

Sensitivity Analysis of Optimal Control Motivated by Model Predictive Control*

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Abstract—We analyze the sensitivity of the extremal equations that arise when concluding first order optimality conditions for time dependent optimization problems. More specifically, we consider parabolic PDEs with a linear quadratic performance criterion. We prove the solutions boundedness with respect to the right-hand side of the first order optimality condition which includes initial data. As a consequence, it can be shown that the influence of a perturbation at a certain time decays exponentially in the temporal distance to the time of perturbation. Moreover, a quantitative turnpike theorem can be derived.

Index Terms—Sensitivity Analysis, Turnpike Property, Model Predictive Control

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I. INTRODUCTION

Model Predictive Control (MPC) is a control method in which the solution of optimal control problems on infinite or indefinitely long horizons is split up into the successive solution of optimal control problems (OCPs) on finite time horizons. Only a first part with given length of each finite horizon solution is implemented as a control for the longer, possibly infinite horizon. This means that in a numerical solution of the optimal control problem only the first part of the optimal control must be computed accurately.

Motivated by this observation, we will study the effect of perturbations and discretization errors near the end of the optimization horizon on the initial part of the control. It will turn out that, considering linear quadratic optimal control problems, their influence decays exponentially in time. Thus, they are indeed negligible if the horizon is long enough.

An in depth introduction to Model Predictive Control can be found in [1]. A central assumption for the approximation of the infinite horizon solution by a solution controlled with an MPC-feedback is that the dynamics of the optimal control problem exhibits turnpike behavior, i.e. the time dependent solution of the optimal control problem remains close to a constant value for the majority of the time [2]. This value is called the turnpike of the OCP. A quantitative turnpike result was proven in [3] for nonlinear finite dimensional problems. This was extended to a general Hilbert space setting in [4], where the assumptions made to show turnpike behavior are stabilizability and detectability of the dynamics. The interested reader is also referred to [5], [6].

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As depicted in Fig. 1, one might conjecture that for systems showing turnpike behaviour, perturbations at the end of the trajectory do not affect the initial part if T is large.

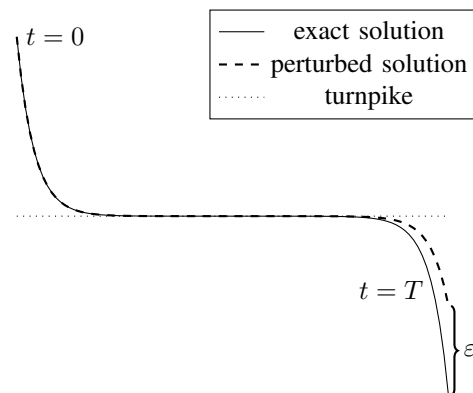


Fig. 1. Turnpike behavior of the OCP's solution and possible perturbation ε at the end of the optimization horizon.

In this paper, we will show that this property can be proven rigorously for linear-quadratic optimal control problems with stabilizable and detectable dynamics.

II. PRELIMINARIES AND SETTING

Let $(V, \|\cdot\|_V)$ be a separable and reflexive Banach space and $[0, T]$ a bounded proper interval on \mathbb{R} . By $L_2(0, T; V)$ we denote the space of Bochner integrable functions $f :]0, T[\rightarrow V$, see [7] and [8, Section 23.2ff] for basic results on these spaces. It holds that $L_2(0, T; V)^* \cong L_2(0, T; V^*)$.

Let H be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle_H$ and $V \hookrightarrow H$ continuously and densely. The spaces form a so called Gelfand triple (or evolution triple) $V \hookrightarrow H \hookrightarrow V^*$. Moreover, by $W[0, T]$ we denote the space of $L_2(0, T; V)$ functions with weak derivative in $L_2(0, T; V^*)$.

