)^{\alpha-1}d\tau\,[s(s_1)-t_1]^{1-\nu-i}+\int_{s(s_2)}^{s_1}[(s_1-\tau)^{\alpha-1}-(s_2-\tau)^{\alpha-1}]\,d\tau\,[s(s_1)-t_1]^{1-\nu-i}\\
&\quad+\int_{s_1}^{s_2}(s_2-\tau)^{\alpha-1}d\tau\,(s_1-t_1)^{1-\nu-i}\Big\}.
\end{aligned}$$

Fix s_1 and let $s_2 \to s_1$, then $\| \int_{s(s_1)}^{s_1} (s_1 - \tau)^{\alpha-1} d^i f(\tau, x(\tau))/d\tau^i d\tau - \int_{s(s_2)}^{s_2} (s_2 - \tau)^{\alpha-1} d^i f(\tau, x(\tau))/d\tau^i d\tau \|_1 \to 0$, which implies that the integral term of $(T_2 x)^{(q)}(t)$ is also continuous on $(t_1, t_2]$. Thus, $(T_2 x)^{(q)}(t) \in C(t_1, t_2]$.

Therefore, for any $x \in B$, $(Tx)^{(i)}(t) \in C(t_1, t_2], i = 1, 2, ..., q$. Moreover, for sufficiently small $t_2 - t_1$ such that $t - t_1 < 1$,

$$
\begin{aligned}
\|(Tx)^{(i)}(t)\|_1 &= \|(T_1 x)^{(i)}(t) + (T_2 x)^{(i)}(t)\|_1 \\
&= \|c_i \int_{t_1}^{s(t)} (t-\tau)^{\alpha-i-1} f(\tau, (x(\tau)) d\tau + \sum_{j=0}^{i-1} c_{ij} (t - t_1)^{\alpha-i+j} \frac{d^j}{d\tau^j} f(\tau, x(\tau))|_{\tau=s(t)} \\
&\quad + d_1 \int_{s(t)}^{t} (t-\tau)^{\alpha-1} \frac{d^i}{d\tau^i} f(\tau, x(\tau)) d\tau + \sum_{j=0}^{i-1} d_{ij} (t-t_1)^{\alpha-i+j} \frac{d^j}{d\tau^j} f(\tau, x(\tau))|_{\tau=s(t)} \|_1 \\
&= \|c_i \int_{t_1}^{s(t)} (t-\tau)^{\alpha-i-1} f(\tau, (x(\tau)) d\tau + d_1 \int_{s(t)}^{t} (t-\tau)^{\alpha-1} \frac{d^i}{d\tau^i} f(\tau, x(\tau)) d\tau + \sum_{j=0}^{i-1} e_{ij} (t-t_1)^{\alpha-i+j} \frac{d^j}{d\tau^j} f(\tau, x(\tau))|_{\tau=s(t)} \|_1 \\
&\le |c_i| \int_{t_1}^{s(t)} (t-\tau)^{\alpha-i-1} \|f(\tau, (x(\tau))\|_1 d\tau + d_1 \int_{s(t)}^{t} (t-\tau)^{\alpha-1} \|\frac{d^i}{d\tau^i} f(\tau, x(\tau))\|_1 d\tau \\
&\quad + \sum_{j=1}^{i-1} |e_{ij}| (t-t_1)^{\alpha-i+j} \|\frac{d^j}{d\tau^j} f(\tau, x(\tau))|_{\tau=s(t)}\|_1 + |e_{i0}| (t-t_1)^{\alpha-i} \|f(\tau, x(\tau))|_{\tau=s(t)}\|_1 \\
&\le |c_i| M \int_{t_1}^{s(t)} (t-\tau)^{\alpha-i-1} d\tau + d_1 c_{f,i} \int_{s(t)}^{t} (t-\tau)^{\alpha-1} (\tau - t_1)^{1-\nu-i} d\tau + \sum_{j=1}^{i-1} |e_{ij}| c_{f,j} (t-t_1)^{\alpha-i+j} (\frac{t-t_1}{2})^{1-\nu-j} \\
&\quad + |e_{i0}| M (t-t_1)^{\alpha-i} \\
&\le |c_i| M (\frac{t-t_1}{2})^{\alpha-i} + d_1 c_{f,i} \frac{-1}{\alpha} [(t-\tau)^{\alpha} (\tau - t_1)^{1-\nu-i}|_{\tau=s(t)}^{\tau=t} - (1-\nu-i) \int_{s(t)}^{t} (t-\tau)^{\alpha} (\tau-t_1)^{1-\nu-i-1} d\tau] \\
&\quad + \sum_{j=1}^{i-1} |e_{ij}| c_{f,j} 2^{\nu-1+j} (t-t_1)^{\alpha+1-\nu-i} + |e_{i0}| M (t-t_1)^{\alpha-i} \\
&\le |c_i| M (\frac{t-t_1}{2})^{\alpha-i} + d_1 c_{f,i} \frac{1}{\alpha} [(\frac{t-t_1}{2})^{\alpha+1-\nu-i} + (1-\nu-i)(\frac{t-t_1}{2})^{\alpha+1-\nu-i}] + \sum_{j=1}^{i-1} |e_{ij}| c_{f,j} 2^{\nu-1+j} (t-t_1)^{\alpha+1-\nu-i} \\
&\quad + |e_{i0}| M (t-t_1)^{\alpha-i} \\
&= c_{T,i} (t-t_1)^{\alpha-i} + c_{T,i,1} (t-t_1)^{\alpha+1-\nu-i} \\
&\le c_{T,i} (t-t_1)^{1-\nu-i} + c_{T,i,1} (t-t_1)^{\alpha+1-\nu-i}, t \in (t_1, t_2], i = 1, 2, ..., q.
\end{aligned}
$$

where $c_{T,i} := |c_i| M / 2^{\alpha-i} + |e_{i0}| M$. Thus, $(Tx)(t) \in C^{q,\nu}(t_1, t_2]$. This together with $z(t) \in C^{q,\nu}(t_1, t_2]$ implies that for any $x \in B$, $(Sx)(t) \in C^{q,\nu}(t_1, t_2]$.

In the following, we shall show that for any $x \in B$, $\|(Sx)(t)\|_{1,q,\nu,B} \leq c$ so that $(Sx)(t) \in B$. For any $x \in B$, $t \in (t_1, t_2]$ and sufficiently small $t_2 - t_1$,

$$\|(Tx)(t)\|_{1,\infty,B} = \frac{1}{\Gamma(\alpha)} \| \int_{t_1}^{t} (t-\tau)^{\alpha-1} f(\tau, x(\tau)) d\tau \|_{1,\infty,B} \leq \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-\tau)^{\alpha-1} \|f(\tau, x(\tau))\|_{1,\infty,B} d\tau \leq \frac{M}{\Gamma(\alpha+1)} (t_2 - t_1)^{1-\nu},$$

then

$$\begin{aligned}
\|(Sx)(t)\|_{1,q,\nu,B} &= (W+1)\|(Sx)(t)\|_{1,\infty,B} + \sum_{i=1}^{q} \sup_{t\in(t_1,t_2]} (t-t_1)^{\nu-1+i} \|(Sx)^{(i)}(t)\|_1 \\
&\leq (W+1)\|(Tx)(t)\|_{1,\infty,B} + (W+1)\|z(t)\|_{1,\infty,B} + \sum_{i=1}^{q} \sup_{t\in(t_1,t_2]} (t-t_1)^{\nu-1+i} \|z^{(i)}(t)\|_1 \\
&\quad + \sum_{i=1}^{q} \sup_{t\in(t_1,t_2]} (t-t_1)^{\nu-1+i} \|(Tx)^{(i)}(t)\|_1 \\
&\leq \|z\|_{1,q,\nu,B} + (W+1)\frac{M}{\Gamma(\alpha+1)}(t_2-t_1)^{1-\nu} + \sum_{i=1}^{q} c_{T,i} + \sum_{i=1}^{q} c_{T,i,1}(t_2-t_1)^{1-\nu}.
\end{aligned}$$

Thus, $\|(Sx)(t)\|_{1,q,\nu,B} \leq c$. This results in $(Sx)(t) \in B$, for any $x \in B$.

In the next, we shall show that S is a contraction mapping on B. At first, we shall show that for any $(t, x), (t, y) \in S$ and all possible nonnegative integers $u_0, u_1, ..., u_n$ with $u_0 + u_1 + ... + u_n \leq q$,

$$\|\frac{\partial^{u_0+u_1+...+u_n}}{\partial t^{u_0} \partial {x_1}^{u_1} ... \partial {x_n}^{u_n}} f(t, x) - \frac{\partial^{u_0+u_1+...+u_n}}{\partial t^{u_0} \partial {y_1}^{u_1} ... \partial {y_n}^{u_n}} f(t, y)\|_1 \leq \max\{nM_d, L_d\} \|x - y\|_1 \begin{cases} 1 & \text{if } u_0 = 0 \\ (t-t_0)^{1-\nu-u_0} & \text{if } u_0 \geq 1 \end{cases}. \tag{3.12}$$

Let $u_0 + u_1 + ... + u_n = 1$ and $u_0 = 0$, then it follows from (3.7) that for any $(t, x) \in S$ and $l, r = 1, 2, ..., n$, $|\partial f_r(t, x)/\partial x_l| \leq M_d$. According to the generalized mean value theorem, for any $(t, x), (t, y) \in S$, $(t, w) \in S$, where $w = y + \theta(x - y)$, $\theta \in [0, 1]$, then

$$\|f(t, x) - f(t, y)\|_1 = \| \int_0^1 Df(t, w) d\theta (x - y)\|_1 \leq \int_0^1 \|Df(t, w)\|_1 d\theta \|(x - y)\|_1 \leq nM_d \|x - y\|_1,$$

where Df denotes the Jacobian matrix of f. This implies $\lim_{t \to t_0^+} \|f(t, x) - f(t, y)\|_1 \leq nM_d\|x - y\|_1$. Since f is continuous on $\bar{S}$, $\|f(t_0, x) - f(t_0, y)\|_1 \leq nM_d\|x - y\|_1$. Thus, for any $(t, x), (t, y) \in \bar{S}$,

$$\|f(t, x) - f(t, y)\|_1 \leq nM_d\|x - y\|_1. \tag{3.13}$$

Let $u_0 + u_1 + ... + u_n = i$, $1 \leq i \leq q - 1$, then it follows from (3.7) that for any $(t, x) \in S$ and $l, r = 1, 2, ..., n$,

$$|\frac{\partial^{i+1}}{\partial t^{u_0} \partial {x_1}^{u_1} ... {x_l}^{u_l+1} ... \partial {x_n}^{u_n}} f_r(t, x)| \leq M_d \begin{cases} 1 & \text{if } u_0 = 0 \\ (t-t_0)^{1-\nu-u_0} & \text{if } u_0 \geq 1 \end{cases}.$$

According to the mean value theorem, for any $(t, x), (t, y) \in S$,

$$\begin{aligned}\|\frac{\partial^i}{\partial t^{u_0}\partial {x_1}^{u_1}...\partial {x_n}^{u_n}} f(t,x) - \frac{\partial^i}{\partial t^{u_0}\partial {y_1}^{u_1}...\partial {y_n}^{u_n}} f(t,y)\|_1 &= \| \int_0^1 D(\frac{\partial^i}{\partial t^{u_0}\partial {w_1}^{u_1}...\partial {w_n}^{u_n}} f)(t,w) d\theta (x-y)\|_1 \\ &\leq \int_0^1 \|D(\frac{\partial^i}{\partial t^{u_0}\partial {w_1}^{u_1}...\partial {w_n}^{u_n}} f)(t,w)\|_1 d\theta \|(x-y)\|_1 \\ &\leq nM_d \|x-y\|_1 \begin{cases} 1 & \text{if } u_0 = 0 \\ (t-t_0)^{1-\nu-u_0} & \text{if } u_0 \geq 1 \end{cases}.\end{aligned}$$

This together with (3.8) and (3.13) proves (3.12). Therefore, for any $x, y \in B$ and $\tau \in (t_1, t_2]$, $(\tau, x(\tau)), (\tau, y(\tau)) \in S$ so that for all possible nonnegative integers $u_0, u_1, ..., u_n$ with $u_0 + u_1 + ... + u_n \leq q$,

$$\|\frac{\partial^{u_0+u_1+...+u_n}}{\partial \tau^{u_0}\partial {x_1}^{u_1}...\partial {x_n}^{u_n}} f(\tau, x(\tau)) - \frac{\partial^{u_0+u_1+...+u_n}}{\partial \tau^{u_0}\partial {y_1}^{u_1}...\partial {y_n}^{u_n}} f(\tau, y(\tau))\|_1 \leq \max\{nM_d, L_d\} \|x(\tau) - y(\tau)\|_1 \begin{cases} 1 & \text{if } u_0 = 0 \\ (\tau - t_0)^{1-\nu-u_0} & \text{if } u_0 \geq 1 \end{cases}. \tag{3.14}$$

Especially, it follows from (3.13) that for any $x, y \in B$ and $\tau \in [t_1, t_2]$,

$$\|f(\tau, x(\tau)) - f(\tau, y(\tau))\|_1 \leq nM_d \|x(\tau) - y(\tau)\|_1. \tag{3.15}$$

It follows from (3.15) that for any $x, y \in B$, $t \in [t_1, t_2]$ and $t_2 - t_1 < 1$,

$$\begin{aligned}\|(Sx)(t) - (Sy)(t)\|_{1,\infty,B} &= \|(Tx)(t) - (Ty)(t)\|_{1,\infty,B} \\ &\leq \frac{1}{\Gamma(\alpha)} \max_{t\in[t_1,t_2]} \| \int_{t_1}^t (t-\tau)^{\alpha-1} f(\tau, x(\tau)) - f(\tau, y(\tau)) d\tau\|_1 \\ &\leq \frac{1}{\Gamma(\alpha)} \max_{t\in[t_1,t_2]} \int_{t_1}^t (t-\tau)^{\alpha-1} \|f(\tau, x(\tau)) - f(\tau, y(\tau))\|_1 d\tau \\ &\leq \frac{1}{\Gamma(\alpha)} \max_{t\in[t_1,t_2]} \int_{t_1}^t (t-\tau)^{\alpha-1} nM_d \|x(\tau) - y(\tau)\|_1 d\tau \\ &\leq \frac{nM_d}{\Gamma(\alpha)} \max_{t\in[t_1,t_2]} \int_{t_1}^t (t-\tau)^{\alpha-1} d\tau \|x(t) - y(t)\|_{1,\infty,B} \\ &\leq \frac{nM_d}{\Gamma(\alpha+1)} (t_2 - t_1)^{1-\nu} \|x(t) - y(t)\|_{1,\infty,B}.\end{aligned}$$

According to Lemma 3.3.1, for any $x, y \in B$ and $\tau \in (t_1, t_2]$,

$$\begin{aligned}\|\frac{d^i}{d\tau^i}[f(\tau, (x(\tau)) - f(\tau, (y(\tau))]\|_1 = \|\sum_0 \sum_1 ... \sum_i c_{i,k,n,v} \Big\{ &\frac{\partial^k}{\partial \tau^{u_0}\partial {x_1}^{u_1}...\partial {x_n}^{u_n}} f(\tau, x(\tau)) \prod_{j=1}^i [x_1^{(j)}(\tau)]^{v_{j1}} [x_2^{(j)}(\tau)]^{v_{j2}} ... [x_n^{(j)}(\tau)]^{v_{jn}} \\ &- \frac{\partial^k}{\partial \tau^{u_0}\partial {y_1}^{u_1}...\partial {y_n}^{u_n}} f(\tau, y(\tau)) \prod_{j=1}^i [y_1^{(j)}(\tau)]^{v_{j1}} [y_2^{(j)}(\tau)]^{v_{j2}} ... [y_n^{(j)}(\tau)]^{v_{jn}} \Big\}\|_1.\end{aligned}$$

Minus and plus one term as follows, then a factor $[x_n^{(i)}(\tau) - y_n^{(i)}(\tau)]$ appears, and after a common $y_n^{(i)}(\tau)$ is factored out the power of $x_n^{(i)}(\tau)$ and $y_n^{(i)}(\tau)$ behind the multiplication sign in the inner braces is reduced by one.

$$\begin{aligned}
&\| \sum_0 \sum_1 \cdots \sum_i c_{i,k,n,v} \Big\{ \frac{\partial^k}{\partial \tau^{u_0} \partial {x_1}^{u_1} \ldots \partial {x_n}^{u_n}} f(\tau, x(\tau)) \prod_{j=1}^{i-1} [x_1^{(j)}(\tau)]^{v_{j1}} [x_2^{(j)}(\tau)]^{v_{j2}} \ldots [x_n^{(j)}(\tau)]^{v_{jn}} \\
&\times [x_1^{(i)}(\tau)]^{v_{i1}} [x_2^{(i)}(\tau)]^{v_{i2}} \ldots [x_n^{(i)}(\tau)]^{v_{in}-1} x_n^{(i)}(\tau) - \frac{\partial^k}{\partial \tau^{u_0} \partial {x_1}^{u_1} \ldots \partial {x_n}^{u_n}} f(\tau, x(\tau)) \prod_{j=1}^{i-1} [x_1^{(j)}(\tau)]^{v_{j1}} [x_2^{(j)}(\tau)]^{v_{j2}} \ldots [x_n^{(j)}(\tau)]^{v_{jn}} \\
&\times [x_1^{(i)}(\tau)]^{v_{i1}} [x_2^{(i)}(\tau)]^{v_{i2}} \ldots [x_n^{(i)}(\tau)]^{v_{in}-1} y_n^{(i)}(\tau) + \frac{\partial^k}{\partial \tau^{u_0} \partial {x_1}^{u_1} \ldots \partial {x_n}^{u_n}} f(\tau, x(\tau)) \prod_{j=1}^{i-1} [x_1^{(j)}(\tau)]^{v_{j1}} [x_2^{(j)}(\tau)]^{v_{j2}} \ldots [x_n^{(j)}(\tau)]^{v_{jn}} \\
&\times [x_1^{(i)}(\tau)]^{v_{i1}} [x_2^{(i)}(\tau)]^{v_{i2}} \ldots [x_n^{(i)}(\tau)]^{v_{in}-1} y_n^{(i)}(\tau) - \frac{\partial^k}{\partial \tau^{u_0} \partial {y_1}^{u_1} \ldots \partial {y_n}^{u_n}} f(\tau, y(\tau)) \prod_{j=1}^{i-1} [y_1^{(j)}(\tau)]^{v_{j1}} [y_2^{(j)}(\tau)]^{v_{j2}} \ldots [y_n^{(j)}(\tau)]^{v_{jn}} \\
&\times [y_1^{(i)}(\tau)]^{v_{i1}} [y_2^{(i)}(\tau)]^{v_{i2}} \ldots [y_n^{(i)}(\tau)]^{v_{in}-1} y_n^{(i)}(\tau) \Big\} \|_1 \\
&= \| \sum_0 \sum_1 \cdots \sum_i c_{i,k,n,v} \Big\{ \frac{\partial^k}{\partial \tau^{u_0} \partial {x_1}^{u_1} \ldots \partial {x_n}^{u_n}} f(\tau, x(\tau)) \prod_{j=1}^{i-1} [x_1^{(j)}(\tau)]^{v_{j1}} [x_2^{(j)}(\tau)]^{v_{j2}} \ldots [x_n^{(j)}(\tau)]^{v_{jn}} \\
&\times [x_1^{(i)}(\tau)]^{v_{i1}} [x_2^{(i)}(\tau)]^{v_{i2}} \ldots [x_n^{(i)}(\tau)]^{v_{in}-1} [x_n^{(i)}(\tau) - y_n^{(i)}(\tau)] \\
&+ y_n^{(i)}(\tau) \Big\{ \frac{\partial^k}{\partial \tau^{u_0} \partial {x_1}^{u_1} \ldots \partial {x_n}^{u_n}} f(\tau, x(\tau)) \prod_{j=1}^{i-1} [x_1^{(j)}(\tau)]^{v_{j1}} [x_2^{(j)}(\tau)]^{v_{j2}} \ldots [x_n^{(j)}(\tau)]^{v_{jn}} \times [x_1^{(i)}(\tau)]^{v_{i1}} [x_2^{(i)}(\tau)]^{v_{i2}} \ldots [x_n^{(i)}(\tau)]^{v_{in}-1} \\
&- \frac{\partial^k}{\partial \tau^{u_0} \partial {y_1}^{u_1} \ldots \partial {y_n}^{u_n}} f(\tau, y(\tau)) \prod_{j=1}^{i-1} [y_1^{(j)}(\tau)]^{v_{j1}} [y_2^{(j)}(\tau)]^{v_{j2}} \ldots [y_n^{(j)}(\tau)]^{v_{jn}} \times [y_1^{(i)}(\tau)]^{v_{i1}} [y_2^{(i)}(\tau)]^{v_{i2}} \ldots [y_n^{(i)}(\tau)]^{v_{in}-1} \Big\} \Big\} \|_1 .
\end{aligned}$$

Keep doing this till the power of $x_n^{(i)}(\tau)$ and $y_n^{(i)}(\tau)$ there is reduced to zero, then repeat the similar process for $x_{n-1}^{(i)}(\tau)$ and $y_{n-1}^{(i)}(\tau)$ as following,

$$\begin{aligned}
&\| \sum_0 \sum_1 \cdots \sum_i c_{i,k,n,v} \Big\{ \sum_{l=0}^{v_{in}-1} [y_n^{(i)}(\tau)]^l \frac{\partial^k}{\partial \tau^{u_0} \partial {x_1}^{u_1} \ldots \partial {x_n}^{u_n}} f(\tau, x(\tau)) \prod_{j=1}^{i-1} [x_1^{(j)}(\tau)]^{v_{j1}} [x_2^{(j)}(\tau)]^{v_{j2}} \ldots [x_n^{(j)}(\tau)]^{v_{jn}} \\
&\times [x_1^{(i)}(\tau)]^{v_{i1}} [x_2^{(i)}(\tau)]^{v_{i2}} \ldots [x_n^{(i)}(\tau)]^{v_{in}-1-l} [x_n^{(i)}(\tau) - y_n^{(i)}(\tau)] \\
&+ [y_n^{(i)}(\tau)]^{v_{in}} \Big\{ \frac{\partial^k}{\partial \tau^{u_0} \partial {x_1}^{u_1} \ldots \partial {x_n}^{u_n}} f(\tau, x(\tau)) \prod_{j=1}^{i-1} [x_1^{(j)}(\tau)]^{v_{j1}} [x_2^{(j)}(\tau)]^{v_{j2}} \ldots [x_n^{(j)}(\tau)]^{v_{jn}} \times [x_1^{(i)}(\tau)]^{v_{i1}} [x_2^{(i)}(\tau)]^{v_{i2}} \ldots [x_{n-1}^{(i)}(\tau)]^{v_{i(n-1)}} \\
&- \frac{\partial^k}{\partial \tau^{u_0} \partial {y_1}^{u_1} \ldots \partial {y_n}^{u_n}} f(\tau, y(\tau)) \prod_{j=1}^{i-1} [y_1^{(j)}(\tau)]^{v_{j1}} [y_2^{(j)}(\tau)]^{v_{j2}} \ldots [y_n^{(j)}(\tau)]^{v_{jn}} \times [y_1^{(i)}(\tau)]^{v_{i1}} [y_2^{(i)}(\tau)]^{v_{i2}} \ldots [y_{n-1}^{(i)}(\tau)]^{v_{i(n-1)}} \Big\} \Big\} \|_1
\end{aligned}$$

$$= \| \sum_0 \sum_1 \cdots \sum_i c_{i,k,n,v} \Big\{ \sum_{l=0}^{v_{in}-1} [y_n^{(i)}(\tau)]^l \frac{\partial^k}{\partial \tau^{u_0} \partial {x_1}^{u_1} \ldots \partial {x_n}^{u_n}} f(\tau, x(\tau)) \prod_{j=1}^{i-1} [x_1^{(j)}(\tau)]^{v_{j1}} [x_2^{(j)}(\tau)]^{v_{j2}} \ldots [x_n^{(j)}(\tau)]^{v_{jn}}$$

$$\times [x_1^{(i)}(\tau)]^{v_{i1}} [x_2^{(i)}(\tau)]^{v_{i2}} \ldots [x_n^{(i)}(\tau)]^{v_{in}-1-l} [x_n^{(i)}(\tau) - y_n^{(i)}(\tau)]$$

$$+ [y_n^{(i)}(\tau)]^{v_{in}} \Big\{ \sum_{l=0}^{v_{i(n-1)}-1} [y_{n-1}^{(i)}(\tau)]^l \frac{\partial^k}{\partial \tau^{u_0} \partial {x_1}^{u_1} \ldots \partial {x_n}^{u_n}} f(\tau, x(\tau)) \prod_{j=1}^{i-1} [x_1^{(j)}(\tau)]^{v_{j1}} [x_2^{(j)}(\tau)]^{v_{j2}} \ldots [x_n^{(j)}(\tau)]^{v_{jn}}$$

$$\times [x_1^{(i)}(\tau)]^{v_{i1}} [x_2^{(i)}(\tau)]^{v_{i2}} \ldots [x_{n-1}^{(i)}(\tau)]^{v_{i(n-1)}-1-l} [x_{n-1}^{(i)}(\tau) - y_{n-1}^{(i)}(\tau)]$$

$$+ [y_{n-1}^{(i)}(\tau)]^{v_{i(n-1)}} \Big\{ \frac{\partial^k}{\partial \tau^{u_0} \partial {x_1}^{u_1} \ldots \partial {x_n}^{u_n}} f(\tau, x(\tau)) \prod_{j=1}^{i-1} [x_1^{(j)}(\tau)]^{v_{j1}} \ldots [x_n^{(j)}(\tau)]^{v_{jn}} \times [x_1^{(i)}(\tau)]^{v_{i1}} [x_2^{(i)}(\tau)]^{v_{i2}} \ldots [x_{n-2}^{(i)}(\tau)]^{v_{i(n-2)}}$$

$$- \frac{\partial^k}{\partial \tau^{u_0} \partial {y_1}^{u_1} \ldots \partial {y_n}^{u_n}} f(\tau, y(\tau)) \prod_{j=1}^{i-1} [y_1^{(j)}(\tau)]^{v_{j1}} [y_2^{(j)}(\tau)]^{v_{j2}} \ldots [y_n^{(j)}(\tau)]^{v_{jn}} \times [y_1^{(i)}(\tau)]^{v_{i1}} [y_2^{(i)}(\tau)]^{v_{i2}} \ldots [y_{n-2}^{(i)}(\tau)]^{v_{i(n-2)}} \Big\}\Big\}\Big\} \|_1.$$

Continue the similar processes for $x_{n-2}^{(i)}(\tau)$ and $y_{n-2}^{(i)}(\tau)$, ..., $x_1^{(i)}(\tau)$ and $y_1^{(i)}(\tau)$, till the power of all these behind the multiplication sign in the inmost braces is reduced to zero, then the above becomes

$$\| \sum_0 \sum_1 \cdots \sum_i c_{i,k,n,v} \Big\{ \sum_{r=1}^{n} [y_{r+1}^{(i)}(\tau)]^{v_{i(r+1)}} [y_{r+2}^{(i)}(\tau)]^{v_{i(r+2)}} \ldots [y_n^{(i)}(\tau)]^{v_{in}} \sum_{l=0}^{v_{ir}-1} [y_r^{(i)}(\tau)]^l \frac{\partial^k}{\partial \tau^{u_0} \partial {x_1}^{u_1} \ldots \partial {x_n}^{u_n}} f(\tau, x(\tau))$$

$$\times \prod_{j=1}^{i-1} [x_1^{(j)}(\tau)]^{v_{j1}} [x_2^{(j)}(\tau)]^{v_{j2}} \ldots [x_n^{(j)}(\tau)]^{v_{jn}} \times [x_1^{(i)}(\tau)]^{v_{i1}} [x_2^{(i)}(\tau)]^{v_{i2}} \ldots [x_r^{(i)}(\tau)]^{v_{ir}-1-l} [x_r^{(i)}(\tau) - y_r^{(i)}(\tau)]$$

$$+ [y_1^{(i)}(\tau)]^{v_{i1}} [y_2^{(i)}(\tau)]^{v_{i2}} \ldots [y_n^{(i)}(\tau)]^{v_{in}} \Big\{ \frac{\partial^k}{\partial \tau^{u_0} \partial {x_1}^{u_1} \ldots \partial {x_n}^{u_n}} f(\tau, x(\tau)) \prod_{j=1}^{i-1} [x_1^{(j)}(\tau)]^{v_{j1}} [x_2^{(j)}(\tau)]^{v_{j2}} \ldots [x_n^{(j)}(\tau)]^{v_{jn}}$$

$$- \frac{\partial^k}{\partial \tau^{u_0} \partial {y_1}^{u_1} \ldots \partial {y_n}^{u_n}} f(\tau, y(\tau)) \prod_{j=1}^{i-1} [y_1^{(j)}(\tau)]^{v_{j1}} [y_2^{(j)}(\tau)]^{v_{j2}} \ldots [y_n^{(j)}(\tau)]^{v_{jn}} \Big\}\Big\} \|_1.$$

As we observe, after these processes, the upper index of the product operator in the inner braces above is reduced by one. We now repeat these processes again so that the index is reduced by one more, see the following,

$$\| \sum_0 \sum_1 \cdots \sum_i c_{i,k,n,v} \Big\{ \sum_{r=1}^{n} [y_{r+1}^{(i)}(\tau)]^{v_{i(r+1)}} [y_{r+2}^{(i)}(\tau)]^{v_{i(r+2)}} \ldots [y_n^{(i)}(\tau)]^{v_{in}} \sum_{l=0}^{v_{ir}-1} [y_r^{(i)}(\tau)]^l \frac{\partial^k}{\partial \tau^{u_0} \partial {x_1}^{u_1} \ldots \partial {x_n}^{u_n}} f(\tau, x(\tau))$$

$$\times \prod_{j=1}^{i-1} [x_1^{(j)}(\tau)]^{v_{j1}} [x_2^{(j)}(\tau)]^{v_{j2}} \ldots [x_n^{(j)}(\tau)]^{v_{jn}} \times [x_1^{(i)}(\tau)]^{v_{i1}} [x_2^{(i)}(\tau)]^{v_{i2}} \ldots [x_r^{(i)}(\tau)]^{v_{ir}-1-l} [x_r^{(i)}(\tau) - y_r^{(i)}(\tau)]$$

$$+ [y_1^{(i)}(\tau)]^{v_{i1}} [y_2^{(i)}(\tau)]^{v_{i2}} \ldots [y_n^{(i)}(\tau)]^{v_{in}} \Big\{ \sum_{r=1}^{n} [y_{r+1}^{(i-1)}(\tau)]^{v_{(i-1)(r+1)}} [y_{r+2}^{(i-1)}(\tau)]^{v_{(i-1)(r+2)}} \ldots [y_n^{(i-1)}(\tau)]^{v_{(i-1)n}}$$

$$\times \sum_{l=0}^{v_{(i-1)r}-1} [y_r^{(i-1)}(\tau)]^l \frac{\partial^k}{\partial\tau^{u_0}\partial x_1{}^{u_1}...\partial x_n{}^{u_n}} f(\tau, x(\tau)) \prod_{j=1}^{i-2} [x_1^{(j)}(\tau)]^{v_{j1}} [x_2^{(j)}(\tau)]^{v_{j2}} ... [x_n^{(j)}(\tau)]^{v_{jn}}$$

$$\times [x_1^{(i-1)}(\tau)]^{v_{(i-1)1}} [x_2^{(i-1)}(\tau)]^{v_{(i-1)2}} ... [x_r^{(i-1)}(\tau)]^{v_{(i-1)r}-1-l} [x_r^{(i-1)}(\tau) - y_r^{(i-1)}(\tau)]$$

$$+ [y_1^{(i-1)}(\tau)]^{v_{(i-1)1}} [y_2^{(i-1)}(\tau)]^{v_{(i-1)2}} ... [y_n^{(i-1)}(\tau)]^{v_{(i-1)n}} \Big\{ \frac{\partial^k}{\partial\tau^{u_0}\partial x_1{}^{u_1}...\partial x_n{}^{u_n}} f(\tau, x(\tau)) \prod_{j=1}^{i-2} [x_1^{(j)}(\tau)]^{v_{j1}} [x_2^{(j)}(\tau)]^{v_{j2}} ... [x_n^{(j)}(\tau)]^{v_{jn}}$$

$$- \frac{\partial^k}{\partial\tau^{u_0}\partial y_1{}^{u_1}...\partial y_n{}^{u_n}} f(\tau, y(\tau)) \prod_{j=1}^{i-2} [y_1^{(j)}(\tau)]^{v_{j1}} [y_2^{(j)}(\tau)]^{v_{j2}} ... [y_n^{(j)}(\tau)]^{v_{jn}} \Big\}\Big\}\Big\} \|_1.$$

Continue repeating those processes, till the index there is reduced to zero. Then the above can be further rewritten as following

$$\| \sum_0 \sum_1 ... \sum_i c_{i,k,n,v} \Big\{ \sum_{o=1}^{i} \prod_{p=o+1}^{i} [y_1^{(p)}(\tau)]^{v_{p1}} [y_2^{(p)}(\tau)]^{v_{p2}} ... [y_n^{(p)}(\tau)]^{v_{pn}} \sum_{r=1}^{n} [y_{r+1}^{(o)}(\tau)]^{v_{o(r+1)}} [y_{r+2}^{(o)}(\tau)]^{v_{o(r+2)}} ... [y_n^{(o)}(\tau)]^{v_{on}}$$

$$\times \sum_{l=0}^{v_{or}-1} [y_r^{(o)}(\tau)]^l \frac{\partial^k}{\partial\tau^{u_0}\partial x_1{}^{u_1}...\partial x_n{}^{u_n}} f(\tau, x(\tau)) \prod_{j=1}^{o-1} [x_1^{(j)}(\tau)]^{v_{j1}} [x_2^{(j)}(\tau)]^{v_{j2}} ... [x_n^{(j)}(\tau)]^{v_{jn}} [x_1^{(o)}(\tau)]^{v_{o1}} [x_2^{(o)}(\tau)]^{v_{o2}} ... [x_r^{(o)}(\tau)]^{v_{or}-1-l}$$

$$\times [x_r^{(o)}(\tau) - y_r^{(o)}(\tau)]$$

$$+ \prod_{g=1}^{i} [y_1^{(g)}(\tau)]^{v_{g1}} [y_2^{(g)}(\tau)]^{v_{g2}} ... [y_n^{(g)}(\tau)]^{v_{gn}} [\frac{\partial^k}{\partial\tau^{u_0}\partial x_1{}^{u_1}...\partial x_n{}^{u_n}} f(\tau, x(\tau)) - \frac{\partial^k}{\partial\tau^{u_0}\partial y_1{}^{u_1}...\partial y_n{}^{u_n}} f(\tau, y(\tau))] \Big\} \|_1.$$

Now it is ready to apply the inequality (3.14). For any $x, y \in B$, $\tau \in (t_1, t_2]$ and sufficiently small $t_2 - t_1$ such that $\tau - t_1 < 1$,

$$\| \frac{d^i}{d\tau^i} [f(\tau, (x(\tau)) - f(\tau, (y(\tau))] \|_1$$

$$\leq \sum_0 \sum_1 ... \sum_i c_{i,k,n,v} \Big\{ \sum_{o=1}^{i} \prod_{p=o+1}^{i} |[y_1^{(p)}(\tau)]^{v_{p1}} [y_2^{(p)}(\tau)]^{v_{p2}} ... [y_n^{(p)}(\tau)]^{v_{pn}}| \sum_{r=1}^{n} |[y_{r+1}^{(o)}(\tau)]^{v_{o(r+1)}} [y_{r+2}^{(o)}(\tau)]^{v_{o(r+2)}} ... [y_n^{(o)}(\tau)]^{v_{on}}|$$

$$\times \sum_{l=0}^{v_{or}-1} |[y_r^{(o)}(\tau)]^l| \, \| \frac{\partial^k}{\partial\tau^{u_0}\partial x_1{}^{u_1}...\partial x_n{}^{u_n}} f(\tau, x(\tau)) \|_1 \prod_{j=1}^{o-1} |[x_1^{(j)}(\tau)]^{v_{j1}} [x_2^{(j)}(\tau)]^{v_{j2}} ... [x_n^{(j)}(\tau)]^{v_{jn}}|$$

$$\times |[x_1^{(o)}(\tau)]^{v_{o1}} [x_2^{(o)}(\tau)]^{v_{o2}} ... [x_r^{(o)}(\tau)]^{v_{or}-1-l}| \, |[x_r^{(o)}(\tau) - y_r^{(o)}(\tau)]|$$

$$+ \prod_{g=1}^{i} |[y_1^{(g)}(\tau)]^{v_{g1}} [y_2^{(g)}(\tau)]^{v_{g2}} ... [y_n^{(g)}(\tau)]^{v_{gn}}| \, \| [\frac{\partial^k}{\partial\tau^{u_0}\partial x_1{}^{u_1}...\partial x_n{}^{u_n}} f(\tau, x(\tau)) - \frac{\partial^k}{\partial\tau^{u_0}\partial y_1{}^{u_1}...\partial y_n{}^{u_n}} f(\tau, y(\tau))] \|_1 \Big\}$$

$$\leq \sum_0 \sum_1 \cdots \sum_i c_{i,k,n,v}\Big\{c^{-1+\sum_{p=1}^i v_{p1}+v_{p2}+\ldots+v_{pn}} \sum_{o=1}^{i}(\tau-t_1)^{\sum_{p=o+1}^i(1-\nu-p)(v_{p1}+v_{p2}+\ldots+v_{pn})} \sum_{r=1}^{n}(\tau-t_1)^{(1-\nu-o)(v_{o(r+1)}+v_{o(r+2)}+\ldots+v_{on})}$$

$$\times \sum_{l=0}^{v_{or}-1}(\tau-t_1)^{(1-\nu-o)l} M_d\begin{cases}1 & \text{if } u_0=0\\ (\tau-t_0)^{1-\nu-u_0} & \text{if } u_0\geq 1\end{cases} \times(\tau-t_1)^{\sum_{j=1}^{o-1}(1-\nu-j)(v_{j1}+v_{j2}+\ldots+v_{jn})}$$

$$\times(\tau-t_1)^{(1-\nu-o)[(v_{o1}+v_{o2}+\ldots+v_{o(r-1)})+v_{or}-1-l]}|[x_r^{(o)}(\tau)-y_r^{(o)}(\tau)]|$$

$$+ c^{\sum_{g=1}^i v_{g1}+v_{g2}+\ldots+v_{gn}}(\tau-t_1)^{\sum_{g=1}^i(1-\nu-g)(v_{g1}+v_{g2}+\ldots+v_{gn})}\max\{nM_d,L_d\}\,\|x(\tau)-y(\tau)\|_1\begin{cases}1 & \text{if } u_0=0\\ (\tau-t_0)^{1-\nu-u_0} & \text{if } u_0\geq 1\end{cases}\Big\}$$

$$= \sum_0 \sum_1 \cdots \sum_i \bar{c}_{i,k,n,v}\Big\{\sum_{o=1}^{i}(\tau-t_1)^{-(1-\nu-o)+\sum_{p=1}^i(1-\nu-p)(v_{p1}+v_{p2}+\ldots+v_{pn})}\sum_{r=1}^{n} v_{or}|[x_r^{(o)}(\tau)-y_r^{(o)}(\tau)]|M_d\begin{cases}1 & \text{if } u_0=0\\ (\tau-t_0)^{1-\nu-u_0} & \text{if } u_0\geq 1\end{cases}$$

$$+(\tau-t_1)^{\sum_{g=1}^i(1-\nu-g)(v_{g1}+v_{g2}+\ldots+v_{gn})}\max\{nM_d,L_d\}\,\|x(\tau)-y(\tau)\|_1\begin{cases}1 & \text{if } u_0=0\\ (\tau-t_0)^{1-\nu-u_0} & \text{if } u_0\geq 1\end{cases}\Big\}$$

$$\leq \sum_0 \sum_1 \cdots \sum_i \bar{c}_{i,k,n,v}\Big\{\sum_{o=1}^{i} k_o\|[x^{(o)}(\tau)-y^{(o)}(\tau)]\|_1 M_d\begin{cases}(\tau-t_1)^{-(1-\nu-o)-i+(1-\nu)\sum_{p=1}^i k_p} & \text{if } u_0=0\\ (\tau-t_1)^{-(1-\nu-o)-i+u_0}(\tau-t_0)^{1-\nu-u_0} & \text{if } u_0\geq 1\end{cases}$$

$$+\max\{nM_d,L_d\}\,\|x(\tau)-y(\tau)\|_1\begin{cases}(\tau-t_1)^{-i+(1-\nu)\sum_{g=1}^i k_g} & \text{if } u_0=0\\ (\tau-t_1)^{-i+u_0}(\tau-t_0)^{1-\nu-u_0} & \text{if } u_0\geq 1\end{cases}\Big\}$$

$$\leq \sum \bar{c}_{i,k,n,v}\begin{cases}M_d\sum_{o=1}^i k_o(\tau-t_1)^{o-i}\,\|[x^{(o)}(\tau)-y^{(o)}(\tau)]\|_1+(\tau-t_1)^{1-\nu-i}\max\{nM_d,L_d\}\,\|x(\tau)-y(\tau)\|_1 & \text{if } u_0=0\\ M_d\sum_{o=1}^i k_o(\tau-t_1)^{o-i}\,\|[x_r^{(o)}(\tau)-y_r^{(o)}(\tau)]\|_1+(\tau-t_1)^{1-\nu-i}\max\{nM_d,L_d\}\,\|x(\tau)-y(\tau)\|_1 & \text{if } u_0\geq 1\end{cases}$$

$$\leq \sum \bar{c}_{i,k,n,v}[M_d k\sum_{o=1}^{i}(\tau-t_1)^{o-i}\|x^{(o)}(\tau)-y^{(o)}(\tau)\|_1+(\tau-t_1)^{1-\nu-i}\max\{nM_d,L_d\}\,\|x(\tau)-y(\tau)\|_1]$$

$$\leq c_{f,i,1}(\tau-t_1)^{1-\nu-i}\|x(\tau)-y(\tau)\|_1+c_{f,i,2}\sum_{o=1}^{i}(\tau-t_1)^{o-i}\|x^{(o)}(\tau)-y^{(o)}(\tau)\|_1,$$

where $c_{f,i,1}$ and $c_{f,i,2}$ are both independent of t_1 and t_2. Then for any $x, y\in B$, $t\in(t_1,t_2]$ and sufficiently small t_2-t_1 such that $t-t_1<1$,

$$(t-t_1)^{\nu-1+i}\|(Sx)^{(i)}(t)-(Sy)^{(i)}(t)\|_1=(t-t_1)^{\nu-1+i}\|(Tx)^{(i)}(t)-(Ty)^{(i)}(t)\|_1$$

$$=(t-t_1)^{\nu-1+i}\|c_i\int_{t_1}^{s(t)}(t-\tau)^{\alpha-i-1}[f(\tau,(x(\tau))-f(\tau,(y(\tau))]d\tau+d_1\int_{s(t)}^{t}(t-\tau)^{\alpha-1}\frac{d^i}{d\tau^i}[f(\tau,(x(\tau))-f(\tau,(y(\tau))]d\tau$$

$$+\sum_{j=0}^{i-1}e_{ij}(t-t_1)^{\alpha-i+j}\frac{d^j}{d\tau^j}[f(\tau,(x(\tau))-f(\tau,(y(\tau))]|_{\tau=s(t)}\|_1$$

$$\leq(t-t_1)^{\nu-1+i}\{|c_i|\int_{t_1}^{s(t)}(t-\tau)^{\alpha-i-1}\|[f(\tau,(x(\tau))-f(\tau,(y(\tau))]\|_1d\tau+d_1\int_{s(t)}^{t}(t-\tau)^{\alpha-1}\|\frac{d^i}{d\tau^i}[f(\tau,(x(\tau))-f(\tau,(y(\tau))]\|_1d\tau$$

$$
\begin{aligned}
&+\sum_{j=1}^{i-1}|e_{ij}|(t-t_1)^{\alpha-i+j}\|\frac{d^j}{d\tau^j}[f(\tau,(x(\tau))-f(\tau,(y(\tau))]|_{\tau=s(t)}\|_1+|e_{i0}|(t-t_1)^{\alpha-i}\|[f(\tau,(x(\tau))-f(\tau,(y(\tau))]|_{\tau=s(t)}\|_1\}\\
&\le(t-t_1)^{\nu-1+i}\{|c_i|nM_d\int_{t_1}^{s(t)}(t-\tau)^{\alpha-i-1}\|x(\tau)-y(\tau)\|_1d\tau+|e_{i0}|nM_d(t-t_1)^{\alpha-i}\|[(x(\tau)-y(\tau)]|_{\tau=s(t)}\|_1\\
&+d_1\int_{s(t)}^{t}(t-\tau)^{\alpha-1}[c_{f,i,1}(\tau-t_1)^{1-\nu-i}\|x(\tau)-y(\tau)\|_1+c_{f,i,2}\sum_{o=1}^{i}(\tau-t_1)^{o-i}\|x^{(o)}(\tau)-y^{(o)}(\tau)\|_1]d\tau\\
&+\sum_{j=1}^{i-1}|e_{ij}|(t-t_1)^{\alpha-i+j}[c_{f,j,1}(\tau-t_1)^{1-\nu-j}\|x(\tau)-y(\tau)\|_1+c_{f,j,2}\sum_{o=1}^{j}(\tau-t_1)^{o-j}\|x^{(o)}(\tau)-y^{(o)}(\tau)\|_1]|_{\tau=s(t)}\}\\
&\le|c_i|nM_d\int_{t_1}^{s(t)}(t-\tau)^{\alpha-i-1}(t-t_1)^{\nu-1+i}d\tau\|x(t)-y(t)\|_{1,\infty,B}+|e_{i0}|nM_d(t-t_1)^{\alpha+\nu-1}\|(x(t)-y(t)\|_{1,\infty,B}\\
&+d_1\,c_{f,i,1}\int_{s(t)}^{t}(t-\tau)^{\alpha-1}(\tau-t_1)^{1-\nu-i}(t-t_1)^{\nu-1+i}d\tau\|x(t)-y(t)\|_{1,\infty,B}\\
&+d_1\,c_{f,i,2}\int_{s(t)}^{t}(t-\tau)^{\alpha-1}(t-t_1)^{\nu-1+i}(\tau-t_1)^{1-\nu-i}\sum_{o=1}^{i}(\tau-t_1)^{\nu-1+o}\|x^{(o)}(\tau)-y^{(o)}(\tau)\|_1d\tau\\
&+\sum_{j=1}^{i-1}|e_{ij}|c_{f,j,1}(t-t_1)^{\alpha-i+j}(\frac{t-t_1}{2})^{1-\nu-j}(t-t_1)^{\nu-1+i}\|x(t)-y(t)\|_{1,\infty,B}\\
&+\sum_{j=1}^{i-1}|e_{ij}|c_{f,j,2}(t-t_1)^{\alpha+\nu-1+j}(\frac{t-t_1}{2})^{1-\nu-j}\sum_{o=1}^{j}[s(t)-t_1]^{\nu-1+o}\|x^{(o)}(s(t))-y^{(o)}(s(t))\|_1\\
&\le|c_i|nM_d2^{-\alpha+i}\|x(t)-y(t)\|_{1,\infty,B}+|e_{i0}|nM_d\|(x(t)-y(t)\|_{1,\infty,B}+d_1\,c_{f,i,1}\frac{1}{\alpha}2^{-\alpha+\nu-1+i}(t-t_1)^{1-\nu}\|x(t)-y(t)\|_{1,\infty,B}\\
&+d_1\,c_{f,i,2}\frac{1}{\alpha}2^{-\alpha+\nu-1+i}(t-t_1)^{1-\nu}\sum_{o=1}^{q}\sup_{t\in(t_1,t_2]}(t-t_1)^{\nu-1+o}\|x^{(o)}(t)-y^{(o)}(t)\|_1\\
&+\sum_{j=1}^{i-1}|e_{ij}|c_{f,j,1}2^{\nu-1+j}(t-t_1)^{1-\nu}\|x(t)-y(t)\|_{1,\infty,B}\\
&+\sum_{j=1}^{i-1}|e_{ij}|c_{f,j,2}2^{\nu-1+j}(t-t_1)^{1-\nu}\sum_{o=1}^{q}\sup_{t\in(t_1,t_2]}(t-t_1)^{\nu-1+o}\|x^{(o)}(t)-y^{(o)}(t)\|_1\\
&=[c_{S,i,1,1}+c_{S,i,1,2}\,(t-t_1)^{1-\nu}]\|x(t)-y(t)\|_{1,\infty,B}+c_{S,i,2}\,(t-t_1)^{1-\nu}\sum_{o=1}^{q}\sup_{t\in(t_1,t_2]}(t-t_1)^{\nu-1+o}\|x^{(o)}(t)-y^{(o)}(t)\|_1,
\end{aligned}
$$

where $c_{S,i,1,1}:=|c_i|nM_d2^{-\alpha+i}+|e_{i0}|nM_d$.

Therefore,

$$\|(Sx)(t)-(Sy)(t)\|_{1,q,\nu,B} = (W+1)\|(Sx)(t)-(Sy)(t)\|_{1,\infty,B} + \sum_{i=1}^{q} \sup_{t\in(t_1,t_2]} (t-t_1)^{\nu-1+i}\|(Sx)^{(i)}(t)-(Sy)^{(i)}(t)\|_1$$

$$\le (W+1)\frac{nM_d}{\Gamma(\alpha+1)}(t_2-t_1)^{1-\nu}\|x(t)-y(t)\|_{1,\infty,B} + \sum_{i=1}^{q}\{[c_{S,i,1,1}+c_{S,i,1,2}\,(t_2-t_1)^{1-\nu}]\|x(t)-y(t)\|_{1,\infty,B}$$

$$+ c_{S,i,2}\,(t_2-t_1)^{1-\nu}\sum_{o=1}^{q}\sup_{t\in(t_1,t_2]}(t-t_1)^{\nu-1+o}\|x^{(o)}(t)-y^{(o)}(t)\|_1\}$$

$$= [\frac{\sum_{i=1}^{q} c_{S,i,1,1}}{W+1} + \frac{nM_d}{\Gamma(\alpha+1)}(t_2-t_1)^{1-\nu} + \frac{\sum_{i=1}^{q} c_{S,i,1,2}}{W+1}\,(t_2-t_1)^{1-\nu}](W+1)\|x(t)-y(t)\|_{1,\infty,B}$$

$$+ \sum_{i=1}^{q} c_{S,i,2}\,(t_2-t_1)^{1-\nu}\sum_{o=1}^{q}\sup_{t\in(t_1,t_2]}(t-t_1)^{\nu-1+o}\|x^{(o)}(t)-y^{(o)}(t)\|_1.$$

As we see, there exists sufficiently small t_2-t_1 such that $\|(Sx)(t)-(Sy)(t)\|_{1,q,\nu,B} \le \lambda\|x(t)-y(t)\|_{1,q,\nu,B}$, for some $0<\lambda<1$. Then S is a contraction mapping on B. According to the contraction mapping theorem, S has a unique fixed point in B, i.e. the equation (3.9) has a unique solution in B. Thus, this solution coincides with $x_*(t)$, $t\in(t_1,t_2]$. Therefore, $x_*(t)\in C^{q,\nu}(t_1,t_2]$.

Finally, we shall show $x_*(t)\in C^{q,\nu}(t_0,t_0+h]$, by the arbitrariness of t_1 and t_2, see also [14]. Since all parameters are independent of t_1 and t_2, we can use a uniform δ to denote t_2-t_1, i.e. $\delta=t_2-t_1$. First select $t_1=t_0$, then $x_*(t)\in C^{q,\nu}(t_0,t_0+\delta]$. Then for any $t\in(t_0+\delta,t_0+h]$, select $t_2=t$. It follows from $x_*(t)\in B\subset C^{q,\nu}(t_1,t_2]$ that $x_*(t)$ is q times differentiable at t_2 and $\|x_*^{(i)}(t_2)\|_1\le c(t_2-t_1)^{1-\nu-i}=c\delta^{1-\nu-i}$, $i=1,2,...,q$. Thus, for all $t\in(t_0+\delta,t_0+h]$, x_* is q times differentiable at t and $\|x_*^{(i)}(t)\|_1\le c\delta^{1-\nu-i}\le c[\delta(t-t_0)/h]^{1-\nu-i}=c(\delta/h)^{1-\nu-i}(t-t_0)^{1-\nu-i}$, $i=1,2,...,q$. Therefore, $x_*(t)\in C^{q,\nu}(t_0,t_0+h]$. This completes the proof. □

In the next, we move to the local smoothness theorem for $\alpha>1$. The proof is similar to but more generalized than that of Theorem 3.3.2 ($0<\alpha<1$).

Theorem 3.3.3. *Let $\alpha>1$. Assume that f is continuous in t and x on $\bar{S}$, and $q-m$ times continuously differentiable with respect to t and x on $S=\{(t,x):t\in(t_0,t_0+a],\|x-\sum_{k=0}^{m}(t-t_0)^k x_{0,k}/k!\|_1\le b\}$, and there exist constants $\nu\in[1-(\alpha-m),1)$, M_d and L_d such that, for any $(t,x)\in S$ and all possible nonnegative integers $u_0,u_1,...,u_n$ with $1\le u_0+u_1+...+u_n\le q-m$,*

$$\|\frac{\partial^{u_0+u_1+...+u_n}}{\partial t^{u_0}\partial {x_1}^{u_1}...\partial {x_n}^{u_n}}f(t,x)\|_1 \le M_d\begin{cases}1 & \text{if } u_0=0\\ (t-t_0)^{1-\nu-u_0} & \text{if } u_0\ge 1\end{cases}; \tag{3.16}$$

and for any (t,x), $(t,y)\in S$ and those $u_0,u_1,...,u_n$ with $u_0+u_1+...+u_n=q-m$,

$$\|\frac{\partial^{q-m}}{\partial t^{u_0}\partial {x_1}^{u_1}...\partial {x_n}^{u_n}}f(t,x)-\frac{\partial^{q-m}}{\partial t^{u_0}\partial {y_1}^{u_1}...\partial {y_n}^{u_n}}f(t,y)\|_1 \le L_d\,\|x-y\|_1\begin{cases}1 & \text{if } u_0=0\\ (t-t_0)^{1-\nu-u_0} & \text{if } u_0\ge 1\end{cases}. \tag{3.17}$$

Then (3.1) has a unique solution $x(t)\in C^{q,m,\nu}(t_0,t_0+h]$.

Proof. Since f is continuous on $\bar{S}$, according to Theorem 3.1.1, (3.1) has a continuous solution on $[t_0, t_0+h]$. Let $x_*(t)$ denote this solution, then $x_*(t) \in C[t_0, t_0+h]$, and according to Lemma 3.1.1,

$$x_*(t) = \sum_{k=0}^{m} \frac{(t-t_0)^k}{k!} x_{0,k} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau, x_*(\tau)) d\tau, \ t \in [t_0, t_0+h].$$

Fix two arbitrary different points in $[t_0, t_0+h]$: t_1, t_2 such that $t_0 \le t_1 < t_2 \le t_0+h$, then consider the integral equation

$$x(t) = (Tx)(t) + z(t), \ t \in (t_1, t_2], \tag{3.18}$$

where

$$(Tx)(t) = \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-\tau)^{\alpha-1} f(\tau, x(\tau)) d\tau, \ t \in (t_1, t_2],$$

and

$$z(t) = \sum_{k=0}^{m} \frac{(t-t_0)^k}{k!} x_{0,k} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t-\tau)^{\alpha-1} f(\tau, x_*(\tau)) d\tau, \ t \in (t_1, t_2].$$

Clearly, $x_*(t)$, $t \in (t_1, t_2]$ is a solution to (3.18).

We now begin to show $z(t) \in C^{q,m,\nu}(t_1, t_2]$. For $i = 1, 2, ..., m-1$, $\alpha - i - 1 > 0$ so that

$$z^{(i)}(t) = \sum_{k=i}^{m} \frac{(t-t_0)^{k-i}}{(k-i)!} x_{0,k} + \frac{(\alpha-1)...(\alpha-i)}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t-\tau)^{\alpha-i-1} f(\tau, x_*(\tau)) d\tau,$$

is continuous on $[t_1, t_2]$. For $i = m$ and any $t_1 \le s_1 \le s_2 \le t_2$,

$$\|z^{(m)}(s_1) - z^{(m)}(s_2)\|_1 = \frac{(\alpha-1)...(\alpha-m+1)M}{\Gamma(\alpha)} [(s_2-t_1)^{\alpha-m} - (s_1-t_1)^{\alpha-m} + (s_1-t_0)^{\alpha-m} - (s_2-t_0)^{\alpha-m}]$$

As $s_1 \to s_2$, $\|z(s_1) - z(s_2)\|_1 \to 0$ due to $\alpha > m$. Thus, $z(t) \in C^m[t_1, t_2]$. For $i = m+1, m+2, ..., q$,

$$z^{(i)}(t) = \frac{(\alpha-1)...(\alpha-i)}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t-\tau)^{\alpha-i-1} f(\tau, x_*(\tau)) d\tau, \ t \in (t_1, t_2],$$

which is continuous due to $t \ne \tau$. Thus, for sufficiently small $t_2 - t_1$ such that $t - t_1 < 1$, we can estimate

$$\|z^{(i)}(t)\|_1 \le |\frac{(\alpha-1)...(\alpha-i+1)}{\Gamma(\alpha)}| M (t-t_1)^{1-\nu-(i-m)}, \ t \in (t_1, t_2], \ i = m+1, ..., q.$$

Therefore, $z(t) \in C^{q,m,\nu}(t_1, t_2]$.

Define $(Sx)(t) = (Tx)(t) + z(t)$, $t \in (t_1, t_2]$. We shall show that S maps the following closed ball

$$B = \{x \in C^{q,m,\nu}(t_1, t_2] : \|x - \sum_{k=0}^{m} (t-t_0)^k x_{0,k}/k!\|_{1,\infty,B} \le b \text{ and } \|x\|_{1,q,m,\nu,B} \le c\},$$

where $\|x\|_{1,q,m,\nu,B} = (W+1)\|x\|_{1,\infty,B} + \sum_{i=1}^{m}\|x^{(i)}\|_{1,\infty,B} + \sum_{i=m+1}^{q}\sup_{t\in(t_1,t_2]}(t-t_1)^{\nu-1+(i-m)}\|x^{(i)}(t)\|_1$, $W > \max\{\sum_{i=m+1}^{q} c_{T,i}, \sum_{i=m+1}^{q} c_{S,i,1,1}\}$ and $c > c_z + W$, into itself. Here c_z denotes the right hand side (constant) of the following inequality

$$\|z\|_{1,q,m,\nu,B} \leq (W+1)[\sum_{k=0}^{m}\frac{h^k}{k!}\|x_{0,k}\|_1 + \frac{Mh^\alpha}{\Gamma(\alpha)\alpha}] + \sum_{i=1}^{m}[\sum_{k=i}^{m}\frac{h^{k-i}}{(k-i)!}\|x_{0,k}\|_1 + \frac{(\alpha-1)...(\alpha-i+1)Mh^{\alpha-i}}{\Gamma(\alpha)}]$$
$$+ \sum_{i=m+1}^{q}\frac{|(\alpha-1)...(\alpha-i+1)|M}{\Gamma(\alpha)}.$$

According to Lemma 3.3.3, B equipped with $\|\cdot\|_{1,q,m,\nu,B}$ is nonempty and complete.

For $i = 1, 2, ..., m-1$, $\alpha - i - 1 > 0$ so that

$$(Tx)^{(i)}(t) = \frac{(\alpha-1)...(\alpha-i)}{\Gamma(\alpha)}\int_{t_1}^{t}(t-\tau)^{\alpha-i-1}f(\tau, x(\tau))d\tau,$$

is continuous on $[t_1, t_2]$. For $i = m$ and any $x \in B$, $t_1 \leq s_1 \leq s_2 \leq t_2$,

$$\|(Tx)^{(m)}(s_1) - (Tx)^{(m)}(s_2)\|_1 \leq \frac{(\alpha-1)...(\alpha-m+1)M}{\Gamma(\alpha)}[(s_1-t_1)^{\alpha-m} - (s_2-t_1)^{\alpha-m} + 2(s_2-s_1)^{\alpha-m}]. \tag{3.19}$$

As $s_1 \to s_2$, $\|(Tx)(s_1) - (Tx)(s_2)\|_1 \to 0$ due to $\alpha > m$. Thus, $(Tx)(t) \in C^m[t_1, t_2]$. Moreover, for any $x \in B$,

$$\|(Sx)(t) - \sum_{k=0}^{m}\frac{(t-t_0)^k}{k!}x_{0,k}\|_{1,\infty,B} \leq b, \ t \in [t_1, t_2]. \tag{3.20}$$

We need to further show that for any $x \in B$, $(Sx)(t) \in C^{q,m,\nu}(t_1, t_2]$ and $\|(Sx)(t)\|_{1,q,m,\nu,B} \leq c$. Let $s(t) = t_1 + (t-t_1)/2$, then for $i = m+1, m+2, ..., q$,

$$(Tx)^{(i)}(t) = \frac{(\alpha-1)...(\alpha-m)}{\Gamma(\alpha)}\frac{d^{i'}}{dt^{i'}}\int_{t_1}^{t}(t-\tau)^{\alpha-m-1}f(\tau, x(\tau))d\tau$$
$$= (\alpha-1)...(\alpha-m)[(T_{m1}x)^{(i')}(t) + (T_{m2}x)^{(i')}(t)], t \in (t_1, t_2], i' = 1, ..., q-m,$$

where

$$(T_{m1}x)(t) = \frac{1}{\Gamma(\alpha)}\int_{t_1}^{s(t)}(t-\tau)^{\alpha-m-1}f(\tau, x(\tau))d\tau, \ t \in (t_1, t_2],$$

and

$$(T_{m2}x)(t) = \frac{1}{\Gamma(\alpha)}\int_{s(t)}^{t}(t-\tau)^{\alpha-m-1}f(\tau, x(\tau))d\tau, \ t \in (t_1, t_2].$$

We can derive

$$(T_{m1}x)^{(i')}(t) = c_{i'}\int_{t_1}^{s(t)}(t-\tau)^{\alpha-m-i'-1}f(\tau, (x(\tau))d\tau + \sum_{j'=0}^{i'-1}c_{i'j'}(t-t_1)^{\alpha-m-i'+j'}\frac{d^{j'}}{d\tau^{j'}}f(\tau, x(\tau))|_{\tau=s(t)}, t \in (t_1, t_2], i' = 1, ..., q-m.$$

Since $x \in B$ (then $(\tau, x(\tau)) \in S, \tau \in (t_1, t_2]$) and $f(t, x)$ is $q - m$ times continuously differentiable on S, according to Lemma 3.3.1, $d^{j'} f(\tau, x(\tau))/d\tau^{j'}|_{\tau=s(t)}, t \in (t_1, t_2], i' = 1, 2, ..., q-m$, is continuous. Moreover, $(t-\tau)^{\alpha-m-i'-1} f(\tau, (x(\tau))$ is continuous due to $t > s(t)$ implied by $t > t_1$. Thus, $(T_{m1}x)^{(i')}(t) \in C(t_1, t_2], i' = 1, 2, ..., q-m$. Also, we can derive

$$(T_{m2}x)^{(i')}(t) = d_{1'}\int_{s(t)}^{t}(t-\tau)^{\alpha-m-1}\frac{d^{i'}}{d\tau^{i'}}f(\tau, x(\tau))d\tau+\sum_{j'=0}^{i'-1}d_{i'j'}(t-t_1)^{\alpha-m-i'+j'}\frac{d^{j'}}{d\tau^{j'}}f(\tau, x(\tau))|_{\tau=s(t)}, t \in (t_1, t_2], i' = 1, ..., q-m.$$

For $i' = 1, 2, ..., q-m-1$, we can easily conclude $(T_{2m}x)^{(i')}(t) \in C(t_1, t_2]$, by the integration by parts and the differentiability of $d^{i'} f(\tau, x(\tau))/d\tau^{i'}$. But this does not work for $i' = q-m$, because $f(\tau, x(\tau))$ is not $q-m+1$ times continuously differentiable. In $(T_{2m}x)^{(q-m)}(t)$, the sum term is obviously continuous on $(t_1, t_2]$. Thus, we only need to show that the integral term is also continuous on $(t_1, t_2]$. For sufficiently small $t_2 - t_1$ such that $\tau - t_1 < 1$, if $i' \le m$,

$$\begin{aligned}
\|\frac{d^{i'}}{d\tau^{i'}}f(\tau, x(\tau))\|_1 &= \|\sum_0\sum_1...\sum_{i'}\frac{i'!}{\prod_{j=1}^{i'}(j!)^{k_j}\prod_{j=1}^{i'}\prod_{l=0}^{n}v_{jl}!}\frac{\partial^k}{\partial\tau^{u_0}\partial{x_1}^{u_1}..\partial{x_n}^{u_n}}f(\tau, x(\tau))\prod_{j=1}^{i'}[x_1^{(j)}(\tau)]^{v_{j1}}[x_2^{(j)}(\tau)]^{v_{j2}}...[x_n^{(j)}(\tau)]^{v_{jn}}\|_1 \\
&\le \sum_0\sum_1...\sum_{i'} c_{i',k,n,v}\, M_d\begin{cases}1 & \text{if } u_0 = 0\\ (\tau-t_0)^{1-\nu-u_0} & \text{if } u_0 \ge 1\end{cases} \times c^{\sum_{j=1}^{i'} v_{j1}+v_{j2}+...+v_{jn}} \\
&\le \sum_0\sum_1...\sum_{i'} \tilde{c}_{i',k,n,v}\, M_d(\tau-t_0)^{1-\nu-i'} \\
&\le c_{f,i'}(\tau-t_1)^{1-\nu-i'}, \tau \in (t_1, t_2];
\end{aligned}$$

if $i' \ge m+1$, according to Lemma 3.3.1 and Proposition 3.3.1,

$$\sum_{j=m+1}^{i'}[1-\nu-(j-m)](v_{j1}+...+v_{jn}) \ge \sum_{j=m+1}^{i'}-j(k_j-v_{j0}) = \sum_{j=m+1}^{i'}-jk_j+jv_{j0}+\sum_{j=1}^{m}-jk_j+jk_j \ge -i'+u_0,$$

especially, $\sum_{j=m+1}^{i'}[1-\nu-(j-m)](v_{j1}+...+v_{jn}) = \sum_{j=m+1}^{i'}[1-\nu-(j-m)]k_j = \sum_{j=m+1}^{i'}(1-\nu)k_j + \sum_{j=1}^{m}(1-\nu)k_j - \sum_{j=m+1}^{i'}(j-m)k_j - \sum_{j=1}^{m}(1-\nu)k_j \ge 1-\nu-\sum_{j=1}^{i'} jk_j = 1-\nu-i'$ if $u_0 = 0$, then

$$\begin{aligned}
\|\frac{d^{i'}}{d\tau^{i'}}f(\tau, x(\tau))\|_1 &= \|\sum_0\sum_1...\sum_{i'}\frac{i'!}{\prod_{j=1}^{i'}(j!)^{k_j}\prod_{j=1}^{i'}\prod_{l=0}^{n}v_{jl}!}\frac{\partial^k}{\partial\tau^{u_0}\partial{x_1}^{u_1}..\partial{x_n}^{u_n}}f(\tau, x(\tau))\prod_{j=1}^{i'}[x_1^{(j)}(\tau)]^{v_{j1}}[x_2^{(j)}(\tau)]^{v_{j2}}...[x_n^{(j)}(\tau)]^{v_{jn}}\|_1 \\
&\le \sum_0\sum_1...\sum_{i'} c_{i',k,n,v}\, M_d\begin{cases}1 & \text{if } u_0 = 0\\ (\tau-t_0)^{1-\nu-u_0} & \text{if } u_0 \ge 1\end{cases} \times c^{\sum_{j=1}^{i'} v_{j1}+v_{j2}+...+v_{jn}}(\tau-t_1)^{\sum_{j=m+1}^{i'}[1-\nu-(j-m)](v_{j1}+v_{j2}+...+v_{jn})} \\
&\le \sum_0\sum_1...\sum_{i} \tilde{c}_{i',k,n,v}\, M_d\begin{cases}(\tau-t_1)^{1-\nu-i'} & \text{if } u_0 = 0\\ (\tau-t_0)^{1-\nu-u_0}(\tau-t_1)^{-i'+u_0} & \text{if } u_0 \ge 1\end{cases} \\
&= \sum_0\sum_1...\sum_{i} \tilde{c}_{i',k,n,v}\, M_d\begin{cases}(\tau-t_1)^{1-\nu-i'} & \text{if } u_0 = 0\\ (\tau-t_1)^{1-\nu-i'} & \text{if } u_0 \ge 1\end{cases} \\
&\le c_{f,i'}(\tau-t_1)^{1-\nu-i'}, \tau \in (t_1, t_2].
\end{aligned}$$

Thus, for sufficiently small t_2-t_1 such that $\tau-t_1 < 1$, $\|d^{i'} f(\tau, x(\tau))/d\tau^{i'}\|_1 \leq c_{f,i'}(\tau-t_1)^{1-\nu-i'}$, $\tau \in (t_1, t_2]$, $i' = 1, ..., q-m$, where $c_{f,i'}$ is independent of t_1 and t_2. For $t_1 < s_1 \leq s_2 \leq t_2$, $s_1 - s(s_2) = (s_1 - s_2 + s_1 - t_1)/2 > 0$ if s_2 is sufficiently close to s_1. In this situation,

$$
\begin{aligned}
&\| \int_{s(s_1)}^{s_1} (s_1 - \tau)^{\alpha-m-1} \frac{d^{i'}}{d\tau^{i'}} f(\tau, x(\tau))d\tau - \int_{s(s_2)}^{s_2} (s_2 - \tau)^{\alpha-m-1} \frac{d^{i'}}{d\tau^{i'}} f(\tau, x(\tau))d\tau\|_1 \\
&\leq c_{f,i'}\Big\{ \int_{s(s_1)}^{s(s_2)} (s_1 - \tau)^{\alpha-m-1} d\tau\, [s(s_1) - t_1]^{1-\nu-i'} + \int_{s(s_2)}^{s_1} [(s_1 - \tau)^{\alpha-m-1} - (s_2 - \tau)^{\alpha-m-1}]\, d\tau\, [s(s_1) - t_1]^{1-\nu-i'} \\
&\quad + \int_{s_1}^{s_2} (s_2 - \tau)^{\alpha-m-1} d\tau\, (s_1 - t_1)^{1-\nu-i'} \Big\}.
\end{aligned}
$$

Fix s_1 and let $s_2 \to s_1$, then $\| \int_{s(s_1)}^{s_1} (s_1 - \tau)^{\alpha-m-1} d^{i'} f(\tau, x(\tau))/d\tau^{i'} d\tau - \int_{s(s_2)}^{s_2} (s_2 - \tau)^{\alpha-m-1} d^{i'} f(\tau, x(\tau))/d\tau^{i'} d\tau\|_1 \to 0$. This implies that the integral term of $(T_{m2}x)^{(q-m)}(t)$ is also continuous on $(t_1, t_2]$. Thus, $(T_{m2}x)^{(q-m)}(t) \in C(t_1, t_2]$.

Therefore, for any $x \in B$, $(Tx)^{(i)}(t) \in C(t_1, t_2]$, $i = m + 1, m + 2, ..., q$. Moreover, for sufficiently small $t_2 - t_1$ such that $t - t_1 < 1$,

$$
\begin{aligned}
\|(Tx)^{(i)}(t)\|_1 &= (\alpha - 1)...(\alpha - m)\|(T_{m1}x)^{(i')}(t) + (T_{m2}x)^{(i')}(t)\|_1 \\
&\leq (\alpha - 1)...(\alpha - m)\, [c_{T_m,i'}\, (t - t_1)^{\alpha-m-i'} + c_{T_m,i',1}\, (t - t_1)^{\alpha-m+1-\nu-i'}] \\
&\leq c_{T,i}(t - t_1)^{1-\nu-(i-m)} + c_{T,i,1}(t - t_1)^{\alpha-m+1-\nu-(i-m)}, t \in (t_1, t_2], i = m + 1, ..., q,
\end{aligned}
$$

where $c_{T,i} := (\alpha - 1)...(\alpha - m)(|c_{i'}|M/2^{\alpha-m-i'} + |e_{i'0}|M)$. Thus, $(Tx)(t) \in C^{q,m,\nu}(t_1, t_2]$. This together with $z(t) \in C^{q,m,\nu}(t_1, t_2]$ implies that for any $x \in B$, $(Sx)(t) \in C^{q,m,\nu}(t_1, t_2]$. For any $x \in B$, $t \in (t_1, t_2]$ and sufficiently small $t_2 - t_1$,

$$
\|(Tx)(t)\|_{1,\infty,B} \leq \frac{M}{\Gamma(\alpha + 1)}(t_2 - t_1)^{1-\nu+m},
$$

$$
\|(Tx)^{(i)}(t)\|_{1,\infty,B} \leq \frac{(\alpha - 1)...(\alpha - i + 1)M}{\Gamma(\alpha)}(t_2 - t_1)^{1-\nu-i+m}, i = 1, ..., m,
$$

then

$$
\begin{aligned}
&\|(Sx)(t)\|_{1,q,m,\nu,B} = (W + 1)\|(Sx)(t)\|_{1,\infty,B} + \sum_{i=1}^{m} \|(Sx)^{(i)}(t)\|_{1,\infty,B} + \sum_{i=m+1}^{q} \sup_{t\in(t_1,t_2]} (t - t_1)^{\nu-1+(i-m)}\|(Sx)^{(i)}(t)\|_1 \\
&\leq (W + 1)\|(Tx)(t)\|_{1,\infty,B} + (W + 1)\|z(t)\|_{1,\infty,B} + \sum_{i=1}^{m} \|(Tx)^{(i)}(t)\|_{1,\infty,B} + \sum_{i=1}^{m} \|z^{(i)}(t)\|_{1,\infty,B} \\
&\quad + \sum_{i=m+1}^{q} \sup_{t\in(t_1,t_2]} (t - t_1)^{\nu-1+(i-m)}\|(Tx)^{(i)}(t)\|_1 + \sum_{i=m+1}^{q} \sup_{t\in(t_1,t_2]} (t - t_1)^{\nu-1+(i-m)}\|z^{(i)}(t)\|_1 \\
&\leq (W + 1)\frac{M}{\Gamma(\alpha+1)}(t_2 - t_1)^{1-\nu+m} + \sum_{i=1}^{m} \frac{(\alpha - 1)...(\alpha - i + 1)M}{\Gamma(\alpha)}(t_2 - t_1)^{1-\nu-i+m} + \|z\|_{1,q,m,\nu,B} + \sum_{i=m+1}^{q} c_{T,i} + \sum_{i=m+1}^{q} c_{T,i,1}(t_2 - t_1)^{1-\nu}.
\end{aligned}
$$

Thus, $\|(Sx)(t)\|_{1,q,m,\nu,B} \le c$. It then turns out, $(Sx)(t) \in B$, for any $x \in B$.