We will consider the parabolic PDE in weak form. For this, we define the operator

$$D_{[0, T]} : W([0, T]) \rightarrow L_2(0, T; V)^* \times H^*,$$

where

$$(D_{[0, T]} w)(v, v_0) := \int_0^T w'(t)(v(t)) dt + \langle w(0), v_0 \rangle_H,$$

which constitutes a time derivative operator with initial value testing. Additionally, we introduce operators

$\Lambda_{[0,T]} : L_2(0, T; V) \rightarrow L_2(0, T; V^*)$ and $\underline{\Lambda} : V \rightarrow V^*$ linear, bounded with

$$(\Lambda_{[0,T]}v)w = \int_0^T (\underline{\Lambda}v)w \, dt,$$

where $\underline{\Lambda}$ could be e.g. the weak form of the Laplace operator. For the influence of the control, we define $B_{[0,T]} : L_2([0, T] \times \Omega) \rightarrow L_2(0, T; H^*)$, $\underline{B} : L_2(\Omega) \rightarrow H^*$ linear, bounded with

$$(B_{[0,T]}u)v = \int_0^T (\underline{B}u)v \, dt.$$

For details on the solution theory of parabolic PDEs formulated as operator equations, see [9]. In this paper, we will study the sensitivity of the first order optimality condition of the optimal control problem

$$\min_{y,u} \frac{1}{2} \|C(y - y_d)\|_{L_2(0,T;H)}^2 + \frac{\alpha}{2} \|R(u - u_d)\|_{L_2([0,T] \times \Omega)}^2 \quad (1)$$

subject to the parabolic PDE

$$D_{[0,T]}y - \Lambda_{[0,T]}y - B_{[0,T]}u = \langle y_0, \cdot \rangle$$

for linear bounded operators $C : L_2(0, T; H) \rightarrow L_2(0, T; H)$ and $R : L_2([0, T] \times \Omega) \rightarrow L_2([0, T] \times \Omega)$ additionally elliptic. If we denote the Riesz isomorphisms of the state space by $M_y : L_2(0, T; H) \rightarrow L_2(0, T; H^*)$, the first order optimality condition is given by the following equation in dual spaces

$$M \begin{pmatrix} y \\ (\lambda, \lambda_0) \end{pmatrix} = \begin{pmatrix} 0 \\ \langle y_0, \cdot \rangle \end{pmatrix} + \begin{pmatrix} C^* M_y C y_d \\ B_{[0,T]} u_d \end{pmatrix}, \quad (2)$$

where

$$M := \begin{pmatrix} C^* M_y C & (D_{[0,T]} - \Lambda_{[0,T]})^* \\ (D_{[0,T]} - \Lambda_{[0,T]}) & -B_{[0,T]} Q^{-1} B_{[0,T]}^* \end{pmatrix}. \quad (3)$$

Here we eliminated the control via $u = Q^{-1} B_{[0,T]}^* \lambda + u_d$, where $Q := \alpha R^* M_u R$ and $M_u : L_2([0, T] \times \Omega) \rightarrow L_2([0, T] \times \Omega)^*$ is the Riesz isomorphism of the control space.

III. ANALYSIS OF THE EXTREMAL EQUATIONS' SOLUTION OPERATOR

Our main analysis consists of proving the bounded invertibility of the linear operator M defined in (3) with constants independent of T . The approach taken is inspired by the stability estimate performed in [4, Lemma 2] and [5, Lemma 3.5]. The main result in this section reads as follows.

Theorem 1: Let $(\underline{\Lambda}, C)$ satisfy an exponential detectability and $(\underline{\Lambda}, \underline{B})$ satisfy an exponential stabilizability assumption. Then there exists a constant $c \geq 0$ independent of T , such that

$$\|M^{-1}\|_{(L_2(0,T;H^*) \times H^*)^2 \rightarrow L_2(0,T;H)^2} \leq c. \quad (4)$$

The constant c is indirectly proportional to the grade of exponential stabilizability and exponential detectability. The

assumptions of this theorem are similar to standard assumptions under which turnpike behavior can be shown in infinite dimension, cf. [4].