In the next, we shall show that S is a contraction mapping on B. It follows from (3.16) and (3.17), for any $x, y \in B$ and $\tau \in (t_1, t_2]$, $(\tau, x(\tau)), (\tau, y(\tau)) \in S$ so that for all possible nonnegative integers $u_0, u_1, ..., u_n$ with $u_0 + u_1 + ... + u_n \le q - m$,

$$\|\frac{\partial^{u_0+u_1+...+u_n}}{\partial\tau^{u_0}\partial x_1{}^{u_1}...\partial x_n{}^{u_n}} f(\tau, x(\tau)) - \frac{\partial^{u_0+u_1+...+u_n}}{\partial t^{u_0}\partial y_1{}^{u_1}...\partial y_n{}^{u_n}} f(\tau, y(\tau))\|_1 \le \max\{nM_d, L_d\}\,\|x(\tau) - y(\tau)\|_1 \begin{cases} 1 & \text{if } u_0 = 0 \\ (\tau - t_0)^{1-\nu-u_0} & \text{if } u_0 \ge 1 \end{cases}, \tag{3.21}$$

especially,

$$\|f(\tau, x(\tau)) - f(\tau, y(\tau))\|_1 \le nM_d\|x(\tau) - y(\tau)\|_1. \tag{3.22}$$

It follows from (3.22) that for any $x, y \in B$, $t \in [t_1, t_2]$ and $t_2 - t_1 < 1$,

$$\|(Sx)(t) - (Sy)(t)\|_{1,\infty,B} = \|(Tx)(t) - (Ty)(t)\|_{1,\infty,B} \le \frac{nM_d}{\Gamma(\alpha+1)}(t_2 - t_1)^{1-\nu+m}\|x(t) - y(t)\|_{1,\infty,B},$$

and

$$\begin{aligned}\|(Sx)^{(i)}(t) - (Sy)^{(i)}(t)\|_{1,\infty,B} &= \|(Tx)^{(i)}(t) - (Ty)^{(i)}(t)\|_{1,\infty,B} \\ &\le \frac{(\alpha-1)...(\alpha-i+1)nM_d}{\Gamma(\alpha)}(t_2 - t_1)^{1-\nu-i+m}\|x(t) - y(t)\|_{1,\infty,B}, i = 1, ..., m.\end{aligned}$$

It follows from (3.21) that for any $x, y \in B$, $\tau \in (t_1, t_2]$ and sufficiently small $t_2 - t_1$ such that $\tau - t_1 < 1$, if $i' \le m$,

$$\begin{aligned}
&\|\frac{d^{i'}}{d\tau^{i'}}[f(\tau, (x(\tau)) - f(\tau, (y(\tau))]\|_1 \\
&= \|\sum_0\sum_1 ... \sum_{i'} c_{i',k,n,v}\Big\{\sum_{o=1}^{i'}\prod_{p=o+1}^{i'}[y_1^{(p)}(\tau)]^{v_{p1}}[y_2^{(p)}(\tau)]^{v_{p2}}...[y_n^{(p)}(\tau)]^{v_{pn}}\sum_{r=1}^{n}[y_{r+1}^{(o)}(\tau)]^{v_{o(r+1)}}[y_{r+2}^{(o)}(\tau)]^{v_{o(r+2)}}...[y_n^{(o)}(\tau)]^{v_{on}} \\
&\quad\times\sum_{l=0}^{v_{or}-1}[y_r^{(o)}(\tau)]^l\frac{\partial^k}{\partial\tau^{u_0}\partial x_1{}^{u_1}...\partial x_n{}^{u_n}}f(\tau, x(\tau))\prod_{j=1}^{o-1}[x_1^{(j)}(\tau)]^{v_{j1}}[x_2^{(j)}(\tau)]^{v_{j2}}...[x_n^{(j)}(\tau)]^{v_{jn}}[x_1^{(o)}(\tau)]^{v_{o1}}[x_2^{(o)}(\tau)]^{v_{o2}}...[x_r^{(o)}(\tau)]^{v_{or}-1-l} \\
&\quad\times[x_r^{(o)}(\tau) - y_r^{(o)}(\tau)] \\
&\quad+\prod_{g=1}^{i'}[y_1^{(g)}(\tau)]^{v_{g1}}[y_2^{(g)}(\tau)]^{v_{g2}}...[y_n^{(g)}(\tau)]^{v_{gn}}[\frac{\partial^k}{\partial\tau^{u_0}\partial x_1{}^{u_1}...\partial x_n{}^{u_n}}f(\tau, x(\tau)) - \frac{\partial^k}{\partial\tau^{u_0}\partial y_1{}^{u_1}...\partial y_n{}^{u_n}}f(\tau, y(\tau))]\Big\}\|_1 \\
&\le \sum_0\sum_1 ... \sum_{i'} c_{i',k,n,v}\Big\{\sum_{o=1}^{i'}\prod_{p=o+1}^{i'}|[y_1^{(p)}(\tau)]^{v_{p1}}[y_2^{(p)}(\tau)]^{v_{p2}}...[y_n^{(p)}(\tau)]^{v_{pn}}|\sum_{r=1}^{n}|[y_{r+1}^{(o)}(\tau)]^{v_{o(r+1)}}[y_{r+2}^{(o)}(\tau)]^{v_{o(r+2)}}...[y_n^{(o)}(\tau)]^{v_{on}}| \\
&\quad\times\sum_{l=0}^{v_{or}-1}|[y_r^{(o)}(\tau)]^l|\,\|\frac{\partial^k}{\partial\tau^{u_0}\partial x_1{}^{u_1}...\partial x_n{}^{u_n}}f(\tau, x(\tau))\|_1\prod_{j=1}^{o-1}|[x_1^{(j)}(\tau)]^{v_{j1}}[x_2^{(j)}(\tau)]^{v_{j2}}...[x_n^{(j)}(\tau)]^{v_{jn}}|
\end{aligned}$$

$$\times\,|[x_1^{(o)}(\tau)]^{v_{o1}}[x_2^{(o)}(\tau)]^{v_{o2}}...[x_r^{(o)}(\tau)]^{v_{or}-1-l}|\,|[x_r^{(o)}(\tau)-y_r^{(o)}(\tau)]|$$

$$+\prod_{g=1}^{i'}|[y_1^{(g)}(\tau)]^{v_{g1}}[y_2^{(g)}(\tau)]^{v_{g2}}...[y_n^{(g)}(\tau)]^{v_{gn}}|\,\|[\frac{\partial^k}{\partial\tau^{u_0}\partial x_1{}^{u_1}...\partial x_n{}^{u_n}}f(\tau,x(\tau))-\frac{\partial^k}{\partial\tau^{u_0}\partial y_1{}^{u_1}...\partial y_n{}^{u_n}}f(\tau,y(\tau))]\|_1\Big\}$$

$$\leq\sum_0\sum_1...\sum_{i'}c_{i',k,n,v}\Big\{\sum_{o=1}^{i'}\prod_{p=o+1}^{i'}(\|y^{(p)}\|_{1,\infty,B})^{v_{p1}+...+v_{pn}}\sum_{r=1}^{n}(\|y^{(o)}\|_{1,\infty,B})^{v_{o(r+1)}+...+v_{on}}\sum_{l=0}^{v_{or}-1}(\|y^{(o)}\|_{1,\infty,B})^{l}$$

$$\times M_d\begin{cases}1 & \text{if } u_0=0\\ (\tau-t_0)^{1-\nu-u_0} & \text{if } u_0\geq 1\end{cases}\times\prod_{j=1}^{o-1}(\|x^{(j)}\|_{1,\infty,B})^{v_{j1}+...+v_{jn}}\times(\|x^{(o)}\|_{1,\infty,B})^{v_{o1}+...+v_{o(r-1)}+v_{or}-1-l}|[x_r^{(o)}(\tau)-y_r^{(o)}(\tau)]|$$

$$+\prod_{g=1}^{i'}(\|y^{(g)}\|_{1,\infty,B})^{v_{g1}+...+v_{gn}}\ \max\{nM_d,L_d\}\,\|x(\tau)-y(\tau)\|_1\begin{cases}1 & \text{if } u_0=0\\ (\tau-t_0)^{1-\nu-u_0} & \text{if } u_0\geq 1\end{cases}\Big\}$$

$$\leq\sum_0\sum_1...\sum_{i'}c_{i',k,n,v}\Big\{c^{-1+\sum_{p=1}^{i'}v_{p1}+...+v_{pn}}\sum_{o=1}^{i'}M_d\begin{cases}1 & \text{if } u_0=0\\ (\tau-t_0)^{1-\nu-u_0} & \text{if } u_0\geq 1\end{cases}\times\sum_{r=1}^{n}\sum_{l=0}^{v_{or}-1}|[x_r^{(o)}(\tau)-y_r^{(o)}(\tau)]|$$

$$+c^{\sum_{g=1}^{i'}v_{g1}+...+v_{gn}}\max\{nM_d,L_d\}\,\|x(\tau)-y(\tau)\|_1\begin{cases}1 & \text{if } u_0=0\\ (\tau-t_0)^{1-\nu-u_0} & \text{if } u_0\geq 1\end{cases}\Big\}$$

$$\leq\sum_0\sum_1...\sum_{i'}\bar{c}_{i',k,n,v}\Big\{\sum_{o=1}^{i'}(\tau-t_1)^{1-\nu-i'}\|x^{(o)}(\tau)-y^{(o)}(\tau)\|_1+(\tau-t_1)^{1-\nu-i'}\,\|x(\tau)-y(\tau)\|_1\Big\}$$

$$\leq c_{f,i',1}(\tau-t_1)^{1-\nu-i'}\|x(\tau)-y(\tau)\|_1+c_{f,i',2}\sum_{o=1}^{i'}(\tau-t_1)^{1-\nu-i'}\|x^{(o)}(\tau)-y^{(o)}(\tau)\|_1;$$

if $i'\geq m+1$,

$$\|\frac{d^{i'}}{d\tau^{i'}}[f(\tau,(x(\tau))-f(\tau,(y(\tau))]\|_1$$

$$\leq\sum_0\sum_1...\sum_{i'}c_{i',k,n,v}\Big\{\sum_{o=1}^{i'}\prod_{p=o+1}^{i'}|[y_1^{(p)}(\tau)]^{v_{p1}}[y_2^{(p)}(\tau)]^{v_{p2}}...[y_n^{(p)}(\tau)]^{v_{pn}}|\sum_{r=1}^{n}|[y_{r+1}^{(o)}(\tau)]^{v_{o(r+1)}}[y_{r+2}^{(o)}(\tau)]^{v_{o(r+2)}}...[y_n^{(o)}(\tau)]^{v_{on}}|$$

$$\times\sum_{l=0}^{v_{or}-1}|[y_r^{(o)}(\tau)]^{l}|\,\|\frac{\partial^k}{\partial\tau^{u_0}\partial x_1{}^{u_1}...\partial x_n{}^{u_n}}f(\tau,x(\tau))\|_1\prod_{j=1}^{o-1}|[x_1^{(j)}(\tau)]^{v_{j1}}[x_2^{(j)}(\tau)]^{v_{j2}}...[x_n^{(j)}(\tau)]^{v_{jn}}|$$

$$\times\,|[x_1^{(o)}(\tau)]^{v_{o1}}[x_2^{(o)}(\tau)]^{v_{o2}}...[x_r^{(o)}(\tau)]^{v_{or}-1-l}|\,|[x_r^{(o)}(\tau)-y_r^{(o)}(\tau)]|$$

$$+\prod_{g=1}^{i'}|[y_1^{(g)}(\tau)]^{v_{g1}}[y_2^{(g)}(\tau)]^{v_{g2}}...[y_n^{(g)}(\tau)]^{v_{gn}}|\,\|[\frac{\partial^k}{\partial\tau^{u_0}\partial x_1{}^{u_1}...\partial x_n{}^{u_n}}f(\tau,x(\tau))-\frac{\partial^k}{\partial\tau^{u_0}\partial y_1{}^{u_1}...\partial y_n{}^{u_n}}f(\tau,y(\tau))]\|_1\Big\}$$

$$\leq\sum_0\sum_1...\sum_{i'}c_{i',k,n,v}\Big\{\sum_{o=1}^{m}\prod_{p=o+1}^{m}(\|y^{(p)}\|_{1,\infty,B})^{v_{p1}+...+v_{pn}}c^{\sum_{p=m+1}^{i'}v_{p1}+...+v_{pn}}(\tau-t_1)^{\sum_{p=m+1}^{i'}[1-\nu-(p-m)](v_{p1}+...+v_{pn})}$$

$$\times \sum_{r=1}^{n}(\|y^{(o)}\|_{1,\infty,B})^{v_{o(r+1)}+\ldots+v_{on}} \sum_{l=0}^{v_{or}-1}(\|y^{(o)}\|_{1,\infty,B})^{l} M_d\begin{cases}1 & \text{if } u_0=0\\ (\tau-t_0)^{1-\nu-u_0} & \text{if } u_0\geq 1\end{cases} \times \prod_{j=1}^{o-1}(\|x^{(j)}\|_{1,\infty,B})^{v_{j1}+\ldots+v_{jn}}$$

$$\times (\|x^{(o)}\|_{1,\infty,B})^{v_{o1}+\ldots+v_{or}-1-l}\|[x_r^{(o)}(\tau)-y_r^{(o)}(\tau)]\| + \sum_{o=m+1}^{i'} c^{\sum_{p=o+1}^{i'} v_{p1}+\ldots+v_{pn}}(\tau-t_1)^{\sum_{p=o+1}^{i'}[1-\nu-(p-m)](v_{p1}+\ldots+v_{pn})}$$

$$\times \sum_{r=1}^{n} c^{v_{o(r+1)}+\ldots+v_{on}}(\tau-t_1)^{[1-\nu-(o-m)](v_{o(r+1)}+\ldots+v_{on})} \sum_{l=0}^{v_{or}-1} c^{l}(\tau-t_1)^{[1-\nu-(o-m)]l} M_d\begin{cases}1 & \text{if } u_0=0\\ (\tau-t_0)^{1-\nu-u_0} & \text{if } u_0\geq 1\end{cases}$$

$$\times \prod_{j=1}^{m}(\|x^{(j)}\|_{1,\infty,B})^{v_{j1}+\ldots+v_{jn}} c^{\sum_{j=m+1}^{o-1} v_{j1}+\ldots+v_{jn}}(\tau-t_1)^{\sum_{j=m+1}^{o-1}[1-\upsilon-(j-m)](v_{j1}+\ldots+v_{jn})} c^{v_{o1}+\ldots+v_{or}-1-l}(\tau-t_1)^{[1-\nu-(o-m)](v_{o1}+\ldots+v_{or}-1-l)}$$

$$\times \|[x_r^{(o)}(\tau)-y_r^{(o)}(\tau)]\| + \prod_{g=1}^{m}(\|y^{(g)}\|_{1,\infty,B})^{v_{g1}+\ldots+v_{gn}} c^{\sum_{g=m+1}^{i'} v_{g1}+\ldots+v_{gn}}(\tau-t_1)^{\sum_{g=m+1}^{i'}[1-\nu-(g-m)](v_{g1}+\ldots+v_{gn})}$$

$$\times \max\{nM_d, L_d\}\|x(\tau)-y(\tau)\|_1\begin{cases}1 & \text{if } u_0=0\\ (\tau-t_0)^{1-\nu-u_0} & \text{if } u_0\geq 1\end{cases}\Bigg\}$$

$$\leq \sum_{0}\sum_{1}\cdots\sum_{i'} c_{i',k,n,v}\Bigg\{c^{-1+\sum_{p=1}^{i'} v_{p1}+\ldots+v_{pn}} \sum_{o=1}^{m}(\tau-t_1)^{\sum_{p=m+1}^{i'}[1-\nu-(p-m)](v_{p1}+\ldots+v_{pn})} M_d\begin{cases}1 & \text{if } u_0=0\\ (\tau-t_0)^{1-\nu-u_0} & \text{if } u_0\geq 1\end{cases}$$

$$\times \sum_{r=1}^{n}\sum_{l=0}^{v_{or}-1}\|[x_r^{(o)}(\tau)-y_r^{(o)}(\tau)]\| + c^{-1+\sum_{p=1}^{i'} v_{p1}+\ldots+v_{pn}} \sum_{o=m+1}^{i'}(\tau-t_1)^{-[1-\nu-(o-m)]+\sum_{p=m+1}^{i'}[1-\nu-(p-m)](v_{p1}+\ldots+v_{pn})}$$

$$\times M_d\begin{cases}1 & \text{if } u_0=0\\ (\tau-t_0)^{1-\nu-u_0} & \text{if } u_0\geq 1\end{cases} \times \sum_{r=1}^{n}\sum_{l=0}^{v_{or}-1}\|[x_r^{(o)}(\tau)-y_r^{(o)}(\tau)]\| + c^{\sum_{g=1}^{i'} v_{g1}+\ldots+v_{gn}}(\tau-t_1)^{\sum_{g=m+1}^{i'}[1-\nu-(g-m)](v_{g1}+\ldots+v_{gn})}$$

$$\times \max\{nM_d, L_d\}\|x(\tau)-y(\tau)\|_1\begin{cases}1 & \text{if } u_0=0\\ (\tau-t_0)^{1-\nu-u_0} & \text{if } u_0\geq 1\end{cases}\Bigg\}$$

$$\leq \sum_{0}\sum_{1}\cdots\sum_{i'} \bar{c}_{i',k,n,v}\Bigg\{\sum_{o=1}^{m}(\tau-t_1)^{1-\nu-i'}\|x^{(o)}(\tau)-y^{(o)}(\tau)\|_1 + \sum_{o=m+1}^{i'}(\tau-t_1)^{(o-m)-i'}\|x^{(o)}(\tau)-y^{(o)}(\tau)\|_1 + (\tau-t_1)^{1-\nu-i'}\|x(\tau)-y(\tau)\|_1\Bigg\}$$

$$\leq c_{f,i',1}(\tau-t_1)^{1-\nu-i'}\|x(\tau)-y(\tau)\|_1 + c_{f,i',2}\sum_{o=1}^{m}(\tau-t_1)^{1-\nu-i'}\|x^{(o)}(\tau)-y^{(o)}(\tau)\|_1 + c_{f,i',3}\sum_{o=m+1}^{i'}(\tau-t_1)^{(o-m)-i'}\|x^{(o)}(\tau)-y^{(o)}(\tau)\|_1.$$

As we see, $c_{f,i',1}, c_{f,i',2}$ and $c_{f,i',3}$ are all independent of t_1 and t_2 in both cases. Now it is ready to estimate $(t-t_1)^{\nu-1+(i-m)}\|(Sx)^{(i)}(t)-(Sy)^{(i)}(t)\|_1$. If $i'\leq m$,

$$\begin{aligned}
&(t-t_1)^{\nu-1+(i-m)}\|(Sx)^{(i)}(t)-(Sy)^{(i)}(t)\|_1 = (t-t_1)^{\nu-1+(i-m)}\|(Tx)^{(i)}(t)-(Ty)^{(i)}(t)\|_1\\
&= (t-t_1)^{\nu-1+i'}(\alpha-1)\ldots(\alpha-m)\|(T_{m1}x)^{(i')}(t)+(T_{m2}x)^{(i')}(t)-(T_{m1}y)^{(i')}(t)-(T_{m2}y)^{(i')}(t)\|_1\\
&= (\alpha-1)\ldots(\alpha-m)(t-t_1)^{\nu-1+i'}\|c_{i'}\int_{t_1}^{s(t)}(t-\tau)^{\alpha-m-i'-1}[f(\tau,(x(\tau))-f(\tau,(y(\tau))]d\tau
\end{aligned}$$

$$
\begin{aligned}
&+ d_{1'}\int_{s(t)}^{t}(t-\tau)^{\alpha-m-1}\frac{d^{i'}}{d\tau^{i'}}[f(\tau,(x(\tau))-f(\tau,(y(\tau))]d\tau+\sum_{j'=0}^{i'-1}e_{i'j'}(t-t_1)^{\alpha-m-i'+j'}\frac{d^{j'}}{d\tau^{j'}}[f(\tau,(x(\tau))-f(\tau,(y(\tau))]|_{\tau=s(t)}\|_1 \\
&\leq (t-t_1)^{\nu-1+i'}\{|\bar{c}_{i'}|\int_{t_1}^{s(t)}(t-\tau)^{\alpha-m-i'-1}\|[f(\tau,(x(\tau))-f(\tau,(y(\tau))]\|_1 d\tau+\bar{d}_{1'}\int_{s(t)}^{t}(t-\tau)^{\alpha-m-1}\|\frac{d^{i'}}{d\tau^{i'}}[f(\tau,(x(\tau))-f(\tau,(y(\tau))]\|_1 d\tau \\
&+\sum_{j'=1}^{i'-1}|\bar{e}_{i'j'}|(t-t_1)^{\alpha-m-i'+j'}\|\frac{d^{j'}}{d\tau^{j'}}[f(\tau,(x(\tau))-f(\tau,(y(\tau))]|_{\tau=s(t)}\|_1+|\bar{e}_{i'0}|(t-t_1)^{\alpha-m-i'}\|[f(\tau,(x(\tau))-f(\tau,(y(\tau))]|_{\tau=s(t)}\|_1\} \\
&\leq (t-t_1)^{\nu-1+i'}\{|\bar{c}_{i'}|nM_d\int_{t_1}^{s(t)}(t-\tau)^{\alpha-m-i'-1}\|x(\tau)-y(\tau)\|_1 d\tau+|\bar{e}_{i'0}|nM_d(t-t_1)^{\alpha-m-i'}\|[(x(\tau)-y(\tau)]|_{\tau=s(t)}\|_1 \\
&+\bar{d}_{1'}\int_{s(t)}^{t}(t-\tau)^{\alpha-m-1}[c_{f,i',1}(\tau-t_1)^{1-\nu-i'}\|x(\tau)-y(\tau)\|_1+c_{f,i',2}\sum_{o=1}^{i'}(\tau-t_1)^{1-\nu-i'}\|x^{(o)}(\tau)-y^{(o)}(\tau)\|_1]d\tau \\
&+\sum_{j'=1}^{i'-1}|\bar{e}_{i'j'}|(t-t_1)^{\alpha-m-i'+j'}[c_{f,j',1}(\tau-t_1)^{1-\nu-j'}\|x(\tau)-y(\tau)\|_1+c_{f,j',2}\sum_{o=1}^{j'}(\tau-t_1)^{1-\nu-j'}\|x^{(o)}(\tau)-y^{(o)}(\tau)\|_1]|_{\tau=s(t)}\} \\
&\leq |\bar{c}_{i'}|nM_d\int_{t_1}^{s(t)}(t-\tau)^{\alpha-m-i'-1}(t-t_1)^{\nu-1+i'}d\tau\|x(t)-y(t)\|_{1,\infty,B}+|\bar{e}_{i'0}|nM_d(t-t_1)^{\alpha-m+\nu-1}\|(x(t)-y(t)\|_{1,\infty,B} \\
&+\bar{d}_{1'}\,c_{f,i',1}\int_{s(t)}^{t}(t-\tau)^{\alpha-m-1}(\tau-t_1)^{1-\nu-i'}(t-t_1)^{\nu-1+i'}d\tau\|x(t)-y(t)\|_{1,\infty,B} \\
&+\bar{d}_{1'}\,c_{f,i',2}\int_{s(t)}^{t}(t-\tau)^{\alpha-m-1}(t-t_1)^{\nu-1+i'}(\tau-t_1)^{1-\nu-i'}\sum_{o=1}^{i'}\|x^{(o)}(\tau)-y^{(o)}(\tau)\|_1 d\tau \\
&+\sum_{j'=1}^{i'-1}|\bar{e}_{i'j'}|c_{f,j',1}(t-t_1)^{\alpha-m-i'+j'}(\frac{t-t_1}{2})^{1-\nu-j'}(t-t_1)^{\nu-1+i'}\|x(t)-y(t)\|_{1,\infty,B} \\
&+\sum_{j'=1}^{i'-1}|\bar{e}_{i'j'}|c_{f,j',2}(t-t_1)^{\alpha-m-i'+j'}(\frac{t-t_1}{2})^{1-\nu-j'}(t-t_1)^{\nu-1+i'}\sum_{o=1}^{j'}\|x^{(o)}(s(t))-y^{(o)}(s(t))\|_1 \\
&\leq |\bar{c}_{i'}|nM_d 2^{-\alpha+m+i'}\|x(t)-y(t)\|_{1,\infty,B}+|\bar{e}_{i'0}|nM_d\|(x(t)-y(t)\|_{1,\infty,B}+\bar{d}_{1'}\,c_{f,i',1}\frac{2^{-\alpha+m+\nu-1+i'}}{\alpha-m}(t-t_1)^{1-\nu}\|x(t)-y(t)\|_{1,\infty,B} \\
&+\bar{d}_{1'}\,c_{f,i',2}\frac{2^{-\alpha+m+\nu-1+i'}}{\alpha-m}(t-t_1)^{1-\nu}\sum_{o=1}^{i'}\|x^{(o)}(t)-y^{(o)}(t)\|_{1,\infty,B}+\sum_{j'=1}^{i'-1}|\bar{e}_{i'j'}|c_{f,j',1}2^{\nu-1+j'}(t-t_1)^{1-\nu}\|x(t)-y(t)\|_{1,\infty,B} \\
&+\sum_{j'=1}^{i'-1}|\bar{e}_{i'j'}|c_{f,j',2}2^{\nu-1+j'}(t-t_1)^{1-\nu}\sum_{o=1}^{j'}\|x^{(o)}(t)-y^{(o)}(t)\|_{1,\infty,B} \\
&=[c_{S,i,1,1}+c_{S,i,1,2}\,(t-t_1)^{1-\nu}]\|x(t)-y(t)\|_{1,\infty,B}+c_{S,i,2}\,(t-t_1)^{1-\nu}\sum_{o=1}^{m}\|x^{(o)}(t)-y^{(o)}(t)\|_{1,\infty,B},\, i=m+1,...,q,
\end{aligned}
$$

where $c_{S,i,1,1} := |\bar{c}_{i'}|nM_d 2^{-\alpha+m+i'} + |\bar{e}_{i'0}|nM_d$; if $i' \geq m+1$,

$$(t-t_1)^{\nu-1+(i-m)}\|(Sx)^{(i)}(t) - (Sy)^{(i)}(t)\|_1$$

$$\leq (t-t_1)^{\nu-1+i'}\{|\bar{c}_{i'}|\int_{t_1}^{s(t)}(t-\tau)^{\alpha-m-i'-1}\|[f(\tau,(x(\tau))-f(\tau,(y(\tau))]\|_1 d\tau + \bar{d}_{1'}\int_{s(t)}^{t}(t-\tau)^{\alpha-m-1}\|\frac{d^{i'}}{d\tau^{i'}}[f(\tau,(x(\tau))-f(\tau,(y(\tau))]\|_1 d\tau$$

$$+ \sum_{j'=1}^{i'-1}|\bar{e}_{i'j'}|(t-t_1)^{\alpha-m-i'+j'}\|\frac{d^{j'}}{d\tau^{j'}}[f(\tau,(x(\tau))-f(\tau,(y(\tau))]_{\tau=s(t)}\|_1 + |\bar{e}_{i'0}|(t-t_1)^{\alpha-m-i'}\|[f(\tau,(x(\tau))-f(\tau,(y(\tau))]_{\tau=s(t)}\|_1\}$$

$$\leq (t-t_1)^{\nu-1+i'}\{|\bar{c}_{i'}|nM_d\int_{t_1}^{s(t)}(t-\tau)^{\alpha-m-i'-1}\|x(\tau)-y(\tau)\|_1 d\tau + |\bar{e}_{i'0}|nM_d(t-t_1)^{\alpha-m-i'}\|[(x(\tau)-y(\tau)]_{\tau=s(t)}\|_1$$

$$+ \bar{d}_{1'}\int_{s(t)}^{t}(t-\tau)^{\alpha-m-1}[c_{f,i',1}(\tau-t_1)^{1-\nu-i'}\|x(\tau)-y(\tau)\|_1 + c_{f,i',2}\sum_{o=1}^{m}(\tau-t_1)^{1-\nu-i'}\|x^{(o)}(\tau)-y^{(o)}(\tau)\|_1$$

$$+ c_{f,i',3}\sum_{o=m+1}^{i'}(\tau-t_1)^{(o-m)-i'}\|x^{(o)}(\tau)-y^{(o)}(\tau)\|_1]d\tau$$

$$+ \sum_{j'=1}^{i'-1}|\bar{e}_{i'j'}|(t-t_1)^{\alpha-m-i'+j'}[c_{f,j',1}(\tau-t_1)^{1-\nu-j'}\|x(\tau)-y(\tau)\|_1 + c_{f,j',2}\sum_{o=1}^{m}(\tau-t_1)^{1-\nu-j'}\|x^{(o)}(\tau)-y^{(o)}(\tau)\|_1$$

$$+ c_{f,j',3}\sum_{o=m+1}^{j'}(\tau-t_1)^{(o-m)-j'}\|x^{(o)}(\tau)-y^{(o)}(\tau)\|_1]|_{\tau=s(t)}\}$$

$$\leq |\bar{c}_{i'}|nM_d\int_{t_1}^{s(t)}(t-\tau)^{\alpha-m-i'-1}(t-t_1)^{\nu-1+i'}d\tau\|x(t)-y(t)\|_{1,\infty,B} + |\bar{e}_{i'0}|nM_d(t-t_1)^{\alpha-m+\nu-1}\|(x(t)-y(t)\|_{1,\infty,B}$$

$$+ \bar{d}_{1'}c_{f,i',1}\int_{s(t)}^{t}(t-\tau)^{\alpha-m-1}(\tau-t_1)^{1-\nu-i'}(t-t_1)^{\nu-1+i'}d\tau\|x(t)-y(t)\|_{1,\infty,B}$$

$$+ \bar{d}_{1'}c_{f,i',2}\int_{s(t)}^{t}(t-\tau)^{\alpha-m-1}(t-t_1)^{\nu-1+i'}(\tau-t_1)^{1-\nu-i'}\sum_{o=1}^{m}\|x^{(o)}(\tau)-y^{(o)}(\tau)\|_1 d\tau$$

$$+ \bar{d}_{1'}c_{f,i',3}\int_{s(t)}^{t}(t-\tau)^{\alpha-m-1}(t-t_1)^{\nu-1+i'}(\tau-t_1)^{1-\nu-i'}\sum_{o=m+1}^{i'}(\tau-t_1)^{\nu-1+(o-m)}\|x^{(o)}(\tau)-y^{(o)}(\tau)\|_1 d\tau$$

$$+ \sum_{j'=1}^{i'-1}|\bar{e}_{i'j'}|c_{f,j',1}(t-t_1)^{\alpha-m-i'+j'}(\frac{t-t_1}{2})^{1-\nu-j'}(t-t_1)^{\nu-1+i'}\|x(t)-y(t)\|_{1,\infty,B}$$

$$+ \sum_{j'=1}^{i'-1}|\bar{e}_{i'j'}|c_{f,j',2}(t-t_1)^{\alpha-m-i'+j'}(\frac{t-t_1}{2})^{1-\nu-j'}(t-t_1)^{\nu-1+i'}\sum_{o=1}^{m}\|x^{(o)}(s(t))-y^{(o)}(s(t))\|_1$$

$$+ \sum_{j'=1}^{i'-1}|\bar{e}_{i'j'}|c_{f,j',3}(t-t_1)^{\alpha-m+\nu-1+j'}(\frac{t-t_1}{2})^{1-\nu-j'}\sum_{o=m+1}^{j'}[s(t)-t_1]^{\nu-1+(o-m)}\|x^{(o)}(s(t))-y^{(o)}(s(t))\|_1$$

$$\leq |\bar{c}_{i'}|nM_d 2^{-\alpha+m+i'}\|x(t)-y(t)\|_{1,\infty,B} + |\bar{e}_{i'0}|nM_d\|(x(t)-y(t)\|_{1,\infty,B} + \bar{d}_{1'}\, c_{f,i',1}\frac{2^{-\alpha+m+\nu-1+i'}}{\alpha-m}(t-t_1)^{1-\nu}\|x(t)-y(t)\|_{1,\infty,B}$$

$$+ \bar{d}_{1'}\, c_{f,i',2}\frac{2^{-\alpha+m+\nu-1+i'}}{\alpha-m}(t-t_1)^{1-\nu}\sum_{o=1}^{m}\|x^{(o)}(t)-y^{(o)}(t)\|_{1,\infty,B}$$

$$+ \bar{d}_{1'}\, c_{f,i',3}\frac{2^{-\alpha+m+\nu-1+i'}}{\alpha-m}(t-t_1)^{1-\nu}\sum_{o=m+1}^{i'}\sup_{t\in(t_1,t_2]}(t-t_1)^{\nu-1+(o-m)}\|x^{(o)}(t)-y^{(o)}(t)\|_1$$

$$+ \sum_{j'=1}^{i'-1}|\bar{e}_{i'j'}|c_{f,j',1}2^{\nu-1+j'}(t-t_1)^{1-\nu}\|x(t)-y(t)\|_{1,\infty,B} + \sum_{j'=1}^{i'-1}|\bar{e}_{i'j'}|c_{f,j',2}2^{\nu-1+j'}(t-t_1)^{1-\nu}\sum_{o=1}^{m}\|x^{(o)}(t)-y^{(o)}(t)\|_{1,\infty,B}$$

$$+ \sum_{j'=1}^{i'-1}|\bar{e}_{i'j'}|c_{f,j',3}2^{\nu-1+j'}(t-t_1)^{1-\nu}\sum_{o=m+1}^{j'}\sup_{t\in(t_1,t_2]}(t-t_1)^{\nu-1+(o-m)}\|x^{(o)}(t)-y^{(o)}(t)\|_1$$

$$= [c_{S,i,1,1} + c_{S,i,1,2}\,(t-t_1)^{1-\nu}]\|x(t)-y(t)\|_{1,\infty,B} + c_{S,i,2}\,(t-t_1)^{1-\nu}\sum_{o=1}^{m}\|x^{(o)}(t)-y^{(o)}(t)\|_{1,\infty,B}$$

$$+ c_{S,i,3}\,(t-t_1)^{1-\nu}\sum_{o=m+1}^{q}\sup_{t\in(t_1,t_2]}(t-t_1)^{\nu-1+(o-m)}\|x^{(o)}(t)-y^{(o)}(t)\|_1,\ i=m+1,...,q,$$

where $c_{S,i,1,1} := |\bar{c}_{i'}|nM_d 2^{-\alpha+m+i'} + |\bar{e}_{i'0}|nM_d$. Thus,

$\|(Sx)(t)-(Sy)(t)\|_{1,q,m,\nu,B}$

$$= (W+1)\|(Sx)(t)-(Sy)(t)\|_{1,\infty,B} + \sum_{i=1}^{m}\|(Sx)^{(i)}(t)-(Sy)^{(i)}(t)\|_{1,\infty,B} + \sum_{i=m+1}^{q}\sup_{t\in(t_1,t_2]}(t-t_1)^{\nu-1+(i-m)}\|(Sx)^{(i)}(t)-(Sy)^{(i)}(t)\|_1$$

$$\leq (W+1)\frac{nM_d}{\Gamma(\alpha+1)}(t_2-t_1)^{1-\nu+m}\|x(t)-y(t)\|_{1,\infty,B} + \sum_{i=1}^{m}\frac{(\alpha-1)...(\alpha-i+1)nM_d}{\Gamma(\alpha)}(t_2-t_1)^{1-\nu-i+m}\|x(t)-y(t)\|_{1,\infty,B}$$

$$+ \sum_{i=m+1}^{q}\{[c_{S,i,1,1} + c_{S,i,1,2}\,(t_2-t_1)^{1-\nu}]\|x(t)-y(t)\|_{1,\infty,B} + c_{S,i,2}\,(t_2-t_1)^{1-\nu}\sum_{o=1}^{m}\|x^{(o)}(t)-y^{(o)}(t)\|_{1,\infty,B}$$

$$+ c_{S,i,3}(t_2-t_1)^{1-\nu}\sum_{o=m+1}^{q}\sup_{t\in(t_1,t_2]}(t-t_1)^{\nu-1+(o-m)}\|x^{(o)}(t)-y^{(o)}(t)\|_1\}$$

$$= [\frac{nM_d}{\Gamma(\alpha+1)}(t_2-t_1)^{1-\nu+m} + \frac{1}{W+1}\sum_{i=1}^{m}\frac{(\alpha-1)...(\alpha-i+1)nM_d}{\Gamma(\alpha)}(t_2-t_1)^{1-\nu-i+m} + \frac{\sum_{i=m+1}^{q}c_{S,i,1,1}}{W+1}$$

$$+ \frac{\sum_{i=m+1}^{q}c_{S,i,1,2}}{W+1}\,(t_2-t_1)^{1-\nu}](W+1)\|x(t)-y(t)\|_{1,\infty,B} + \sum_{i=m+1}^{q}c_{S,i,2}(t_2-t_1)^{1-\nu}\sum_{o=1}^{m}\|x^{(o)}(t)-y^{(o)}(t)\|_{1,\infty,B}$$

$$+ \sum_{i=m+1}^{q}c_{S,i,3}\,(t_2-t_1)^{1-\nu}\sum_{o=m+1}^{q}\sup_{t\in(t_1,t_2]}(t-t_1)^{\nu-1+(o-m)}\|x^{(o)}(t)-y^{(o)}(t)\|_1.$$

We can select $t_2 - t_1$ sufficiently small such that $\|(Sx)(t) - (Sy)(t)\|_{1,q,m,\nu,B} \le \lambda \|x(t) - y(t)\|_{1,q,m,\nu,B}$, for some $0 < \lambda < 1$, then S is a contraction mapping on B. Thus, S has a unique fixed point in B, i.e. the equation (3.18) has a unique solution in B. Thus, the solution coincides with $x_*(t)$ on $(t_1, t_2]$. Therefore, $x_*(t) \in C^{q,m,\nu}(t_1, t_2]$.