One will observe that the range of the adjoint equation was restricted. Originally, the equation for the adjoint variable is posed in $W[0, T]^*$. However, this results in an adjoint state tuple $(\lambda, \lambda_0) \in L_2(0, T; V) \times H$. By restricting the range of the adjoint operator $D_{[0,T]}^*$ to $L_2(0, T; H^*) \times H^*$, one can show that $\lambda \in W([0, T])$ and $\lambda_0 = \lambda(0)$ (see [9, Proposition 3.8]). As a result, we can treat the variables y and λ in the same way. In the remainder of this paper, the two variables (λ, λ_0) will therefore be denoted by one variable λ , as $\lambda_0 = \lambda(0)$.

IV. SENSITIVITY WITH RESPECT TO L_2 - PERTURBATIONS

A. Theoretical considerations

An application of the above analysis is the property that the effect of perturbations decays exponentially in time.

In the following we consider sensitivity of solutions of (2) with respect to perturbations of the right hand side. In addition to the exact solution (y, λ) of (2) we consider a perturbed solution $(\tilde{y}, \tilde{\lambda})$ that solves

$$M \begin{pmatrix} \tilde{y} \\ \tilde{\lambda} \end{pmatrix} = \begin{pmatrix} 0 \\ \langle y_0, \cdot \rangle \end{pmatrix} + \begin{pmatrix} C^* M_y C y_d \\ B_{[0,T]} u_d \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}.$$

The perturbation variables ε_1 and ε_2 model the error performed by e.g. a discretization in time or space. Hence, defining the difference of exact and perturbed solution $(\delta y, \delta \lambda) := (\tilde{y} - y, \tilde{\lambda} - \lambda)$ it follows using linearity that

$$M \begin{pmatrix} \delta y \\ \delta \lambda \end{pmatrix} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}. \quad (5)$$

Theorem 1 directly yields an estimate of $(\delta y, \delta \lambda)$ in terms of $\|(\varepsilon_1, \varepsilon_2)\|_{L_2(0,T;H^*)^2}$. In the following result, we will refine this estimate by introducing an exponential scaling in time.

Theorem 2: Assume $(\delta y, \delta \lambda) \in W([0, T])^2$ solves (5) and let the assumptions of Theorem 1 hold.

Furthermore assume that for $\rho \geq 0$ and $0 \leq \mu < \frac{1}{\|M^{-1}\|}$

$$\begin{aligned} \|e^{-\mu t} \varepsilon_1(t)\|_{L_2(0,T;H^*)} &\leq \rho \\ \|e^{-\mu t} \varepsilon_2(t)\|_{L_2(0,T;H^*)} &\leq \rho. \end{aligned} \quad (6)$$

Then there exists a constant $c_1 \geq 0$ independent of T , such that

$$\|e^{-\mu t} \delta y\|_Y + \|e^{-\mu t} \delta u\|_U + \|e^{-\mu t} \delta \lambda\|_P \leq c_1 \rho,$$

where $Y = P = L_2(0, T; H)$ and $U = L_2([0, T] \times \Omega)$.

For $\mu > 0$, all inequalities involve weighted norms, where the functions are scaled by $e^{-\mu t}$. We interpret this result for the error in the state $\delta y = \tilde{y} - y$: Assume we solve for the state and adjoint with perturbations that are bounded by (6). Then, $\|e^{-\mu t} \delta y\|$ is bounded by a constant independent of T . This implies that at the beginning of the time horizon, i.e. for small t , $\|\delta y(t)\| \simeq \rho$, where ρ is the scaling factor of the perturbations bound in (6). Therefore, only local (in a temporal sense) perturbations play a role and perturbations e.g. at the end of the horizon only slightly affect the initial part, even if they grow exponentially in time.