Since all parameters are independent of t_1 and t_2, we can select any t_1, t_2 with a uniform distance δ, $\delta = t_2 - t_1$, on $[t_0, t_0 + h]$, then it follows, $x_*(t) \in C^{q,m,\nu}(t_1, t_2]$. We first select $t_1 = t_0$, then $x_*(t) \in C^{q,m,\nu}(t_0, t_0 + \delta]$. For any $t \in (t_0 + \delta, t_0 + h]$, we select $t_2 = t$, then $x_*(t) \in C^{q,m,\nu}(t_1, t]$, i.e. x_* is q times differentiable at t and $\|x_*^{(i)}(t)\|_1 \le c\delta^{1-\nu-(i-m)} \le c[\delta(t - t_0)/h]^{1-\nu-(i-m)} = c(\delta/h)^{1-\nu-(i-m)}(t - t_0)^{1-\nu-(i-m)}$, $i = m+1, m+2, ..., q$. Therefore, $x_*(t) \in C^{q,m,\nu}(t_0, t_0 + h]$. □

Proof of Corollary 3.3.1

Proof. Since f is $q - m$ times continuously differentiable on $\bar{S}$ and $\bar{S}$ is closed and bounded, there exists $M_d > 0$ such that for any $(t, x) \in S$ and all possible nonnegative integers $u_0, u_1, ..., u_n$ with $1 \le u_0 + u_1 + ... + u_n \le q - m$,

$$\|\frac{\partial^{u_0+u_1+...+u_n}}{\partial t^{u_0} \partial x_1{}^{u_1} ... \partial x_n{}^{u_n}} f(t, x)\|_1 \le M_d. \tag{3.23}$$

Compared to (3.4) and (3.5), (3.6) and (3.23) are without the "if" part. It can be shown as follows that without this part we can still derive the same estimation for $\|d^{i'} f(\tau, x(\tau))/d\tau^{i'}\|_1$ and $\|d^{i'}[f(\tau, (x(\tau)) - f(\tau, (y(\tau))]/d\tau^{i'}\|_1$. If $i' \le m$,

$$\begin{aligned}
\|\frac{d^{i'}}{d\tau^{i'}} f(\tau, x(\tau))\|_1 =& \|\sum_0 \sum_1 ... \sum_{i'} \frac{i'!}{\prod_{j=1}^{i'} (j!)^{k_j} \prod_{j=1}^{i'} \prod_{l=0}^{n} v_{jl}!} \frac{\partial^k}{\partial \tau^{u_0} \partial x_1{}^{u_1} ... \partial x_n{}^{u_n}} f(\tau, x(\tau)) \prod_{j=1}^{i'} [x_1^{(j)}(\tau)]^{v_{j1}} [x_2^{(j)}(\tau)]^{v_{j2}} ... [x_n^{(j)}(\tau)]^{v_{jn}} \|_1 \\
&\le \sum_0 \sum_1 ... \sum_{i'} c_{i',k,n,v} M_d \, c^{\sum_{j=1}^{i'} v_{j1} + v_{j2} + ... + v_{jn}} \\
&\le \sum_0 \sum_1 ... \sum_{i'} \tilde{c}_{i',k,n,v} M_d (\tau - t_1)^{1-\nu-i'} \\
&\le c_{f,i'} (\tau - t_1)^{1-\nu-i'},
\end{aligned}$$

and

$$\begin{aligned}
&\|\frac{d^{i'}}{d\tau^{i'}} [f(\tau, (x(\tau)) - f(\tau, (y(\tau))]\|_1 \\
&\le \sum_0 \sum_1 ... \sum_{i'} c_{i',k,n,v} \Big\{ c^{-1+\sum_{p=1}^{i'} v_{p1} + ... + v_{pn}} \sum_{o=1}^{i'} M_d \sum_{r=1}^{n} \sum_{l=0}^{v_{or}-1} \|[x_r^{(o)}(\tau) - y_r^{(o)}(\tau)]\| + c^{\sum_{g=1}^{i'} v_{g1} + ... + v_{gn}} \max\{n M_d, L_d\} \|x(\tau) - y(\tau)\|_1 \Big\} \\
&\le \sum_0 \sum_1 ... \sum_{i'} \bar{c}_{i',k,n,v} \Big\{ \sum_{o=1}^{i'} \sum_{r=1}^{n} \|[x_r^{(o)}(\tau) - y_r^{(o)}(\tau)]\| + \|x(\tau) - y(\tau)\|_1 \Big\} \\
&\le \sum_0 \sum_1 ... \sum_{i'} \bar{c}_{i',k,n,v} \Big\{ \sum_{o=1}^{i'} (\tau - t_1)^{1-\nu-i'} \|x^{(o)}(\tau) - y^{(o)}(\tau)\|_1 + (\tau - t_1)^{1-\nu-i'} \|x(\tau) - y(\tau)\|_1 \Big\} \\
&\le c_{f,i',1} (\tau - t_1)^{1-\nu-i'} \|x(\tau) - y(\tau)\|_1 + c_{f,i',2} \sum_{o=1}^{i'} (\tau - t_1)^{1-\nu-i'} \|x^{(o)}(\tau) - y^{(o)}(\tau)\|_1.
\end{aligned}$$

If $i' \geq m+1$, according to Lemma 3.3.1 and Proposition 3.3.1, $\sum_{j=m+1}^{i'}[1-\nu-(j-m)](v_{j1}+...+v_{jn}) = \sum_{j=m+1}^{i'}[1-\nu-(j-m)]k_j = \sum_{j=m+1}^{i'}(1-\nu)k_j + \sum_{j=1}^{m}(1-\nu)k_j - \sum_{j=m+1}^{i'}(j-m)k_j - \sum_{j=1}^{m}(1-\nu)k_j \geq 1-\nu-\sum_{j=1}^{i'} jk_j = 1-\nu-i'$, then

$$
\begin{aligned}
\|\frac{d^{i'}}{d\tau^{i'}}f(\tau,x(\tau))\|_1 =& \|\sum_0\sum_1...\sum_{i'}\frac{i'!}{\prod_{j=1}^{i'}(j!)^{k_j}\prod_{j=1}^{i'}\prod_{l=0}^{n}v_{jl}!}\frac{\partial^k}{\partial\tau^{u_0}\partial x_1{}^{u_1}...\partial x_n{}^{u_n}}f(\tau,x(\tau))\prod_{j=1}^{i'}[x_1^{(j)}(\tau)]^{v_{j1}}[x_2^{(j)}(\tau)]^{v_{j2}}...[x_n^{(j)}(\tau)]^{v_{jn}}\|_1 \\
&\leq \sum_0\sum_1...\sum_{i'} c_{i',k,n,v}\, M_d\, c^{\sum_{j=1}^{i'} v_{j1}+v_{j2}+...+v_{jn}}(\tau-t_1)^{\sum_{j=m+1}^{i'}[1-\nu-(j-m)](v_{j1}+v_{j2}+...+v_{jn})} \\
&\leq \sum_0\sum_1...\sum_{i} \tilde{c}_{i',k,n,v}\, M_d(\tau-t_1)^{1-\nu-i'} \\
&\leq c_{f,i'}(\tau-t_1)^{1-\nu-i'},
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\frac{d^{i'}}{d\tau^{i'}}[f(\tau,(x(\tau))-f(\tau,(y(\tau))]\|_1 \\
&\leq \sum_0\sum_1...\sum_{i'} c_{i',k,n,v}\Big\{c^{-1+\sum_{p=1}^{i'}v_{p1}+...+v_{pn}}\sum_{o=1}^{m}(\tau-t_1)^{\sum_{p=m+1}^{i'}[1-\nu-(p-m)](v_{p1}+...+v_{pn})}M_d\sum_{r=1}^{n}\sum_{l=0}^{v_{or}-1}|[x_r^{(o)}(\tau)-y_r^{(o)}(\tau)]| \\
&\quad + c^{-1+\sum_{p=1}^{i'}v_{p1}+...+v_{pn}}\sum_{o=m+1}^{i'}(\tau-t_1)^{-[1-\nu-(o-m)]+\sum_{p=m+1}^{i'}[1-\nu-(p-m)](v_{p1}+...+v_{pn})}M_d\sum_{r=1}^{n}\sum_{l=0}^{v_{or}-1}|[x_r^{(o)}(\tau)-y_r^{(o)}(\tau)]| \\
&\quad + c^{\sum_{g=1}^{i'}v_{g1}+...+v_{gn}}(\tau-t_1)^{\sum_{g=m+1}^{i'}[1-\nu-(g-m)](v_{g1}+...+v_{gn})}\max\{nM_d,L_d\}\|x(\tau)-y(\tau)\|_1\Big\} \\
&\leq \sum_0\sum_1...\sum_{i'}\bar{c}_{i',k,n,v}\Big\{\sum_{o=1}^{m}(\tau-t_1)^{1-\nu-i'}\|x^{(o)}(\tau)-y^{(o)}(\tau)\|_1 + \sum_{o=m+1}^{i'}(\tau-t_1)^{(o-m)-i'}\|x^{(o)}(\tau)-y^{(o)}(\tau)\|_1 \\
&\quad + (\tau-t_1)^{1-\nu-i'}\|x(\tau)-y(\tau)\|_1\Big\} \\
&\leq c_{f,i',1}(\tau-t_1)^{1-\nu-i'}\|x(\tau)-y(\tau)\|_1 + c_{f,i',2}\sum_{o=1}^{m}(\tau-t_1)^{1-\nu-i'}\|x^{(o)}(\tau)-y^{(o)}(\tau)\|_1 + c_{f,i',3}\sum_{o=m+1}^{i'}(\tau-t_1)^{(o-m)-i'}\|x^{(o)}(\tau)-y^{(o)}(\tau)\|_1.
\end{aligned}
$$

The other part of proof remains the same as that of Theorem 3.3.3. Thus, (3.1) has a unique solution $x(t) \in C^{q,m,\nu}(t_0, t_0+h]$. Note that here ν is arbitrary in $[1-(\alpha-m), 1)$. Therefore, $x(t) \in C^{q,m,1-(\alpha-m)}(t_0, t_0+h]$. □

3.3.3 Global Smoothness

In this subsection, a main theorem for the smoothness property of global solutions on the full interval $[t_0, t_0+a]$ is first proven, then the continuation results are applied to deduce some useful corollaries for that of solutions on their maximal interval of existence.

Theorem 3.3.4. *Assume that f is continuous in t and x on $\bar{S}_g$, and there exist constants $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ and $0 \leq \mu < 1$ such that for any $(t,x) \in \bar{S}_g$,*

$$\|f(t,x)\|_1 \leq \lambda_1 + \lambda_2 \|x\|_1^{\mu}. \tag{3.24}$$

Moreover, assume that f is $q-m$ times continuously differentiable with respect to t and x on $S_g = \{(t,x) : t \in (t_0, t_0 + a], x \in \mathbb{R}^n\}$, and there exists a constant $\nu \in [1-(\alpha-m), 1)$ and monotonically increasing functions φ, ψ: $[0,\infty) \to [0,\infty)$, such that for any $(t,x) \in S_g$ and all possible nonnegative integers $u_0, u_1, ..., u_n$ with $1 \leq u_0 + u_1 + ... + u_n \leq q-m$,

$$\|\frac{\partial^{u_0+u_1+...+u_n}}{\partial t^{u_0} \partial {x_1}^{u_1} ... \partial {x_n}^{u_n}} f(t,x)\|_1 \leq \varphi(\|x\|_1) \begin{cases} 1 & \text{if } u_0 = 0 \\ (t-t_0)^{1-\nu-u_0} & \text{if } u_0 \geq 1 \end{cases}; \tag{3.25}$$

and for any (t,x), $(t,y) \in S_g$ and those $u_0, u_1, ..., u_n$ with $u_0 + u_1 + ... + u_n = q-m$,

$$\|\frac{\partial^{q-m}}{\partial t^{u_0} \partial {x_1}^{u_1} ... \partial {x_n}^{u_n}} f(t,x) - \frac{\partial^{q-m}}{\partial t^{u_0} \partial {y_1}^{u_1} ... \partial {y_n}^{u_n}} f(t,y)\|_1 \leq \psi(\max\{\|x\|_1, \|y\|_1\}) \|x-y\|_1 \begin{cases} 1 & \text{if } u_0 = 0 \\ (t-t_0)^{1-\nu-u_0} & \text{if } u_0 \geq 1 \end{cases}. \tag{3.26}$$

Then (3.1) has a unique solution $x(t) \in C^{q,m,\nu}(t_0, t_0+a]$.

Proof. According to Remark 3.1.1, it follows from the continuity of f on $\bar{S}_g$ and (3.24), (3.1) has a continuous solution on $[t_0, t_0+a]$. The rest of this proof remains the same as the proof of Theorem 3.3.3, except that the closed ball B is taken to be $B_g = \{x \in C^{q,m,\nu}(t_1,t_2] : \|x\|_{1,\infty,B_g} \leq b_g \text{ and } \|x\|_{1,q,m,\nu,B_g} \leq c_g\}$, where $b_g > \sum_{k=0}^{m} a^k \|x_{0,k}\|_1/k! + a^{\alpha}[\lambda_1 + \lambda_2(\|x_*\|_{1,\infty})^{\mu}]/\Gamma(\alpha+1)$, $\|x\|_{1,q,m,\nu,B_g} = (W+1)\|x\|_{1,\infty,B_g} + \sum_{i=1}^{m} \|x^{(i)}\|_{1,\infty,B_g} + \sum_{i=m+1}^{q} \sup_{t \in (t_1,t_2]} (t-t_1)^{\nu-1+(i-m)} \|x^{(i)}(t)\|_1$, $W > \max\{\sum_{i=m+1}^{q} c_{T,i}, \sum_{i=m+1}^{q} c_{S,i,1,1}\}$ and $c_g > c_z + W$; the inequality (3.20) is replaced by $\|(Sx)(t)\|_{1,\infty,B_g} \leq b_g$ for sufficiently small $t_2 - t_1$; and the parameters h, c, B, S M before (3.19), M in and after (3.19), M_d and L_d are replaced to be a, c_g, B_g, S_g, $\lambda_1 + \lambda_2(\|x_*\|_{1,\infty})^{\mu}$, $\lambda_1 + \lambda_2 b_g^{\mu}$, $\varphi(b_g)$, and $\psi(b_g)$, respectively. □

Remark 3.3.3. *If $q = m$, i.e. only the continuity of f on $\bar{S}$ and (3.24) are assumed, then (3.1) has a solution $x(t) \in C^m[t_0, t_0+a]$.*

Remark 3.3.4. *If follows from Remark 3.1.1 that the sufficient condition (3.24) for the existence of solution on $[t_0, t_0+a]$ can be replaced by the Lipschitz condtion: there exists a constant $L > 0$ such that for any $(t,x), (t,y) \in \bar{S}_g$, $\|f(t,x) - f(t,y)\|_1 \leq L\|x-y\|_1$. Alternatively, we can immediately assume that (3.1) has a solution $x(t) \in C[t_0, t_0+a]$, instead of these sufficient conditions for existence.*

Remark 3.3.5. *It follows that (3.26) is satisfied if for any (t,x), $(t,y) \in S_g$, $l = 1, 2, ..., n$ and those $u_0, u_1, ..., u_n$ with $u_0 + u_1 + ... + u_n = q-m$,*

$$\|\frac{\partial^{q-m+1}}{\partial t^{u_0} \partial {x_1}^{u_1} ... \partial {x_l}^{u_l+1} ... \partial {x_n}^{u_n}} f(t,x)\|_1 \leq \frac{1}{n} \psi(\max\{\|x\|_1, \|y\|_1\}) \begin{cases} 1 & \text{if } u_0 = 0 \\ (t-t_0)^{1-\nu-u_0} & \text{if } u_0 \geq 1 \end{cases}.$$

As in (3.4) and (3.5) of Theorem 3.3.1, the "if" part in (3.25) and (3.26) can be also removed, see the followings.

Corollary 3.3.3. *Assume that f is is $q-m$ times continuously differentiable on $\bar{S}_g$, and there exist constants $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ and $0 \leq \mu < 1$ such that for any $(t,x) \in \bar{S}_g$,*

$$\|f(t,x)\|_1 \leq \lambda_1 + \lambda_2 \|x\|_1^{\mu}. \tag{3.27}$$

Moreover, assume that there exists a monotonically increasing function ψ: $[0,\infty) \to [0,\infty)$, such that for any $(t,x),(t,y) \in \bar{S}_g$ and all possible nonnegative integers $u_0, u_1, ..., u_n$ with $u_0 + u_1 + ... + u_n = q-m$,

$$\|\frac{\partial^{q-m}}{\partial t^{u_0}\partial x_1{}^{u_1}...\partial x_n{}^{u_n}} f(t,x) - \frac{\partial^{q-m}}{\partial t^{u_0}\partial y_1{}^{u_1}...\partial y_n{}^{u_n}} f(t,y)\|_1 \leq \psi(\max\{\|x\|_1, \|y\|_1\})\,\|x-y\|_1. \tag{3.28}$$

Then (3.1) has a unique solution $x(t) \in C^{q,m,1-(\alpha-m)}(t_0, t_0+a]$.

Proof. Since f is $q-m$ times continuously differentiable on $\bar{S}_g$, then the $q-m\,th$ order partial derivatives are continuous on $\bar{S}_g$. Thus, for all possible nonnegative integers $u_0, u_1, ..., u_n$ with $1 \leq u_0 + u_1 + ... + u_n \leq q-m$, $\|\partial^{u_0+u_1+...+u_n} f(t,x)/\partial t^{u_0}\partial x_1{}^{u_1}...\partial x_l{}^{u_l}...\partial x_n{}^{u_n}\|_1$ is also continuous in t and x on $\bar{S}_g$. Let $\varphi(\|x\|_1) = \max_{t\in[t_0,t_0+a],\,\|y\|_1\leq\|x\|_1}$ $\|\partial^{u_0+u_1+...+u_n} f(t,y)/\partial t^{u_0}\partial y_1{}^{u_1}...\partial y_l{}^{u_l}...\partial y_n{}^{u_n}\|_1$, then for any $(t,x) \in \bar{S}_g$ and all possible nonnegative integers $u_0, u_1, ..., u_n$ with $1 \leq u_0 + u_1 + ... + u_n \leq q-m$,

$$\|\frac{\partial^{u_0+u_1+...+u_n}}{\partial t^{u_0}\partial x_1{}^{u_1}...\partial x_n{}^{u_n}} f(t,x)\|_1 \leq \varphi(\|x\|_1). \tag{3.29}$$

Compared to (3.25) and (3.26), (3.28) and (3.29) are without the "if" part. It can be shown as in the proof of Corollary 3.3.1 (with c, M_d and L_d replaced by c_g, $\varphi(b_g)$ and $\psi(b_g)$, respectively) that without this part the estimation for $\|d^{i'} f(\tau, x(\tau))/d\tau^{i'}\|_1$ and $\|d^{i'}[f(\tau,(x(\tau)) - f(\tau,(y(\tau))]/d\tau^{i'}\|_1$ remains the same form. The rest of proof is the same as that of Theorem 3.3.4. Therefore, (3.1) has a unique solution $x(t) \in C^{q,m,\nu}(t_0, t_0+a]$, for any $\nu \in [1-(\alpha-m), 1)$, i.e. $x(t) \in C^{q,m,1-(\alpha-m)}(t_0, t_0+a]$. □

By using the continuation results, we can now prove the following corollary that suggests the smoothness property of solutions on their maximal interval of existence.

Corollary 3.3.4. *Let $D = [t_0, \gamma) \times \mathbb{R}^n$, where $\gamma \leq \infty$. Assume that f is continuous in t and x on D, and $q-m$ times continuously differentiable with respect to t and x on $(t_0,\gamma)\times\mathbb{R}^n$, and for any $\eta \in (t_0, \gamma)$, there exists a common constant $\nu \in [1-(\alpha-m), 1)$ and corresponding monotonically increasing functions φ_η, ψ_η: $[0,\infty) \to [0,\infty)$, such that for any $(t,x) \in (t_0,\eta]\times\mathbb{R}^n$ and all possible nonnegative integers $u_0, u_1, ..., u_n$ with $1 \leq u_0 + u_1 + ... + u_n \leq q-m$,*

$$\|\frac{\partial^{u_0+u_1+...+u_n}}{\partial t^{u_0}\partial x_1{}^{u_1}...\partial x_n{}^{u_n}} f(t,x)\|_1 \leq \varphi_\eta(\|x\|_1)\begin{cases}1 & \text{if } u_0 = 0\\ (t-t_0)^{1-\nu-u_0} & \text{if } u_0 \geq 1\end{cases}; \tag{3.30}$$

and for any (t,x), $(t,y) \in (t_0,\eta]\times\mathbb{R}^n$ and those $u_0, u_1, ..., u_n$ with $u_0 + u_1 + ... + u_n = q-m$,

$$\|\frac{\partial^{q-m}}{\partial t^{u_0}\partial x_1{}^{u_1}...\partial x_n{}^{u_n}} f(t,x) - \frac{\partial^{q-m}}{\partial t^{u_0}\partial y_1{}^{u_1}...\partial y_n{}^{u_n}} f(t,y)\|_1 \leq \psi_\eta(\max\{\|x\|_1, \|y\|_1\})\,\|x-y\|_1\begin{cases}1 & \text{if } u_0 = 0\\ (t-t_0)^{1-\nu-u_0} & \text{if } u_0 \geq 1\end{cases}. \tag{3.31}$$

Then (3.1) has a unique solution $x(t) \in C^{q,m,\nu}(t_0,\beta)$, where $[t_0,\beta)$ is the maximal interval of existence for $x(t)$.

Proof. Since for sufficiently small a, $\bar{S} \subset D$, then f is continuous in t and x on $\bar{S}$, and $q - m$ times continuously differentiable with respect to t and x on S. Choose a sequence of sets in D: $\{D_n\}$, such that $\cup_{n=1}^{\infty} D_n = D$, $\bar{D}_n$ is bounded and $\bar{D}_n \subset D_{n+1}$ for $n = 1, 2...,$. Then there exists $N > 0$ such that $n > N$ implies $\bar{S} \subset D_n$. Clearly, $\bar{S} \subset \bar{D}_n$. Let $\bar{D}_n = [t_0, \gamma_n] \times \bar{\Omega}_n$ and $\eta = \gamma_n$, then there exist $M_d = \varphi_\eta(\sum_{k=0}^{m} a^k \|x_{0,k}\|_1/k + b)$ and $L_d = \psi_\eta(\sum_{k=0}^{m} a^k \|x_{0,k}\|_1/k + b)$, such that (3.4) and (3.5) hold for all (t, x), $(t, y) \in S$. According to Theorem 3.3.1, (3.1) has a unique solution $x(t) \in C^{q,m,\nu}(t_0, t_0 + h]$ and $(t, x(t)) \in \bar{S}$ for $t \in [t_0, t_0 + h]$.

According to Theorem 3.2.3, the solution can be extended out of $\bar{D}_n$, i.e. there exists $\beta_n \le \gamma_{n+1}$ such that $(\beta_n, \tilde{x}(\beta_n)) \notin \bar{D}_n$, where the extended solution $\tilde{x}(t) \in C[t_0, \beta_n]$ and $\tilde{x}(t) = x(t)$, for $t \in [t_0, t_0 + h]$. According to Theorem 3.3.4 and Remark 3.3.4, the continuous solution $\tilde{x}(t) \in C^{q,m,\nu}(t_0, \beta_n]$ and it is unique. Similarly, for D_{n+1}, $n > N$, there exists $\beta_{n+1} \le \gamma_{n+2}$ such that the solution has a smooth extension to $[t_0, \beta_{n+1}]$ and $(\beta_{n+1}, \tilde{x}(\beta_{n+1})) \notin \bar{D}_{n+1}$. Clearly, $\{\beta_n\}$ is a monotone increasing sequence. Let $\beta = \lim_{n\to\infty} \beta_n$, then $\beta \le \gamma$. Thus, $x(t)$ has been extended to $[t_0, \beta)$ and cannot be extended further, since the sequence $\{(\beta_n, x(\beta_n))\}$ is either unbounded or has a limit point on the boundary of D. Therefore, the solution and its smoothness can be extended over its maximal interval of existence $[t_0, \beta)$. □

Note that if $\beta < \infty$, there exists a uniform constant $c > 0$ such that $\|x^{(i)}(t)\| \le c(t - t_0)^{1-\nu-(i-m)}$, $i = m + 1, ..., q$, for any $t \in (t_0, \beta)$; if $\beta = \infty$, there may only exist $c_d > 0$ depending on d such that $\|x^{(i)}(t)\| \le c_d(t - t_0)^{1-\nu-(i-m)}$, $t \in (t_0, d]$, for each $d \in (t_0, \infty)$. This follows from the end of the proof of Theorem 3.3.2 or 3.3.3, where the constant $c(\delta/h)^{1-\nu-(i-m)}$ is bounded for any finite h but blows up as $h \to \infty$. Moreover, the "if" part in this corollary can be removed as before, see Corollary 3.3.5.

Corollary 3.3.5. *Let $D = [t_0, \gamma) \times \mathbb{R}^n$, where $\gamma \le \infty$. Assume that f is $q-m$ times continuously differentiable on $[t_0, \gamma) \times \mathbb{R}^n$, and for any $\eta \in (t_0, \gamma)$, there exists a corresponding monotonically increasing function ψ_η: $[0, \infty) \to [0, \infty)$, such that for any $(t, x), (t, y) \in [t_0, \eta] \times \mathbb{R}^n$ and all possible nonnegative integers $u_0, u_1, ..., u_n$ with $u_0 + u_1 + ... + u_n = q - m$,*

$$\|\frac{\partial^{q-m}}{\partial t^{u_0} \partial x_1{}^{u_1} ... \partial x_n{}^{u_n}} f(t, x) - \frac{\partial^{q-m}}{\partial t^{u_0} \partial y_1{}^{u_1} ... \partial y_n{}^{u_n}} f(t, y)\|_1 \le \psi_\eta(\max\{\|x\|_1, \|y\|_1\}) \|x - y\|_1. \tag{3.32}$$

Then (3.1) has a unique solution $x(t) \in C^{q,m,1-(\alpha-m)}(t_0, \beta)$, where $[t_0, \beta)$ is the maximal interval of existence for $x(t)$.

Proof. The proof is the same as that of Corollary 3.3.4, except that $M_d = \varphi_\eta(\sum_{k=0}^{m} a^k \|x_{0,k}\|_1/k + b)$ is deleted, and Theorem 3.3.1, Theorem 3.3.4 are replaced by Corollary 3.3.1, Corollary 3.3.3, respectively. Note that ν changes to $1 - (\alpha - m)$ in the conclusion, due to its arbitrariness. □

As we shall see in the corollary below, the conditions in the corollary above can be replaced by some stronger ones.

Corollary 3.3.6. *Assume that f is $q - m + 1$ times continuously differentiable with respect to t and x on $[0, \infty) \times \mathbb{R}^n$. Then (3.1) has a unique solution $x(t) \in C^{q,m,1-(\alpha-m)}(t_0, \beta)$, where $[t_0, \beta)$ is the maximal interval of existence for $x(t)$.*

Proof. Since f is $q-m+1$ times continuously differentiable on $[0,\infty)\times\mathbb{R}^n$, i.e. the $q-m+1th$ order partial derivatives are continuous on $[0,\infty)\times\mathbb{R}^n$, then f is $q-m$ times continuously differentiable on $[t_0,\gamma)\times\mathbb{R}^n$. Moreover, for any $\eta\in(t_0,\gamma)$, the $q-m+1th$ order partial derivatives are continuous on $[t_0,\eta]\times\mathbb{R}^n$. Thus, for all possible nonnegative integers $u_0,u_1,...,u_n$ with $1\le u_0+u_1+...+u_n\le q-m+1$, $\|\partial^{u_0+u_1+...+u_n}f(t,x)/\partial t^{u_0}\partial {x_1}^{u_1}...\partial {x_l}^{u_l}...\partial {x_n}^{u_n}\|_1$ is also continuous in t and x on $[t_0,\eta]\times\mathbb{R}^n$. Let $\phi_\eta(\|x\|_1)=\max_{t\in[t_0,\eta],\,\|y\|_1\le\|x\|_1}\|\partial^{u_0+u_1+...+u_n}f(t,y)/\partial t^{u_0}\partial {y_1}^{u_1}...\partial {y_l}^{u_l}...\partial {y_n}^{u_n}\|_1$, then for any $(t,x)\in[t_0,\eta]\times\mathbb{R}^n$ and all possible nonnegative integers $u_0,u_1,...,u_n$ with $1\le u_0+u_1+...+u_n\le q-m+1$,

$$\|\frac{\partial^{u_0+u_1+...+u_n}}{\partial t^{u_0}\partial {x_1}^{u_1}...\partial {x_l}^{u_l}...\partial {x_n}^{u_n}}f(t,x)\|_1\le\phi_\eta(\|x\|_1).$$

Let $u_0+u_1+...+u_n=q-m$, then it follows that for any $(t,x)\in[t_0,\eta]\times\mathbb{R}^n$ and $l,r=1,2,...,n$,

$$|\frac{\partial^{i+1}}{\partial t^{u_0}\partial {x_1}^{u_1}...\,{x_l}^{u_l+1}...\,\partial {x_n}^{u_n}}f_r(t,x)|\le\phi_\eta(\|x\|_1).$$

According to the mean value theorem, for any $(t,x),(t,y)\in[t_0,\eta]\times\mathbb{R}^n$,

$$\begin{aligned}\|\frac{\partial^{q-m}}{\partial t^{u_0}\partial {x_1}^{u_1}...\partial {x_n}^{u_n}}f(t,x)-\frac{\partial^{q-m}}{\partial t^{u_0}\partial {y_1}^{u_1}...\partial {y_n}^{u_n}}f(t,y)\|_1&=\|\int_0^1 D(\frac{\partial^{q-m}}{\partial t^{u_0}\partial {w_1}^{u_1}...\partial {w_n}^{u_n}}f)(t,w)d\theta(x-y)\|_1\\&\le\int_0^1\|D(\frac{\partial^{q-m}}{\partial t^{u_0}\partial {w_1}^{u_1}...\partial {w_n}^{u_n}}f)(t,w)\|_1d\theta\|(x-y)\|_1\\&\le n\phi_\eta(\max\{\|x\|_1,\|y\|_1\})\|x-y\|_1\end{aligned}$$

Let $\psi_\eta=n\phi_\eta$, then (3.32) is satisfied. According to Corollary 3.3.5, the conclusion follows. □

3.4 Lyapunov Stability

When talking about stability, one is interested in the behavior of solutions for $t\to\infty$. Therefore, we only consider those initial value problems whose solutions exist on $[0,\infty)$, see page 157 in [1]. For Lyapunov stability analysis, the Caputo fractional order nonautonomous system can be given by [1] as

$$\begin{cases}{}_0^CD_t^\alpha x=f(t,x)\\x(0)=x_0,\end{cases}\tag{3.33}$$

where $\alpha\in(0,1)$ and $f:G\to\mathbb{R}^n$, $G=[0,\infty)\times\mathbb{R}^n$.

Definition 3.4.1. *The constant x^* is an equilibrium point of (3.33), if and only if ${}_0^CD_t^\alpha x^*=f(t,x^*)$, for any $t\ge0$.*

Note that the constant x^* is an equilibrium point if and only $f(t,x^*)=0$ for all $t\ge0$, due to ${}_0^CD_t^\alpha x^*\equiv0$. Obviously, it is the same as the definition of equilibrium points of integer order systems. We now, referring to Definition 7.2 in [1], introduce the stability concepts for (3.33) in the sense of Lyapunov.

Definition 3.4.2. *Assume $f(t,0) \equiv 0$ and let $x(t) = x(t, x_0)$ denote the solution of (3.33). Then the trivial solution of (3.33) is said to be*

i. *stable, if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\|x_0\|_2 < \delta$ implies $\|x(t)\|_2 < \epsilon$, for all $t \geq 0$;*

ii. *asymptotically stable, if it is stable, and there exists a $\sigma > 0$ such that $\|x_0\|_2 < \sigma$ implies $\lim_{t\to\infty} \|x(t)\|_2 = 0$.*

As we see, the concepts of Lyapunov stability for Caputo fractional order systems are simpler than those for integer order systems. This is mainly because the initial time of the formers must be the same as that of their fractional order differential operators so that there is no concept of uniform stability. Next, quadratic Lyapunov functions will be employed to investigate the Lyapunov stability.

3.4.1 Quadratic Lyapunov Function

Here we work out an estimation for the Caputo fractional order derivative of a general quadratic Lyapunov function by using the smoothness property of solutions. On this estimation, there are already some results, see [16, 17, 18]. However, in all these results, it was assumed that the $x(t)$ involved in the quadratic Lyapunov function is differentiable. As illustrated by a counterexample in Introduction, this assumption is not feasible. Fortunately, our smoothness results enable us to derive the following.

Lemma 3.4.1. *Assume:*

i. *f is continuous in t and x on $[0,\infty) \times \mathbb{R}^n$;*

ii. *f is continuously differentiable with respect to t and x on $(0,\infty) \times \mathbb{R}^n$;*

iii. *for any $h^* > 0$, there exist corresponding monotonically increasing functions φ_{h^*}, $\psi_{h^*} : [0,\infty) \to [0,\infty)$, such that for any $(t,x), (t,y) \in (0,h^*]\times\mathbb{R}^n$ and all possible nonnegative integers $u_0, u_1, ..., u_n$ with $u_0+u_1+...+u_n = 1$,*

$$\|\frac{\partial}{\partial t^{u_0}\partial x_1{}^{u_1}...\partial x_n{}^{u_n}} f(t,x)\|_1 \leq \varphi_{h^*}(\|x\|_1)\begin{cases}1 & \text{if } u_0 = 0\\ t^{\alpha-1} & \text{if } u_0 = 1\end{cases} \tag{3.34}$$

and

$$\|\frac{\partial}{\partial t^{u_0}\partial x_1{}^{u_1}...\partial x_n{}^{u_n}} f(t,x) - \frac{\partial}{\partial t^{u_0}\partial y_1{}^{u_1}...\partial y_n{}^{u_n}} f(t,y)\|_1 \leq \psi_{h^*}(\max\{\|x\|_1, \|y\|_1\})\,\|x-y\|_1\begin{cases}1 & \text{if } u_0 = 0\\ t^{\alpha-1} & \text{if } u_0 = 1\end{cases}. \tag{3.35}$$

Then for any $n \times n$ positive definite matrix P, ${}_0^C D_t^{\alpha}[x^T(t)Px(t)] \in C[0,\beta)$ and

$${}_0^C D_t^{\alpha}[x^T(t)Px(t)] \leq [{}_0^C D_t^{\alpha} x^T(t)]Px(t) + x^T(t)P\,{}_0^C D_t^{\alpha} x(t), \tag{3.36}$$

for all $t \in [0,\beta)$, where $x(t)$ is the solution of (3.33) and $[0,\beta)$ is the maximal interval of existence for $x(t)$.

Proof. According to Corollary 3.3.4, (3.33) has a unique solution $x(t) \in C^{1,1-\alpha}[0,\beta)$. Thus, $x(t) \in C[0,\beta) \cap C^1(0,\beta)$. Moreover, if $\beta < \infty$, then there exists a uniform constant $c > 0$ such that $\|x'(t)\|_1 \le ct^{\alpha-1}$, for any $t \in (0,\beta)$; if $\beta = \infty$, then for each $0 < d < \infty$, there exists a constant $c_d > 0$, depending on d, such that $\|x'(t)\|_1 \le c_d t^{\alpha-1}$, for any $t \in (0,d]$. We shall first consider $\beta = \infty$, then the conclusion for $\beta < \infty$ follows.