The proof of Theorem 2 utilizes scaling of the variables by $e^{-\mu t}$ and the identity

$$(D_{[0,T]}\delta y)e^{-\mu t}v = ((\mu I + D_{[0,T]})e^{-\mu t}\delta y)v.$$

This implies that the scaled variables satisfy a modified equation with M replaced by $M + \mu P$, where

$$P = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Since $M + \mu P = (I + \mu PM^{-1})M$, a Neumann-series argument implies invertibility of this operator for sufficiently small μ , depending only on $\|M^{-1}\|$. The smaller $\|M^{-1}\|$, the larger μ can be chosen, the faster the exponential decay.

B. Numerical Results

In this subsection, the statement of Theorem 2 will be illustrated by a numerical example.

We take a look at solutions of the optimization problem

$$\min_{(y,u)} \frac{1}{2} \|y - y_d\|_{L_2([0,T] \times [0,1]^2)}^2 + \frac{1}{4} \|u\|_{L_2([0,T] \times [0,1]^2)}^2$$

subject to the constraint

$$D_{[0,T]}y + 0.1 \int_0^T \int_{[0,1]^2} \nabla y \nabla \cdot d\omega dt - \int_0^T \int_{[0,1]^2} u \cdot d\omega dt = 0$$

for different temporal grids. We choose $V = H_0^1(\Omega)$ for the adjoint and state space and $H = L_2(\Omega)$ for the initial value. The constraint constitutes a linear heat equation with distributed control and zero initial condition and it obviously satisfies the stabilizability and detectability assumption of Theorem 1, as $\nabla \cdot \nabla \cdot$ is elliptic in $H_0^1(\Omega)$ and hence stable. $y_d \in H_0^1(\Omega)$ is a reference temperature which is depicted in Fig. 2.

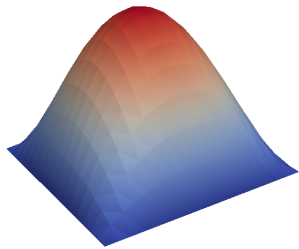


Fig. 2. Plot of the constant (w.r.t. time) reference temperature y_d on $[0, 1]^2$.

We compute two perturbed solutions, the first triple with a uniform grid \mathcal{G}_{uni} consisting of 11 grid points and the second with a grid \mathcal{G}_{exp} with meshsize increasing exponentially in time as depicted in Fig. 3.

To compute the absolute error $\|\delta y(t)\|_{L_2(\Omega)}$, $\|\delta u(t)\|_{L_2(\Omega)}$ and $\|\delta \lambda(t)\|_{L_2(\Omega)}$ as functions of the time t , we will consider a solution triple (y, u, λ) on a fine grid with 101 grid points in $[0, 30]$ as a reference. Fig. 4 shows the norm of the perturbed and exact solution triple's state, control and adjoint. Note that, as stated in the introduction, Model Predictive Control only implements an initial part of the trajectory and the

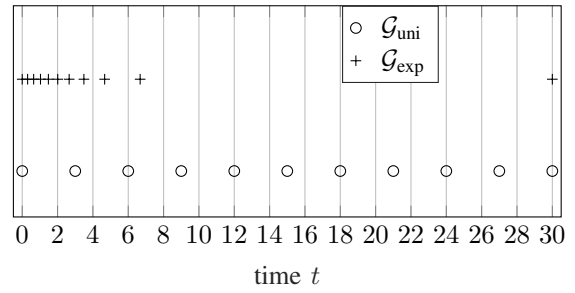


Fig. 3. Grids used for the computations.