For each fixed $t \in (0,\infty)$, $\Delta x^T(\tau)[\Delta x(\tau)]'/(t-\tau)^\alpha$ is integrable on any closed subinterval of $(0,t)$, where $\Delta x(t) = x(t) - x_0$, and there exist constants $d_\Delta > t$ and $c_{d_\Delta} > 0$ depending on d_Δ, such that for $\tau \in (0,t] \subseteq (0,d_\Delta]$, $\|\dot{x}(\tau)\|_1 \le c_{d_\Delta}\tau^{\alpha-1}$ and $|\dot{x}_i(\tau)| \le c_{d_\Delta}\tau^{\alpha-1}$, $i = 1,2,...,n$, where $x^T = [x_1, x_2, ..., x_n]$. Thus, for any $\tau \in (0,t]$,

$$\int_0^\tau -c_{d_\Delta} s^{\alpha-1} ds \le \int_0^\tau \dot{x}_i(s) ds \le \int_0^\tau c_{d_\Delta} s^{\alpha-1} ds,$$

so that $|x_i(\tau) - x_i(0)| \le (c_{d_\Delta}/\alpha)\tau^\alpha$, then $\|\Delta x(\tau)\|_1 \le c_\Delta \tau^\alpha$, $c_\Delta = nc_{d_\Delta}/\alpha$. Thus,

$$\int_0^t \Big|\frac{\Delta x^T(\tau)P[\Delta x(\tau)]'}{(t-\tau)^\alpha}\Big| \mathrm{d}\tau \le \int_0^t \frac{\|\Delta x(\tau)\|_1\|P\|_1\|[\Delta x(\tau)]'\|_1}{(t-\tau)^\alpha}\mathrm{d}\tau \le c_\Delta c_{d_\Delta}\|P\|_1 \int_0^t \frac{\tau^{2\alpha-1}}{(t-\tau)^\alpha}\mathrm{d}\tau = c_\Delta c_{d_\Delta}\|P\|_1 \frac{\Gamma(1-\alpha)\Gamma(2\alpha)}{\Gamma(\alpha+1)} t^\alpha,$$

so that the improper integral ${}_0^C D_t^\alpha[\Delta x^T(t)P\Delta x(t)]$ is absolutely convergent on $(0,\infty)$, i.e. ${}_0^C D_t^\alpha[\Delta x^T(t)P\Delta x(t)]$ exists on $(0,\infty)$. Moreover, ${}_0^C D_{0^+}^\alpha[\Delta x^T(t)P\Delta x(t)]$ exists and it equals zero. Let ${}_0^C D_0^\alpha[\Delta x^T(t)P\Delta x(t)] = {}_0^C D_{0^+}^\alpha[\Delta x^T(t)P\Delta x(t)]$, then ${}_0^C D_0^\alpha[\Delta x^T(t)P\Delta x(t)]$ exists on $[0,\infty)$.

For a given $\delta_1 > 0$, there exist constants $d_1 > \delta_1$ and $c_{d_1} > 0$ depending on d_1 such that for any $t \in (0,\delta_1]$, $\|\dot{x}(t)\|_1 \le c_{d_1}t^{\alpha-1}$, then $\|[\Delta x(t)]'\|_1 = \|\dot{x}(t)\|_1 \le c_{d_1}t^{\alpha-1}$ and $\|\Delta x(t)\|_1 \le c_1 t^\alpha$, $c_1 = nc_{d_1}/\alpha$. Therefore, $t_1, t_2 \in [0,\delta_1]$ implies

$$\begin{aligned}
|{}_0^C D_{t_1}^\alpha[\Delta x^T(t)P\Delta x(t)] - {}_0^C D_{t_2}^\alpha[\Delta x^T(t)P\Delta x(t)]| &= \frac{2}{\Gamma(1-\alpha)}\Big|\int_0^{t_1}\frac{\Delta x^T(\tau)P[\Delta x(\tau)]'}{(t_1-\tau)^\alpha}\mathrm{d}\tau - \int_0^{t_2}\frac{[\Delta x^T(\tau)P[\Delta x(\tau)]'}{(t_2-\tau)^\alpha}\mathrm{d}\tau\Big| \\
&\le \frac{2}{\Gamma(1-\alpha)}\|P\|_1[\int_0^{t_1}\frac{\|\Delta x(\tau)\|_1\|[\Delta x(\tau)]'\|_1}{(t_1-\tau)^\alpha}\mathrm{d}\tau + \int_0^{t_2}\frac{\|\Delta x(\tau)\|_1\|[\Delta x(\tau)]'\|_1}{(t_2-\tau)^\alpha}\mathrm{d}\tau] \\
&\le \frac{2}{\Gamma(1-\alpha)}c_1c_{d_1}\|P\|_1[\int_0^{t_1}\frac{\tau^{2\alpha-1}}{(t_1-\tau)^\alpha}\mathrm{d}\tau + \int_0^{t_2}\frac{\tau^{2\alpha-1}}{(t_2-\tau)^\alpha}\mathrm{d}\tau] \\
&= \frac{2}{\Gamma(1-\alpha)}c_1c_{d_1}\|P\|_1\frac{\Gamma(1-\alpha)\Gamma(2\alpha)}{\Gamma(1+\alpha)}[t_1^\alpha + t_2^\alpha] \\
&\le 4c_1c_{d_1}\|P\|_1\frac{\Gamma(2\alpha)}{\Gamma(1+\alpha)}\delta_1^\alpha \\
&= \hat{c}_1\delta_1^\alpha,
\end{aligned}$$

where $\hat{c}_1 = 4c_1c_{d_1}\|P\|_1\Gamma(2\alpha)/\Gamma(1+\alpha)$. We now, referring to (3.2) – (3.3) in [19], have the following statements.

If $t_0 \in [0,\delta_1/2]$, then $|t-t_0| \le \delta_1/2$ implies $|{}_0^C D_t^\alpha[\Delta x^T(t)P\Delta x(t)] - {}_0^C D_{t_0}^\alpha[\Delta x^T(t)P\Delta x(t)]| \le \hat{c}_1\delta_1^\alpha$.

If $t_0 \in [\delta_1/2,\infty)$, for $|t-t_0| \le \delta_1/2$, we only need to consider the case in which $t \in [\delta_1/2,\infty)$. For the given t_0, there exists $d_2 > t_0+\delta_1/2$ and c_{d_2} depending on d_2 such that for any $t \in (0,t_0+\delta_1/2]$, $\|\dot{x}(t)\|_1 \le c_{d_2}t^{\alpha-1}$ so $\|[\Delta x(t)]'\|_1 =$

$\|\dot{x}(t)\|_1 \le c_{d_2} t^{\alpha-1}$ and $\|\Delta x(t)\|_1 \le c_2 t^{\alpha}$, $c_2 = nc_{d_2}/\alpha$. Assume $t \ge t_0$, then

$$\int_{t_0}^{t} \frac{\tau^{2\alpha-1}}{(t-\tau)^{\alpha}} \mathrm{d}\tau = \max\{t^{2\alpha-1}, t_0^{2\alpha-1}\} \int_{t_0}^{t} \frac{1}{(t-\tau)^{\alpha}} \mathrm{d}\tau = \frac{1}{1-\alpha} \max\{t^{2\alpha-1}, t_0^{2\alpha-1}\}(t-t_0)^{1-\alpha}$$
$$\le \begin{cases} \frac{1}{1-\alpha} t_0^{2\alpha-1} (\frac{\delta_1}{2})^{1-\alpha} & \text{if } \alpha \in (0, 0.5) \\ \frac{1}{1-\alpha} t^{2\alpha-1} (\frac{\delta_1}{2})^{1-\alpha} & \text{if } \alpha \in [0.5, 1) \end{cases} \le \begin{cases} \frac{1}{1-\alpha} (\frac{\delta_1}{2})^{\alpha} & \text{if } \alpha \in (0, 0.5) \\ \frac{1}{1-\alpha} (t_0 + \frac{\delta_1}{2})^{2\alpha-1} (\frac{\delta_1}{2})^{1-\alpha} & \text{if } \alpha \in [0.5, 1) \end{cases},$$

and

$$\int_{0}^{t_0} \tau^{2\alpha-1} [\frac{1}{(t_0-\tau)^{\alpha}} - \frac{1}{(t-\tau)^{\alpha}}] \mathrm{d}\tau = \int_{0}^{\frac{\delta_1}{2}} \tau^{2\alpha-1} [\frac{1}{(t_0-\tau)^{\alpha}} - \frac{1}{(t-\tau)^{\alpha}}] \mathrm{d}\tau + \int_{\frac{\delta_1}{2}}^{t_0} \tau^{2\alpha-1} [\frac{1}{(t_0-\tau)^{\alpha}} - \frac{1}{(t-\tau)^{\alpha}}] \mathrm{d}\tau$$
$$\le \int_{0}^{\frac{\delta_1}{2}} \tau^{2\alpha-1} [\frac{1}{(t_0-\tau)^{\alpha}} + \frac{1}{(t-\tau)^{\alpha}}] \mathrm{d}\tau + \max\{t_0^{2\alpha-1}, (\frac{\delta_1}{2})^{2\alpha-1}\} \int_{\frac{\delta_1}{2}}^{t_0} \frac{1}{(t_0-\tau)^{\alpha}} - \frac{1}{(t-\tau)^{\alpha}} \mathrm{d}\tau$$
$$\le 2 \int_{0}^{\frac{\delta_1}{2}} \frac{\tau^{2\alpha-1}}{(\frac{\delta_1}{2}-\tau)^{\alpha}} \mathrm{d}\tau + \frac{1}{1-\alpha} \max\{t_0^{2\alpha-1}, (\frac{\delta_1}{2})^{2\alpha-1}\} [(t-t_0)^{1-\alpha} + (t_0 - \frac{\delta_1}{2})^{1-\alpha} - (t - \frac{\delta_1}{2})^{1-\alpha}]$$
$$\le \frac{2\Gamma(1-\alpha)\Gamma(2\alpha)}{\Gamma(\alpha+1)} (\frac{\delta_1}{2})^{\alpha} + \frac{1}{1-\alpha} \max\{t_0^{2\alpha-1}, (\frac{\delta_1}{2})^{2\alpha-1}\} (\frac{\delta_1}{2})^{1-\alpha}$$
$$\le \begin{cases} \frac{1}{1-\alpha} [(\frac{\delta_1}{2})^{\alpha} + \frac{2\Gamma(2-\alpha)\Gamma(2\alpha)}{\Gamma(\alpha+1)} (\frac{\delta_1}{2})^{\alpha}] & \text{if } \alpha \in (0, 0.5) \\ \frac{1}{1-\alpha} [t_0^{2\alpha-1} (\frac{\delta_1}{2})^{1-\alpha} + \frac{2\Gamma(2-\alpha)\Gamma(2\alpha)}{\Gamma(\alpha+1)} (\frac{\delta_1}{2})^{\alpha}] & \text{if } \alpha \in [0.5, 1) \end{cases},$$

so that

$$|{}_0^C D_t^{\alpha} [\Delta x^T(t) P \Delta x(t)] - {}_0^C D_{t_0}^{\alpha} [\Delta x^T(t) P \Delta x(t)]| = \frac{2}{\Gamma(1-\alpha)} \Big| \int_0^t \frac{\Delta x^T(\tau) P [\Delta x(\tau)]'}{(t-\tau)^{\alpha}} \mathrm{d}\tau - \int_0^{t_0} \frac{\Delta x^T(\tau) P [\Delta x(\tau)]'}{(t_0-\tau)^{\alpha}} \mathrm{d}\tau \Big|$$
$$\le \frac{2}{\Gamma(1-\alpha)} \{ \int_{t_0}^{t} \frac{\|\Delta x(\tau)\|_1 \|P\|_1 \|[\Delta x(\tau)]'\|_1}{(t-\tau)^{\alpha}} \mathrm{d}\tau + \int_0^{t_0} \|\Delta x(\tau)\|_1 \|P\|_1 \|[\Delta x(\tau)]'\|_1 [\frac{1}{(t_0-\tau)^{\alpha}} - \frac{1}{(t-\tau)^{\alpha}}] \mathrm{d}\tau \}$$
$$\le \frac{2}{\Gamma(1-\alpha)} c_2 c_{d_2} \|P\|_1 \{ \int_{t_0}^{t} \frac{\tau^{2\alpha-1}}{(t-\tau)^{\alpha}} \mathrm{d}\tau + \int_0^{t_0} \tau^{2\alpha-1} [\frac{1}{(t_0-\tau)^{\alpha}} - \frac{1}{(t-\tau)^{\alpha}}] \mathrm{d}\tau \}$$
$$\le \hat{c}_{21} \begin{cases} 2(\frac{\delta_1}{2})^{\alpha} + \hat{c}_{22} (\frac{\delta_1}{2})^{\alpha} & \text{if } \alpha \in (0, 0.5) \\ (t_0 + \frac{\delta_1}{2})^{2\alpha-1} (\frac{\delta_1}{2})^{1-\alpha} + t_0^{2\alpha-1} (\frac{\delta_1}{2})^{1-\alpha} + \hat{c}_{22} (\frac{\delta_1}{2})^{\alpha} & \text{if } \alpha \in [0.5, 1) \end{cases},$$

where $\hat{c}_{21} = 2c_2 c_{d_2} \|P\|_1 / \Gamma(2-\alpha)$ and $\hat{c}_{22} = 2\Gamma(2-\alpha)\Gamma(2\alpha)/\Gamma(\alpha+1)$. For $t \le t_0$, we can derive the same estimation.

For any given $t_0 \in [0, \infty)$, let $d_1 = a$, $d_2 = t_0 + b$, for some constant $a > 0$, $b > 0$, then we have fixed c_{d_1}, c_{d_2} furthermore fixed $\hat{c}_1$, $\hat{c}_{21}$. For a sufficiently small $\epsilon > 0$, δ_1 must be also sufficiently small such that $\hat{c}_1 \delta_1^{\alpha} < \epsilon$ and the right hand side of the inequality above is also less than ϵ. Thus, $\delta_1 < a$, $\delta_1/2 < b$ so that $\delta_1 < d_1$ and $t_0 + \delta_1/2 < d_2$. Then we can derive a constant $\delta_1 = \delta_1(\epsilon, t_0, a, b)$ such that all these inequalities hold. Let $\delta = \delta_1/2$, then $|t - t_0| \le \delta$ implies $|{}_0^C D_t^{\alpha} [\Delta x^T(t) P \Delta x(t)] - {}_0^C D_{t_0}^{\alpha} [\Delta x^T(t) P \Delta x(t)]| < \epsilon$. Thus, ${}_0^C D_t^{\alpha} [\Delta x^T(t) P \Delta x(t)] \in C[0, \infty)$, then ${}_0^C D_t^{\alpha} [x^T(t) P x(t)] \in C[0, \infty)$, due to ${}_0^C D_t^{\alpha} x(t) \in C[0, \infty)$.

For $t = 0$, (3.36) holds, due to ${}_0^C D_0^\alpha[\Delta x^T(t) P \Delta x(t)] = 0$.

For $t \in (0, \infty)$, we can prove (3.36) as follows. According to Caputo's definition, we have

$$x^T(t) P {}_0^C D_t^\alpha x(t) - \frac{1}{2} {}_0^C D_t^\alpha [x^T(t) P x(t)] = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{[x^T(t) - x^T(\tau)] P \dot{x}(\tau)}{(t-\tau)^\alpha} \mathrm{d}\tau. \tag{3.37}$$

Referring to [16], let $y(\tau) = x(t) - x(\tau)$, then $\dot{y}(\tau) = -\dot{x}(\tau)$. (3.37) can be rewritten as

$$x^T(t) P {}_0^C D_t^\alpha x(t) - \frac{1}{2} {}_0^C D_t^\alpha [x^T(t) P x(t)] = -\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{y^T(\tau) P \dot{y}(\tau)}{(t-\tau)^\alpha} \mathrm{d}\tau. \tag{3.38}$$

For any $0 < t_1 < t_2 < t$, $y^T(\tau) P y(\tau)/2$ and $1/(t-\tau)^\alpha$ are continuously differentiable with respect to τ on $[t_1, t_2]$. Integrating by parts,

$$\int_{t_1}^{t_2} \frac{y^T(\tau) P \dot{y}(\tau)}{(t-\tau)^\alpha} \mathrm{d}\tau = \frac{y^T(\tau) P y(\tau)}{2(t-\tau)^\alpha}\Big|_{\tau=t_2} - \frac{y^T(\tau) P y(\tau)}{2(t-\tau)^\alpha}\Big|_{\tau=t_1} - \int_{t_1}^{t_2} \frac{\alpha y^T(\tau) P y(\tau)}{2(t-\tau)^{\alpha+1}} \mathrm{d}\tau.$$

Taking the limit as $t_1 \to 0$ and $t_2 \to t$, then

$$\int_0^t \frac{y^T(\tau) P \dot{y}(\tau)}{(t-\tau)^\alpha} \mathrm{d}\tau = \lim_{\tau \to t} \frac{y^T(\tau) P y(\tau)}{2(t-\tau)^\alpha} - \lim_{\tau \to 0} \frac{y^T(\tau) P y(\tau)}{2(t-\tau)^\alpha} - \int_0^t \frac{\alpha y^T(\tau) P y(\tau)}{2(t-\tau)^{\alpha+1}} \mathrm{d}\tau \tag{3.39}$$

holds, if any three of these four terms exist. It follows from (3.38) and the existence of ${}_0^C D_t^\alpha x(t)$ and ${}_0^C D_t^\alpha [x^T(t) P x(t)]$ that the left side integral exists. In the following, we shall check the existence of those two limits on the right side.

$$\lim_{\tau \to 0} \frac{y^T(\tau) P y(\tau)}{2(t-\tau)^\alpha} = \frac{y^T(0) P y(0)}{2t^\alpha} = \frac{[x(t) - x(0)]^T P [x(t) - x(0)]}{2t^\alpha} \geq 0.$$

Since $y(\tau) \to 0$, $(t-\tau)^\alpha \to 0$ as $\tau \to t$, and $\dot{y}(\tau) = -\dot{x}(\tau)$ exists due to $x(\tau) \in C^1(0, \infty)$, by the L'Hospital rule,

$$\lim_{\tau \to t} \frac{y^T(\tau) P y(\tau)}{2(t-\tau)^\alpha} = \lim_{\tau \to t} \frac{y^T(\tau) P \dot{y}(\tau)}{-\alpha(t-\tau)^{\alpha-1}} = 0.$$

Thus, $\lim_{\tau \to t} y^T(\tau) P y(\tau)/[2(t-\tau)^\alpha] = 0$ and $\lim_{\tau \to 0} y^T(\tau) P y(\tau)/[2(t-\tau)^\alpha] \geq 0$, for $t \in (0, \infty)$. Therefore, (3.39) does hold such that $\int_0^t \alpha y^T(\tau) P y(\tau)/[2(t-\tau)^{\alpha+1}] \mathrm{d}\tau$ is well defined and nonnegative. Then it follows from (3.38) and (3.39) that ${}_0^C D_t^\alpha [x^T(t) P x(t)] \leq 2 x^T(t) P {}_0^C D_t^\alpha x(t)$, for all $t \in (0, \infty)$. The proof is completed. □

Remark 3.4.1. *According to Corollary 3.3.5 and 3.3.6, the assumptions in Lemma 3.4.1 may be replaced by one of the two following stronger conditions:*

i. f is continuously differentiable on $[0, \infty) \times \mathbb{R}^n$, and for any $h^* \in (0, \infty)$, there exists a corresponding monotonically increasing function $\psi_{h^*} : [0, \infty) \to [0, \infty)$, such that for any $(t, x), (t, y) \in [0, h^*] \times \mathbb{R}^n$ and all possible nonnegative integers $u_0, u_1, ..., u_n$ with $u_0 + u_1 + ... + u_n = 1$,

$$\|\frac{\partial}{\partial t^{u_0} \partial x_1^{u_1} ... \partial x_n^{u_n}} f(t, x) - \frac{\partial}{\partial t^{u_0} \partial y_1^{u_1} ... \partial y_n^{u_n}} f(t, y)\|_1 \leq \psi_{h^*}(\max\{\|x\|_1, \|y\|_1\}) \|x - y\|_1;$$

ii. $f \in C^2([0, \infty) \times \mathbb{R}^n)$.

3.4.2 Lyapunov Stability Criteria

We can now use the estimation in Lemma 3.4.1 to prove our Lyapunov stability results.

Theorem 3.4.1. *Let $x = 0$ be an equilibrium point for the Caputo fractional order nonautonomous system (3.33). Assume the hypotheses of Lemma 3.4.1. Then the equilibrium point of system (3.33) is stable if there exist $n \times n$ positive definite matrices P, Q such that for any $(t, x) \in G$,*

$$x^T P f(t, x) + f^T(t, x) P x \leq 0, \tag{3.40}$$

and is asymptotically stable if

$$x^T P f(t, x) + f^T(t, x) P x \leq -x^T Q x. \tag{3.41}$$

Proof. Consider the Lyapunov function candidate $V(x) = x^T P x$, where P is the mentioned positive definite matrix. According to Lemma 3.4.1, the α order fractional derivative of V along the trajectory of (3.33), ${}_0^C D_t^\alpha V[x(t)] \in C[0, \beta)$ and

$${}_0^C D_t^\alpha V[x(t)] \leq [{}_0^C D_t^\alpha x^T(t)] P x(t) + x^T(t) P \, {}_0^C D_t^\alpha x(t) = f^T(t, x(t)) P x(t) + x^T(t) P f(t, x(t)),$$

for all $t \in [0, \beta)$, where $[0, \beta)$ is the maximal interval of existence for $x(t)$.

If (3.40) holds, then ${}_0^C D_t^\alpha V[x(t)] \leq 0$ so that there exists a nonnegative function $r_s(t) \in C[0, \beta)$ such that ${}_0^C D_t^\alpha V[x(t)] = -r_s(t)$. According to Theorem 2.1.4,

$$V[x(t)] = V(x_0) - \int_0^t (t - \tau)^{\alpha - 1} r_s(\tau) d\tau,$$

where the convolution above is continuous and nonnegative. Thus, $V[x(t)] \leq V(x_0)$, for all $t \in [0, \beta)$ so that the solution is bounded. According to Corollary 3.2.1, $\beta = \infty$. Thus, for all $t \geq 0$, $\|x(t)\|_2 \leq \sqrt{\lambda_{max}(P)/\lambda_{min}(P)} \|x_0\|_2$, where $\lambda_{max}(P)$, $\lambda_{min}(P)$ are the maximum, minimum eigenvalues of P. Therefore, the equilibrium point of (3.33) is stable.

If (3.41) holds, then

$${}_0^C D_t^\alpha V[x(t)] \leq -\mu V[x(t)]. \tag{3.42}$$

where $\mu = \lambda_{min}(Q)/\lambda_{max}(P)$. Clearly, it follows from the same proof as above that the equilibrium point of system (3.33) is stable. Due to the global existence and continuity of ${}_0^C D_t^\alpha V[x(t)]$ and $V[x(t)]$, there exists a nonnegative function $r_a(t) \in C[0, \infty)$ such that ${}_0^C D_t^\alpha V[x(t)] + r_a(t) = -\mu V[x(t)]$. According to Theorem 2.1.4,

$$V[x(t)] = V(x_0) E_\alpha(-\mu t^\alpha) - \int_0^t (t - \tau)^{\alpha - 1} E_{\alpha,\alpha}[-\mu (t - \tau)^\alpha] r_a(\tau) d\tau.$$

Since $E_{\alpha,\alpha}(-\mu t^\alpha)$ is smooth and nonnegative, see [20], the convolution is continuous and nonnegative. Thus, for all $t \geq 0$, $V[x(t)] \leq V(x_0) E_\alpha(-\mu t^\alpha)$. It follows from Theorem 4.6(a) in [1] that $\lim_{t \to \infty} E_\alpha(-\mu t^\alpha) = 0$. Therefore, the equilibrium point of (3.33) is asymptotically stable. □

According to Remark 3.4.1, we have the following usefull corollaries.

Corollary 3.4.1. *Let $x = 0$ be an equilibrium point for the Caputo fractional order nonautonomous system (3.33). Assume:*

i. f is continuously differentiable on $[0, \infty) \times \mathbb{R}^n$;

ii. for any $h^* \in (0, \infty)$, there exists a corresponding monotonically increasing function $\psi_{h^*} : [0, \infty) \to [0, \infty)$, such that for any $(t, x), (t, y) \in [0, h^*] \times \mathbb{R}^n$ and all possible nonnegative integers $u_0, u_1, ..., u_n$ with $u_0 + u_1 + ... + u_n = 1$,

$$\|\frac{\partial}{\partial t^{u_0} \partial x_1^{u_1} ... \partial x_n^{u_n}} f(t, x) - \frac{\partial}{\partial t^{u_0} \partial y_1^{u_1} ... \partial y_n^{u_n}} f(t, y)\|_1 \le \psi_{h^*}(\max\{\|x\|_1, \|y\|_1\})\|x - y\|_1.$$

Then the equilibrium point of system (3.33) is stable if there exist $n \times n$ positive definite matrices P, Q such that for any $(t, x) \in G$,

$$x^T P f(t, x) + f^T(t, x) P x \le 0, \tag{3.43}$$

and is asymptotically stable if

$$x^T P f(t, x) + f^T(t, x) P x \le -x^T Q x. \tag{3.44}$$

Corollary 3.4.2. *Let $x = 0$ be an equilibrium point for the Caputo fractional order nonautonomous system (3.33). Assume $f \in C^2(G)$. Then the equilibrium point of system (3.33) is stable if there exist $n \times n$ positive definite matrices P, Q such that for any $(t, x) \in G$,*

$$x^T P f(t, x) + f^T(t, x) P x \le 0, \tag{3.45}$$

and is asymptotically stable if

$$x^T P f(t, x) + f^T(t, x) P x \le -x^T Q x. \tag{3.46}$$

It follows from Remark 3.4.1 that the proofs of these two corollaries are the same as that of Theorem 3.4.1.

In fact, the inequality (3.42) is the second condition in the fractional Lyapunov direct method presented in [21, 22]. However, for $0 < q < 1$,

$${}_a^R D_t^q [f(t) g(t)] = f(t) {}_a^R D_t^q g(t) + \sum_{k=1}^{\infty} \frac{\Gamma(q+1)}{\Gamma(k+1)\Gamma(q-k+1)} {}_a^R D_t^k f(t) {}_a\mathscr{D}_t^{-(k-q)} g(t),$$

$${}_a^C D_t^q [f(t) g(t)] = f(t) {}_a^C D_t^q g(t) + \frac{(t-a)^{-q}}{\Gamma(1-q)} [f(t) - f(a)] g(a) + \sum_{k=1}^{\infty} \frac{\Gamma(q+1)}{\Gamma(k+1)\Gamma(q-k+1)} {}_a^C D_t^k f(t) {}_a\mathscr{D}_t^{-(k-q)} g(t),$$

if f and g are analytic in t [1]. Thus, even if we choose the simplest quadratic Lyapunov function $V = x^T x$ for system (3.33) and assume that the solution is analytic (this is not practical), the fractional derivative is an infinite series as shown above. As we see, it is not very easy to calculate this derivative and make it satisfy (3.42) as required by the fractional Lyapunov direct method. Fortunately, Lemma 3.4.1 provides a simple estimation for the fractional derivative, and Theorem 3.4.1 and Corollary 3.4.1, 3.4.2 present simple sufficient conditions for Lyapunov stability after the smoothness of solutions is guaranteed. All these can be easily checked even by Matlab. These conveniences can be seen from numerical examples in Section 3.7.

3.5 External Stability

For external stability analysis, a general control (input forced) system is necessary to be first introduced. Taking the input into account for (3.33), then it becomes the following Caputo fractional order nonlinear control system

$$\begin{cases} {}_0^C D_t^\alpha x = \bar{f}(t, x, u) \\ y = h(t, x, u) \\ x(0) = x_0, \end{cases} \tag{3.47}$$

where $\bar{f} : G_u \to \mathbb{R}^n$, $G_u = [0, \infty) \times \mathbb{R}^n \times \Omega_u$ and $\Omega_u \subseteq \mathbb{R}^l$ is a domain that contains $u = 0$; $h : G_u \to \mathbb{R}^p$; u, y are the input, output respectively.

Without the explicit presence of u, i.e. $u = 0$, (3.47) reduces to a so-called unforced system in the form of (3.33). Now we give the definition of external stability, see more details in [23].

Definition 3.5.1. *A system is externally stable (or L_2 stable) if, for every input $u \in L_2[0, \infty)$, the zero-state output $y \in L_2[0, \infty)$.*

Note that the L_2 gain of an externally stable system is given by $\gamma = \sup_{u \in L_2, u \neq 0} \|y\|_{L_2} / \|u\|_{L_2}$. Except for the assumption $u \in L_2$, the input $u(t)$ here is further assumed to be continuous such that the existence of solutions to (3.47) is guaranteed.

3.5.1 Diffusive Realization

To investigate the external stability of the Caputo fractional order nonlinear control system, we start from proving the equivalence between (3.47) and its diffusive realization. The so-called diffusive realization, appearing in the following lemma, is referred from [24].

Lemma 3.5.1. *Assume that $\bar{f}$ is continuous in t and Lipschitz in x and u on G_u, and $u \in (C[0, \infty), \Omega_u)$, then there exists a unique solution to system (3.47) $x(t) \in C[0, \infty)$ and*

$$x(t) = x_0 + \int_0^\infty \phi(\omega, t) d\omega,$$

where $\phi(\omega, t)$ is the solution of the initial value problem $\partial \phi(\omega, t)/\partial t = -\omega \phi(\omega, t) + \mu_\alpha(\omega) \bar{f}(t, x(t), u(t))$, $\phi(\omega, 0) = 0$, in which $\mu_\alpha(\omega) = [\sin(\pi\alpha)/\pi] \omega^{-\alpha}$ and $\omega \in (0, \infty)$.

Proof. Since $\bar{f}(t, x, u)$ is continuous in t and Lipschitz in both x and u on G_u, and $u \in (C[0, \infty), \Omega_u)$, then $F(t, x) := \bar{f}(t, x, u(t))$ is continuous in t and Lipschitz in x on $[0, \infty) \times \mathbb{R}^n$. According to Theorem 3.1.1 and Remark 3.1.1, there exists a unique solution $x(t) \in C[0, \infty)$ for (3.47). Moreover, according to Lemma 3.1.1, the solution $x(t)$ must be of the following form

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{F(\tau, x(\tau))}{(t - \tau)^{1-\alpha}} d\tau.$$

Referring to [24], $x(t)$ also takes the form of a convolution of F with a power function of t: $P_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$. That is, $x(t) = x_0 + P_\alpha(t) * F(t, x(t))$. P_α (for $t > 0$) can be rewritten as

$$\begin{aligned} P_\alpha(t) &= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(1-\alpha)}\frac{\Gamma(1-\alpha)}{s^{1-\alpha}}|_{s=t} = \frac{1}{\Gamma(\alpha)}\mathscr{L}[P_{1-\alpha}(\omega)]|_{Re(s)>0, s=t} = \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-t\omega}\frac{\omega^{-\alpha}}{\Gamma(1-\alpha)}d\omega \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\omega^{-\alpha}e^{-\omega t}d\omega = \int_0^\infty \frac{\sin(\alpha\pi)}{\pi}\omega^{-\alpha}e^{-\omega t}d\omega = \int_0^\infty \mu_\alpha(\omega)e^{-\omega t}d\omega. \end{aligned}$$

It follows that

$$\begin{aligned} x(t) &= x_0 + \int_0^t F(\tau, x(\tau))P_\alpha(t-\tau)d\tau \\ &= x_0 + \int_0^t F(\tau, x(\tau))\int_0^\infty \mu_\alpha(\omega)e^{-\omega(t-\tau)}d\omega d\tau \\ &= x_0 + \int_0^t \int_0^\infty e^{-\omega(t-\tau)}\mu_\alpha(\omega)F(\tau, x(\tau))d\omega d\tau. \end{aligned}$$

Since $e^{-\omega(t-\tau)}\mu_\alpha(\omega)F(\tau, x(\tau))$ is continuous in ω and τ on $[\omega_1, \omega_2] \times [0, t]$, for any $[\omega_1, \omega_2] \subset (0, \infty)$, then

$$\int_0^t \int_{\omega_1}^{\omega_2} e^{-\omega(t-\tau)}\mu_\alpha(\omega)F(\tau, x(\tau))d\omega d\tau = \int_{\omega_1}^{\omega_2}\int_0^t e^{-\omega(t-\tau)}\mu_\alpha(\omega)F(\tau, x(\tau))d\tau d\omega.$$

Taking the limit as $\omega_1 \to 0$ and $\omega_2 \to \infty$, then the limit equation

$$\int_0^t \int_0^\infty e^{-\omega(t-\tau)}\mu_\alpha(\omega)F(\tau, x(\tau))d\omega d\tau = \int_0^\infty \int_0^t e^{-\omega(t-\tau)}\mu_\alpha(\omega)F(\tau, x(\tau))d\tau d\omega, \tag{3.48}$$

holds, if any one of the limits exists. Due to the existence of $x(t)$ on $[0, \infty)$, the right-side limit exists so that (3.48) holds. Thus,

$$x(t) = x_0 + \int_0^\infty \int_0^t e^{-\omega(t-\tau)}\mu_\alpha(\omega)F(\tau, x(\tau))d\tau d\omega. \tag{3.49}$$

It follows from the initial value problem (diffusive realization) that

$$\phi(\omega, t) = \int_0^t e^{-\omega(t-\tau)}\mu_\alpha(\omega)\bar{f}(\tau, x(\tau), u(\tau))d\tau = \int_0^t e^{-\omega(t-\tau)}\mu_\alpha(\omega)F(\tau, x(\tau))d\tau.$$

Therefore, $x(t) = x_0 + \int_0^\infty \phi(\omega, t)d\omega, t \in [0, \infty)$. □

Remark 3.5.1. *The expression of $x(t)$ in Theorem 3.5.1 can be replaced by $x(t) = x_0 + \int_0^\infty \mu_\alpha(\omega)\phi(\omega, t)d\omega$, where $\phi(\omega, t)$ is the solution of the initial value problem: $\partial\phi(\omega, t)/\partial t = -\omega\phi(\omega, t) + \bar{f}(t, x(t), u(t))$, $\phi(\omega, 0) = 0$, and $\mu_\alpha(\omega) = [\sin(\pi\alpha)/\pi]\omega^{-\alpha}$, $\omega \in (0, \infty)$, because here $x(t) = x_0 + \int_0^\infty \mu_\alpha(\omega)\int_0^t e^{-\omega(t-\tau)}\bar{f}(\tau, x(\tau), u(\tau))d\tau d\omega$ is the same as (3.49).*

3.5.2 Lyapunov-Like Function

We are now ready to introduce the Lyapunov-like function $V(t) = \int_0^\infty \mu_\alpha(\omega)\phi^T(\omega,t)P\phi(\omega,t)\, d\omega$. It first appeared in [25]. As we see, V is an improper integral not like usual Lyapunov functions, e.g. quadratic ones, and V would be nonnegative if existing. For proving the external stability of the Caputo fractional order nonlinear control system, it is necessary to first guarantee the existence of V.

Lemma 3.5.2. *Assume that $\bar{f}$ is continuous in t and Lipschitz in x and u on G_u, $u \in (C[0,\infty), \Omega_u)$, and $x_0 = 0$, then,*

i. for any $0 \le T < \infty$, $V(T)$ exists, $V(T) \ge 0$ and

$$V(T) \le \int_0^T x^T(t)P\bar{f}(t,x(t),u(t)) + \bar{f}^T(t,x(t),u(t))Px(t)dt;$$

ii. for any $0 \le t < \infty$, $\dot{V}(t)$ exists and

$$\dot{V}(t) \le x^T(t)P\bar{f}(t,x(t),u(t)) + \bar{f}^T(t,x(t),u(t))Px(t),$$

where $V(t) = \int_0^\infty \mu_\alpha(\omega)\phi^T(\omega,t)P\phi(\omega,t)d\omega$, in which $\phi(\omega,t)$ is the solution of the initial value problem in Remark 3.5.1 and P is any positive definite matrix; and $\dot{V}(t)$ denotes the "derivative" function of V such that $V(T) = \int_0^T \dot{V}(t)dt$.

Proof. i. Since the solution of (3.47) $x(t)$ and $\bar{f}(t,x(t),u(t))$ are both continuous on $[0,\infty)$, concluded from the assumption, then for any $0 \le T < \infty$, $\int_0^T x^T(t)P\bar{f}(t,x(t),u(t))dt$ exists. According to Lemma 3.5.1 and Remark 3.5.1,

$$x(t) = \int_0^\infty \mu_\alpha(\omega)\phi(\omega,t)d\omega,$$

due to $x_0 = 0$. Moreover,

$$\phi(\omega,t) = \int_0^t e^{-\omega(t-\tau)}\bar{f}(\tau,x(\tau),u(\tau))d\tau.$$

Clearly, $\phi(\omega,t)$ is differentiable in ω and t. Thus, $\mu_\alpha(\omega)\phi^T(\omega,t)P\bar{f}(t,x(t),u(t))$ is continuous in ω and t on $[\omega_1,\omega_2] \times [0,T]$, for any $[\omega_1,\omega_2] \subset (0,\infty)$. Then,

$$\begin{aligned}\int_0^T x^T(t)P\bar{f}(t,x(t),u(t))dt &= \int_0^T \int_0^\infty \mu_\alpha(\omega)\phi^T(\omega,t)P\bar{f}(t,x(t),u(t))d\omega dt \\ &= \int_0^\infty \int_0^T \mu_\alpha(\omega)\phi^T(\omega,t)P\bar{f}(t,x(t),u(t))dtd\omega.\end{aligned}$$

Since $\bar{f}(t,x(t),u(t))$ is independent of ω, the function $\bar{f}$ in the integral above $\bar{f}(t,x(t),u(t)) = \partial\phi(\omega,t)/\partial t + \omega\phi(\omega,t)$, where ω can be the same as the one in the integral.

Thus, we have

$$\int_0^\infty \int_0^T \mu_\alpha(\omega)\phi^T(\omega,t)P\bar{f}(t,x(t),u(t))dtd\omega = \int_0^\infty \int_0^T \mu_\alpha(\omega)\phi^T(\omega,t)P[\frac{\partial\phi(\omega,t)}{\partial t} + \omega\phi(\omega,t)]dtd\omega$$
$$= \int_0^\infty \int_0^T \mu_\alpha(\omega)\phi^T(\omega,t)P\frac{\partial\phi(\omega,t)}{\partial t}dtd\omega + \int_0^\infty \int_0^T \mu_\alpha(\omega)\phi^T(\omega,t)P\omega\phi(\omega,t)dtd\omega$$
$$= \int_0^\infty \mu_\alpha(\omega)\int_0^T \phi^T(\omega,t)P\frac{\partial\phi(\omega,t)}{\partial t}dtd\omega + \int_0^\infty \int_0^T \mu_\alpha(\omega)\omega\phi^T(\omega,t)P\phi(\omega,t)dtd\omega$$
$$= \frac{1}{2}\int_0^\infty \mu_\alpha(\omega)\phi^T(\omega,T)P\phi(\omega,T)d\omega + \int_0^\infty \int_0^T \mu_\alpha(\omega)\omega\phi^T(\omega,t)P\phi(\omega,t)dtd\omega.$$

Observing the above equation, we can find

$$\int_0^T \mu_\alpha(\omega)\phi^T(\omega,t)P\bar{f}(t,x(t),u(t))dt = \frac{1}{2}\mu_\alpha(\omega)\phi^T(\omega,T)P\phi(\omega,T) + \int_0^T \mu_\alpha(\omega)\omega\phi^T(\omega,t)P\phi(\omega,t)dt.$$

All these three terms, exist and are nonnegative for any $\omega \in (0,\infty)$, and are integrable on any $[\omega_1,\omega_2] \subset (0,\infty)$. Thus, for any $\omega \in (0,\infty)$,

$$|\frac{1}{2}\mu_\alpha(\omega)\phi^T(\omega,T)P\phi(\omega,T)| \leq \int_0^T \mu_\alpha(\omega)\phi^T(\omega,t)P\bar{f}(t,x(t),u(t))dt.$$

Since the improper integral of the right-side function (integral) above from 0 to ∞ equals J, the left-side function is absolutely integrable over $(0,\infty)$. Thus, $V(T)$ exists and $V(T) \geq 0$, for any $0 \leq T < \infty$. It follows that $\int_0^\infty \int_0^T \mu_\alpha(\omega)\omega\phi^T(\omega,t)P\phi(\omega,t)dtd\omega$ exists and is nonnegative as well. Therefore, for any $0 \leq T < \infty$,

$$V(T) = 2\int_0^T x^T(t)P\bar{f}(t,x(t),u(t))dt - 2\int_0^\infty \int_0^T \mu_\alpha(\omega)\omega\phi^T(\omega,t)P\phi(\omega,t)dtd\omega$$
$$\leq \int_0^T x^T(t)P\bar{f}(t,x(t),u(t)) + x(t)P\bar{f}^T(t,x(t),u(t))dt.$$

ii. $\dot{V}(t)$ can be expressed by

$$\dot{V}(t) = \int_0^\infty \mu_\alpha(\omega)\frac{\partial\phi^T(\omega,t)}{\partial t}P\phi(\omega,t) + \mu_\alpha(\omega)\phi^T(\omega,t)P\frac{\partial\phi(\omega,t)}{\partial t}d\omega$$
$$= \int_0^\infty \mu_\alpha(\omega)[-\omega\phi(\omega,t) + \bar{f}(t,x(t),u(t))]^T P\phi(\omega,t)d\omega + \int_0^\infty \mu_\alpha(\omega)\phi^T(\omega,t)P[-\omega\phi(\omega,t) + \bar{f}(t,x(t),u(t))]d\omega$$
$$= \int_0^\infty -2\mu_\alpha(\omega)\omega\phi^T(\omega,t)P\phi(\omega,t)d\omega + \int_0^\infty \bar{f}^T(t,x(t),u(t))P\mu_\alpha(\omega)\phi(\omega,t)d\omega + \int_0^\infty \mu_\alpha(\omega)\phi^T(\omega,t)P\bar{f}(t,x(t),u(t))d\omega$$
$$= -2\int_0^\infty \mu_\alpha(\omega)\omega\phi^T(\omega,t)P\phi(\omega,t)d\omega + \bar{f}^T(t,x(t),u(t))P\int_0^\infty \mu_\alpha(\omega)\phi(\omega,t)d\omega + [\int_0^\infty \mu_\alpha(\omega)\phi(\omega,t)d\omega]^T P\bar{f}(t,x(t),u(t))$$
$$= -2\int_0^\infty \mu_\alpha(\omega)\omega\phi^T(\omega,t)P\phi(\omega,t)d\omega + \bar{f}^T(t,x(t),u(t))Px(t) + x^T(t)P\bar{f}(t,x(t),u(t)).$$

It is necessary to check the existence of the improper integral. For any $\omega_2 \in (0, \infty)$,

$$\begin{aligned}\int_0^\infty \int_0^T \mu_\alpha(\omega)\omega\phi^T(\omega,t)P\phi(\omega,t)dtd\omega &= \lim_{\omega_2\to\infty}\int_0^{\omega_2}\int_0^T \mu_\alpha(\omega)\omega\phi^T(\omega,t)P\phi(\omega,t)dtd\omega\\ &= \lim_{\omega_2\to\infty}\int_0^{T}\int_0^{\omega_2} \mu_\alpha(\omega)\omega\phi^T(\omega,t)P\phi(\omega,t)d\omega dt\\ &= \int_0^{T}\int_0^{\infty} \mu_\alpha(\omega)\omega\phi^T(\omega,t)P\phi(\omega,t)d\omega dt.\end{aligned}$$

This implies that $\int_0^\infty \mu_\alpha(\omega)\omega\phi^T(\omega,t)P\phi(\omega,t)d\omega$ is an integrable function of t on $[0,T]$. Thus, for every fixed $t \in [0,T]$, the improper integral $\int_0^\infty \mu_\alpha(\omega)\omega\phi^T(\omega,t)P\phi(\omega,t)d\omega$ is bounded. Let $\Phi(W) = \int_0^W \mu_\alpha(\omega)\omega\phi^T(\omega,t)P\phi(\omega,t)d\omega$, then for a fixed $t \in [0,T]$, $\Phi(W)$ is a bounded and monotonically increasing function on $[0,\infty)$. Thus, $\lim_{W\to\infty}\Phi(W)$ exists, i.e. $\int_0^\infty \mu_\alpha(\omega)\omega\phi^T(\omega,t)P\phi(\omega,t)d\omega$ exists for every $t \in [0,T]$. Therefore, $\dot{V}(t)$ exists and

$$\dot{V}(t) \le x^T(t)P\bar{f}(t,x(t),u(t)) + \bar{f}^T(t,x(t),u(t))Px(t).$$

□

3.5.3 External Stability Criterion

Using the diffusive realization and Lyapunov-like function, we can now prove the following external stability criterion.

Theorem 3.5.1. *Assume:*

i. $\bar{f}(t,x,u) = \tilde{A}x + \tilde{f}(t,x,u)$, *where* $\tilde{f} : G_u \to \mathbb{R}^n$, *is continuous in t and Lipschitz in x, u with Lipschitz constants* $L_{\tilde{f}x}$, $L_{\tilde{f}u}$ *respectively, and* $\tilde{f}(t,0,0) \equiv 0$;

ii. $h(t,x,u)$ *is continuous in t and Lipschitz in x, u on G_u with Lipschitz constants L_{hx}, L_{hu} respectively, and* $h(t,0,0) \equiv 0$.

Then the control system (3.47) is externally stable, i.e. $\|y\|_{L_2} \le \gamma\|u\|_{L_2}$, *for any* $u \in (C[0,\infty),\Omega_u) \cap L_2[0,\infty)$ *under the zero initial condition, if there exist constants* $\varepsilon_{\tilde{f}} > 0$, $\varepsilon_h > 1$ *and an* $n \times n$ *positive definite matrix P such that*

$$2\varepsilon_{\tilde{f}}L_{\tilde{f}u}^2 + 2\varepsilon_h L_{hu}^2 - \gamma^2 < 0, \tag{3.50}$$

and

$$\begin{bmatrix}\tilde{A}^TP + P\tilde{A} + 2\varepsilon_{\tilde{f}}L_{\tilde{f}x}^2 + 2\varepsilon_h L_{hx}^2 & P\\ * & -\varepsilon_{\tilde{f}}\end{bmatrix} < 0. \tag{3.51}$$

Proof. According to Lemma 3.5.2, for any $0 \le T < \infty$.

$$0 \le V(T) \le \int_0^T x^T(t)P\bar{f}(t, x(t), u(t)) + \bar{f}^T(t, x(t), u(t))Px(t)dt.$$

It follows from i and ii that for any $\varepsilon_{\tilde{f}}, \varepsilon_h > 0$,

$$2\varepsilon_{\tilde{f}} L^2_{\tilde{f}x} x^T(t)x(t) + 2\varepsilon_{\tilde{f}} L^2_{\tilde{f}u} u^T(t)u(t) - \varepsilon_{\tilde{f}} \tilde{f}^T(t, x(t), u(t))\tilde{f}(t, x(t), u(t)) \ge 0$$

and

$$2\varepsilon_h L^2_{hx} x^T(t)x(t) + 2\varepsilon_h L^2_{hu} u^T(t)u(t) - \varepsilon_h h^T(t, x(t), u(t))h(t, x(t), u(t)) \ge 0.$$

Then we can derive the following

$$\begin{aligned}
&\int_0^T y^T(t)y(t)dt - \gamma^2 \int_0^T u^T(t)u(t)dt + V(T) \\
&\le \int_0^T [h^T(t, x(t), u(t))h(t, x(t), u(t)) - \gamma^2 u^T(t)u(t) + x^T(t)P\bar{f}(t, x(t), u(t)) + \bar{f}^T(t, x(t), u(t))Px(t)]dt \\
&\le \int_0^T [h^T(t, x(t), u(t))h(t, x(t), u(t)) - \gamma^2 u^T(t)u(t) + x^T(t)P\tilde{A}x(t) + x^T(t)P\tilde{f}(t, x(t), u(t)) + x^T(t)\tilde{A}^T Px(t) \\
&\quad + \tilde{f}^T(t, x(t), u(t))Px(t) + 2\varepsilon_{\tilde{f}} L^2_{\tilde{f}x} x^T(t)x(t) + 2\varepsilon_{\tilde{f}} L^2_{\tilde{f}u} u^T(t)u(t) - \varepsilon_{\tilde{f}} \tilde{f}^T(t, x(t), u(t))\tilde{f}(t, x(t), u(t)) \\
&\quad + 2\varepsilon_h L^2_{hx} x^T(t)x(t) + 2\varepsilon_h L^2_{hu} u^T(t)u(t) - \varepsilon_h h^T(t, x(t), u(t))h(t, x(t), u(t))]dt \\
&= \int_0^T \eta^T(t) \begin{bmatrix} (1,1) & P & 0 & 0 \\ * & -\varepsilon_{\tilde{f}} & 0 & 0 \\ * & * & 2\varepsilon_{\tilde{f}} L^2_{\tilde{f}u} + 2\varepsilon_h L^2_{hu} - \gamma^2 & 0 \\ * & * & * & 1 - \varepsilon_h \end{bmatrix} \eta(t)dt,
\end{aligned}$$

where $(1,1) = \tilde{A}^T P + P\tilde{A} + 2\varepsilon_{\tilde{f}} L^2_{\tilde{f}x} + 2\varepsilon_h L^2_{hx}$ and $\eta(t) = [x^T(t), \tilde{f}^T(t, x(t), u^T(t), h^T(t)]^T$.

From (3.50), (3.51) and $\varepsilon_h > 1$, we know that the large matrix just appearing above, is negative definite. Thus,

$$\int_0^T y^T(t)y(t)dt \le \gamma^2 \int_0^T u^T(t)u(t)dt.$$

Due to the global existence of $x(t)$, it makes sense to take the limit as

$$\lim_{T\to\infty} \int_0^T y^T(t)y(t)dt \le \lim_{T\to\infty} \gamma^2 \int_0^T u^T(t)u(t)dt.$$

Due to $u \in L_2[0, \infty)$, the right-side limit exists, then it is bounded. Let $Y(T) = \int_0^T y^T(t)y(t)dt$, then $Y(T)$ is bounded and monotonically increasing on $[0, \infty)$. Thus, $\lim_{T\to\infty} Y(T)$ exists, i.e. the left-side limit exists. Therefore, under the zero initial condition, for any $u \in L_2[0, \infty) \cap C[0, \infty)$, $y \in L_2[0, \infty)$ and $\|y\|_{L_2} \le \gamma \|u\|_{L_2}$. □

3.6 Application to H_∞ Control

Here we consider to apply our results on Lyapunov and external stability to H_∞ control. The so-called H_∞ control is named from the H_∞ functions defined on the H_∞ (Hardy) space (see page 1 in [26]): $H_\infty := \{F : \mathbb{C} \to \mathbb{C} | F$ is analytic, $\sup_{Re(s)>0} |F(s)| < \infty\}$ equipped with the norm $\|F\|_\infty := \sup_{Re(s)>0} |F(s)|$. As well known (see page 4 in [27]), transfer functions for finite dimensional linear control systems are rational functions with real coefficients. We may only focus on a subset of the H_∞ space consisting of real-rational functions: $RH_\infty \subset H_\infty$. In fact, a transfer function $F(s) \in RH_\infty$ if and only if F is proper ($\lim_{s\to\infty} F(s) < \infty$) and stable ($F$ has no poles in the closed right half complex plane) [27]. In this case, $\|F\|_\infty = \sup_{\omega\in R} |F(j\omega)| = \sup_{u\in L_2, u\neq 0} \|y\|_{L_2}/\|u\|_{L_2}$, where u, y denotes the input, output of the control system described by F [23]. Therefore, those linear control systems with real-rational H_∞ transfer functions are externally stable, i.e. every L_2 input only excites an L_2 zero-state output. If the input is considered as a disturbance, then the external stability measures the robustness of the zero-state output on the disturbance. It ensures that the zero-state output excited by the energy-bounded (the square of the L_2 norm of a signal can be considered as the energy content of the signal) disturbance will not blow up. For linear control systems, H_∞ control is to find a control such that the norm of the transfer function from the disturbance (input) u_d to the output y (something we want to minimize) $\|F_{d\to y}\|_\infty$ is minimized, i.e. the the zero-state output excited by disturbance y_d is minimized, under the constraint that the overall system is stabilized, see page 17 in [26]. For nonlinear control systems, they do not have transfer functions as the linear ones do, but the same name H_∞ control is employed for the following control objective: to find a control such that the controlled system is asymptotically stable when no disturbances are present, and moreover, has finite L_2 gain from u_d to y, under the zero initial condition (is externally stable from u_d to y), see page 6 in [27]. As we see above, for either linear or nonlinear control systems, H_∞ control has the same physical meaning: the H_∞ controller starts to stabilize the system after the energy-bounded disturbance has already decayed to zero, then maintains the stabilized system to be externally stable from the disturbance to the output such that effect of the disturbance on the output is attenuated during the steady period. This specializes the practical importance of H_∞ control in industrial environments with noises. However, for Caputo fractional order control systems, the implication of H_∞ control may differ, due to the non-locality or memory of Caputo fractional order derivatives. For more details of "non-locality", see [1], pp.87. We shall elucidate this difference in Example 3.7.3 later.

Consider the input forced system (3.47), with both control input u_c and disturbance input u_d, of the following form

$$\begin{cases} {}_0^C D_t^\alpha x = Ax + \hat{f}(t, x, u_d) + Bu_c \\ y = h(t, x, u_d) \\ x(0) = x_0, \end{cases} \tag{3.52}$$

where $\hat{f} : G_u \to \mathbb{R}^n$, is twice continuously differentiable with respect to t and x, and Lipschitz in x, u_d with Lipschitz constants $L_{\hat{f}x}$, $L_{\hat{f}d}$ respectively, $\hat{f}(t, 0, 0) \equiv 0$; $h : G_u \to \mathbb{R}^p$, is continuous in t and Lipschitz in x, u with Lipschitz constants L_{hx}, L_{hd} respectively, $h(t, 0, 0) \equiv 0$; and $u_d \in (C[0, \infty), \Omega_u) \cap L_2[0, \infty)$.

As introduced, the H_∞ control problem is to find a control $u_c = Kx$ such that the controlled system without disturbance: ${}_0^C D_t^\alpha x = \hat{A}x + \hat{f}(t, x, 0)$, where $\hat{A} = A + BK$, is asymptotically stable, and the controlled system rewritten as ${}_0^C D_t^\alpha x = \hat{A}x + \hat{f}(t, x, u_d)$, $y = h(t, x, u_d)$, with $x_0 = 0$, satisfies $\|y\|_{L_2} \leq \gamma \|u_d\|_{L_2}$, for some priori prescribed constant γ.

Since the origin is an equilibrium point of (3.52) without disturbance, and $\hat{A}x + \hat{f}(t,x,0) \in C^2([0,\infty)\times\mathbb{R}^n)$, according to Corollary 3.4.2, if there exists $\varepsilon_{\hat{f}} > 0$, $P > 0$ and $Q > 0$ such that

$$\begin{bmatrix} P\hat{A} + \hat{A}^T P + Q + \varepsilon_{\hat{f}} L^2_{\hat{f}X} & P \\ * & -\varepsilon_{\hat{f}} \end{bmatrix} < 0, \tag{3.53}$$

then the controlled system (3.52) without disturbance is asymptotically stable. This is because

$$\begin{aligned} & x^T P[\hat{A}x + \hat{f}(t,x,0)] + [\hat{A}x + \hat{f}(t,x,0)]^T Px + x^T Qx = x^T(P\hat{A} + \hat{A}^T P + Q)x + x^T P\hat{f}(t,x,0) + \hat{f}^T(t,x,0)Px \\ & \le x^T(P\hat{A} + \hat{A}^T P + Q)x + x^T P\hat{f}(t,x,0) + \hat{f}^T(t,x,0)Px + \varepsilon_{\hat{f}} L^2_{\hat{f}x} x^T x - \varepsilon_{\hat{f}} \hat{f}^T(t,x,0)\hat{f}(t,x,0) \\ & = [x^T, \hat{f}^T(t,x,0)] \begin{bmatrix} P\hat{A} + \hat{A}^T P + Q + \varepsilon_{\hat{f}} L^2_{\hat{f}x} & P \\ * & -\varepsilon_{\hat{f}} \end{bmatrix} \begin{bmatrix} x \\ \hat{f}(t,x,0) \end{bmatrix}. \end{aligned}$$

According to Theorem 3.5.1, if there exists $\varepsilon_{\hat{f}} > 0$, $\varepsilon_h > 1$ and $P > 0$ such that

$$\begin{bmatrix} (1,1) & P & 0 & 0 \\ * & -\varepsilon_{\hat{f}} & 0 & 0 \\ * & * & 2\varepsilon_{\hat{f}} L^2_{\hat{f}d} + 2\varepsilon_h L^2_{hd} - \gamma^2 & 0 \\ * & * & * & 1-\varepsilon_h \end{bmatrix} < 0, \tag{3.54}$$

where $(1,1) = \hat{A}^T P + P\hat{A} + 2\varepsilon_{\hat{f}} L^2_{\hat{f}x} + 2\varepsilon_h L^2_{hx}$, then the controlled system (3.52) with $x_0 = 0$ satisfies $||y||_{L_2} \le \gamma ||u_d||_{L_2}$.

As we see, (3.54) implies (3.53) with $Q := (\varepsilon_{\hat{f}} L^2_{\hat{f}x} + 2\varepsilon_h L^2_{hx})I$. Therefore, the H_∞ control goal is achieved if we can find those parameters such that (3.54) holds. However, there are two nonlinear terms: $K^T B^T P$ and PBK in (3.54). We can left multiply $diag(P^{-1}, I)$ and right multiply its transpose to the first 2×2 block of the matrix in (3.54), and let $Y = KP^{-1}$. According to Schur complement, it then follows,

$$\begin{bmatrix} (1,1) & I & P^{-1} & 0 & 0 \\ * & -\varepsilon_{\hat{f}} & 0 & 0 & 0 \\ * & * & (3,3) & 0 & 0 \\ * & * & * & 2\varepsilon_{\hat{f}} L^2_{\hat{f}d} + 2\varepsilon_h L^2_{hd} - \gamma^2 & 0 \\ * & * & * & * & 1-\varepsilon_h \end{bmatrix} < 0, \tag{3.55}$$

where $(1,1) = AP^{-1} + P^{-1}A^T + BY + Y^T B^T$ and $(3,3) = -(2\varepsilon_{\hat{f}} L^2_{\hat{f}x} + 2\varepsilon_h L^2_{hx})^{-1}$, which is equivalent to (3.54), and the control gain $K = YP$. Now we can state the following theorem for the H_∞ control problem.

Theorem 3.6.1. *The control $u_c = Kx$, $K = YP$, solves the H_∞ control problem of system (3.52) if (3.55) holds.*

Remark 3.6.1. *The assumption on (3.52) that $\hat{f}$ is twice continuously differentiable with respect to t and x on* $[0,\infty)\times$ Ω *may be weakened as the assumptions on f:* i, ii, iii *in Lemma 3.4.1; or* i, ii *in Corollary 3.4.1.*

3.7 Numerical Examples

To illustrate our stability and control results, we shall give three examples with numerical simulations implemented by the algorithm in [28].

Example 3.7.1. *Consider the following Caputo fractional order nonautonomous system*

$$\begin{cases} {}_0^C D_t^\alpha x = -x + y - yt^\alpha \\ {}_0^C D_t^\alpha y = -x - y + xt^\alpha \end{cases}. \tag{3.56}$$

The origin is an equilibrium point of (3.56). Let $X = [x, y]^T$, then the vector field function of (3.56) $f(t, X) = [-x + y - yt^\alpha, -x - y + xt^\alpha]^T$. Clearly, $f \in C[0, \infty) \times \mathbb{R}^2$ and $f \in C^1(0, \infty) \times \mathbb{R}^2$. Moreover, for any $(t, X), (t, Y) \in (0, \infty) \times \mathbb{R}^2$,

$$\|\frac{\partial}{\partial t} f(t, X)\|_1 = (|x| + |y|)\alpha t^{\alpha-1} = \alpha \|X\|_1 t^{\alpha-1},$$

$$\|\frac{\partial}{\partial X_1} f(t, X)\|_1 = \|\frac{\partial}{\partial x} f(t, X)\|_1 = 1 + |-1 + t^\alpha|,$$

$$\|\frac{\partial}{\partial X_2} f(t, X)\|_1 = \|\frac{\partial}{\partial y} f(t, X)\|_1 = |1 - t^\alpha| + 1;$$

and

$$\|\frac{\partial}{\partial t} f(t, X) - \frac{\partial}{\partial t} f(t, Y)\|_1 = \alpha \|X - Y\|_1 t^{\alpha-1},$$

$$\|\frac{\partial}{\partial X_1} f(t, X) - \frac{\partial}{\partial Y_1} f(t, Y)\|_1 = 0,$$

$$\|\frac{\partial}{\partial X_2} f(t, X) - \frac{\partial}{\partial Y_2} f(t, Y)\|_1 = 0.$$

Let $\varphi_{h^*}(\|X\|_1) = \alpha \|X\|_1 + \max\{(h^*)^\alpha, 2\}$ and $\psi_{h^*}(\max\{\|X\|_1, \|Y\|_1\}) = \alpha$, then iii in Lemma 3.4.1 is satisfied. Furthermore, $X^T f(t, X) = -(x^2 + y^2) = -X^T X$, for all $(t, X) \in G$. According to Theorem 3.4.1, the equilibrium point $(0, 0)$ is asymptotically stable.

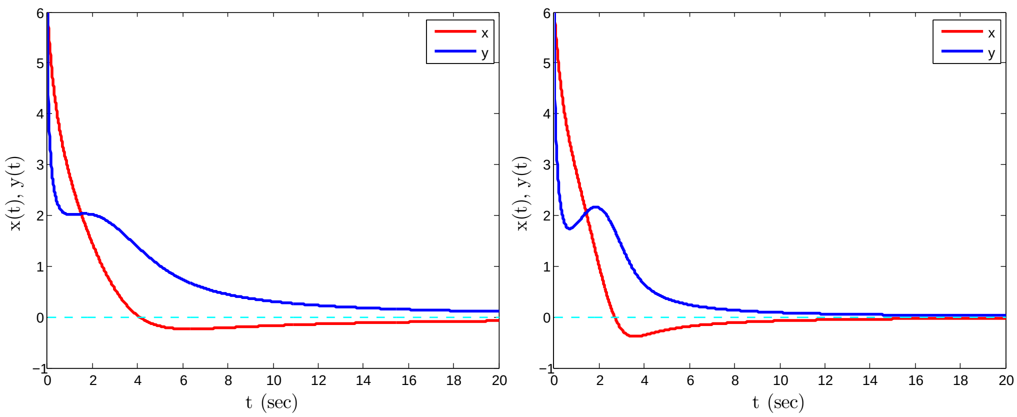
Now we consider a modification on f. Change the vector field function to $f(t, X) = [-x + y - yt, -x - y + xt]^T$, then the origin $(0, 0)$ is still an equilibrium point of (3.56). Moreover, $f \in C^2([0, \infty) \times \mathbb{R}^2)$, and $X^T f(t, X) = -X^T X$, for all $(t, X) \in G$. According to Corollary 3.4.2, the equilibrium point is still asymptotically stable.

The evolutions of system (3.56) and its modified version, with $\alpha = 0.6$ and $(x(0), y(0))= (6, 6)$, are shown in Figure 3.1*a* and Figure 3.1*b*, respectively. As expected from our analysis, the figures show that $(0, 0)$ is asymptotically stable.

Example 3.7.2. *Consider the Caputo fractional order Lorenz system*

$$\begin{cases} {}_0^C D_t^\alpha x = \sigma(y - x) \\ {}_0^C D_t^\alpha y = rx - y - xz \\ {}_0^C D_t^\alpha z = xy - bz \end{cases}, \tag{3.57}$$

where σ, r and b are positive constants.



(a) States of the original system

(b) States of the modified system

Figure 3.1: States of the Caputo fractional order nonautonomous systems

According to Definition 3.4.1, for $0 < r \leq 1$ system (3.57) has an unique equilibrium $(0, 0, 0)$; for $r > 1$ system (3.57) has three equilibrium points:

$$(0, 0, 0), (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1), (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1).$$

It is clear that the vector field function is twice continuously differentiable on the whole space.

To investigate the Lyapunov stability of the Lorenz system (3.57), we select the positive definite matrix as $P = diag(1/\sigma, 1, 1)$, then

$$X^T P f(t, X) + f^T(t, X) P X \leq -2[-(1 + r)xy + x^2 + y^2 + bz^2],$$

where $X = [x, y, z]^T$ and $f(t, X) = [\sigma(y - x), rx - y - xz, xy - bz]^T$.

If $0 < r < 1$, then

$$X^T P f(t, X) + f^T(t, X) P X \leq -c_1(x^2 + y^2 + z^2) = -c_1 X^T X,$$

where $c_1 = \min(2b, 1 - r)$. According to Corollary 3.4.2, the unique equilibrium $(0, 0, 0)$ is asymptotically stable.

If $r = 1$, then

$$X^T P f(t, X) + f^T(t, X) P X \leq -2[(x - y)^2 + bz^2] \leq 0.$$

According to Corollary 3.4.2, the unique equilibrium point is stable.

If $r > 1$, chaos may exist in the Caputo fractional order Lorenz system with appropriate other parameters. To stabilize the Caputo fractional order Lorenz system in this case, from the analysis above, we only need to add one control $u = kx$, $k \leq 1 - r$, on the right hand side of the second equation.

The phase portrait of the Caputo fractional order Lorenz chaos, with $\sigma = 10$, $r = 28$, $b = 8/3$, fractional order $\alpha = 0.996$ and initial condition $(x(0), y(0), z(0)) = (6, 6, 6)$, is shown in Figure 3.2. Note that the fractional order here must be selected sufficiently close to 1 such that the necessary condition [29], for the existence of chaotic attractor, is satisfied. As expected, we can observe the chaotic (or chaotic-like) phenomenon. Design a control $u = -27.2x$, where $k = -27.2 \leq 1 - r = -27$, as required for stabilization, then as shown in Figure 3.3, the system is stabilized.

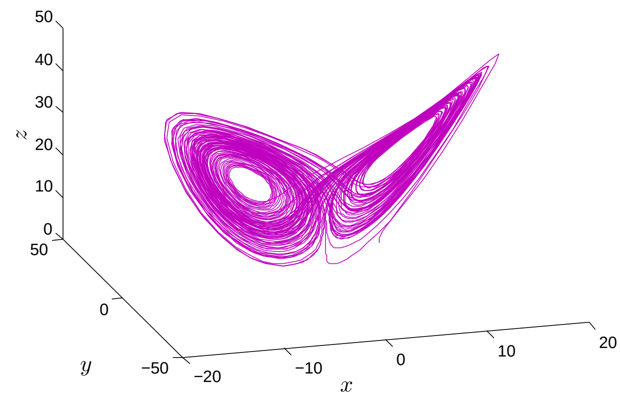


Figure 3.2: Phase portrait of the Caputo fractional order Lorenz chaos in x-y-z plane.

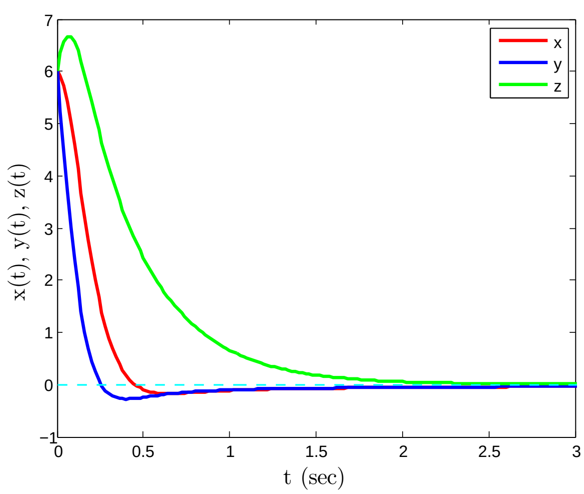


Figure 3.3: States of the controlled Caputo fractional order Lorenz system.

Example 3.7.3. *Consider the Caputo fractional order modified Chua's circuit*

$$\begin{cases} {}_0^C D_t^\alpha x = a(y - g(\epsilon, x)) \\ {}_0^C D_t^\alpha y = x - y + z \\ {}_0^C D_t^\alpha z = -by \end{cases}, \tag{3.58}$$

where $g(\epsilon, x)$*,* $\epsilon \in [0,1]$*, is continuously differentiable in x, and* $g(\epsilon, 0) = 0$*, as defined in [30]; a, b are constants. Its vector form is as* ${}_0^C D_t^\alpha X = AX + F(X)$*, where* $X = [x, y, z]^T$*,* $A = \begin{bmatrix} 0 & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{bmatrix}$ *and* $F(X) = [-ag(\epsilon, x), 0, 0]^T$*.*

We consider the H_∞ control problem of (3.58) with control input u_c and disturbance input $u_d \in (C[0,\infty), \mathbb{R}) \cap L_2[0,\infty)$ of the form: ${}_0^C D_t^\alpha X = AX + F(X) + Bu_c + Du_d$, $y = CX$, $X(0) = X_0$, where $u_c = KX$. This system can be rewritten in the form consistent with (3.52) as

$$\begin{cases} {}_0^C D_t^\alpha X = \hat{A}X + \hat{f}(t, X, u_d) \\ y = h(t, X, u_d) \\ X(0) = X_0, \end{cases} \tag{3.59}$$

where $\hat{A} = A + BK$, $\hat{f}(t, X, u_d) = F(X) + Du_d$ and $h(t, X, u_d) = CX$.

Since $g(\epsilon, x)$ is continuously differentiable; the right and left derivatives of $g'(\epsilon, x)$ exist and bounded for all x, see [30]; and $g(\epsilon, 0) = 0$, then $\hat{f}$ satisfies i and ii as required in Corollary 3.4.1 and $\hat{f}(t, 0, 0) \equiv 0$. In addition, $h(t, 0, 0) = C0 = 0$ and $u_d \in C[0,\infty) \cap L_2[0,\infty)$. According to Remark 3.6.1, all conditions on $\hat{f}$ required by Theorem 3.6.1 are satisfied.

We select the Chua's circuit parameters: $\alpha = 0.98$, $a = 12$, $b = 17$, $m_0 = -1/7$, $m_1 = 2/7$ and $\epsilon = 0.5$; prescribed constant: $\gamma = 0.3$; arbitrary constants: $\varepsilon_{\hat{f}} = 0.01$ and $\varepsilon_h = 1.01$; control matrices: $B = I_3$ (three dimensional identity matrix), $C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$; disturbance (for simulation):

$$u_d(t) = \begin{cases} 10\sin(2\pi t), 0 \le t \le 1 \\ 0, 1 < t \le 2 \\ 10\sin(2\pi(t-2)), 2 < t \le 3 \\ 0, t > 3 \end{cases}$$

which is continuous and square integrable on $[0,\infty)$ as required. Then $L_{\hat{f}X} = |a|\max(|m_0|, |m_1|) = 24/7$, $L_{\hat{f}d} = \|D\|_2 = \sqrt{3}$, and $L_{hX} = \|C\|_2 = \sqrt{3}$. Since h only depends on X, the $(3,3)$, $(4,4)$ entries of the matrix in (3.55) can be reduced to $-(2\varepsilon_{\hat{f}} L_{\hat{f}x}^2 + \varepsilon_h L_{hx}^2)^{-1}$, $2\varepsilon_{\hat{f}} L_{\hat{f}d}^2 - \gamma^2$, respectively. Let $L_{\hat{f}x} = L_{\hat{f}X}$ and $L_{\hat{h}x} = L_{\hat{h}X}$, then solve (3.55) with the reduction by Matlab, we derive

$$K = \begin{bmatrix} -24.5920 & -6.9518 & 0.1181 \\ -6.0482 & -23.5920 & 8.3445 \\ -0.1181 & 7.6555 & -24.5920 \end{bmatrix}.$$

The phase portrait of the Caputo fractional order modified Chua's circuit (or the controlled system with $u_c = 0$ and $u_d = 0$) under initial condition $(x(0), y(0), z(0)) = (0.2, 0.2, 0.2)$, is shown in Figure 3.4. It appears to be chaotic, naturally unstable. As expected, the control $u_c = KX$ stabilizes the unstable states, see Figure 3.5, and meanwhile makes $\gamma(t) = [\int_0^t y^2(t)dt]^{1/2}/[\int_0^t u_d^2(t)dt]^{1/2}$, under the zero initial condition, tend to some constant less than $\gamma = 0.3$, see Figure 3.6. Therefore, the control goal has been achieved.

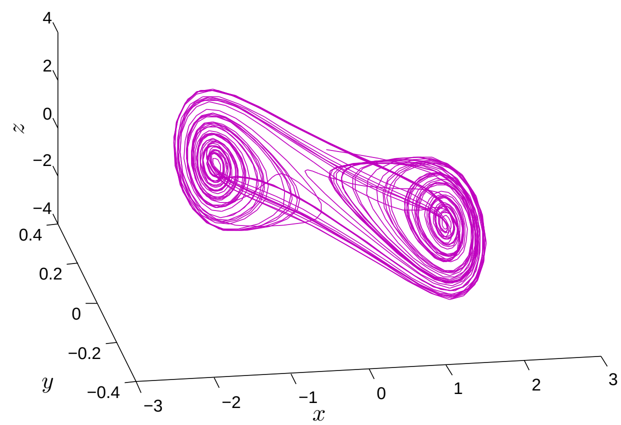


Figure 3.4: Phase portrait of the Caputo fractional order modified Chua's circuit in *x*-*y*-*z* plane.