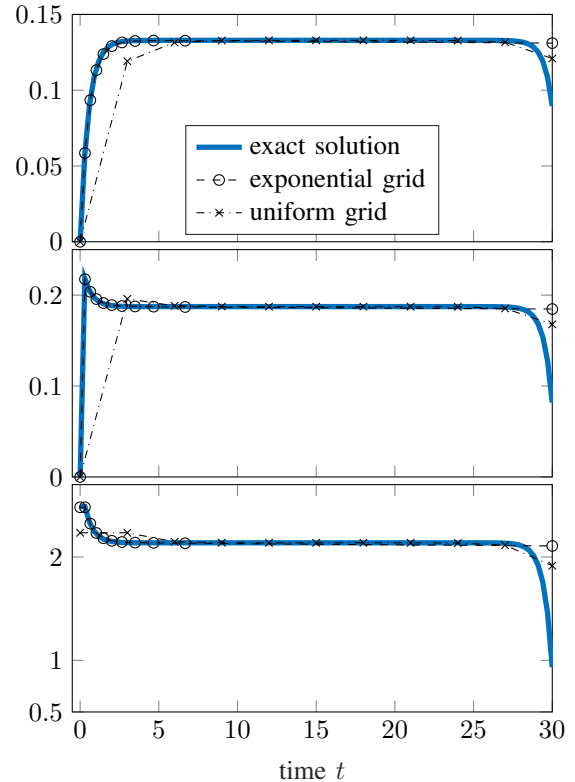


Fig. 4. From top to bottom: Norm of the state $\|y(t)\|_{L_2(\Omega)}$, norm of the control $\|u(t)\|_{L_2(\Omega)}$ and norm of the adjoint $\|\lambda(t)\|_{L_2(\Omega)}$ for every grid point.

remaining part is discarded. Therefore, we are especially interested in this initial part. Additionally, we observe, that the exact solution stays close to a certain quantity. This feature is called turnpike property and will be discussed in Sec. V. Moreover, we note the accuracy of the solution on \mathcal{G}_{exp} in the first part of the horizon, even though it does not follow the so called leaving arc close to the end time.

The absolute error is illustrated in Fig 5. We observe that due to the fine meshsize of the exponential grid \mathcal{G}_{exp} at the beginning of the time interval (9 of 11 grid points are used in $[0, 5]$), the error is lower by several orders of magnitude in comparison to the uniform grid \mathcal{G}_{uni} . For both solutions, we observe an increase of the error when approaching time zero and time $T = 30$. This stems from the faster dynamics

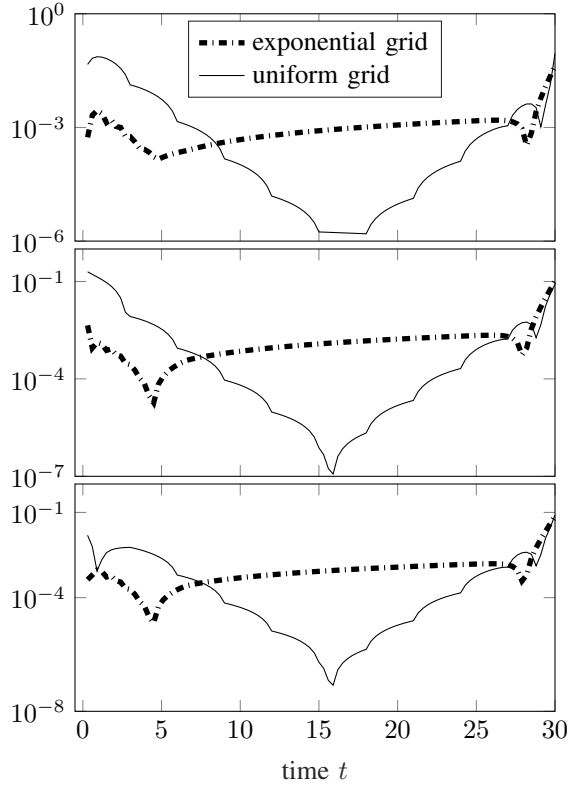


Fig. 5. Absolute error of the perturbed solutions. From top to bottom: $\|\delta y(t)\|_{L_2(\Omega)}$, $\|\delta u(t)\|_{L_2(\Omega)}$ and $\|\delta \lambda(t)\|_{L_2(\Omega)}$.

of the system in this part.