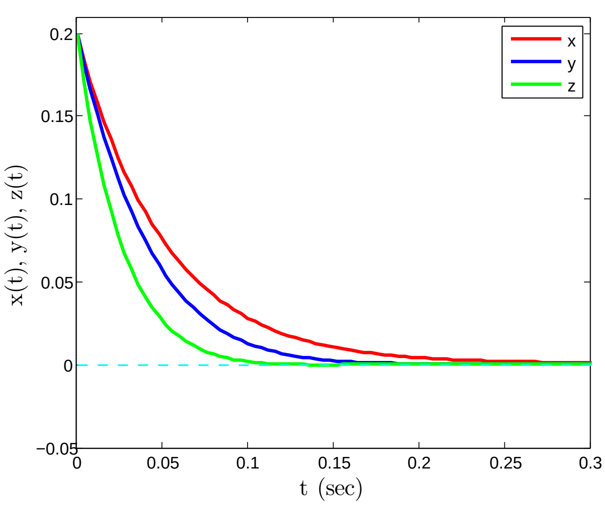


Figure 3.5: States of the controlled Caputo fractional order modified Chua's circuit.

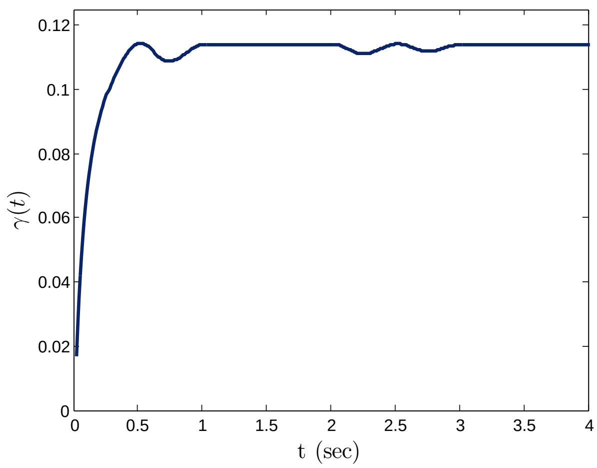


Figure 3.6: L_2 gain of the controlled Caputo fractional order modified Chua's circuit.

With this example, here we explain how the implication of the fractional-version H_∞ control differs from that of the classical one. According to Lemma 3.1.1, the solution of (3.59) can be expressed by

$$X(t) = X_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[\hat{A}X(\tau) + \hat{f}(t, X(\tau), u_d(\tau))]d\tau$$
$$= X_0 + \frac{1}{\Gamma(\alpha)}\int_0^1 (t-\tau)^{\alpha-1}[\hat{A}X(\tau) + \hat{f}(t, X(\tau), 10\sin(2\pi\tau))]d\tau + \frac{1}{\Gamma(\alpha)}\int_1^2 (t-\tau)^{\alpha-1}[\hat{A}X(\tau) + \hat{f}(t, X(\tau), 0)]d\tau$$
$$+ \frac{1}{\Gamma(\alpha)}\int_2^3 (t-\tau)^{\alpha-1}[\hat{A}X(\tau) + \hat{f}(t, X(\tau), 10\sin(2\pi(\tau-2)))]d\tau + \frac{1}{\Gamma(\alpha)}\int_3^t (t-\tau)^{\alpha-1}[\hat{A}X(\tau) + \hat{f}(t, X(\tau), 0)]d\tau.$$

In the classical case of H_∞ control, i.e. the H_∞ control of first order systems as introduced in [26, 27], $\alpha = 1$, then the above can be reduced to $X(t) = X_3 + \int_3^t \hat{A}X(\tau) + \hat{f}(t, X(\tau), 0)d\tau$, where $X_3 = X(3)$, which is the integral equation of

$$\begin{cases} \dot{X} = \hat{A}X + \hat{f}(t, X, 0) \\ X(3) = X_3. \end{cases}$$

The H_∞ controller starts to stabilize this controlled system as $t = 3$. After the state is stabilized, i.e. $\|X(t)\|_1$ is driven sufficiently close to zero, it will maintain the external stability of the stabilized system from possible disturbance to output. With the emergence and disappearance of possible energy-bounded disturbances, the controller works under this mechanism automatically and repeatedly. However, in the fractional case, it is only ensured that the H_∞ controller is able to stabilize the system of the following integral form

$$X(t) = X_3 + \frac{1}{\Gamma(\alpha)}\int_3^t (t-\tau)^{\alpha-1}[\hat{A}X(\tau) + \hat{f}(t, X(\tau), 0)]d\tau.$$

Those irreducible integrals result from the non-locality of the Caputo fractional derivative in (3.59). It is not straightforward to evaluate their effects on the convergence of $x(t)$ to zero.

Chapter 4

Hybrid System

In this chapter, the results on Lyapunov and external stability for nonlinear systems will be extended for Caputo fractional order hybrid systems. The hybrid system focused here consists of a family of subsystems and a running law that determines the switches between these subsystems and the change of the system state at each switching instant. During the period between any two consecutive switching instants, only one subsystem is active. Thus, the fundamental results including the existence, uniqueness, continuation and smoothness of solutions to nonlinear systems in the last chapter are also applicable here so that the Lyapunov stability results can be naturally extended. To extend the external stability results, we discover that the hybrid system state must be reset at each switching instant.

4.1 System Formulation

For Lyapunov stability analysis, we consider the Caputo fractional order switching nonautonomous system as

$$\begin{cases} {}_{t_{k-1}}^{\ C}D_t^{\alpha} x = f_{\sigma(t)}(t, x) \\ x(t_0) = x_0, \end{cases} \tag{4.1}$$

where $\alpha \in (0, 1)$, and $f_\sigma : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, in which $\sigma(t) : [t_0, t_1) \cup [t_1, t_2) \cup ... \cup [t_{k-1}, t_k) \cup ... \to \mathscr{P} = \{1, 2, ...\}$, is the switching signal, and denotes the number of the active subsystem at t, i.e. $\sigma(t) = i$, where $t \in [t_{k-1}, t_k)$ for some $k = 1, 2, ...$ and $i \in \mathscr{P}$, means that the subsystem i is active during $[t_{k-1}, t_k)$. Here t_k, $k = 1, 2, ...$, with $\lim_{k\to\infty} t_k = \infty$ and $0 < T_{min} \leq t_k - t_{k-1} \leq T_{max} < \infty$, is called a switching instant.

Note that the lower bound of the Caputo fractional derivative in (4.1) is updated to t_{k-1} at each switching instant. This avoids the effect of the history of solution $x(t)$ from t_0 to t_{k-1} on its future evolution, i.e. $x(t)$, $t \geq t_{k-1}$, so that there is no irreducible integral terms, as those appearing in Example 3.7.3, affecting the Lyapunov stability of (4.1).

Definition 4.1.1. *The constant x^* is an equilibrium point of (4.1), if and only if $f_{\sigma(t)}(t, x^*) = 0$, for all $t \geq t_0$.*

Consider the effect of input on (4.1), then it becomes the following Caputo fractional order switching control system

$$\begin{cases} {}_{t_{k-1}}^{\ C}D_t^{\alpha} x = \bar{f}_{\sigma(t)}(t,x,u) \\ y = h_{\sigma(t)}(t,x,u) \\ x(t_0) = x_0, \end{cases} \tag{4.2}$$

where $\bar{f}_\sigma : [0,\infty) \times \mathbb{R}^n \times \Omega_u \to \mathbb{R}^n$, $\Omega_u \subseteq \mathbb{R}^l$; $h_\sigma : [0,\infty) \times \mathbb{R}^n \times \Omega_u \to \mathbb{R}^p$; u, y are the input, output respectively.

As we shall see in Section 4.3, in order for the external stability of the control system (4.2), it is necessary to reset its state at each t_k as follows

$$\begin{cases} {}_{t_{k-1}}^{\ C}D_t^{\alpha} x = \bar{f}_{\sigma(t)}(t,x,u), t \in [t_{k-1}, t_k) \\ x(t_k) = 0 \\ y = h_{\sigma(t)}(t,x,u) \\ x(t_0) = x_0, \end{cases} \tag{4.3}$$

where the second equation is equivalent to the difference (impulse) form: $\Delta x = x(t_k) - x(t_k^-) = -x(t_k^-), t = t_k$.

4.2 Lyapunov Stability

The concepts of Lyapunov stability for the hybrid system (4.1) are almost the same as that for the nonlinear system (3.33) in the last chapter, see below.

Definition 4.2.1. *Assume $f_i(t,0) \equiv 0$ for any $i \in \mathscr{P}$, and let $x(t) = x(t,x_0)$ denote the solution of (4.1). Then the trivial solution of (4.1) is said to be*

i. stable, if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\|x_0\|_2 < \delta$ implies $\|x(t)\|_2 < \epsilon$, for all $t \geq t_0$;

ii. asymptotically stable, if it is stable, and there exists a $\sigma > 0$ such that $\|x_0\|_2 < \sigma$ implies $\lim_{t\to\infty} \|x(t)\|_2 = 0$.

As have done for (3.33), here we still use quadratic Lyapunov functions to study the Lyapunov stability for (4.1).

4.2.1 Quadratic Lyapunov Function

We also first prove an estimation for ${}_{t_{k-1}}^{\ C}D_t^{\alpha}[x^T(t)Px(t)]$, by using the smoothness property of $x(t)$,.

Lemma 4.2.1. *Let $x(t)$ be a function: $[t_{k-1}, h_k] \to \mathbb{R}^n$. Assume $x(t) \in C^{1,1-\alpha}(t_{k-1}, h_k]$ and ${}_{t_{k-1}}^{\ C}D_t^{\alpha} x(t) \in C[t_{k-1}, h_k]$, then ${}_{t_{k-1}}^{\ C}D_t^{\alpha}[x^T(t)Px(t)] \in C[t_{k-1}, h_k]$ and*

$${}_{t_{k-1}}^{\ C}D_t^{\alpha}[x^T(t)Px(t)] \leq [{}_{t_{k-1}}^{\ C}D_t^{\alpha} x^T(t)]Px(t) + x^T(t)P\, {}_{t_{k-1}}^{\ C}D_t^{\alpha} x(t), \tag{4.4}$$

for all $t \in [t_{k-1}, h_k]$, where P is any $n \times n$ positive definite matrix.

Proof. It follows from the assumption that $x(t) \in C[t_{k-1}, h_k] \cap C^1(t_{k-1}, h_k]$. Moreover, there exists a constant $c > 0$ such that $\|x'(t)\|_1 \le c(t - t_{k-1})^{\alpha-1}$, for any $t \in (t_{k-1}, h_k]$. Thus, for each fixed $t \in (t_{k-1}, h_k]$, $\Delta x^T(\tau)[\Delta x(\tau)]'/(t-\tau)^\alpha$ is integrable on any closed subinterval of (t_{k-1}, t), where $\Delta x(t) = x(t) - x(t_{k-1})$. Furthermore, for $\tau \in (t_{k-1}, t]$, $\|\dot{x}(\tau)\|_1 \le c(\tau - t_{k-1})^{\alpha-1}$ and $|\dot{x}_i(\tau)| \le c(\tau - t_{k-1})^{\alpha-1}$, $i = 1, 2, ..., n$, where $x^T = [x_1, x_2, ..., x_n]$. Thus, for any $\tau \in (t_{k-1}, t]$,

$$\int_{t_{k-1}}^{\tau} -c(s - t_{k-1})^{\alpha-1} ds \le \int_{t_{k-1}}^{\tau} \dot{x}_i(s) ds \le \int_{t_{k-1}}^{\tau} c(s - t_{k-1})^{\alpha-1} ds,$$

so that $|x_i(\tau) - x_i(t_{k-1})| \le (c/\alpha)(\tau - t_{k-1})^\alpha$, then $\|\Delta x(\tau)\|_1 \le c_\Delta(\tau - t_{k-1})^\alpha$, $c_\Delta = nc/\alpha$. Thus,

$$\begin{aligned}
\int_{t_{k-1}}^{t} \Big|\frac{\Delta x^T(\tau) P [\Delta x(\tau)]'}{(t-\tau)^\alpha}\Big| \mathrm{d}\tau &\le \int_{t_{k-1}}^{t} \frac{\|\Delta x(\tau)\|_1 \|P\|_1 \|[\Delta x(\tau)]'\|_1}{(t-\tau)^\alpha} \mathrm{d}\tau \\
&\le c_\Delta c \|P\|_1 \int_{t_{k-1}}^{t} \frac{(\tau - t_{k-1})^{2\alpha-1}}{(t-\tau)^\alpha} \mathrm{d}\tau \\
&= c_\Delta c \|P\|_1 \frac{\Gamma(1-\alpha)\Gamma(2\alpha)}{\Gamma(\alpha+1)} (t - t_{k-1})^\alpha.
\end{aligned}$$

Therefore, the improper integral ${}^{C}_{t_{k-1}}D^{\alpha}_{t}[\Delta x^T(t) P \Delta x(t)]$ is absolutely convergent on $(t_{k-1}, h_k]$, i.e. ${}^{C}_{t_{k-1}}D^{\alpha}_{t}[\Delta x^T(t) P \Delta x(t)]$ exists on $(t_{k-1}, h_k]$. Moreover, ${}^{C}_{t_{k-1}}D^{\alpha}_{t^+_{k-1}}[\Delta x^T(t) P \Delta x(t)]$ exists and it equals zero. This well defines ${}^{C}_{t_{k-1}}D^{\alpha}_{t}[\Delta x^T(t) P \Delta x(t)]$ at t_{k-1}. It follows that ${}^{C}_{t_{k-1}}D^{\alpha}_{t_{k-1}}[\Delta x^T(t) P \Delta x(t)]$ exists on $[t_{k-1}, h_k]$.

For $\delta_1 \in (0, h_k - t_{k-1}]$, $t_1, t_2 \in [t_{k-1}, t_{k-1} + \delta_1]$ implies

$$\begin{aligned}
&|{}^{C}_{t_{k-1}}D^{\alpha}_{t_1}[\Delta x^T(t) P \Delta x(t)] - {}^{C}_{t_{k-1}}D^{\alpha}_{t_2}[\Delta x^T(t) P \Delta x(t)]| = \frac{2}{\Gamma(1-\alpha)} \Big| \int_{t_{k-1}}^{t_1} \frac{\Delta x^T(\tau) P [\Delta x(\tau)]'}{(t_1-\tau)^\alpha} \mathrm{d}\tau - \int_{t_{k-1}}^{t_2} \frac{[\Delta x^T(\tau) P [\Delta x(\tau)]'}{(t_2-\tau)^\alpha} \mathrm{d}\tau \Big| \\
&\le \frac{2}{\Gamma(1-\alpha)} \|P\|_1 \Big[\int_{t_{k-1}}^{t_1} \frac{\|\Delta x(\tau)\|_1 \|[\Delta x(\tau)]'\|_1}{(t_1-\tau)^\alpha} \mathrm{d}\tau + \int_{t_{k-1}}^{t_2} \frac{\|\Delta x(\tau)\|_1 \|[\Delta x(\tau)]'\|_1}{(t_2-\tau)^\alpha} \mathrm{d}\tau\Big] \\
&\le \frac{2}{\Gamma(1-\alpha)} c_\Delta c \|P\|_1 \Big[\int_{t_{k-1}}^{t_1} \frac{(\tau - t_{k-1})^{2\alpha-1}}{(t_1-\tau)^\alpha} \mathrm{d}\tau + \int_{t_{k-1}}^{t_2} \frac{(\tau - t_{k-1})^{2\alpha-1}}{(t_2-\tau)^\alpha} \mathrm{d}\tau\Big] \\
&= \frac{2}{\Gamma(1-\alpha)} c_\Delta c \|P\|_1 \frac{\Gamma(1-\alpha)\Gamma(2\alpha)}{\Gamma(1+\alpha)} [(t_1 - t_{k-1})^\alpha + (t_2 - t_{k-1})^\alpha] \\
&\le 4 c_\Delta c \|P\|_1 \frac{\Gamma(2\alpha)}{\Gamma(1+\alpha)} \delta_1^\alpha \\
&= \hat{c}_1 \delta_1^\alpha,
\end{aligned}$$

where $\hat{c}_1 = 4 c_\Delta c \|P\|_1 \Gamma(2\alpha)/\Gamma(1+\alpha)$. Now we are ready to have the following statements.

If $t_* \in [t_{k-1}, t_{k-1} + \delta_1/2]$, then $|t - t_*| \le \delta_1/2$ and $t \ge t_{k-1}$ together imply $|{}^{C}_{t_{k-1}}D^{\alpha}_{t}[\Delta x^T(t) P \Delta x(t)] - {}^{C}_{t_{k-1}}D^{\alpha}_{t_*}[\Delta x^T(t) P \Delta x(t)]| \le \hat{c}_1 \delta_1^\alpha$.

If $t_* \in [t_{k-1} + \delta_1/2, h_k]$, for $|t - t_*| \le \delta_1/2$ and $t_{k-1} \le t \le h_k$, we only need to consider the case $t \in [t_{k-1} + \delta_1/2, h_k]$. This is because for $t \in [t_{k-1}, t_{k-1} + \delta_1/2)$, $|t - t_*| \le \delta_1/2$ implies $t, t_* \in [t_{k-1}, t_{k-1} + \delta_1]$, then $|{}^{C}_{t_{k-1}}D^{\alpha}_{t}[\Delta x^T(t) P \Delta x(t)] -$

${}_{t_{k-1}}^{\;C}D_{t_*}^{\alpha}[\Delta x^T(t)P\Delta x(t)]| \leq \hat{c}_1\delta_1^{\alpha}$, which is the same as above. Without loss of generality, assume $t \geq t_*$, then

$$\begin{aligned}
\int_{t_*}^{t}\frac{(\tau-t_{k-1})^{2\alpha-1}}{(t-\tau)^{\alpha}}\mathrm{d}\tau &= \max\{(t-t_{k-1})^{2\alpha-1},(t_*-t_{k-1})^{2\alpha-1}\}\int_{t_*}^{t}\frac{1}{(t-\tau)^{\alpha}}\,\mathrm{d}\tau \\
&= \frac{1}{1-\alpha}\max\{(t-t_{k-1})^{2\alpha-1},(t_*-t_{k-1})^{2\alpha-1}\}(t-t_*)^{1-\alpha} \\
&\leq \begin{cases} \frac{1}{1-\alpha}(t_*-t_{k-1})^{2\alpha-1}(\frac{\delta_1}{2})^{1-\alpha} & \text{if } \alpha\in(0,0.5) \\ \frac{1}{1-\alpha}(t-t_{k-1})^{2\alpha-1}(\frac{\delta_1}{2})^{1-\alpha} & \text{if } \alpha\in[0.5,1) \end{cases} \\
&\leq \begin{cases} \frac{1}{1-\alpha}(\frac{\delta_1}{2})^{\alpha} & \text{if } \alpha\in(0,0.5) \\ \frac{1}{1-\alpha}(h_k-t_{k-1})^{2\alpha-1}(\frac{\delta_1}{2})^{1-\alpha} & \text{if } \alpha\in[0.5,1) \end{cases};
\end{aligned}$$

$$\begin{aligned}
&\int_{t_{k-1}}^{t_*}(\tau-t_{k-1})^{2\alpha-1}[\frac{1}{(t_*-\tau)^{\alpha}}-\frac{1}{(t-\tau)^{\alpha}}]\,\mathrm{d}\tau \\
&= \int_{t_{k-1}}^{t_{k-1}+\frac{\delta_1}{2}}(\tau-t_{k-1})^{2\alpha-1}[\frac{1}{(t_*-\tau)^{\alpha}}-\frac{1}{(t-\tau)^{\alpha}}]\,\mathrm{d}\tau + \int_{t_{k-1}+\frac{\delta_1}{2}}^{t_*}(\tau-t_{k-1})^{2\alpha-1}[\frac{1}{(t_*-\tau)^{\alpha}}-\frac{1}{(t-\tau)^{\alpha}}]\,\mathrm{d}\tau \\
&\leq \int_{t_{k-1}}^{t_{k-1}+\frac{\delta_1}{2}}(\tau-t_{k-1})^{2\alpha-1}[\frac{1}{(t_*-\tau)^{\alpha}}+\frac{1}{(t-\tau)^{\alpha}}]\,\mathrm{d}\tau + \max\{(t_*-t_{k-1})^{2\alpha-1},(\frac{\delta_1}{2})^{2\alpha-1}\}\int_{t_{k-1}+\frac{\delta_1}{2}}^{t_*}\frac{1}{(t_*-\tau)^{\alpha}}-\frac{1}{(t-\tau)^{\alpha}}\,\mathrm{d}\tau \\
&\leq 2\int_{t_{k-1}}^{t_{k-1}+\frac{\delta_1}{2}}\frac{(\tau-t_{k-1})^{2\alpha-1}}{(t_{k-1}+\frac{\delta_1}{2}-\tau)^{\alpha}}\,\mathrm{d}\tau + \frac{1}{1-\alpha}\max\{(t_*-t_{k-1})^{2\alpha-1},(\frac{\delta_1}{2})^{2\alpha-1}\} \\
&\quad \times[(t-t_*)^{1-\alpha}+(t_*-t_{k-1}-\frac{\delta_1}{2})^{1-\alpha}-(t-t_{k-1}-\frac{\delta_1}{2})^{1-\alpha}] \\
&\leq \frac{2\Gamma(1-\alpha)\Gamma(2\alpha)}{\Gamma(\alpha+1)}(\frac{\delta_1}{2})^{\alpha}+\frac{1}{1-\alpha}\max\{(t_*-t_{k-1})^{2\alpha-1},(\frac{\delta_1}{2})^{2\alpha-1}\}(\frac{\delta_1}{2})^{1-\alpha} \\
&\leq \begin{cases} \frac{1}{1-\alpha}[(\frac{\delta_1}{2})^{\alpha}+\frac{2\Gamma(2-\alpha)\Gamma(2\alpha)}{\Gamma(\alpha+1)}(\frac{\delta_1}{2})^{\alpha}] & \text{if } \alpha\in(0,0.5) \\ \frac{1}{1-\alpha}[(h_k-t_{k-1})^{2\alpha-1}(\frac{\delta_1}{2})^{1-\alpha}+\frac{2\Gamma(2-\alpha)\Gamma(2\alpha)}{\Gamma(\alpha+1)}(\frac{\delta_1}{2})^{\alpha}] & \text{if } \alpha\in[0.5,1) \end{cases},
\end{aligned}$$

so that

$$\begin{aligned}
&|{}_{t_{k-1}}^{\;C}D_{t}^{\alpha}[\Delta x^T(t)P\Delta x(t)] - {}_{t_{k-1}}^{\;C}D_{t_*}^{\alpha}[\Delta x^T(t)P\Delta x(t)]| \\
&= \frac{2}{\Gamma(1-\alpha)}\Big|\int_{t_{k-1}}^{t}\frac{\Delta x^T(\tau)P[\Delta x(\tau)]'}{(t-\tau)^{\alpha}}\,\mathrm{d}\tau - \int_{t_{k-1}}^{t_*}\frac{\Delta x^T(\tau)P[\Delta x(\tau)]'}{(t_*-\tau)^{\alpha}}\,\mathrm{d}\tau\Big| \\
&\leq \frac{2}{\Gamma(1-\alpha)}\{\int_{t_*}^{t}\frac{\|\Delta x(\tau)\|_1\|P\|_1\|[\Delta x(\tau)]'\|_1}{(t-\tau)^{\alpha}}\,\mathrm{d}\tau + \int_{t_{k-1}}^{t_*}\|\Delta x(\tau)\|_1\|P\|_1\|[\Delta x(\tau)]'\|_1[\frac{1}{(t_*-\tau)^{\alpha}}-\frac{1}{(t-\tau)^{\alpha}}]\,\mathrm{d}\tau\} \\
&\leq \frac{2}{\Gamma(1-\alpha)}c_{\Delta}c\|P\|_1\{\int_{t_*}^{t}\frac{(\tau-t_{k-1})^{2\alpha-1}}{(t-\tau)^{\alpha}}\,\mathrm{d}\tau + \int_{t_{k-1}}^{t_*}(\tau-t_{k-1})^{2\alpha-1}[\frac{1}{(t_*-\tau)^{\alpha}}-\frac{1}{(t-\tau)^{\alpha}}]\,\mathrm{d}\tau\} \\
&\leq \hat{c}_{21}\begin{cases} 2(\frac{\delta_1}{2})^{\alpha}+\hat{c}_{22}(\frac{\delta_1}{2})^{\alpha} & \text{if } \alpha\in(0,0.5) \\ 2(h_k-t_{k-1})^{2\alpha-1}(\frac{\delta_1}{2})^{1-\alpha}+\hat{c}_{22}(\frac{\delta_1}{2})^{\alpha} & \text{if } \alpha\in[0.5,1) \end{cases},
\end{aligned}$$

where $\hat{c}_{21} = 2c_{\Delta}c\|P\|_1/\Gamma(2-\alpha)$ and $\hat{c}_{22} = 2\Gamma(2-\alpha)\Gamma(2\alpha)/\Gamma(\alpha+1)$.

As we see, for any $\epsilon > 0$, there exists δ_1 such that $\hat{c}_1\delta_1^{\alpha} < \epsilon$ and the right hand side of the inequality above is also less than ϵ. Let $\delta = \delta_1/2$, then for any $t_* \in [t_{k-1}, h_k]$, $|t-t_*| \le \delta$ implies $|{}_{t_{k-1}}^{\;\;C}D_t^{\alpha}[\Delta x^T(t)P\Delta x(t)] - {}_{t_{k-1}}^{\;\;C}D_{t_*}^{\alpha}[\Delta x^T(t)P\Delta x(t)]| < \epsilon$. Thus, ${}_{t_{k-1}}^{\;\;C}D_t^{\alpha}[\Delta x^T(t)P\Delta x(t)] \in C[t_{k-1}, h_k]$, then ${}_{t_{k-1}}^{\;\;C}D_t^{\alpha}[x^T(t)Px(t)] \in C[t_{k-1}, h_k]$, due to ${}_{t_{k-1}}^{\;\;C}D_t^{\alpha}x(t) \in C[t_{k-1}, h_k]$.

For $t = t_{k-1}$, (4.4) holds, due to ${}_{t_{k-1}}^{\;\;C}D_{t_{k-1}}^{\alpha}[\Delta x^T(t)P\Delta x(t)] = 0$.

For $t \in (t_{k-1}, h_k]$, we can also prove (4.4). According to Caputo's definition, we have

$$x^T(t)P{}_{t_{k-1}}^{\;\;C}D_t^{\alpha}x(t) - \frac{1}{2}\,{}_{t_{k-1}}^{\;\;C}D_t^{\alpha}[x^T(t)Px(t)] = \frac{1}{\Gamma(1-\alpha)}\int_{t_{k-1}}^{t}\frac{[x^T(t)-x^T(\tau)]P\dot{x}(\tau)}{(t-\tau)^{\alpha}}\mathrm{d}\tau. \tag{4.5}$$

Let $y(\tau) = x(t) - x(\tau)$, then $\dot{y}(\tau) = -\dot{x}(\tau)$. (4.5) can be rewritten as

$$x^T(t)P{}_{t_{k-1}}^{\;\;C}D_t^{\alpha}x(t) - \frac{1}{2}\,{}_{t_{k-1}}^{\;\;C}D_t^{\alpha}[x^T(t)Px(t)] = -\frac{1}{\Gamma(1-\alpha)}\int_{t_{k-1}}^{t}\frac{y^T(\tau)P\dot{y}(\tau)}{(t-\tau)^{\alpha}}\,\mathrm{d}\tau. \tag{4.6}$$

For any $t_{k-1} < t_1 < t_2 < t$, $y^T(\tau)Py(\tau)/2$ and $1/(t-\tau)^{\alpha}$ are continuously differentiable with respect to τ on $[t_1, t_2]$. Integrating by parts yields,

$$\int_{t_1}^{t_2}\frac{y^T(\tau)P\dot{y}(\tau)}{(t-\tau)^{\alpha}}\,\mathrm{d}\tau = \frac{y^T(\tau)Py(\tau)}{2(t-\tau)^{\alpha}}\Big|_{\tau=t_2} - \frac{y^T(\tau)Py(\tau)}{2(t-\tau)^{\alpha}}\Big|_{\tau=t_1} - \int_{t_1}^{t_2}\frac{\alpha y^T(\tau)Py(\tau)}{2(t-\tau)^{\alpha+1}}\,\mathrm{d}\tau.$$

We take the limit for $t_1 \to t_{k-1}$ and $t_2 \to t$, then

$$\int_{t_{k-1}}^{t}\frac{y^T(\tau)P\dot{y}(\tau)}{(t-\tau)^{\alpha}}\,\mathrm{d}\tau = \lim_{\tau\to t}\frac{y^T(\tau)Py(\tau)}{2(t-\tau)^{\alpha}} - \lim_{\tau\to t_{k-1}}\frac{y^T(\tau)Py(\tau)}{2(t-\tau)^{\alpha}} - \int_{t_{k-1}}^{t}\frac{\alpha y^T(\tau)Py(\tau)}{2(t-\tau)^{\alpha+1}}\,\mathrm{d}\tau \tag{4.7}$$

holds, if any three of these four terms exist. It follows from (4.6) and the existence of both ${}_{t_{k-1}}^{\;\;C}D_t^{\alpha}x(t)$ and ${}_{t_{k-1}}^{\;\;C}D_t^{\alpha}[x^T(t)P\,x(t)]$ that the left side integral above exists. In the following, we shall check the existence of those two limits on the right side.

$$\lim_{\tau\to t_{k-1}}\frac{y^T(\tau)Py(\tau)}{2(t-\tau)^{\alpha}} = \frac{y^T(t_{k-1})Py(t_{k-1})}{2(t-t_{k-1})^{\alpha}} = \frac{[x(t)-x(t_{k-1})]^TP[x(t)-x(t_{k-1})]}{2(t-t_{k-1})^{\alpha}} \ge 0.$$

Since $y(\tau) \to 0$, $(t-\tau)^{\alpha} \to 0$ as $\tau \to t$, and $\dot{y}(\tau) = -\dot{x}(\tau)$ exists due to $x(\tau) \in C^1(t_{k-1}, h_k]$, by the L'Hospital rule,

$$\lim_{\tau\to t}\frac{y^T(\tau)Py(\tau)}{2(t-\tau)^{\alpha}} = \lim_{\tau\to t}\frac{y^T(\tau)P\dot{y}(\tau)}{-\alpha(t-\tau)^{\alpha-1}} = 0.$$

Thus, $\lim_{\tau\to t}y^T(\tau)Py(\tau)/[2(t-\tau)^{\alpha}] = 0$ and $\lim_{\tau\to t_{k-1}}y^T(\tau)Py(\tau)/[2(t-\tau)^{\alpha}] \ge 0$, for $t \in (t_{k-1}, h_k]$. Therefore, (4.7) holds such that $\int_{t_{k-1}}^{t}\alpha y^T(\tau)Py(\tau)/[2(t-\tau)^{\alpha+1}]\,\mathrm{d}\tau$ is well defined and nonnegative. It then follows from (4.6) and (4.7) that ${}_{t_{k-1}}^{\;\;C}D_t^{\alpha}[x^T(t)Px(t)] \le x^T(t)P\,{}_{t_{k-1}}^{\;\;C}D_t^{\alpha}x(t) + {}_{t_{k-1}}^{\;\;C}D_t^{\alpha}x^T(t)Px(t)$, for all $t \in (t_{k-1}, h_k]$. □

4.2.2 Lyapunov Stability Criteria

Based on (4.4), we are then ready to prove our Lyapunov stability criteria for hybrid system (4.1).

Theorem 4.2.1. *Let $x = 0$ be an equilibrium point for the Caputo fractional order switching nonautonomous system (4.1). Assume:*

i. for any $i \in \mathscr{P}$, f_i is continuously differentiable with respect to t and x on $[0, \infty) \times \mathbb{R}^n$;

ii. for any $i \in \mathscr{P}$ and any $h^* > 0$, there exists a monotonically increasing function $\psi_{ih^*} : [0, \infty) \to [0, \infty)$, such that for any $(t, x), (t, y) \in [0, h^*] \times \mathbb{R}^n$ and all possible nonnegative integers $u_0, u_1, ..., u_n$ with $u_0 + u_1 + ... + u_n = 1$,

$$\|\frac{\partial}{\partial t^{u_0} \partial {x_1}^{u_1} ... \partial {x_n}^{u_n}} f_i(t, x) - \frac{\partial}{\partial t^{u_0} \partial {y_1}^{u_1} ... \partial {y_n}^{u_n}} f_i(t, y)\|_1 \leq \psi_{ih^*}(\max\{\|x\|_1, \|y\|_1\}) \, \|x - y\|_1.$$

Then the equilibrium point of system (4.1) is stable if for any $i \in \mathscr{P}$, there exist $n \times n$ positive definite matrices P, Q such that for any $(t, x) \in [0, \infty) \times \mathbb{R}^n$,

$$x^T P f_i(t, x) + f_i^T(t, x) P x \leq 0, \tag{4.8}$$

and is asymptotically stable if

$$x^T P f_i(t, x) + f_i^T(t, x) P x \leq -x^T Q x. \tag{4.9}$$

Proof. Suppose $t \in [t_{k-1}, t_k)$ and $\sigma(t) = i$. It follows from the continuation result Theorem 3.2.3, i implies that the solution of (4.1) $x(t)$ exists and is continuous on either $[t_{k-1}, t_k]$ or $[t_{k-1}, \beta_k)$, where $\beta_k \leq t_k$ such that $\lim_{t \to \beta_k^-} x(t) = \infty$. Moreover, i suffices that f_i is continuously differentiable with respect to t and x on $[t_{k-1}, t_k] \times \mathbb{R}^n$. From ii, for any $(t, x), (t, y) \in [t_{k-1}, t_k] \times \mathbb{R}^n$ and all possible nonnegative integers $u_0, u_1, ..., u_n$ with $u_0 + u_1 + ... + u_n = 1$,

$$\|\frac{\partial}{\partial t^{u_0} \partial {x_1}^{u_1} ... \partial {x_n}^{u_n}} f_i(t, x) - \frac{\partial}{\partial t^{u_0} \partial {y_1}^{u_1} ... \partial {y_n}^{u_n}} f_i(t, y)\|_1 \leq \psi_{it_k}(\max\{\|x\|_1, \|y\|_1\}) \, \|x - y\|_1.$$

Thus, according to Corollary 3.3.3 and Remark 3.3.4, the solution of (4.1) is unique on $[t_{k-1}, \beta_k)$ and $x(t)|_{t \in [t_{k-1}, \beta_k)} \in C^{1,1-\alpha}(t_{k-1}, \beta_k)$, provided the latter case. Moreover, ${}_{t_{k-1}}^{C}D_t^{\alpha} x(t) \in C[t_{k-1}, \beta_k)$ due to $f_i \in C^1[t_{k-1}, t_k] \times \mathbb{R}^n$. Consider the Lyapunov function candidate $V(x) = x^T P x$, then according to Lemma 4.2.1, ${}_{t_{k-1}}^{C}D_t^{\alpha} V[x(t)] \in C[t_{k-1}, \beta_k)$ and

$${}_{t_{k-1}}^{C}D_t^{\alpha} V[x(t)] \leq x^T(t) P f_i(t, x(t)) + f_i^T(t, x(t)) P x(t),$$

for $t \in [t_{k-1}, \beta_k)$.

If (4.8) holds, then ${}_{t_{k-1}}^{C}D_t^{\alpha} V[x(t)] \leq 0$. Thus, there exists a nonnegative function $r_{si}(t) \in C[t_{k-1}, \beta_k)$ such that ${}_{t_{k-1}}^{C}D_t^{\alpha} V[x(t)] = -r_{si}(t)$. According to Theorem 2.1.4,

$$V[x(t)] = V[x(t_{k-1})] - \int_{t_{k-1}}^{t} (t - \tau)^{\alpha - 1} r_{si}(\tau) d\tau, \tag{4.10}$$

where the convolution exists and is nonnegative. Thus, $V[x(t)] \leq V[x(t_{k-1})]$, for $t \in [t_{k-1}, \beta_k)$. This contradicts $\lim_{t\to\beta_k^-} x(t) = \infty$. Thus, the unique solution exists on $[t_{k-1}, t_k]$ and $x(t)|_{t\in[t_{k-1},t_k]} \in C^{1,1-\alpha}(t_{k-1}, t_k]$. Following the similar derivation, we can conclude that (4.10) with $r_{si}(t) \in C[t_{k-1}, t_k]$ holds for all $t \in [t_{k-1}, t_k]$. Thus, $V[x(t)] \leq V(x_0)$, for all $t \geq t_0$. Therefore, the zero equilibrium point is stable.