These results motivate the construction of algorithms for adaptive grid generation which is subject to future research. In this example, only the time grid was altered. The spatial grid was kept constant over all timesteps. However, to fully exploit the results of Theorem 2, the spatial grids can follow the same refinement pattern, leading to fine spatial grids on the initial part with meshsize increasing in time. This will lead to a fast solution of the problems arising in applications of Model Predictive Control for PDEs.

V. A QUANTITATIVE TURNPIKE THEOREM

A second application of the results presented in the first part of this paper is a turnpike result for infinite dimensional linear quadratic optimization problems. The proof in the references put forward in the Introduction includes the solution of the Algebraic Riccati Equation. We present a different approach, using only the assumptions of Theorem 1.

We introduce the solution of a steady state optimization problem, namely $(\bar{y}, \bar{u}, \bar{\lambda})$ that solves the operator equation

$$\begin{pmatrix} C^*C & -\underline{\Lambda}^* \\ -\underline{\Lambda} & -\underline{B}Q^{-1}\underline{B}^* \end{pmatrix} \begin{pmatrix} \bar{y} \\ \bar{\lambda} \end{pmatrix} = \begin{pmatrix} C^*C y_d \\ \underline{B}u_d \end{pmatrix},$$

where $Q = \alpha R^*R$ and $\bar{u} = Q^{-1}\underline{B}^*\bar{\lambda} + u_d$. This resembles the first order optimality condition of the optimization

problem

$$\begin{aligned} \min_{\bar{y}, \bar{u}} \frac{1}{2} \|C(\bar{y} - y_d)\|_H^2 + \frac{\alpha}{2} \|R(\bar{u} - u_d)\|_{L_2(\Omega)}^2 \\ \text{s.t.} \quad \underline{\Lambda}\bar{y} - \underline{B}\bar{u} = 0. \end{aligned}$$

Similar to the perturbation result presented in the previous section, we consider the distance of the solution of (2) to this variable: $(\delta y, \delta \lambda) := (y - \bar{y}, \lambda - \bar{\lambda})$. A straightforward computation shows that this variable solves

$$M \begin{pmatrix} \delta y \\ \delta \lambda \end{pmatrix} = \begin{pmatrix} -\langle \bar{\lambda}, \cdot \rangle \\ \langle y_0 - \bar{y}, \cdot \rangle \end{pmatrix}. \quad (7)$$

We conclude a result similar to Theorem 2.

Theorem 3: Assume $(\delta y, \delta \lambda)$ solves (7) and let the assumptions of Theorem 1 hold. Then, there exists $\mu > 0$ and $c_2 \geq 0$ independent of T , such that

$$\begin{aligned} \left\| \frac{1}{e^{-\mu t} + e^{-\mu(T-t)}} \delta y \right\|_Y + \left\| \frac{1}{e^{-\mu t} + e^{-\mu(T-t)}} \delta u \right\|_U \\ + \left\| \frac{1}{e^{-\mu t} + e^{-\mu(T-t)}} \delta \lambda \right\|_P \leq c_2 (\|y_0 - \bar{y}\|_H + \|\bar{\lambda}\|_H), \end{aligned} \quad (8)$$

where $Y = P = L_2(0, T; H)$ and $U = L_2([0, T] \times \Omega)$.

The scaling factor's denominator is near zero for times t in the middle of the optimization horizon. Therefore, to fulfill inequality (8), the solution triple (y, u, λ) has to be close to the solution $(\bar{y}, \bar{u}, \bar{\lambda})$ of the static problem in the middle part of the optimization horizon. As the constants involved are independent of T , this resembles a turnpike property. Fig. 4 shows this effect for the three variables. Although the variables are scaled by a factor different from that in Theorem 2, the proof is very similar.

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