If (4.9) holds, then for $t \in [t_{k-1}, \beta_k)$,

$$ {}_{t_{k-1}}^{C}D_t^{\alpha} V[x(t)] \leq -\mu V[x(t)], \tag{4.11} $$

where $\mu = \frac{\lambda_{min}(Q)}{\lambda_{max}(P)}$. Here $\lambda_{min}(Q)$, $\lambda_{max}(P)$ denote the minimum, maximum eigenvalues of Q, P, respectively. Clearly, ${}_{t_{k-1}}^{C}D_t^{\alpha} V[x(t)] \leq 0$. Thus, as shown above, the equilibrium point is stable. Moreover, (4.11) holds for all $t \in [t_{k-1}, t_k]$. According to Theorem 2.1.4, for $t \in [t_{k-1}, t_k]$,

$$ V[x(t)] = V[x(t_{k-1})]E_\alpha[-\mu(t-t_{k-1})^\alpha] - \int_{t_{k-1}}^{t} (t-\tau)^{\alpha-1} E_{\alpha,\alpha}[-\mu(t-\tau)^\alpha] r_{ai}(\tau) d\tau, $$

where $r_{ai}(t) \in C[t_{k-1}, t_k]$ is the nonnegative function such that ${}_{t_{k-1}}^{C}D_t^{\alpha} V[x(t)] = -\mu V[x(t)] - r_{ai}(t)$, and the convolution exists. Since $E_{\alpha,\alpha}(-\mu t^\alpha)$ is nonnegative (decreasing and between 0 and 1), see [20], then

$$ V[x(t)] \leq V[x(t_{k-1})]E_\alpha[-\mu(t-t_{k-1})^\alpha]. $$

Due to $0 < T_{min} \leq t_k - t_{k-1} \leq T_{max}$,

$$ V[x(t_k)] \leq V[x(t_{k-1})]E_\alpha[-\mu(t_k - t_{k-1})^\alpha] \leq V[x(t_{k-1})]E_\alpha(-\mu T_{min}^\alpha), $$

which implies that $V[x(t)] \leq V(x_0)[E_\alpha(-\mu T_{min}^\alpha)]^{k-1}$, for $t \in [t_{k-1}, t_k)$. Due to $0 < E_\alpha(-\mu T_{min}^\alpha) < 1$, $\lim_{t\to\infty} V[x(t)] = 0$. Therefore, the equilibrium point of (4.1) is asymptotically stable. □

As we see, the key of the proof above is that the solutions of (4.1) have required smoothness property so that Lemma 4.2.1 is applicable here. Thus, applying our results on the smoothness of solutions, we can have a useful corollary of the theorem above, see below.

Corollary 4.2.1. *Let $x = 0$ be an equilibrium point for the Caputo fractional order switching nonautonomous system (4.1). Assume $f_i \in C^2([0,\infty) \times \mathbb{R}^n)$, for any $i \in \mathscr{P}$. Then the equilibrium point of system (4.1) is stable if for any $i \in \mathscr{P}$, there exist $n \times n$ positive definite matrices P, Q such that for any $(t, x) \in [0,\infty) \times \mathbb{R}^n$,*

$$ x^T P f_i(t,x) + f_i^T(t,x) P x \leq 0, $$

and is asymptotically stable if

$$ x^T P f_i(t,x) + f_i^T(t,x) P x \leq -x^T Q x. $$

Proof. It follows from Theorem 3.2.3, $f_i \in C^2([0,\infty) \times \mathbb{R}^n)$ for any $i \in \mathscr{P}$ implies that the solution of (4.1) $x(t)$ exists and is continuous on either $[t_{k-1}, t_k]$ or $[t_{k-1}, \beta_k)$, where $\beta_k \leq t_k$ such that $\lim_{t\to\beta_k^-} x(t) = \infty$. According to Corollary 3.3.6, this assumption also suffices that the solution is unique on $[t_{k-1}, \beta_k)$ and $x(t)|_{t\in[t_{k-1},\beta_k)} \in C^{1,1-\alpha}(t_{k-1}, \beta_k)$, provided the latter case. The rest of proof straightforwardly follows the proof of Theorem 4.2.1. □

4.3 External Stability

As for nonlinear control systems in Section 3.5, the Lyapunov-like function will be also used to investigate the external stability for Caputo fractional order hybrid control systems here. As we can see, the definition of external stability, Definition 3.5.1, is independent of types of control systems. We shall not introduce it here again.

4.3.1 Lyapunov-Like Function

The properties of the Lyapunov-like function based on the diffusive realization of hybrid control system (4.2) are shown in the following lemma. We shall see, in the conclusion ii, the Lyapunov-like function involves the system state at each t_{k-1}, for $k \geq 2$.

Lemma 4.3.1. *Assume:*

i. $u : [0,\infty) \to \Omega_u$, *is continuous;*

ii. for any $i \in \mathscr{P}$, $\bar{f}_i$ *is continuous on* $[0,\infty) \times \mathbb{R}^n \times \Omega_u$, *and for any* $h^* > 0$, *there exists a constant* $L_{h^*} > 0$, *such that for any* (t, x, u), $(t, y, u) \in \{(t, x, u) : t \in [0, h^*], x \in \mathbb{R}^n, u \in \Omega_u\}$,

$$\|\bar{f}_i(t, x, u) - \bar{f}_i(t, y, u)\|_1 \leq L_{h^* i}\|x - y\|_1.$$

Then

i. for each k, *the solution of switching control system* (4.2) $x(t)$ *exists and is unique on* $[t_{k-1}, t_k)$. *Moreover,* $x(t)|_{t\in[t_{k-1},t_k)} \in C[t_{k-1},\ t_k)$, *and*

$$x(t) = x(t_{k-1}) + \int_0^\infty \mu_\alpha(\omega)\phi_{k-1}(\omega, t)d\omega,$$

where $\phi_{k-1}(\omega, t)$ *is the solution of the initial value problem:* $\partial\phi_{k-1}(\omega, t)/\partial t = -\omega\phi_{k-1}(\omega, t) + \bar{f}_{\sigma(t_{k-1})}(t, x(t), u(t))$, $\phi_{k-1}(\omega, t_{k-1}) = 0$; $\mu_\alpha(\omega) = [\sin(\pi\alpha)/\pi]\omega^{-\alpha}$ *and* $\omega \in (0,\infty)$.

ii. for any $T \in [t_{k-1}, t_k)$, $V_{k-1}(T)$ *exists,* $V_{k-1}(T) \geq 0$ *and*

$$\begin{aligned} V_{k-1}(T) = &\int_{t_{k-1}}^{T} [x(t) - x(t_{k-1})]^T P\bar{f}_{\sigma(t_{k-1})}(t, x(t), u(t)) + \bar{f}^T_{\sigma(t_{k-1})}(t, x(t), u(t))P[x(t) - x(t_{k-1})]dt \\ &- 2\int_0^\infty \int_0^T \mu_\alpha(\omega)\omega\phi^T_{k-1}(\omega, t)P\phi_{k-1}(\omega, t)dtd\omega, \end{aligned}$$

where $V_{k-1}(t) := \int_0^\infty \mu_\alpha(\omega)\phi^T_{k-1}(\omega, t)P\phi_{k-1}(\omega, t)d\omega$, *in which* $\phi_{k-1}(\omega, t)$ *is the solution of the initial value problem in* i *above and* P *is any positive definite matrix.*

Proof. i. Suppose $t \in [t_{k-1}, t_k)$ and $\sigma(t) = i$. It follows from i and ii that $\bar{f}_i$ is continuous in t and Lipschitz in x on $[t_{k-1}, t_k] \times \mathbb{R}^n$. According to Theorem 3.1.1 and Remark 3.1.1, the solution of (4.2) uniquely exists on $[t_{k-1}, t_k)$ and $x(t)|_{t\in[t_{k-1},t_k)} \in C[t_{k-1}, t_k)$. According to Lemma 3.1.1, the solution $x(t)$ must be of the following form

$$x(t) = x(t_{k-1}) + \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^{t} \frac{\bar{f}_i(\tau, x(\tau), u(\tau))}{(t-\tau)^{1-\alpha}} d\tau,$$

for $t \in [t_{k-1}, t_k)$. Let $P_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha), t > 0$, then rewrite it as

$$\begin{aligned} P_\alpha(t) &= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(1-\alpha)} \frac{\Gamma(1-\alpha)}{s^{1-\alpha}}|_{s=t} = \frac{1}{\Gamma(\alpha)} \mathscr{L}[P_{1-\alpha}(\omega)]|_{Re(s)>0, s=t} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-t\omega} \frac{\omega^{-\alpha}}{\Gamma(1-\alpha)} d\omega \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \omega^{-\alpha} e^{-\omega t} d\omega = \int_0^\infty \frac{\sin(\alpha\pi)}{\pi} \omega^{-\alpha} e^{-\omega t} d\omega = \int_0^\infty \mu_\alpha(\omega) e^{-\omega t} d\omega. \end{aligned}$$

It follows that

$$\begin{aligned} x(t) &= x(t_{k-1}) + \int_{t_{k-1}}^{t} \bar{f}_i(\tau, x(\tau), u(\tau)) P_\alpha(t-\tau) d\tau \\ &= x(t_{k-1}) + \int_{t_{k-1}}^{t} \bar{f}_i(\tau, x(\tau), u(\tau)) \int_0^\infty \mu_\alpha(\omega) e^{-\omega(t-\tau)} d\omega d\tau \\ &= x(t_{k-1}) + \int_{t_{k-1}}^{t} \int_0^\infty e^{-\omega(t-\tau)} \mu_\alpha(\omega) \bar{f}_i(\tau, x(\tau), u(\tau)) d\omega d\tau. \end{aligned}$$

For any $[\omega_1, \omega_2] \subset (0, \infty)$, $e^{-\omega(t-\tau)}\mu_\alpha(\omega)\bar{f}_i(\tau, x(\tau), u(\tau))$ is continuous on $[\omega_1, \omega_2] \times [t_{k-1}, t]$. Thus,

$$\int_{t_{k-1}}^{t} \int_{\omega_1}^{\omega_2} e^{-\omega(t-\tau)} \mu_\alpha(\omega) \bar{f}_i(\tau, x(\tau), u(\tau)) d\omega d\tau = \int_{\omega_1}^{\omega_2} \int_{t_{k-1}}^{t} e^{-\omega(t-\tau)} \mu_\alpha(\omega) \bar{f}_i(\tau, x(\tau), u(\tau)) d\tau d\omega.$$

Let $\omega_1 \to 0$ and $\omega_2 \to \infty$, then the existence of $x(t)$ on $[t_{k-1}, t]$ suffices that the limit equation above holds. Thus,

$$\begin{aligned} x(t) &= x(t_{k-1}) + \int_0^\infty \int_{t_{k-1}}^{t} e^{-\omega(t-\tau)} \mu_\alpha(\omega) \bar{f}_i(\tau, x(\tau), u(\tau)) d\tau d\omega \\ &= x(t_{k-1}) + \int_0^\infty \mu_\alpha(\omega) \int_{t_{k-1}}^{t} e^{-\omega(t-\tau)} \bar{f}_i(\tau, x(\tau), u(\tau)) d\tau d\omega. \end{aligned}$$

The solution of the initial value problem is

$$\phi_{k-1}(\omega, t) = \int_{t_{k-1}}^{t} e^{-\omega(t-\tau)} \bar{f}_i(\tau, x(\tau), u(\tau)) d\tau.$$

Therefore, $x(t) = x(t_{k-1}) + \int_0^\infty \mu_\alpha(\omega)\phi_{k-1}(\omega, t) d\omega$, for any $t \in [t_{k-1}, t_k)$.

ii. Since both $x(t)$ and $\bar{f}_{\sigma(t_{k-1})}(t, x(t), u(t))$ are continuous on $[t_{k-1}, t_k)$, then $\int_{t_{k-1}}^{T} x^T(t)P\bar{f}_{\sigma(t_{k-1})}(t, x(t), u(t))dt$ exists, for any $T \in [t_{k-1}, t_k)$. Moreover,

$$\int_{t_{k-1}}^{T} x^T(t)P\bar{f}_{\sigma(t_{k-1})}(t, x(t), u(t))dt = \int_{t_{k-1}}^{T} x^T(t_{k-1})P\bar{f}_i(t, x(t), u(t))dt + \int_{t_{k-1}}^{T}\int_{0}^{\infty} \mu_\alpha(\omega)\phi_{k-1}^T(\omega, t)P\bar{f}_i(t, x(t), u(t))d\omega dt.$$

Clearly, the double integral exists. Since $\mu_\alpha(\omega)\phi_{k-1}^T(\omega, t)P\bar{f}_i(t, x(t), u(t))$ is continuous on $[\omega_1, \omega_2] \times [t_{k-1}, T]$ for any $[\omega_1, \omega_2] \subset (0, \infty)$, then

$$\int_{t_{k-1}}^{T}\int_{\omega_1}^{\omega_2} \mu_\alpha(\omega)\phi_{k-1}^T(\omega, t)P\bar{f}_i(t, x(t), u(t))d\omega dt = \int_{\omega_1}^{\omega_2}\int_{t_{k-1}}^{T} \mu_\alpha(\omega)\phi_{k-1}^T(\omega, t)P\bar{f}_i(t, x(t), u(t))dtd\omega.$$

Let $\omega_1 \to 0$ and $\omega_2 \to \infty$, then the limit equation

$$\int_{t_{k-1}}^{T}\int_{0}^{\infty} \mu_\alpha(\omega)\phi_{k-1}^T(\omega, t)P\bar{f}_i(t, x(t), u(t))d\omega dt = \int_{0}^{\infty}\int_{t_{k-1}}^{T} \mu_\alpha(\omega)\phi_{k-1}^T(\omega, t)P\bar{f}_i(t, x(t), u(t))dtd\omega,$$

holds, since the left side limit exists. Because the function $\bar{f}_i(t, x(t), u(t))$ in the integral above is independent of ω, it can be replaced by $\partial\phi_{k-1}(\omega, t)/\partial t + \omega\phi_{k-1}(\omega, t)$, where ω is the same as that of $\mu_\alpha(\omega)$ in the integral, see below.

$$\begin{aligned}
&\int_{0}^{\infty}\int_{t_{k-1}}^{T} \mu_\alpha(\omega)\phi_{k-1}^T(\omega, t)P\bar{f}_i(t, x(t), u(t))dtd\omega = \int_{0}^{\infty}\int_{t_{k-1}}^{T} \mu_\alpha(\omega)\phi_{k-1}^T(\omega, t)P[\frac{\partial\phi_{k-1}(\omega, t)}{\partial t} + \omega\phi_{k-1}(\omega, t)]dtd\omega \\
&= \int_{0}^{\infty}\int_{t_{k-1}}^{T} \mu_\alpha(\omega)\phi_{k-1}^T(\omega, t)P\frac{\partial\phi_{k-1}(\omega, t)}{\partial t}dtd\omega + \int_{0}^{\infty}\int_{t_{k-1}}^{T} \mu_\alpha(\omega)\phi_{k-1}^T(\omega, t)P\omega\phi_{k-1}(\omega, t)dtd\omega \\
&= \int_{0}^{\infty} \mu_\alpha(\omega)\int_{t_{k-1}}^{T} \phi_{k-1}^T(\omega, t)P\frac{\partial\phi_{k-1}(\omega, t)}{\partial t}dtd\omega + \int_{0}^{\infty}\int_{t_{k-1}}^{T} \mu_\alpha(\omega)\omega\phi_{k-1}^T(\omega, t)P\phi_{k-1}(\omega, t)dtd\omega \\
&= \frac{1}{2}\int_{0}^{\infty} \mu_\alpha(\omega)\phi_{k-1}^T(\omega, T)P\phi_{k-1}(\omega, T)d\omega + \int_{0}^{\infty}\int_{t_{k-1}}^{T} \mu_\alpha(\omega)\omega\phi_{k-1}^T(\omega, t)P\phi_{k-1}(\omega, t)dtd\omega.
\end{aligned}$$

As we observe, the equation in the conclusion holds, if the two terms on the right side of the equation above both exist. Another important observation is

$$\int_{t_{k-1}}^{T} \mu_\alpha(\omega)\phi_{k-1}^T(\omega, t)P\bar{f}_i(t,x(t),u(t))dt = \frac{1}{2}\mu_\alpha(\omega)\phi_{k-1}^T(\omega,T)P\phi_{k-1}(\omega,T) + \int_{t_{k-1}}^{T} \mu_\alpha(\omega)\omega\phi_{k-1}^T(\omega, t)P\phi_{k-1}(\omega, t)dt.$$

These three terms above, exist and are nonnegative for any $\omega \in (0, \infty)$, and are integrable on any $[\omega_1, \omega_2] \subset (0, \infty)$. Thus, for any $\omega \in (0, \infty)$,

$$|\frac{1}{2}\mu_\alpha(\omega)\phi_{k-1}^T(\omega,T)P\phi_{k-1}(\omega, T)| \leq \int_{t_{k-1}}^{T} \mu_\alpha(\omega)\phi_{k-1}^T(\omega,t)P\bar{f}_i(t,x(t),u(t))dt.$$

Since the improper integral of the right-side function of ω above from 0 to ∞ exists, the left-side function is absolutely integrable over $(0, \infty)$. Thus, for any $T \in [t_{k-1}, t_k)$, $V_{k-1}(T)$ exists and $V_{k-1}(T) \geq 0$. Then $\int_0^{\infty}\int_{t_{k-1}}^{T} \mu_\alpha(\omega)\omega\phi_{k-1}^T(\omega, t) P\phi_{k-1}(\omega, t)dtd\omega$ exists and is nonnegative as well. This completes the proof. □

4.3.2 External Stability Criterion

In order to prove the external stability, the state $x(t_{k-1})$ involved in the Lyapunov-like function $V_{k-1}(T)$ must be removed. Otherwise, we cannot derive the inequality (4.14). This is why we need to reset the state of (4.2) at each t_k. After reset, (4.2) becomes the reset switching control system (4.3), for which the external stability criterion is stated as follows.

Theorem 4.3.1. *Assume:*

i. $u : [0,\infty) \to \Omega_u$, *is continuous;*

ii. *for any* $i \in \mathscr{P}$, $\bar{f}_i(t,x,u) = \tilde{A}_i x + \tilde{f}_i(t,x,u)$, *where* $\tilde{f}_i : [0,\infty)\times\mathbb{R}^n\times\Omega_u \to \mathbb{R}^n$, *is continuous in t and Lipschitz in x, u with Lipschitz constants* $L_{\tilde{f}xi}$, $L_{\tilde{f}ui}$ *respectively, and* $\tilde{f}_i(t,0,0) \equiv 0$*;*

iii. *for any* $i \in \mathscr{P}$, $h_i(t,x,u)$ *is continuous in t and Lipschitz in x, u on* $[0,\infty)\times\mathbb{R}^n\times\Omega_u$ *with Lipschitz constants* L_{hxi}, L_{hui} *respectively, and* $h_i(t,0,0) \equiv 0$.

Then the reset switching control system (4.3) is externally stable, i.e. $\|y\|_{L_2} \le \gamma\|u\|_{L_2}$, *for* $u \in L_2[t_0,\infty)$ *under the zero initial condition, if for each* $i \in \mathscr{P}$, *there exist constants* $\varepsilon_{\tilde{f}i} > 0$, $\varepsilon_{hi} > 1$ *and a common* $n\times n$ *positive definite matrix P such that*

$$2\varepsilon_{\tilde{f}i}L^2_{\tilde{f}ui} + 2\varepsilon_{hi}L^2_{hui} - \gamma^2 < 0, \tag{4.12}$$

and

$$\begin{bmatrix} \tilde{A}_i^T P + P\tilde{A}_i + 2\varepsilon_{\tilde{f}_i}L^2_{\tilde{f}x_i} + 2\varepsilon_{hi}L^2_{hxi} & P \\ * & -\varepsilon_{\tilde{f}i} \end{bmatrix} < 0. \tag{4.13}$$

Proof. Suppose $t \in [t_{k-1}, t_k)$ and $\sigma(t) = i$. According to Lemma 4.3.1, for any $T \in [t_{k-1}, t_k)$, $V_{k-1} \ge 0$ and

$$\begin{aligned} V_{k-1}(T) = &\int_{t_{k-1}}^{T} [x(t) - x(t_{k-1})]^T P \bar{f}_{\sigma(t_{k-1})}(t, x(t), u(t)) + \bar{f}^T_{\sigma(t_{k-1})}(t, x(t), u(t)) P [x(t) - x(t_{k-1})] dt \\ &- 2\int_0^\infty \int_0^T \mu_\alpha(\omega)\omega\phi^T_{k-1}(\omega,t)P\phi_{k-1}(\omega,t)dtd\omega. \end{aligned}$$

Due to $x_0 = 0$, $x(t_k) = 0$ for each k, and the nonnegativeness of the double integral above, then $x(t_{k-1}) = 0$ and

$$V_{k-1}(T) \le \int_{t_{k-1}}^{T} x^T(t)P\bar{f}_i(t,x(t),u(t)) + \bar{f}_i^T(t,x(t),u(t))Px(t)dt. \tag{4.14}$$

Moreover, viewing the proof of Lemma 4.3.1, since $x(t) \in C[t_{k-1}, t_k)$, and both $x(t_k^-)$ and $V_{k-1}(t_k^-)$ exist, the inequality above also holds at $T = t_k^-$.

It follows from the Lipschitz conditions in ii and iii, for any $\varepsilon_{\tilde{f}i}, \varepsilon_{hi} > 0$,

$$2\varepsilon_{\tilde{f}i}L_{\tilde{f}xi}^2 x^T(t)x(t) + 2\varepsilon_{\tilde{f}i}L_{\tilde{f}ui}^2 u^T(t)u(t) - \varepsilon_{\tilde{f}i}\tilde{f}_i^T(t,x(t),u(t))\tilde{f}_i(t,x(t),u(t)) \geq 0$$

and

$$2\varepsilon_{hi}L_{hxi}^2 x^T(t)x(t) + 2\varepsilon_{hi}L_{hui}^2 u^T(t)u(t) - \varepsilon_{hi}h_i^T(t,x(t),u(t))\,h_i(t,x(t),u(t)) \geq 0.$$

Thus, we have the following,

$$\begin{aligned}
&\int_{t_{k-1}}^{t_k} y^T(t)y(t)dt - \gamma^2\int_{t_{k-1}}^{t_k} u^T(t)u(t)dt + V_{k-1}(t_k^-)\\
&\leq \int_{t_{k-1}}^{t_k} [h_i^T(t,x(t),u(t))h_i(t,x(t),u(t)) - \gamma^2 u^T(t)u(t) + x^T(t)P\bar{f}_i(t,x(t),u(t)) + \bar{f}_i^T(t,x(t),u(t))Px(t)]dt\\
&\leq \int_{t_{k-1}}^{t_k}[h_i^T(t,x(t),u(t))h_i(t,x(t),u(t)) - \gamma^2 u^T(t)u(t) + x^T(t)P\tilde{A}_i x(t) + x^T(t)P\tilde{f}_i(t,x(t),u(t)) + x^T(t)\tilde{A}_i^T Px(t)\\
&\quad + \tilde{f}_i^T(t,x(t),u(t))Px(t) + 2\varepsilon_{\tilde{f}i}L_{\tilde{f}xi}^2 x^T(t)x(t) + 2\varepsilon_{\tilde{f}i}L_{\tilde{f}ui}^2 u^T(t)u(t) - \varepsilon_{\tilde{f}i}\tilde{f}_i^T(t,x(t),u(t))\tilde{f}_i(t,x(t),u(t))\\
&\quad + 2\varepsilon_{hi}L_{hxi}^2 x^T(t)x(t) + 2\varepsilon_{hi}L_{hui}^2 u^T(t)u(t) - \varepsilon_{hi}h_i^T(t,x(t),u(t))h_i(t,x(t),u(t))]dt\\
&= \int_{t_{k-1}}^{t_k}\eta^T(t)\begin{bmatrix}(1,1) & P & 0 & 0\\ * & -\varepsilon_{\tilde{f}i} & 0 & 0\\ * & * & 2\varepsilon_{\tilde{f}i}L_{\tilde{f}ui}^2 + 2\varepsilon_{hi}L_{hui}^2 - \gamma^2 & 0\\ * & * & * & 1-\varepsilon_{hi}\end{bmatrix}\eta(t)dt,
\end{aligned}$$

where $(1,1) = \tilde{A}_i^T P + P\tilde{A}_i + 2\varepsilon_{\tilde{f}i}L_{\tilde{f}xi}^2 + 2\varepsilon_{hi}L_{hxi}^2$ and $\eta(t) = [x^T(t), \tilde{f}_i^T(t,x(t),u^T(t),h_i^T(t)]^T$. From (4.12), (4.13) and $\varepsilon_{hi} > 1$, we can conclude that the matrix above is negative definite. Thus,

$$\int_{t_{k-1}}^{t_k} y^T(t)y(t)dt \leq \gamma^2\int_{t_{k-1}}^{t_k} u^T(t)u(t)dt.$$

This implies

$$\int_{t_0}^{t_k} y^T(t)y(t)dt \leq \gamma^2\int_{t_0}^{t_k} u^T(t)u(t)dt.$$

Due to the global existence of $x(t)$ concluded from ii, we can take the following limit

$$\lim_{t_k\to\infty}\int_{t_0}^{t_k} y^T(t)y(t)dt \leq \lim_{t_k\to\infty}\gamma^2\int_{t_0}^{t_k} u^T(t)u(t)dt.$$

Therefore, for any continuous (as assumed) $u \in L_2[t_0,\infty)$, $y \in L_2[t_0,\infty)$ and $\|y\|_{L_2} \leq \gamma\|u\|_{L_2}$, under the zero initial condition. □

4.4 Numerical Examples

We shall provide two numerical examples to illustrate the stability results for Caputo fractional order switching systems. The numerical implementations are also base on the algorithm proposed in [28].

Example 4.4.1. *Consider the Caputo fractional order switching nonautonomous system consisting of two following subsystems*

$$subsystem\,1:\begin{cases} {}^{C}_{t_{k-1}}D^{\alpha}_{t}x=-x-yt \\ {}^{C}_{t_{k-1}}D^{\alpha}_{t}y=-y+xt \end{cases};\ subsystem\,2:\begin{cases} {}^{C}_{t_{k-1}}D^{\alpha}_{t}x=-x-xy^2t^2 \\ {}^{C}_{t_{k-1}}D^{\alpha}_{t}y=-y+x^2yt^2 \end{cases},$$

and the switching signal $\sigma(t)$ *with* $T_{max}=2$ *and* $T_{min}=0.5$ *shown in Figure 4.1.*

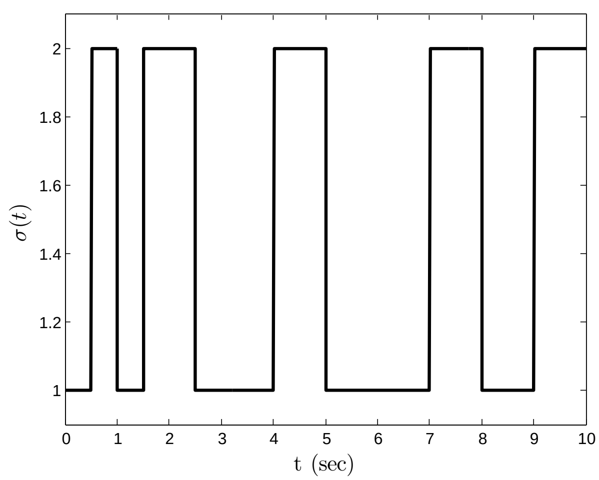


Figure 4.1: Switching signal $\sigma(t)$ with $T_{max}=2$ and $T_{min}=0.5$.

The origin $(0,0)$ is an equilibrium point of the switching system. Let $X=[x,y]^T$, then $f_1(t,X)=[-x-yt,-y+xt]^T$ and $f_2(t,X)=[-x-xy^2t^2,-y+x^2yt^2]^T$. Clearly, $f_1\in C^2([0,\infty)\times\mathbb{R}^2)$ and $f_2\in C^2([0,\infty)\times\mathbb{R}^2)$. Moreover, for any $(t,X)\in[0,\infty)\times\mathbb{R}^n$, $X^Tf_1(t,X)=-x^2-y^2=-X^TX$ and $X^Tf_2(t,X)=-X^TX$. According to Corollary 4.2.1, the equilibrium point is asymptotically stable.

The trajectory of the system with $\alpha=0.5$ and $(x(0),y(0))=(1,-1)$ shown in Figure 4.2 verifies the asymptotic stability.

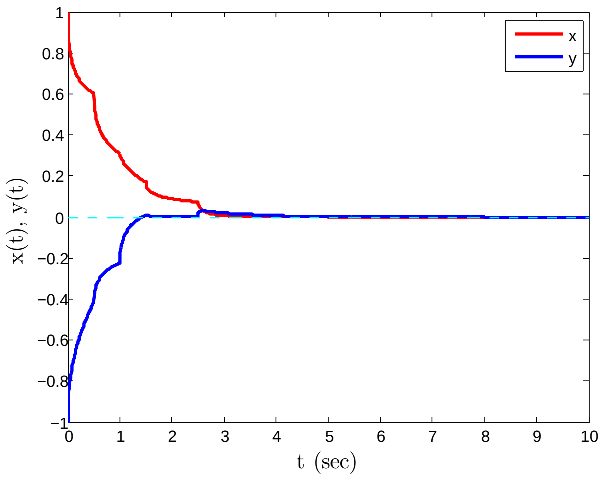


Figure 4.2: States of the Caputo fractional order switching nonautonomous system.

Example 4.4.2. *Consider the Caputo fractional order reset switching control system consisting of two Caputo fractional order simplest dissipative circuits, see [31],*

$$subsystem\,1:\begin{cases} {}^{C}_{t_{k-1}}D^{\alpha}_{t}x = y - g(a,b,x) \\ {}^{C}_{t_{k-1}}D^{\alpha}_{t}y = -\beta(y+x) + F_1u(t) \\ x(t_k) = y(t_k) = 0 \\ h_1(t,x,y) = x+y \end{cases};\ subsystem\,2:\begin{cases} {}^{C}_{t_{k-1}}D^{\alpha}_{t}x = y - g(a,b,x) \\ {}^{C}_{t_{k-1}}D^{\alpha}_{t}y = -\beta(y+x) + F_2u(t) \\ x(t_k) = y(t_k) = 0 \\ h_2(t,x,y) = x+y \end{cases},$$

where a, b, β, F_1 and F_2 are constants and $g(a,b,x) = bx + 0.5(a-b)(|x+1| - |x-1|)$, and the switching signal $\sigma(t)$ with $T_{max} = 1.5$ and $T_{min} = 0.5$ shown in Figure 4.3.

Let $X = [x,y]^T$, then $\bar{f}_1(t,X,u) = \tilde{A}_1X + \tilde{f}_1(t,X,u)$ and $\bar{f}_1(t,X,u) = \tilde{A}_2X + \tilde{f}_2(t,X,u)$, where $\tilde{A}_1 = \tilde{A}_2 = \begin{bmatrix} 0 & 1 \\ -\beta & -\beta \end{bmatrix}$, $\tilde{f}_1(t,X,u) = [g(a,b,x), F_1u]^T$, $\tilde{f}_2(t,X,u) = [g(a,b,x), F_2u]^T$, $h_1(t,X,u) = [1,1]X$ and $h_2(t,X,u) = [1,1]X$. It follows, $L_{\tilde{f}X1} = L_{\tilde{f}X2} = \max\{|a|,|b|\}$, $L_{\tilde{f}u1} = |F_1|$, $L_{\tilde{f}u2} = |F_2|$, $L_{hx1} = L_{hx2} = \sqrt{2}$ and $L_{hu1} = L_{hu2} = 0$. Moreover, $\tilde{f}_1(t,0,0) = \tilde{f}_2(t,0,0) = 0$ and $h_1(t,0,0) = h_2(t,0,0) = 0$.

For simulation, we select the circuit parameters: $\alpha = 0.8, \beta = 5$, $a = -1.27$, $b = -0.68$, $F_1 = 0.2$ and $F_2 = 0.25$; prescribed constant: $\gamma = 0.04$; arbitrary constants: $\varepsilon_{\tilde{f}1} = \varepsilon_{\tilde{f}2} = 0.01$ and $\varepsilon_{h1} = \varepsilon_{h2} = 1.01$; input: $u(t) = \sin(2\pi t/3)\times [H(t) - H(t-3)]$, where H is the Heaviside function. As required, $u \in C[0,\infty) \cap L_2[0,\infty)$. Using the Matlab LMI tool box, we find that there exists a positive definite matrix

$$P = \begin{bmatrix} 0.0246 & 0.0125 \\ 0.0125 & 0.0282 \end{bmatrix},$$

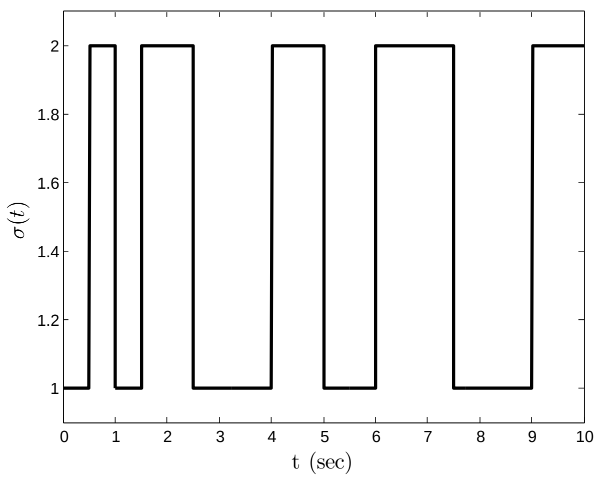


Figure 4.3: Switching signal $\sigma(t)$ with $T_{max} = 1.5$ and $T_{min} = 0.5$.

such that both (4.12) and (4.13) hold. According to Theorem 4.3.1, the reset switching control system here is externally stable.

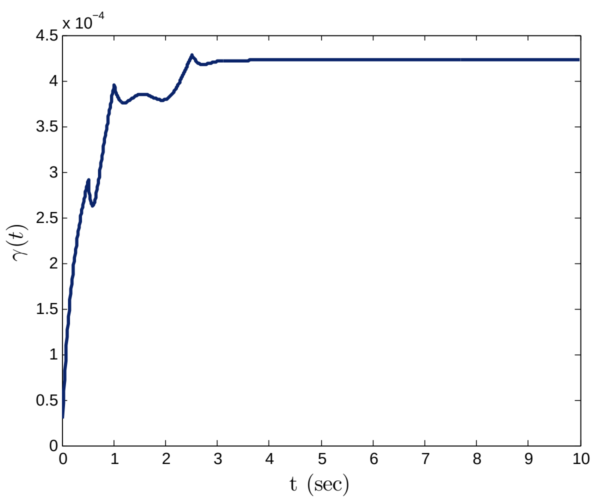


Figure 4.4: L_2 gain of the Caputo fractional order switching nonautonomous system.

As shown in Figure 4.4, $\gamma(t) = [\int_0^t y^2(t)dt]^{1/2}/[\int_0^t u^2(t)dt]^{1/2}$, under the zero initial condition, is really less than the prescribed $\gamma = 0.04$. This validates the theoretical conclusion.

Chapter 5

Conclusion and Future Research

The studies of three main aspects centering on the stability of Caputo fractional order systems have been presented in this thesis. They are the frequency-domain designs, fundamental theory and stability analysis.

The frequency-domain designs based on the BIBO stability of Caputo fractional order linear control systems, including the fractional-version pole placement, internal model principle and model matching, have been developed in Chapter 2, where fractional order polynomials, and their root distribution, coprimeness, properness and $\rho - \kappa$ polynomials, as prerequisites of the designs, have been defined and explored. However, our designs are only for SISO systems. We may intuitively think of to extend the present results to multiple-input-multiple-output (MIMO) systems. Moreover, the linear system designs, in fact, are not limited in the frequency domain or in the classical control theory. That is for why authors developed the modern control theory, by which system properties such as controllability and observability are used for state-space designs (designs based on state-space equations). Thus, another potential direction is to develop the fractional-version modern control theory. Recently, there have been some preliminary results on the fractional-version controllability and observability, see [3].

As for the fundamental theory, we have generalized the existing results on existence and uniqueness of solutions from Caputo fractional order scalar differential equations with zero initial time to Caputo fractional order systems with arbitrary initial time, and have developed the continuation and smoothness of solutions to Caputo fractional order nonlinear systems. The continuation enables us get rid of assumptions for global existence of solutions and the smoothness suffices to yield simple estimations for Caputo fractional order derivatives of quadratic Lyapunov functions, which have established the foundation of our research on stability, especially on Lyapunov stability. One point here deserving attentions is that our global smoothness results are built on those assumptions holding for the whole state space $\mathbb{R}^n$. This might be reduced to only the domains of vector field functions in the future. If done, it would complement the fractional-version fundamental theory in the part of smoothness.

It is also a significant research direction for Lyapunov stability analysis, since the reduction would further complete the fractional-version Lyapunov direct method to some extent. Specifically, the Lyapunov stability results of this thesis on Caputo fractional order nonlinear and hybrid systems could be generalized and localized. When coming to the Lyapunov direct method, we may consider to develop the estimations for the Caputo fractional order derivatives of other positive definite Lyapunov function candidates more than the quadratic ones so that we could use various Lyapunov function candidates for stability analysis. Besides, we may consider the Lyapunov stability of those systems with fractional orders larger than one. Except for the Lyapunov stability, we have also studied the external stability in the thesis, for which the equivalence between Caputo fractional order (nonlinear and hybrid) control systems and their diffusive realizations has been proven and the Lyapunov-like functions based on the diffusive realizations have been also well investigated. As first introduced in Subsection 3.5.1, the diffusive realizations have continuous frequency from zero to infinity. A potential direction here is that the diffusive realizations might be discretized in terms of frequency, which would create new ideas for the numerical approximation of Caputo fractional order control systems. Finally, it would be also meaningful to develop similar realizations and Lyapunov-like functions for those control systems with larger-than-one fractional orders, then their external stability problems would be solvable.

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